Midterm Review

1. Nominal and effective rate

ex. Find the nominal rate of discount $d^{(2)}$ that is equivalent to a nominal rate of interest $i^{(12)} = 8\%$.

Soln: "equivalent to" means effective rate of interest / discount is exactly the same.

Step 1: $i^{(12)} \to i$

formula $1+i = \left(1 + \frac{i^{(12)}}{12}\right)^{12}$

$i = \left(1+\frac{8\%}{12}\right)^{12} - 1 = 8.3\%$

Step 2: $i \to d$

formula $1-d = \frac{1}{1+i} \Rightarrow d = \frac{i}{1+i} = 7.66\%$

Step 3: $d \to d^{(2)}$

formula $1-d = \left(1 - \frac{d^{(2)}}{2}\right)^{2}$

$\Rightarrow d^{(2)} = \left\{1 - (1-d)^{1/2}\right\} \times 2 = 7.8166\%$ //

2. Continuous Interest

ex. T deposit w. $i^{(2)} = 10\%$

F deposit 1000, simple interest rate j.

At yr 5, force of interest $\delta_T = \delta_F$.

Q: Determine $Z = \{AV$ for F's deposit$\}$.

Soln: ① For T, $a_T(t) = \left(1 + \frac{i^{(2)}}{2}\right)^{2t} = 1.05^{2t}$

$$\delta_T(t) = \frac{a_T'(t)}{a_T(t)} = \frac{1.05^{2t} \cdot \log(1.05^2)}{1.05^{2t}} = 2\log(1.05)$$

② For F, $a_F(t) = 1 + jt$

$$\delta_F(t) = \frac{a_F'(t)}{a_F(t)} = \frac{j}{1+jt}$$

$$2\log(1.05) = \left.\frac{j}{1+jt}\right|_{t=5} \Rightarrow j = 5.377\%$$

$$Z = 1000(1+5j) = 1953$$ //

3. Increasing / Decreasing Annuity

Increasing: $1, 2, \cdots, n$

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \qquad (Is)_{\overline{n|}} = (1+i)^n (Ia)_{\overline{n|}}$$

$$(I\ddot{a})_{\overline{n|}} = (1+i)(Ia)_{\overline{n|}} \qquad (I\ddot{s})_{\overline{n|}} = (1+i)^n (I\ddot{a})_{\overline{n|}}$$

Decreasing: $n, n-1, \cdots, 1$

$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i} \qquad (Ds)_{\overline{n|}} = (1+i)^n (Da)_{\overline{n|}}$$

$$(D\ddot{a})_{\overline{n|}} = (1+i)(Ds)_{\overline{n|}} \qquad (D\ddot{s})_{\overline{n|}} = (1+i)^n (Ds)_{\overline{n|}}$$

ex. A perpetuity w- $PV = 77.1$, $i = 10.5\%$

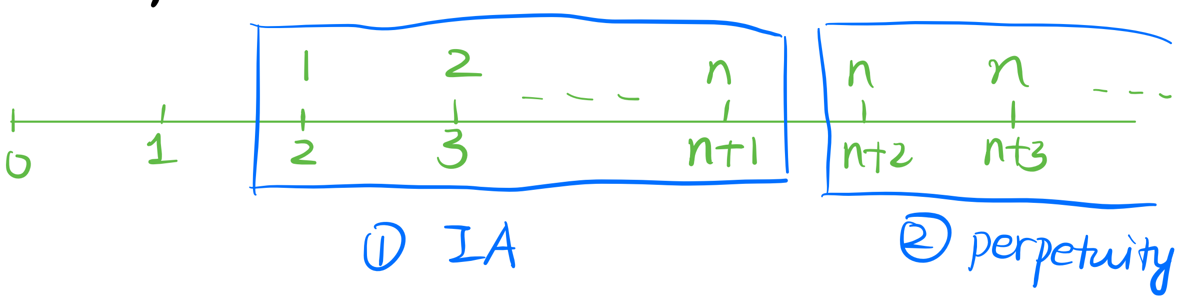
— pay 1 at the end of yr 2

— pay 2 〃 〃 〃 〃 〃 3

⋮

— pay n 〃 〃 〃 〃 〃 n+1

— after yr n+1, pay n at the end of each yr



①: $PV_1 = (Ia)_{\overline{n}|i} \cdot v = \dfrac{a_{\overline{n}|i} - n v^{n+1}}{i}$

②: $PV_2 = n\, a_{\overline{\infty}|i} \cdot v^{n+1} = \dfrac{n v^{n+1}}{i}$

$\left.\begin{aligned} &PV = 77.1 = PV_1 + PV_2 = \tfrac{1}{i} a_{\overline{n}|i} \\ &i = .105 \end{aligned}\right\} \Rightarrow n = 19$ //

ex. S: make a deposit for 7 yrs

At the end of each yr $\begin{cases} \text{1. invest } Z \text{ for } i = 5\% \\ \text{2. withdraw and reinvest interest for } j = 6\% \end{cases}$

Q: Find AV at yr 7?

