Week 2: Annuity

Def: An annuity is a series of payments over time.

1. Annuity Immediate: payments are made at the end of each period.

Payment		\$1	\$1		\$1	\$1
Time	0	1	2	$\cdots$	$n-1$	n
PV		v	v^2		v^{n-1}	v^n
FV		$(1+i)^{n-1}$	$(1+i)^{n-2}$		$(1+i)^1$	1

Accumulated PV $\quad a_{\overline{n}|} = v + v^2 + \cdots + v^n = \dfrac{1-v^n}{i}$

Accumulated FV $\quad s_{\overline{n}|} = (1+i)^{n-1} + \cdots + 1 = \dfrac{(1+i)^n - 1}{i}$

$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$ or $a_{\overline{n}|} = v^n s_{\overline{n}|}$

2. Annuity Due: Payments are made at the beginning of each period

Payment	\$1	\$1	\$1		\$1	
Time	0	1	2	$\cdots$	$n-1$	n
PV	1	v	v^2		v^{n-1}	
FV	$(1+i)^n$	$(1+i)^{n-1}$	$(1+i)^{n-2}$		$(1+i)$	

Accumulated PV $\ddot{a}_{\overline{n}|} = 1+v+v^2+\cdots+v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$

Accumulated FV $\ddot{s}_{\overline{n}|} = (1+i)^n+(1+i)^{n-1}+\cdots+(1+i) = \frac{(1+i)^n-1}{d}$

$$\ddot{a}_{\overline{n}|} = (1+i)\,a_{\overline{n}|}, \quad \ddot{s}_{\overline{n}|} = (1+i)\,s_{\overline{n}|}$$

3. <u>Perpetuity</u> is an annuity in which $n \to \infty$.

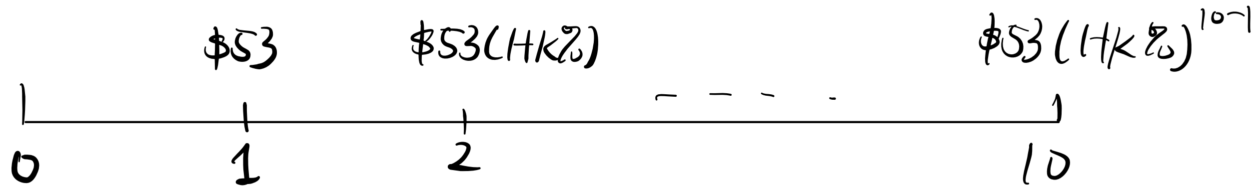
$$\begin{cases} \text{Perpetuity Immediate: } a_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i} \\ \text{Perpetuity Due: } \ddot{a}_{\overline{\infty}|} = (1+i)\,a_{\overline{\infty}|} = \frac{1+i}{i} \end{cases}$$

<u>Problem 10</u>: Two annuities, both cost \$X.

Jeff: <u>perpetuity immediate</u> pays \$30 annually

Jason: 10-yr annuity immediate

payment at yr n: $\$53(1+k\%)^{n-1}$



Effective interest rate $k\%$, $k=?$

<u>Soln</u>: $PV_{Jason} = PV_{Jeff} = X$

$$PV_{Jason} = \frac{53}{1+k\%} + \frac{53(1+k\%)}{(1+k\%)^2} + \cdots + \frac{53(1+k\%)^9}{(1+k\%)^{10}}$$

$$= \frac{53}{1+k\%} + \frac{53}{1+k\%} + \cdots + \frac{53}{1+k\%}$$

$$\left.\begin{aligned} X &= \frac{530}{1+k\%} \\ PU_{Jeff} &= \frac{30}{k\%} \end{aligned}\right\} \Rightarrow k = 6$$

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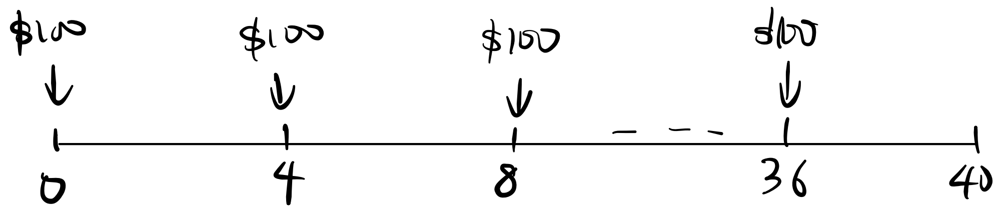
Problem 7: Annuity for 40 yrs

Deposit \$100, at the beginning of each 4-yr period.



