№ VIII . 1.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = f(x,t) \quad , -\infty < x < \infty$$

$$u(x,0) = \varphi(x) \quad , 0 < t < T$$

д) $l-1 \quad l \quad l+1 \quad n+1$

n

Отн-но точки $(l; n+1)$

Разностная схема: $\dfrac{u_l^{n+1} - u_l^n}{\tau} + \dfrac{u_{l+1}^{n+1} - u_{l-1}^{n+1}}{2h} = f_l^{n+1}$

$$u_l^0 = \varphi_l$$

I. Аппроксимация

$$\delta f^{(h)} = L_h [u]_h - f^{(h)} = \begin{cases} \dfrac{[u]_l^{n+1} - [u]_l^n}{\tau} + \dfrac{[u]_{l+1}^{n+1} - [u]_{l-1}^{n+1}}{2h} - f_l^{n+1} \\ [u]_l^0 - \varphi_l \end{cases} =$$

$$[u]_l^n = [u]_l^{n+1} - [\dot{u}_t]_l^{n+1} \tau + [\ddot{u}_{tt}]_l^{n+1} \frac{\tau^2}{2} + O(\tau^3)$$

$$[u]_{l \pm 1}^{n+1} = [u]_l^{n+1} \pm [u'_x]_l^{n+1} h + [u''_{xx}]_l^{n+1} \frac{h^2}{2} \pm$$

$$\pm [u'''_{xxx}] \frac{h^3}{6} + O(h^4)$$

$$= \begin{cases} [\dot{u}_t]_l^{n+1} + [\ddot{u}_{tt}]_l^{n+1} \frac{\tau}{2} + \cancel{[u'_x]_l^{n+1}} + [u'''_{xxx}] \frac{h^2}{6} - \cancel{f_l^{n+1}} \\ 0 \end{cases}$$

$$= \begin{matrix} (C_\tau)_l^{n+1} \tau + (C_h)_l^{n+1} h^2 \\ 0 \end{matrix}$$

$$\|\delta f^{(h)}\| \le C_\tau \tau + C_h h^2$$

II. Устойчивость.

$$u_l^n \Rightarrow \lambda_\alpha^n e^{i\alpha l}, \quad \alpha = \frac{2\pi k}{L+1}.$$

$$\frac{\lambda_\alpha^{n+1} - \lambda_\alpha^n}{\tau} e^{i\alpha l} + \lambda_\alpha^{n+1} \frac{e^{i\alpha(l+1)} - e^{i\alpha(l-1)}}{2h} = 0$$

$$\lambda_\alpha^{n+1} / \lambda_\alpha^n = \lambda_\alpha$$

$$\frac{\lambda_\alpha - 1}{\tau} + \lambda_\alpha \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0$$

$$\frac{\lambda_\alpha - 1}{\tau} + \lambda_\alpha \frac{i\sin\alpha}{h} = 0$$

$$\lambda_\alpha \left(1 + \frac{i\sin\alpha\tau}{h}\right) = 1$$

$$\lambda_\alpha = \frac{1}{\left(1 + \tau i \frac{\sin\alpha}{h}\right)}$$

$$|\lambda_\alpha|^2 \le 1 : \quad \frac{1}{\left(1 + \frac{\tau^2 \sin^2\alpha}{h^2}\right)} \le 1.$$

$$\tau^2 / h^2 \geq 0$$ – безусловно устойчива

a)

$n+1$

n

$l-1 \quad l$

$$\frac{u_l^{n+1} - u_l^n}{\tau} + \frac{u_l^n - u_{l-1}^n}{h} = f_l^n$$

$$u_n^0 = \psi_l$$

I. Аппроксимация

$$\delta f^{(h)} = \begin{cases} \frac{[u]_l^{n+1} - [u]_l^n}{\tau} + \frac{[u]_l^n - [u]_{l-1}^n}{h} - [f]_l^n \\ [u]_l^0 - \psi_l \end{cases} \ominus$$

$$[u]_l^{n+1} = [u]_l^n + [\dot{u}_t]\tau + [\ddot{u}_{tt}] \tau^2/2 + O(\tau^3)$$

