1. Term Structure

(a) Spot rate: yield rate on an zero-coupon bond.

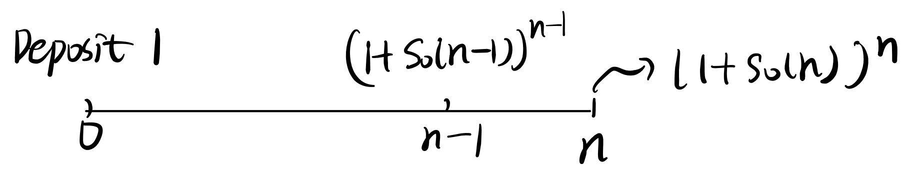
$S_0(t)$: spot rate at time t

C_t: cash flows at time t

$$PV_{CF} = \frac{C_t}{(1+S_0(t))^t}$$

(b) Forward rate of interest

$i(n-1,n) :=$ effective annual interest rate from $n-1$ to n.



$$1 + i_0(n-1,n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}}$$

Worksheet P5 (b):

Term structure:

$S_0(1) = S_0(2) = \cdots = S_0(n-1) < S_0(n)$

n-yr bond: coupon rate $r > 0$, yield rate j

Prove that $S_0(n-1) < j < S_0(n)$

Pf: $$P = \sum_{k=0}^{n} \frac{Fr}{(1+S_0(k))^k} + \frac{C}{(1+S_0(n))^n}$$

$$= \sum_{k=0}^{n-1} \frac{Fr}{(1+S_0(n-1))^k} + \frac{C+Fr}{(1+S_0(n))^n}$$

3 eq/ineq:

① $$P < \sum_{k=0}^{n-1} \frac{Fr}{(1+S_0(n-1))^k} + \frac{C+Fr}{(1+S_0(n-1))^k}$$

② $$P = \sum_{k=0}^{n-1} \frac{Fr}{(1+j)^k} + \frac{C+Fr}{(1+j)^k}$$

③ $$P > \sum_{k=0}^{n-1} \frac{Fr}{(1+S_0(n))^k} + \frac{C+Fr}{(1+S_0(n))^n}$$

①–③ $\Rightarrow$ $S_0(n-1) < j < S_0(n)$

Worksheet P6 a&d:

a. $$i_0(k-1, k) = \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} - 1$$

d. Given $S_0(k-1) < S_0(k)$

$$(1+S_0(k))^k = (1+S_0(k-1))^{k-1}(1+i_0(k-1,k))$$

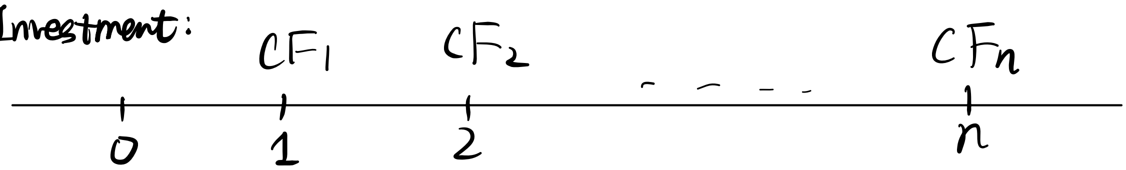
$$< (1+S_0(k))^{k-1}(1+i_0(k-1,k))$$

$$\Rightarrow (1+S_0(k))^1 < 1+i_0(k-1,k)$$

$$S_0(k) < i_0(k-1,k)$$ //

2. Duration

Investment:



$$P = \sum_{k=1}^{n} CF_k \cdot v^k$$

(a) Macaulay Duration: weight-average duration at which the investment paid.

"time"

$$D = \sum_{k=1}^{n} k \cdot w_k$$

weight $w_k = \dfrac{v^k CF_k}{P}$

(b) Modified Duration:

$$DM = -\frac{dP/di}{P}$$

<u>Rmk</u>: $D = (1+i) \cdot DM$

<u>Proof</u>: $P = \sum_{k=1}^{n} CF_k \cdot (1+i)^{-k}$

$$\frac{dP}{di} = \sum_{k=1}^{n} (-k)\, CF_k\, (1+i)^{-k-1}$$

$$= -v \sum_{k=1}^{n} k\, CF_k \cdot v^k$$

$$\frac{dP/di}{P} = -v \, \frac{\sum_{k=1}^{n} k \cdot CF_k \cdot v^k}{P}$$

$$\underbrace{-(1+i)\,\frac{dP/di}{P}}_{(1+i)\,DM} = \underbrace{\frac{\sum_{k=1}^{n} k \cdot CF_k \cdot v^k}{P}}_{D}$$

//

e.g. 12/31/1991 : an investment will pay $X in total w/o interest.

1992 : accumulate percentages paid 60%

1993 : 〃 〃 〃 90%

1994 : 〃 〃 〃 100%

$i = 8\%$, PMT : made in the middle of yrs.

$$PV = (.6)X\,v^{.5} + (.3)X\,v^{1.5} + (.1)X\,v^{2.5}$$

$$= (.895856)X$$

$$D = \frac{1}{PV}\left\{(.5)(.6)X\,v^{.5} + (1.5)(.3)X\,v^{1.5} + (2.5)(.1)X\,v^{2.5}\right\}$$

$$= \frac{.927140}{.895856}$$

$$DM = \frac{D}{1+i} = .966$$

3. Immunization

Assets: Cash inflows at some specific time

Liability: What the company owes / needs to pay at some specific time

<u>Def</u> (Redington Immunization):

A_t: assets cashflows at time t

L_t: liability cashflows at time t

$PV_A(i) = \sum_{t=1}^{n} A_t v^t \qquad PV_L(i) = \sum_{t=1}^{n} L_t v^t$

Valuation rate $= i_0$

Then, three criteria:

$PV_A(i_0) = PV_L(i_0) \Rightarrow$ Asset - Liability Match

$$\frac{d}{di} PV_A(i_0) = \frac{d}{di} PV_L(i_0)$$

$$\frac{d^2}{di^2} PV_A(i_0) > \frac{d^2}{di^2} PV_L(i_0)$$

<u>Idea for Redington Immunization:</u>

When the interest rate changes, we need to avoid the situation in which the liability due exceeds the value of asset cash inflows, resulting in the negative surplus.

Worksheet P8:

Liability PMTs \$100 in yr 2, 4, 6

Asset cashflows A_1 (yr 1) and A_5 (yr 5)

yield rate $i = 10\%$.

<u>Goal</u>: have asset CFs immunize liability CFs

a. Find A_1 & A_5.

<u>Soln</u>: $PV_A(i) = A_1 v + A_5 v^5$

$PV_L(i) = 100v^2 + 100v^4 + 100v^6$

$$\begin{cases} PV_A(.1) = PV_L(.1) \\ \frac{d}{di} PV_A(.1) = \frac{d}{di} PV_L(.1) \end{cases} \Rightarrow \begin{cases} A_1 = 71.44 \\ A_5 = 229.41 \end{cases}$$

b. Determine whether or not the condn'ts for Redington immunization are satisfied?

Yes. Just check

$$\frac{d^2}{di^2} PV_A(.1) > \frac{d^2}{di^2} PV_L(.1).$$