$A(t)$ = the accumulated amount at time t.

$A(40) = 5 \cdot A(20)$. Q: Calculate $X = A(40)$.

Soln: i^*, v^*, d^* — effective rate for 4 yrs.

$$100 \cdot \ddot{s}_{\overline{10}|i^*} = A(40) = 5A(20) = 5 \cdot 100 \cdot \ddot{s}_{\overline{5}|i^*}$$

$$\frac{(1+i^*)^{10}-1}{d^*} = 5 \cdot \frac{(1+i^*)^5-1}{d^*}$$

Let $q = (1+i^*)^5$, then

$$q^2 - 1 = 5(q-1) \Rightarrow q = 4$$

$$i^* = 4^{1/5} - 1 \qquad d^* = \frac{i^*}{1+i^*} = \frac{4^{1/5}-1}{4^{1/5}}$$

$$X = A(40) = 100 \cdot \frac{(1+i^*)^{10}-1}{d^*} = 6195$$

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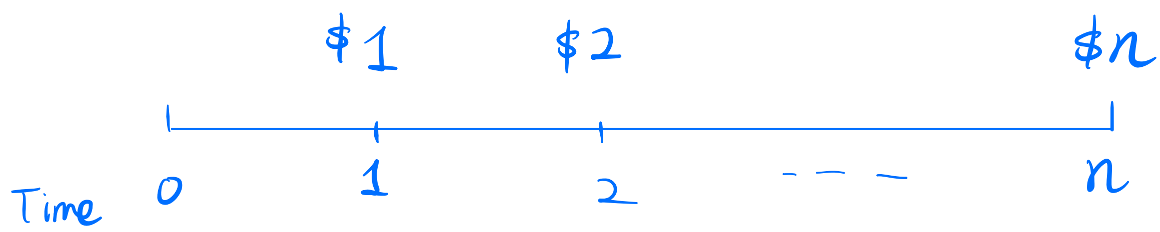
4. Increasing Annuity



Accumulated PV:

(*) $$(Ia)_{\overline{n}|} = v + 2v^2 + \cdots + nv^n = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

In addition:

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

$$(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$$(I\ddot{s})_{\overline{n}|} = (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

5. Decreasing Annuity

(**) $$(Da)_{\overline{n}|} = nv + (n-1)v^2 + \cdots + v^n = \frac{n - a_{\overline{n}|}}{i}$$

$$(D\ddot{a})_{\overline{n}|} = (1+i)(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

$$(Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|}, \quad (D\ddot{s})_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|}$$

Proof of (*) and (**):

$$(Ia)_{\overline{n}|} = v + 2v^2 + \cdots + (n-1)v^{n-1} + nv^n$$

$$-\left[\frac{1}{v}(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + nv^{n-1}\right]$$

$$\left(1 - \frac{1}{v}\right)(Ia)_{\overline{n}|} = -1 - v - v^2 - \cdots - v^{n-1} + nv^n$$

$$= \underbrace{-(1+v+v^2+\cdots+v^{n-1})}+nv^n$$

$$-i\,(Ia)_{\overline{n}|} = -\ddot{a}_{\overline{n}|} + nv^n$$

$$1-\frac{1}{v} = 1-(1+i) = -i$$

$$\Rightarrow \quad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \quad (*)$$