$$[u]_{l-1}^n = [u]_l^n - [u_x]_l^n h + [u_{xx}]_l^n h^2/2 + O(h^3)$$

$$\ominus \begin{cases} (c_\tau)_l^n \tau + (c_h)_l^n h \\ 0 \end{cases}$$

$$\|\delta f^{(h)}\| \le c_\tau \tau + c_h h$$

II. Устойчивость:

$$u_l^n \to \lambda_\alpha^n e^{i\alpha l}, \quad \alpha = \frac{2\pi k}{L+1}$$

$$\frac{\lambda_\alpha^{n+1} - \lambda_\alpha^n}{\tau} e^{i\alpha l} + e^{i\alpha l} \lambda_\alpha^n \frac{1 - e^{-i\alpha}}{h} = 0$$

$$\frac{\lambda_\alpha - 1}{\tau} + \lambda_\alpha \frac{1 - e^{-i\alpha}}{h} = 0$$

$$\lambda_\alpha\left(1+\frac{1-e^{-i\alpha}}{h/\tau}\right)=1.$$

$$\lambda_\alpha=\frac{1}{1+\frac{1-e^{-i\alpha}}{h/\tau}}=\frac{1}{1+\frac{2i\sin\alpha/2}{h/\tau}}$$

$$|\lambda_\alpha|^2\le 1 \;\Rightarrow\; 1\le 1+\frac{2^2\sin^2\alpha/2}{h^2/\tau^2}$$

$\Longrightarrow$ безусловно устойчива.

№ VIII.4

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad \begin{matrix}0<x<1\\ 0<t<T\end{matrix}$$

$u(x,0)=\varphi(x)$; $\frac{\partial u}{\partial t}(0,x)=\psi(x)$; $u(0,t)=t$,

$u(1,t)=t^2$

a) $n+1$; $l-1$, l, $l+1$ n; $n-1$ $\qquad (n,l)$

I Аппроксимация.

$$\delta f^{(h)}=\begin{cases}\frac{[u]_l^{n+1}-2[u]_l^n+[u]_l^{n-1}}{h^2}-\frac{[u]_{l+1}^n-2[u]_l^n+[u]_{l-1}^n}{\tau^2}\\ [u]_l^0-\varphi_l\\ \frac{[u]_l^1-[u]_l^0}{\tau}-\psi_l\\ [u]_0^n-t^n\\ [u]_1^n-(t^2)_l^n\end{cases}$$

=

$$[u]_l^{n\pm1}=[\dot u]_l^n\pm[\dot u_t]_l^n\tau+[\ddot u_{tt}]_l^n\frac{\tau^2}{2}\pm[\dddot u_{ttt}]_l^n\frac{\tau^3}{6}+O(\tau^4)$$

$$[u]_{l\pm1}^n=[u]_l^n\pm[u'_x]h+[u''_{xx}]\frac{h^2}{2}\pm[u'''_{xxx}]\frac{h^3}{6}+O(h^4)$$

$$=\begin{cases}[\ddot u_{tt}]_l^n\frac{1}{2}\cdot 2+2[\ddddot u]_l^n\frac{h^2}{24}-[\ddot u]_l^n\frac{1}{2}\cdot 2-2[u''''_x]\frac{h^2}{24}\\ 0\\ (C\tau)_l^0\,\tau\\ 0\\ 0\end{cases}$$

Т.о. $\|\delta f^{(h)}\|\le C_\tau\tau+C_h h^2$

№ VIII.5

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=f(x,t)$$

$$\frac{\partial v}{\partial t}+\frac{\partial u}{\partial x}=g(x,t)$$

$u(x,0)=\varphi(x)$, $v(x,0)=\psi(x)$.

Запишем систему в характеристическом виде.

$$\frac{\partial\vec w}{\partial t}+\vec A\frac{\partial\vec w}{\partial x}=\vec F,\quad A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

с.з. $\lambda=\pm1$.

Найдем с.в.

1) $\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0$

$\Rightarrow \vec{\omega}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2) $\lambda = -1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0$

$\Rightarrow \vec{\omega}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Составим Ω: $\Omega = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A = \Omega^{-1}\Lambda\Omega$$

Домножим слева на Ω:

$$\Omega\frac{\partial\overline{\omega}}{\partial t} + \Omega\Omega^{-1}\Lambda\Omega\frac{\partial\overline{\omega}}{\partial x} = \Omega\overline{F}$$

$$\Omega\overline{\omega} = J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} u+v \\ u-v \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial J_1}{\partial t} + 1\,\dfrac{\partial J_1}{\partial x} = f + g & \text{с.з.} > 0 \\ \dfrac{\partial J_2}{\partial t} - 1\,\dfrac{\partial J_2}{\partial x} = f - g. & \text{с.з.} < 0 \end{cases}$$

