Section 1 : Theory of Interest

Interest : time value of money

$$\begin{cases} P: \text{ principal (the amount you deposit)} \\ i: \text{ interest rate } \left(\dfrac{\text{interest}}{P}\right) \end{cases}$$

2 types $\begin{cases} \text{simple interest : only principal can generate interest} \\ \text{compounded interest : both principal and interest will accure interest} \end{cases}$

e.g. $P = \$100$, $i = 6\%$, year 2

Simple : $P(1+2i) = 100(1+2\times 6\%) = \$112$

Compound : $P(1+i)(1+i) = 100(1.06)^2 > \$112$

$\Downarrow$

Future Value

e.g. $i = 6\%$, How much P you need to get $\$100$ in year 2?

Simple : $P(1+2i) = \$100 \Rightarrow P = \dfrac{100}{1.12}$

Compound : $P(1+i)^2 = \$100 \Rightarrow P = \dfrac{100}{(1.06)^2}$

$\Downarrow$

Present Value.

Now, we focus on compounded interest.

— Deposit P, interest rate i, year n

Future value : $F = P(1+i)^n$

— Future value F, interest rate i, year n

Present value : $P = \frac{F}{(1+i)^n} = F\nu^n$

$\nu = \frac{1}{1+i}$ — discount factor

Discount Rate

You deposit $\underbrace{\$94.34}_{PV}$ now and get $\underbrace{\$100}_{FV}$ after 1yr.

$\$100 = \$94.34(1+i)$ $\quad i = 6\%$ — interest rate

$\$94.34 = \$100(1-d)$ $\quad d = 5.66\%$ — discount rate.

Remark : • interest rate i is used to get FV given PV, but ν (discount factor) and d (discount rate) are used to get PV given FV.

• $\nu = \frac{1}{1+i}$, $d = 1 - \frac{1}{1+i} = 1 - \nu$

$$= \frac{i}{1+i} = i\nu$$

Nominal and effective rate of Interest

$a(t) :=$ the amount of money at time t given an initial investment of \$1.

e.g. 1 : nominal interest rate $i^{(2)} = 10\%$, compound semi-annually (twice a year)

0	1/2	1
1	$1+\frac{10\%}{2}$	$\left(1+\frac{10\%}{2}\right)^2$

$$a(1) = (1+.05)^2$$

e.g. 2 : nominal interest rate $i^{(4)} = 10\%$, compound quarterly (four times a year)

0	1/4	1/2	3/4	1
1	$1+\frac{10\%}{4}$	$\left(1+\frac{10\%}{4}\right)^2$	$\left(1+\frac{10\%}{4}\right)^3$	$\left(1+\frac{10\%}{4}\right)^4$

$$a(1) = (1+.025)^4$$

Def : The effective interest rate i_{t+1} over $[t, t+1]$ is defined as

$$i_{t+1} = \frac{\overbrace{a(t+1) - a(t)}^{\text{amount earned}}}{\underbrace{a(t)}_{\text{beginning amount}}}$$

e.g. 1 : $i_1 = \frac{a(1)-a(0)}{a(0)} = (1+.05)^2 - 1 = 10.25\%$

e.g. 2 : $i_1 = (1+.025)^4 - 1 = 10.3813\%$

General case : $i_1 = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$

$$\begin{cases} i^{(m)}: \text{ nominal interest rate} \\ m: \text{ \# of times I compound each year} \end{cases}$$

Similarly, d: effective discount rate

$d^{(m)}$: nominal discount rate

$$1-d = \left(1-\frac{d^{(m)}}{m}\right)^m \Rightarrow d = 1-\left(1-\frac{d^{(m)}}{m}\right)^m$$

Exercise: Find the nominal discount rate $d^{(2)}$ equivalent to nominal interest rate $i^{(12)} = 8\%$.

Soln: Step 1 $(i^{(12)} \to i)$:

$$i = \left(1+\frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1+\frac{.08}{12}\right)^{12} - 1$$

$$= 8.3\%$$

Step 2 $(i \to d)$:

$$d = \frac{i}{1+i} = 7.66\%$$

Step 3 $(d \to d^{(2)})$:

$$1-d = \left(1-\frac{d^{(2)}}{2}\right)^2 \Rightarrow d^{(2)} = \left(1-\sqrt{1-d}\right) \times 2$$

$$= 7.8166\%$$

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Section 2:

Review of lec/sec 1:

2 ways to generate interest $\begin{cases} \text{simple} \\ \text{compound (☆)} \end{cases}$

For compound interest:

i — effective interest rate

d — effective discount rate

v — discount factor

For one year period: $FV = PV(1+i)$

or $PV = FV \cdot v = FV \cdot (1-d)$

$$v = \frac{1}{1+i}, \quad d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i} = iv$$

$i^{(m)} / d^{(m)}$ — nominal interest/discount rate

m — number of time for compounding interest

$$1+i = \left(1+\frac{i^{(m)}}{m}\right)^m, \quad 1-d = \left(1-\frac{d^{(m)}}{m}\right)^m$$

Problem 16: invest \$1000, at least \$1500 after 4 yrs,

Q: What is the promising effective rate?