Note: $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|}$

$$= (v + 2v^2 + \cdots + (n-1)v^{n-1} + nv^n)$$

$$+ (nv + (n-1)v^2 + \cdots + 2v^{n-1} + v^n)$$

$$= (n+1)(v+v^2+\cdots+v^n) = (n+1)\,a_{\overline{n}|}$$

$$(Da)_{\overline{n}|} = (n+1)a_{\overline{n}|} - \frac{(1+i)\,a_{\overline{n}|} - nv^n}{i}$$

$$= \frac{ni\,a_{\overline{n}|} - a_{\overline{n}|} + nv^n}{i} = \frac{n - a_{\overline{n}|}}{i} \quad \text{since}$$

$$nv^n + ni\,a_{\overline{n}|} = nv^n + ni\,\frac{1-v^n}{i} = n \qquad //$$

Problem 9: $i^{(4)} = 10\%$, compound quarterly

Scott deposits:

yr k: \$$k$ at the begining of each quarter

$k = 1, 2, 3, \cdots, 8$

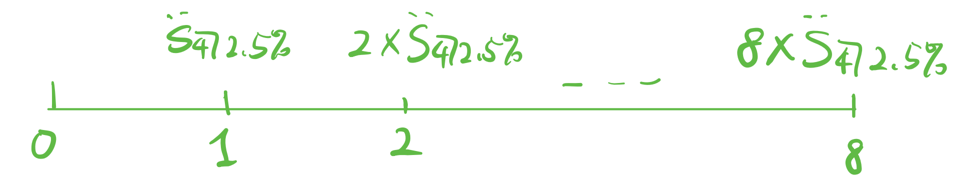
Q1: Calculate the accumulated value of the fund at the end of yr 8? $\frac{i^{(4)}}{4} = 2.5\%$

Deposit in yr 1 $\longrightarrow$ obtain $\ddot{s}_{\overline{4|}2.5\%}$ in yr 1

Deposit in yr 2 $\longrightarrow$ obtain $2\ddot{s}_{\overline{4|}2.5\%}$ in yr 2

...

Deposit in yr 8 $\longrightarrow$ obtain $8\ddot{s}_{\overline{4|}2.5\%}$ in yr 8



(Increasing Annuity)

$$AV = \ddot{s}_{\overline{4|}2.5\%} \times (Ia)_{\overline{8|}i} = 196.7661$$

where i — effective interest rate: $i = \left(1+\frac{i^{(4)}}{4}\right)^4 - 1 = 10.38\%$

Q2: At the end of yr 8, all payments are used to buy a perpetuity - immediate with payment \$X at the end of each yr, calculate X.

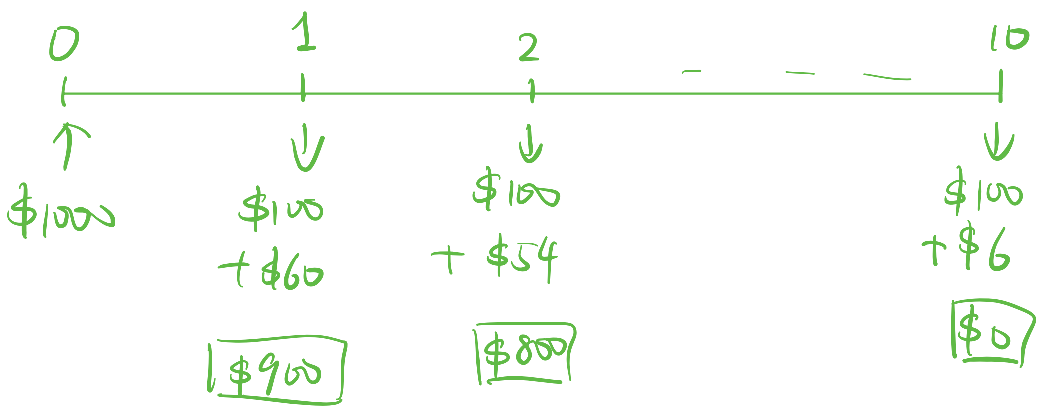
$$PV = \frac{\text{Payments}}{i} \qquad i \text{ — effective rate}$$

$$196.7661 = \frac{X}{10.38\%} \Rightarrow X = 20.4 \quad //$$

Problem 8: Deposit \$1000, $i = 6\%$

At the end of each yr:

withdraw interest plus \$100.



Deposit annual withdrawals are deposited in to fund Y, effective rate $i' = 9\%$.