$n+1$, $i-1$, i, n

$n+1$, n, i, $i+1$

$$\frac{(J_1)_i^{n+1} - (J_1)_i^n}{\tau} + 1\,\frac{(J_1)_i^n - (J_1)_{i-1}^n}{h} = f_i + g_i$$

$$\frac{(J_2)_i^{n+1} - (J_2)_i^n}{\tau} - 1\cdot\frac{(J_2)_{i+1}^n - (J_2)_i^n}{h} = f_i - g_i$$

§ 10.1.4

$$\frac{\partial^2 u}{\partial t^2} + a\,\frac{\partial^2 u}{\partial t\partial x} = f(t,x)$$

$$u(0,x) = \varphi(x)$$

$$\partial u(0,x)/\partial t = \psi(x)$$

Разностная схема:

$$(u_l^{n+1} - 2u_l^n + u_l^{n-1})/\tau^2 + a\,\frac{u_l^{n+1} - u_{l-1}^{n+1} - u_l^n + u_{l-1}^n}{\tau h} = f_l^n$$

$$u_l^0 = \varphi_l; \quad -3u_l^0 + 4u_l^1 - u_l^2 = 2\tau\psi_l$$

$$\delta f^{(h)} = \begin{cases} \dfrac{[u]_l^{n+1} - 2[u]_l^n + [u]_l^{n-1}}{\tau^2} + a\,\dfrac{[u]_l^{n+1} - [u]_{l-1}^{n+1} - [u]_l^n + [u]_{l-1}^n}{\tau h} - f_l^n \\ [u]_l^0 - \varphi_l \\ -3[u]_l^0 + 4[u]_l^1 - [u]_l^2 - 2\tau\psi_l \end{cases} =$$

$=$

$$[u]_l^{n\pm1} = [u]_l^n \pm [\dot{u}_t]_l^n\tau + [\ddot{u}_{tt}]\frac{\tau^2}{2} \pm [\dddot{u}_{ttt}]\frac{\tau^3}{6} + O(\tau^4)$$

$$[u]_{l-1}^{n+1} = [u]_l^n + [\dot{u}_t]_l^n\tau - [u'_x]_l^n h \mp [\dot{u}_x]_l^n h\tau + O(\tau^2 + h^2)$$

$$[u]_{l-1}^n = [u]_l^n - [u'_x]_l^n h + [u''_{xx}]_l^n\frac{h^2}{2} - [u'''_{xxx}]\frac{h^3}{6} + O(h^4)$$

$$[u]_l^1 = [u]_l^0 + [u]_l^0 2\tau + [u]_l^0\frac{4\tau^2}{2} + O(\tau^3)$$

$$[u]_l^2 = [u]_l^0 + [u_l^0]2\tau + [u]_l^0\frac{4\tau^2}{2} + O(\tau^3)$$

$\ominus$ $2[\ddot{u}_{tt}]\frac{1}{2} + 2[\overset{....}{u}]^n_l \frac{\tau^2}{24} + a[\ddot{u}]\frac{\tau^2}{h\tau} +$
$+ a[\dot{u}_x]^n_l \frac{h\tau}{h\tau} - f^n_l + [u'''_{xxx}]\frac{h^3}{6h\tau}$

$\ominus$

$3[u]^n_l\tau$

$$\|\delta f^{(h)}\| \le C_\tau \cdot \tau + C_{\tau h}\frac{\tau}{h} + C_{h\tau}\frac{h^2}{\tau}$$

№10.1.14.

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial y} = f(t,x,y)$$

$u(0,x,y) = \varphi(x,y)$
$u(t,0,y) = \psi_0(t,y)$
$u(t,1,y) = \psi_1(t,y)$
$u(t,x,0) = \chi(t,x)$

$$\delta f^{(h)} = \begin{cases} \dfrac{[u]^{n+1}_{l,m} - [u]^n_{l,m}}{\tau} - a\dfrac{[u]^{n+1}_{l+1,m} - 2[u]^{n+1}_{l,m} + [u]^{n+1}_{l-1,m}}{h_x^2} + b\dfrac{[u]^n_{l,m+1} - [u]^n_{l,m-1}}{2h_y} - \\ \quad - f^{n+1}_{l,m} \\ [u]^n_{0,m} - (\psi_0)^n_m \\ [u]^n_{L,m} - (\psi_1)^n_m \\ [u]^n_{l,0} - \chi^n_l \\ [u]^n_{l,M} - (\chi_1)^n_l \end{cases}$$

