

1. Term Structure

(a) Spot rate: yield rate on an zero-coupon bond.

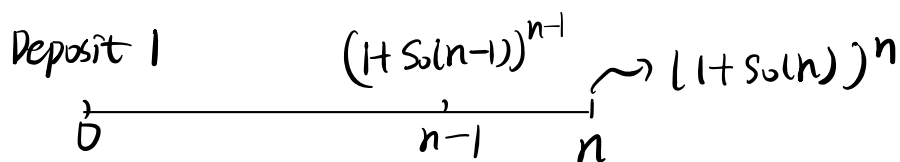
$S_0(t)$: spot rate at time t

C_t : cash flows at time t

$$PV_{CF} = \frac{C_t}{(1+S_0(t))^t}$$

(b) Forward rate of interest

$\bar{z}(n-1, n)$: effective annual interest rate
from $n-1$ to n .



$$1 + \bar{z}(n-1, n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}}$$

Worksheet P5 (b):

Term structure:

$$S_0(1) = S_0(2) = \dots = S_0(n-1) < S_0(n)$$

n -yr bond: coupon rate $r > 0$, yield rate \bar{j}

Prove that $S_0(n-1) < \bar{j} < S_0(n)$

$$\text{Pf: } P = \sum_{k=0}^n \frac{Fr}{(1+s_0(k))^k} + \frac{c}{(1+s_0(n))^n}$$

$$= \sum_{k=0}^{n-1} \frac{Fr}{(1+s_0(n-1))^k} + \frac{c+Fr}{(1+s_0(n))^n}$$

3 eq/ineq:

$$\textcircled{1} P < \sum_{k=0}^{n-1} \frac{Fr}{(1+s_0(n-1))^k} + \frac{c+Fr}{(1+s_0(n-1))^k}$$

$$\textcircled{2} P = \sum_{k=0}^{n-1} \frac{Fr}{(1+j)^k} + \frac{c+Fr}{(1+j)^k}$$

$$\textcircled{3} P > \sum_{k=0}^{n-1} \frac{Fr}{(1+s_0(n))^k} + \frac{c+Fr}{(1+s_0(n))^n}$$

$$\textcircled{1}-\textcircled{3} \Rightarrow s_0(n-1) < j < s_0(n)$$

Worksheet P6 a&d:

$$a. \hat{v}_0(k-1, k) = \frac{(1+s_0(k))^k}{(1+s_0(k-1))^{k-1}} - 1$$

$$d. \text{ Given } s_0(k-1) < s_0(k)$$

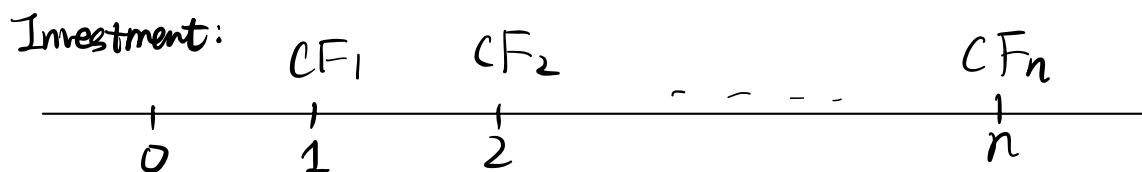
$$(1+s_0(k))^k \geq (1+s_0(k-1))^{k-1} (1+\hat{v}_0(k-1, k))$$

$$< (1+s_0(k))^{k-1} (1+\hat{v}_0(k-1, k))$$

$$\Rightarrow (1+s_0(k))^1 < 1+\hat{v}_0(k-1, k)$$

$$s_0(k) < \hat{v}_0(k-1, k)$$

2. Duration



$$P = \sum_{k=1}^n CF_k \cdot v^k$$

(a) Macaulay Duration: weight-average duration
at which the investment paid.))
"time"

$$D = \sum_{k=1}^n k \cdot w_k$$

$$\text{weight } w_k = \frac{v^k CF_k}{P}$$

(b) Modified Duration:

$$DM = - \frac{dP/di}{P}$$

Rmk: $D = (1+i) \cdot DM$

Proof: $P = \sum_{k=1}^n CF_k \cdot (1+i)^{-k}$

$$\frac{dP}{di} = \sum_{k=1}^n (-k) CF_k (1+i)^{-k-1}$$

$$= -v \sum_{k=1}^n k CF_k \cdot v^k$$

$$\frac{dP/di}{P} = -v \frac{\sum_{k=1}^n k \cdot CF_k \cdot v^k}{P}$$

$$\underbrace{-(1+\bar{i}) \frac{dP/di}{P}}_{(1+\bar{i}) DM} = \underbrace{\frac{\sum_{k=1}^n k \cdot CF_k \cdot v^k}{P}}_D$$

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e.g. 12/31/1991: an investment will pay \$X in total w/o interest.