Q: Calculate AV at the end of yr 10?

<u>Soln</u>: $AV = \underbrace{160(1+i')^9}_{\text{yr } 1} + \underbrace{154(1+i')^8}_{\text{yr } 2} + \cdots + \underbrace{106}_{\text{yr } 10}$

$$= \left(100(1+i')^9 + 100(1+i')^8 + \cdots + 100\right) + \left(60(1+i')^9 + 54(1+i')^8 + \cdots + 6\right)$$

$$= 100\, s_{\overline{10}|i'} + 6\cdot(Ds)_{\overline{10}|i'} = 2084.67$$

$$(Ds)_{\overline{n}|i} = (1+i)^n \frac{n - a_{\overline{n}|}}{i} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

6. <u>Defered Annuity</u>

e.g. A perpetuity: $\underbrace{\text{cost } \$77.1}_{}$ PV, $i = 10.5\%$

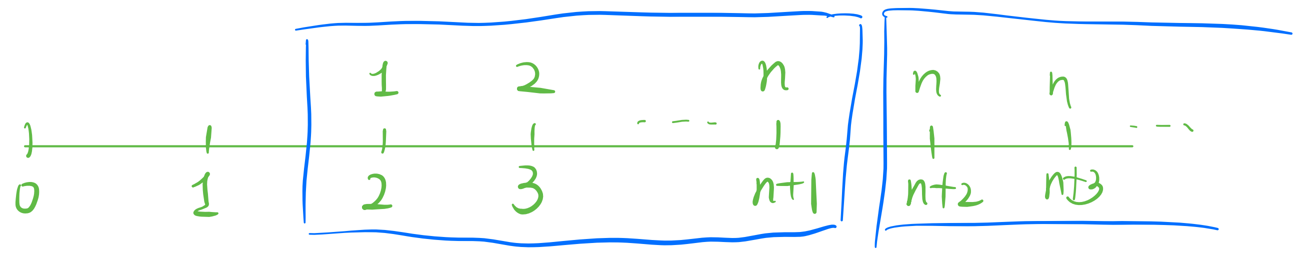
— pays 1 at the end of yr 2

— pays 2 at the end of yr 3

⋮

— pays n at the end of yr $n+1$

— after yr $n+1$, payment $= n$



Increasing annuity ①   perpetuity ②

① $\underbrace{(Ia)_{\overline{n|}i}}_{\text{PV in yr 1}} \times v = v \dfrac{\ddot{a}_{\overline{n|}i} - nv^n}{i} = \dfrac{a_{\overline{n|}i} - nv^{n+1}}{i}$

② $\underbrace{n\, a_{\overline{\infty|}i}}_{\text{PV in yr } n+1} \times v^{n+1} = \dfrac{n}{i} \cdot v^{n+1}$

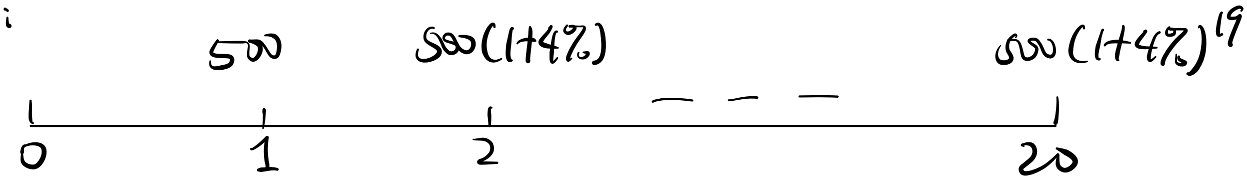
$① + ② = \dfrac{a_{\overline{n|}i} - n v^{n+1}}{i} + \dfrac{n v^{n+1}}{i} = \dfrac{a_{\overline{n|}i}}{i} = 77.1$

$a_{\overline{n|}i} = (77.1)(.105) \Rightarrow$ solve for $n = 19$.

Extra Exercise (NOT covered in the section):

A 20-year annuity immediate with annual payments is calculated at 6.2%. The first payment is 500 and increases at 4% annually. Find the present value of this annuity.

