3. Immunization

Assets: Cash inflows at some specific time

Liability: What the company owes / needs to pay at some specific time

<u>Def</u> (Redington Immunization):

A_t : assets cashflows at time t

L_t : liability cashflows at time t

$$PV_A(i) = \sum_{k=1}^{n} A_k v^k \qquad PV_L(i) = \sum_{k=1}^{n} L_k v^k$$

Valuation rate $= i_0$

Then, three criteria:

(1) $PV_A(i_0) = PV_L(i_0) \Rightarrow$ Asset - Liability Match

(2) $\frac{d}{di} PV_A(i_0) = \frac{d}{di} PV_L(i_0) \Rightarrow$ Duration Matching

(3) $\frac{d^2}{di^2} PV_A(i_0) > \frac{d^2}{di^2} PV_L(i_0) \Rightarrow$ Greater Convexity for Assets

<u>Facts</u>: ① If $PV_A(i) = PV_L(i)$ (Eqn (1) holds), then

$$PV_A'(i) = PV_L'(i) \Leftrightarrow \sum_k k A_k v^k = \sum_k k L_k v^k$$

<u>Pf</u>: $PV_A'(i) = PV_L'(i) \Leftrightarrow DM_A = DM_B \quad \left(DM = -\frac{dP/di}{P} \right)$

$\Leftrightarrow D_A = D_B \quad (D = (1+i) DM)$

$\Leftrightarrow \sum_k k A_k v^k = \sum_k k L_k v^k$ (Numerator of D)

② If (1) and (2) are satisfied, then

$$\frac{d^2}{di^2} PV_A(i) > \frac{d^2}{di^2} PV_L(i) \iff \sum k^2 A_k v^k > \sum k^2 L_k v^k$$

Pf: (exercises)

Idea for Redington Immunization:

When the interest rate changes, we need to avoid the situation in which the liability due exceeds the value of asset cash inflows, resulting in the negative surplus.

Worksheet P8:

Liability PMTs \$100 in yr 2, 4, 6

Asset cashflows A_1 (yr 1) and A_5 (yr 5)

yield rate $i = 10\%$.

Goal: have asset CFs immunize liability CFs

a. Find A_1 & A_5.

Soln: $PV_A(i) = A_1 v + A_5 v^5$

$PV_L(i) = 100v^2 + 100v^4 + 100v^6$

(1) Asset-Liability Matching: $PV_A(.1) = PV_L(.1)$

$$\frac{A_1}{1.1} + \frac{A_5}{1.1^5} = \frac{100}{1.1^2} + \frac{100}{1.1^4} + \frac{100}{1.1^6}$$

(2) Duration Matching:

$$\frac{1\,A_1}{1.1} + \frac{5A_5}{1.1^5} = 2\frac{100}{1.1^2} + 4\frac{100}{1.1^4} + 6\frac{100}{1.1^6}$$

Solve for $\begin{cases} A_1 = 71.44 \\ A_5 = 229.41 \end{cases}$

b. Determine whether or not the condn's for Redington immunization are satisfied?

Yes. Since the equations (1) & (2) are satisfied, we only need to check (3):

Asset: $A_1 v + 5^2 A_5 v^5 = 3626.0845$

Liability: $100\left(2^2 v^2 + 4^2 v^4 + 6^2 v^6\right)$

$= 3455.5062$

So we have

$$\sum k^2 A_k v^k > \sum k^2 L_k v^k$$

4. Equity

Dividend Discount Model: Perpetuity Immediate

You buy a stock at time 0,

d_t: dividend at time t

i: long-term effective annual rate

Price $P = \sum_{t=1}^{\infty} \frac{d_t}{(1+i)^t}$

ex. 1: You buy a stock which pays dividends at the end of each yr perpectually.

Assumptions: $\begin{cases} 1. \text{ Divident increases by } g\% \text{ each yr} \\ 2. \text{ effective rate } i\% \; (i>g) \end{cases}$

The first dividend PMT $d_1 = 1$, Calculate P.

Soln: $P = \sum_{k=1}^{\infty} \frac{d_k}{(1+i\%)^k}$ $\quad d_k = (1+g\%)^{k-1}$

$$= \sum_{k=1}^{\infty} \frac{(1+g\%)^{k-1}}{(1+i\%)^k} = \frac{1}{1+i} \sum_{k=1}^{\infty} \left(\frac{1+g\%}{1+i\%}\right)^{k-1}$$

$$= \frac{1}{(1+i\%)\left(1 - \frac{1+g\%}{1+i\%}\right)} \qquad \left(\sum_{k=1}^{\infty} p^{k-1} = \frac{1}{1-p}, \; |p|<1 \right)$$

$$= \frac{1}{i\% - g\%}$$ (You need to memorize this) //

ex 2: Continuing from ex 1, assume that $i\% = 5\%$, $g\% = 2\%$. Calculate the duration of the stock.

Soln: $D \cdot P = \sum_{k=1}^{\infty} k \cdot d_k v^k$

$$= \sum_{k=1}^{\infty} k (1.02)^{k-1} (1.05)^{-k}$$

$$= \frac{1}{1.02} \boxed{\sum_{k=1}^{\infty} k r^k}$$

increasing annuity w. $n = \infty$ and discount factor $r = 1.02/1.05$

$$= \frac{1}{1.02} \lim_{n \to \infty} (Ia)_{\overline{n}|j} \quad \text{where } j = \frac{1}{r} - 1 = \frac{1}{34}$$

$$= \frac{1}{1.02} \lim_{n \to \infty} \frac{\ddot{a}_{\overline{n}|j} - n r^n}{j} \qquad n r^n \xrightarrow{n \to \infty} 0$$

$$= \frac{34}{1.02} \cdot \ddot{a}_{\overline{\infty}|j} = \frac{34}{1.02} \cdot \frac{1+j}{j} = \frac{3500}{3}$$

$$P = \frac{1}{i\% - g\%} = \frac{1}{.05 - .02} = \frac{100}{3}$$

$$\Rightarrow D = \frac{DP}{P} = \frac{3500/3}{100/3} = 35 \qquad //$$

ex 3. The duration of a perpetuity-due w. level PMT is 25, calculate effective rate i.

Soln: Assume $PMT = 1$

Numerator of Duration:

$$0 + v + 2v^2 + 3v^3 + \cdots = \sum_{k=1}^{\infty} k v^k = (Ia)_{\overline{\infty}|i} = \frac{\ddot{a}_{\overline{\infty}|i}}{i} = \frac{1+i}{i^2}$$

Denominator of Duration: $\ddot{a}_{\overline{\infty}|\bar{i}} = \dfrac{1+\bar{i}}{\bar{i}}$

Duration: $\dfrac{(1+\bar{i})/\bar{i}^2}{(1+\bar{i})/\bar{i}} = \dfrac{1}{\bar{i}} = 25 \Rightarrow \bar{i} = 4\%$ //