1992: accumulate percentages paid 60%

1993: \ \ \ \ \ \ \ \ \ \ 90%

1994: \ \ \ \ \ \ \ \ \ \ 100%

$\bar{i} = 8\%$, PMT: made in the middle of yrs.

$$PV = (.6)X v^{.5} + (.3)X v^{1.5} + (.1)X v^{2.5}$$

$$= (.895856)X$$

$$D = \frac{1}{PV} \left\{ (.5)(.6)X v^{.5} + (1.5)(.3)X v^{1.5} + (2.5)(.1)X v^{2.5} \right\}$$

$$= \frac{.927140}{.895856}$$

$$DM = \frac{D}{1+\bar{i}} = .966$$

3. Immunization

Assets: Cash inflows at some specific time

Liability: What the company owes / needs to pay
at some specific time

Def (Redington Immunization):

A_t : assets cashflows at time t

L_t : liability cashflows at time t

$$PV_A(i) = \sum_{k=1}^n A_k v^k \quad PV_L(i) = \sum_{k=1}^n L_k v^k$$

Valuation rate $= i_0$

Then, three criteria:

$$(1) PV_A(i_0) = PV_L(i_0) \Rightarrow \text{Asset-Liability Match}$$

$$(2) \frac{d}{di} PV_A(i_0) = \frac{d}{di} PV_L(i_0) \Rightarrow \text{Duration Matching}$$

$$(3) \frac{d^2}{di^2} PV_A(i_0) > \frac{d^2}{di^2} PV_L(i_0) \Rightarrow \text{Greater Convexity for Assets}$$

Facts: ① If $PV_A(i) = PV_L(i)$ (Eqn (1) holds), then

$$PV'_A(i) = PV'_L(i) \Leftrightarrow \sum_k k A_k v^k = \sum_k k L_k v^k$$

$$\underline{\text{Pf}}: PV'_A(i) = PV'_L(i) \Leftrightarrow DM_A = DM_B \quad (DM = -\frac{dP/di}{P})$$

$$\Leftrightarrow D_A = D_B \quad (D = (1+i) DM)$$

$$\Leftrightarrow \sum_k k A_k v^k = \sum_k k L_k v^k \quad (\text{Numerator of } D)$$

② If (1) and (2) are satisfied, then

$$\frac{d^2}{di^2} PV_A(i) > \frac{d^2}{di^2} PV_L(i) \Leftrightarrow \sum k^2 A_k v^k > \sum k^2 L_k v^k$$

Pf: (exercises)

Idea for Redington Immunization:

When the interest rate changes, we need to avoid the situation in which the liability due exceeds the value of asset cash inflows, resulting in the negative surplus.

Worksheet P8:

Liability PMTs \$100 in yr 2, 4, 6

Asset cashflows A_1 (yr1) and A_5 (yr5)

yield rate $\bar{i} = 10\%$.

Goal: have asset CFs immunize liability CFs

a. Find A_1 & A_5 .

Soln: $PV_A(i) = A_1 v + A_5 v^5$

$$PV_L(i) = 100 v^2 + 100 v^4 + 100 v^6$$

(1) Asset-Liability Matching: $PV_A(.1) = PV_L(.1)$

$$\frac{A_1}{1.1} + \frac{A_5}{1.1^5} = \frac{100}{1.1^2} + \frac{100}{1.1^4} + \frac{100}{1.1^6}$$

(2) Duration Matching:

$$\frac{1 A_1}{1.1} + \frac{5 A_5}{1.1^5} = 2 \frac{100}{1.1^2} + 4 \frac{100}{1.1^4} + 6 \frac{100}{1.1^6}$$

$$\text{Solve for } \begin{cases} A_1 = 71.44 \\ A_5 = 229.41 \end{cases}$$

b. Determine whether or not the condn'ts for Redington immunization are satisfied?

Yes. Since the equations (1) & (2) are satisfied,

we only need to check (3):

$$\text{Asset: } A_1 v + 5^2 A_5 v^5 = 3626.0845$$

$$\text{Liability: } 100 (2^2 v^2 + 4^2 v^4 + 6^2 v^6)$$

$$= 3455.5062$$

So we have

$$\sum k^2 A_k v^k > \sum k^2 L_k v^k$$