Soln: $PV = 1000$, $FV = 1500$, $n = 4$

$$1500 = 1000(1+i)^4 \Rightarrow i = 10.67\%$$

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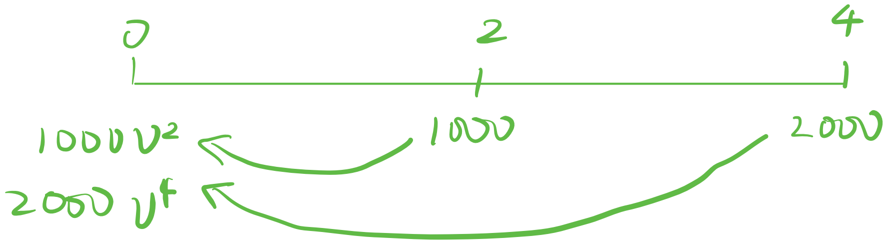
Problem 18: You deposit money, $i = 5\%$, withdraw \$1000 in 2 yrs and \$2000 in 4 yrs.

FV (yr 2)

FV (yr 4)

Q: How much should you deposit?

Soln: $v = \frac{1}{1+i} = \frac{1}{1.05}$; Goal: FV $\rightarrow$ PV



$$PV = 1000v^2 + 2000v^4$$

Continuous Interest

Useful result: If $a_n \to a$ as $n \to \infty$, then

$$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = e^a$$

Fix effective interest rate i

- $m=2$ (semi-annually): $1+i = \left(1+\frac{i^{(2)}}{2}\right)^2$
- $m=4$ (quarterly): $1+i = \left(1+\frac{i^{(4)}}{4}\right)^4$
- $m=12$ (monthly): $1+i = \left(1+\frac{i^{(12)}}{12}\right)^{12}$

Q: What happen when $m \to \infty$?

Def: The force of interest $\delta \overset{def}{=} \lim_{m\to\infty} i^{(m)}$

Since $i^{(m)} \to \delta$ as $m \to \infty$ and effective rate i is fixed,

$$1+i = \lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^m = e^{\delta} \quad (\delta = \ln(1+i))$$

$$a(t) = (1+i)^t = e^{\delta t}$$

Moreover, $\delta = \dfrac{\delta e^{\delta t}}{e^{\delta t}} = \dfrac{a'(t)}{a(t)}$

Therefore, δ can also be interpreted as proportional rate of change of the accumulation func

In general, the effective interest rate i is not necessarily unchanged and depends on time t, which implies that the force of interest δ is a function of t.

i.e.
$$\begin{cases} \delta(t) = \dfrac{a'(t)}{a(t)} \\ a(0) = 1 \end{cases}$$

Soln to IVP : $a(t) = \exp\left\{\int_0^t \delta(s)\,ds\right\}$

Problem 26:

(Varying) force of interest $\delta(t) = \dfrac{2t}{1+t^2}$, make a deposit in yr 2. Q: How many yrs we need to double the fund?

Soln: $a(2) = 1$ — deposit starts in yr 2.

$$a(t) = a(2)\exp\left\{\int_2^t \frac{2s}{1+s^2}\,ds\right\}$$

$$= a(2)\exp\left\{\ln(s^2+1)\Big|_{s=2}^{s=t}\right\}$$

$$= a(2)\exp\{\ln(t^2+1) - \ln 5\}$$

$$= a(2)\cdot\frac{t^2+1}{5}$$

choose t s.t. $a(t) = 2a(2) = 2 \Rightarrow t = 3$

Thus, we need $t - 2 = 1$ more year to double the deposit. //

Sequence and Series

1. Arithmetic Progression (simple interest):

formula: $1+2+\cdots+n = \frac{n(n+1)}{2}$

Pf: $\begin{aligned} S &= 1 + 2 + \cdots + n \\ +\ S &= n + (n-1) + \cdots + 1 \\ \hline 2S &= (n+1) + (n+1) + \cdots + (n+1) = n(n+1) \end{aligned}$

so $S = \frac{n(n+1)}{2}$ //

In general, sequence: $a_1, a_2, a_3, \cdots$ with

$a_k = P + (k-1)Q$

Partial sum:

$$\begin{aligned} S_n := {} & a_1 + a_2 + \cdots + a_n \\ = {} & P + (P+Q) + \cdots + (P+(n-1)Q) \\ = {} & nP + \{1+2+\cdots+(n-1)\}Q = nP + \frac{n(n-1)}{2}Q \end{aligned}$$

2. Geometric Progression (compound interest):

formula: $1 + v + \cdots + v^n = \frac{1-v^{n+1}}{1-v}$

Pf: $\begin{aligned} T &= 1 + v + \cdots + v^n \\ -vT &= v + \cdots + v^n + v^{n+1} \\ \hline (1-v)T &= 1 + 0 + \cdots + 0 - v^{n+1} \end{aligned}$

so $T = \frac{1-v^{n+1}}{1-v}$ //

In general, sequence: $a_k = cv^{k-1}$

Partial sum:

$$T_n = a_1 + a_2 + \cdots + a_n$$

$$= c + cv + \cdots + cv^{n-1} = c\,\frac{1-v^n}{1-v}$$