Soln:



$i = 6.2\%$ $\quad v = \frac{1}{1.062}$

$$PV = 500\left(v + 1.04v^2 + \cdots + 1.04^{19}v^{20}\right)$$

$$= 500\,v\left(1 + 1.04v + \cdots + (1.04v)^{19}\right)$$

$$= 500\,v \cdot \frac{1-(1.04v)^{20}}{1-1.04v}$$

$$= 7774.43$$

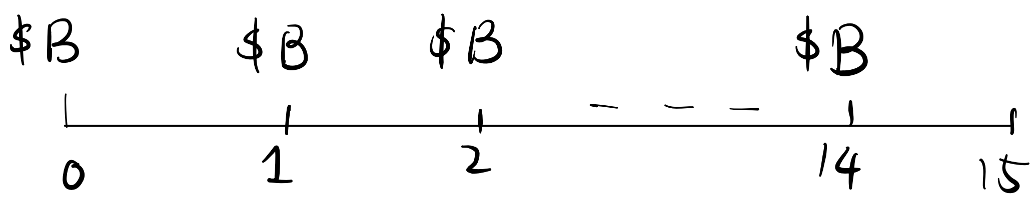
On January 1, an insurance company has 100,000 which is due to Linden as a life insurance death benefit. He chooses to receive the benefit annually over a period of 15 years, with the first payment immediately. The benefit he receives is based on an effective interest rate of 4% per annum.

The insurance company earns interest at an effective rate of 5% per annum. Every July 1, the company pays 100 in expenses and taxes to maintain the policy.

At the end of nine years, the company has X remaining.

Calculate X.

Soln: Step 1: Calculate the annual Benefit \$B.



$$PV = 100000 = B\ \ddot{a}_{\overline{15}|4\%} = B \cdot \frac{1-(1/1.04)^{15}}{.04/1.04}$$

$$B = 8648$$

Step 2: X = Income − cost

$$\text{income} = 100000(1.05)^9 = 100000(1.5513)$$

$$\text{cost} = B\ \ddot{s}_{\overline{9}|.05} + 100\ \ddot{s}_{\overline{9}|.05}\ v^{1/2}$$

why?

Because the payments (\$100) happen in the mid-year

$$X = 53870.$$

A 10-year annuity-immediate pays 100 quarterly for the first year. In each subsequent year, each payment is increased by 5% over the payment for the previous year. There is a nominal annual interest of 8% convertible quarterly. Find the present value of this annuity.

Soln: nominal interest rate : $i^{(4)} = 8\%$, quarter: $\frac{i^{(4)}}{4} = 2\%$

yr 1:

0	1/4	1/2	3/4	1
	100	100	100	100

$AV = 100\, s_{\overline{4}|2\%}$

yr 2:

0	1/4	1/2	3/4	1
	100(1.05)	100(1.05)	100(1.05)	100(1.05)

$AV = 100(1.05)\, s_{\overline{4}|2\%}$

- - -

yr 10:

0	1/4	1/2	3/4	1
	$100(1.05)^9$	$100(1.05)^9$	$100(1.05)^9$	$100(1.05)^9$

$AV = 100(1.05)^9\, s_{\overline{4}|2\%}$

$$PV = \frac{100\, s_{\overline{4}|2\%}}{1.02^4} + \frac{100(1.05)\, s_{\overline{4}|2\%}}{1.02^8} + \cdots + \frac{100(1.05)^9\, s_{\overline{4}|2\%}}{1.02^{40}}$$

$$\left(1 + \frac{i^{(4)}}{4}\right)^4$$

$$= \frac{100\, s_{\overline{4}|2\%}}{1.02^4}\left(1 + \left(\frac{1.05}{1.02^4}\right) + \cdots + \left(\frac{1.05}{1.02^4}\right)^9\right)$$

$$= \frac{100\, s_{\overline{4}|2\%}}{1.02^4} \cdot \frac{1 - \left(\frac{1.05}{1.02^4}\right)^{10}}{1 - \frac{1.05}{1.02^4}} =$$

$$s_{\overline{4}|2\%} = \frac{1.02^4 - 1}{.02} = 3333.28$$