# PSTAT 171 Week 3 Section 1

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## Problem 1

Sylvia is repaying a loan of $X$ by making 17 annual payments of 100. Each payment consists of interest on the loan at 5% effective and an amount that is deposit in a sinking fund earning 4% effective. At the end of 17 years, the amount in the sinking fund is equal to the original loan amount accumulated with interest at the sinking fund rate. Determine $X$.

# Notes

payment: \$100 — interest $5\% \cdot X$; sinking fund $100-(.05)X$

sinking fund $\Downarrow$ $4\%$

FV of sinking fund : $\left(100-(.05)X\right)\cdot s_{\overline{17}|.04}$

$\shortparallel$ $\qquad\qquad$ $\shortparallel$

original loan X accumulated w. 4% : $X(1.04)^{17}$

# Notes (Continued)

$$X = \frac{100\, s_{\overline{17}|.04}}{(1.04)^{17} + (.05)\, s_{\overline{17}|.04}}$$

$$= 756.44.$$

# Written Answer Questions

## Problem 2

Henry is repaying a loan at an effective rate of 5% a year. The payments at the end of each year for 10 years are 1000 each. In addition to the loan payments, Henry pays premiums for loan insurance at the beginning of each year. The first premium is 0.5% of the original balance, the second premium is 0.5% of the loan balance ~~premium~~ immediately after the first loan payment, etc., and the tenth premium is 0.5% of the loan balance after the ninth payment. The present value of the premiums at 5% is $X$. Determine $X$.

0 1 2 - - - 9 10

Loan

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DB: $1000\, a_{\overline{10}|.05}$ $\quad 1000\, a_{\overline{9}|.05}$ $\quad 1000\, a_{\overline{8}|.05}$ $\quad 1000\, a_{\overline{1}|.05}$ $\quad 0$

9 payments remaining after the 1st payment

8 - - -

Premium: $(.5\%) 1000\, a_{\overline{10}|.05}$ $\quad (.5\%) 1000\, a_{\overline{9}|.05}$ $\quad \cdots$ $\quad (.5\%) 1000\, a_{\overline{1}|.05}$ $\quad 0$

# Notes (Continued)

$$\text{PV of premiums} = 5\left(a_{\overline{10}|.05} + v\,a_{\overline{9}|.05} + \cdots + v^9 a_{\overline{1}|.05} + v^{10}\cdot 0\right)$$

$$= 5\left(\frac{1-v^{10}}{i} + \frac{v-v^{10}}{i} + \cdots + \frac{v^9-v^{10}}{i}\right)$$

$$= \frac{5}{i}\left(1 + v + \cdots + v^9 - 10v^{10}\right)$$

$$= \frac{5}{.05}\left(\frac{1-v^{10}}{1-v} - 10v^{10}\right) \approx 196.87$$

## Problem 3

A house purchase for $\$200,000$ and a 20% down payment is made. The balance is financed by a 30-year adjustable rate mortgage with monthly payments. The initial interest rate is 12% per annum, compounded monthly. Just after the 240th payment, the interest rate is increased to 14% compounded monthly. The payments remain at the original amount until a final smaller payment fully repays the loan. What is the total number of monthly mortgage payments made over the life of the loan, including the final smaller payment?

# Notes

Price: 200000, 20% down $\Rightarrow$ loan: $200000(1-20\%) = 160000$

$$160000 = X a_{\overline{360}|12\%/12} = X \cdot \frac{1 - 1.01^{-360}}{i}$$

After the 240th payment:

$$OB_{240} = X a_{\overline{120}|.01} \quad (1)$$

$i^{(12)} = 12\% \nearrow 14\%$, $X$: unchanged

$$OB_{240} = X a_{\overline{N}|(.14/12)} \quad (2)$$

# Notes (Continued)

(1), (2): $a_{\overline{120}|.01} = a_{\overline{N}|(.14/12)}$

$$\frac{1 - 1.01^{-120}}{.01} = \frac{1 - \left(1 + \frac{.14}{12}\right)^{-N}}{.14/12}$$

$\Rightarrow N = 144.6$ months

\# of payments: $240 + 144 + 1 = 385$

# Multiple Choice

## Problem 2

A ten year loan of $10,000$ at $8\%$ annual effective can be repaid using any of the four following methods:

(i) Amortization Method, with annual payments at the end of each year.

(ii) Repay the principal at the end of ten years which paying $8\%$ annual effective interest on the loan at the end of each year. In addition, make equal annual deposits at the end of each year into a sinking fund earning $6\%$ annual effective so that the sinking fund accumulates to $10,000$ at the end of 10th year.

(iii) Same as (*ii*), except the sinking fund earns $8\%$ annual effective.

(iv) Same as (*ii*), except the sinking funds earns $12\%$ effective.

Rank the annual payment amounts of each method.

# Notes

Fact 1: The higher interest rate the sinking fund earns, the less you need to repay.

Fact 2: rate for sinking fund = rate for loan,

$\Rightarrow$ Amount to repay for Amortization

$\quad\quad ||$

Amount to repay for sinking fund.

(iv) < (i) = (iii) < (ii)

# Notes (Continued)

Pf: (Fact 1): Sinking fund $X$ satisfies

$$\text{Loan} = X s_{\overline{n}|i} = X + X(1+i) + \cdots + X(1+i)^{n-1}$$

Given Loan, as $i$ grows larger, $X$ goes lower.

(Fact 2): If rate for sinking fund = rate for Loan = $i$

Sinking fund: $$\frac{\text{Loan}}{s_{\overline{n}|i}} + \text{Loan} \cdot i = \frac{\text{Loan}\,(1 + i s_{\overline{n}|i})}{s_{\overline{n}|i}} = \frac{\text{Loan}\,(1+i)^n}{s_{\overline{n}|i}}$$

Amortization: $$\frac{\text{Loan}}{a_{\overline{n}|i}} = \frac{\text{Loan}\,(1+i)^n}{s_{\overline{n}|i}}$$ b/c $s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}$

# Extra Practice

## Problem 1

Smith can repay a loan of $250,000$ in one of the two ways:

i. 30 annual payments based on amortization at $i = .12$;

ii. 30 annual interest payments to the lender at rate $i = .1$, along with 30 level annual deposit to a sinking fund earning rate $j$.

Find the level of $j$ to make the scheme equivalent.

# Extra Practice

## Problem 2

A homebuyer borrows $\$250,000$ to be repaid over a $30-$year period with level monthly payments beginning one month after the loan is made. The interest rate on the loan is a nominal annual rate of 9% compounded monthly. Find each of the following:

(a) the amount of interest and the amount of principal paid in the first year.

(b) the amount of interest and the amount of principal paid in the 30th year.

# Extra Practice

## Problem 3

A 20-year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%. Calculate X.