Soln:

Time	Payment	Interest
0		
1	Z	
2	$Z+Z$	Zi
3	$2Z+Z$	$2Zi$
⋮		
7	$6Z+Z$	$6Zi$

increasing annuity

$$AV = 7Z + Zi\,(Ia)_{\overline{6}|i} = 8.1653\,Z \;//$$

4. Amortization

3 formulas:

$$\begin{cases} K_t = I_t + PR_t \\ I_t = OB_{t-1} \cdot i \\ PR_t = OB_{t-1} - OB_t \end{cases}$$

OB_t — Outstanding Balance at time t

I_t — interest paid at time t

K_t — Payments at time t

ex. Loan 250000, 30 yrs, $i^{(12)} = 9\%$, monthly PMT K.

$$250000 = K\, \underbrace{a_{\overline{360}|}}_{30\times 12}\, \underbrace{.0075}_{i^{(12)}/12} \Rightarrow K = 2011.56$$

(a) The amount of interest and principal paid in the first yr.

yr 0: $OB_0 = 250000$

yr 1: $OB_{12} = K\, \underbrace{a_{\overline{348}|}}_{29\times 12 \text{ PMTs remaining}}\, .0075 = 248292.0073$

Amount Paid $= 12K = 24138.6785$

$PR = OB_0 - OB_{12} = 250000 - 248292.0073 = 1707.9927$

Interest Paid = Amount Paid − PR

$= 22430.6858$

(b) The amount of interest and principal paid in the last yr.

yr 29: $OB_{348} = K a_{\overline{12}|}\, .0075 = 23001.9734$

yr 30: $OB_{360} = 0 \Rightarrow PR = OB_{348}$

Interest Paid = Amount Paid − PR

$= 24138.6785 - 23001.9734$

$= 1136.7051$

In general, $OB_{j-1} = K\, a_{\overline{n-j+1}|i}$

$$OB_j = K\, a_{\overline{n-j}|i}$$

$$\overset{\text{check}}{\Longrightarrow} \quad PR_j = OB_{j-1} - OB_j = K\, v^{n-j+1}$$

If you buy a bond w. $P > F$ (premium),

$(BV = OB)$

$$BV_{j-1} = Fr\, a_{\overline{n-j+1}|i} + F v^{n-j+1}$$

$$BV_j = Fr\, a_{\overline{n-j}|i} + F v^{n-j}$$

$\overset{\text{check}}{\Longrightarrow}$ Premium for amortization in the j th coupon

$$= BV_{j-1} - BV_j = F(r-i)\, v^{n-j+1}$$

5. Sinking Fund

Idea: Loan P w. interest rate i

each payment $<$ pay the interest Pi / deposit K as a sinking fund w. j

$$P = K\, s_{\overline{n}|j}$$

Fact 1: If the interest rate j increases, K goes down.

Fact 2: If $i = j$, then

Payment for amortization = Payment for sinking fund $(Pi + K)$

ex. see HW3 Multiple choice P2.

If time permits:

3. (Chapter 2, May.2003.15) John borrows 1000 for 10 years at an annual effective interest rate of 10%. He can repay this loan using the amortization method with payments of P at the end of each year. Instead, John repays the 1000 using a sinking fund that pays an annual effective rate of 14%. The deposits to the sinking fund are equal to P minus the interest on the loan and are made at the end of each year for 10 years. Determine the balance in the sinking fund immediately after repayment of the loan.
(A) 213 (B) 218 (C) 223 (D) 230 (E) 237

SOLUTION: $1000 = Pa_{\overline{10}|.10} = P\dfrac{1-1.1^{-10}}{.10}$ so $P = \dfrac{.10(1000)}{1-1.1^{-10}} = 162.7454$.
Amount in sinking fund after payment of loan is
$(P-100)s_{\overline{10}|.14} - 1000 = 62.7454\dfrac{1.14^{10}-1}{.14} - 1000 = 213.3263$. A.