$[u]^n_{l,m} = [u]^{n+1}_{l,m} - [\dot{u}]^{n+1}_{l,m}\tau + [\ddot{u}]^{n+1}_{l,m}\frac{\tau^2}{2} + O(\tau^3)$

$[u]^n_{l,m\pm1} = [u]^{n+1}_{l,m} \pm [u'_y]^{n+1}_{l,m}h_y + [u''_{yy}]^{n+1}_{l,m}\frac{h_y^2}{2} \pm [u'''_{yyy}]^n_{l,m}\frac{h_y^3}{6}$
$+ O(h_y^4) + O(\tau)$

$[u]^{n+1}_{l\pm1,m} = [u]^{n+1}_{l,m} \pm [u'_x]^{n+1}_{l,m}h_x + [u''_{xx}]^{n+1}_{l,m}\frac{h_x^2}{2} \pm$
$\pm [u'''_{xxx}]\frac{h_x^3}{6} + O(h_x^4)$

$\ominus$ $[\dot{u}]^{n+1}_{l,m} - [\ddot{u}]^{n+1}_{l,m}\frac{\tau}{2} - a\left([u''_{xx}]^{n+1}_{l,m} + [u^{IV}_{xxxx}]\frac{h_x^2}{12}\right)$
$+ b\left([u'_y]^{n+1}_{l,m} + [u'''_{yyy}]\frac{h_y^2}{3}\right) - f^{n+1}_{l,m}$

$\ominus$

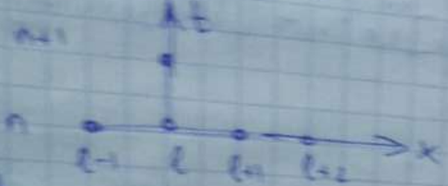
т.о.

$$\|\delta f^{(h)}\| \le C_\tau\tau + C_{h_y}h_y^2 + C_{h_x}h_x^2$$

Пример

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f(t,x)$$

$$a = const > 0$$



$O(\tau, h^3)$ – ?

Метод неопределённых коэффициентов.

$$\alpha u_l^{n+1} + \beta u_{l-1}^n + \gamma u_l^n + \delta u_{l+1}^n + \varepsilon u_{l+2}^n = f_l^n$$

$$\delta f_l^n = \alpha [u]_l^{n+1} + \beta [u]_{l-1}^n + \delta [u]_{l+1}^n + \gamma [u]_l^n + \varepsilon [u]_{l+2}^n - f_l^n$$

α | $[u]_l^{n+1} = [u]_l^n + [\dot u_t]_l^n \tau + [\ddot u_{tt}] \tau^2/2 + O(\tau^3)$

δ, β | $[u]_{l\pm1}^n = [u]_l^n \pm [u'_x]_l^n h + [u''_{xx}]_l^n h^2/2 \pm [u'''_{xxx}]_l^n h^3/6 + [u^{IV}_{xxxx}] h^4/24 + O(h^5)$

ε | $[u]_{l+2}^n = [u]_l^n + [u'_x]_l^n 2h + [u''_{xx}] \frac{(2h)^2}{2} + [u'''_{xxx}] \frac{(2h)^3}{6} + [u^{IV}_{xxxx}] \frac{(2h)^4}{24} + O(h^5)$

(+)

$$[u]_l^n(\alpha+\beta+\gamma+\delta+\varepsilon) + [\dot u_t]_l^n \alpha\tau + [\ddot u_{tt}]_l^n \frac{\tau^2}{2}\alpha$$

$$+ [u'_x]_l^n \{h(\delta-\beta+2\varepsilon)\}$$

$$+ [u''_{xx}]_l^n \{\tfrac{h^2}{2}(\delta+\beta+4\varepsilon)\} + [u'''_{xxx}]_l^n \{\tfrac{h^3}{6}(\delta-\beta+8\varepsilon\}$$

$$+ [u^{IV}_{xxxx}]_l^n (\varepsilon\, \tfrac{2}{3}h^4 + \delta h^4/24 + \beta h^4/24)$$

$\alpha+\beta+\gamma+\delta+\varepsilon = 0$

$\alpha\tau = 1$

$h(\delta-\beta+2\varepsilon) = a$

$$\left.\begin{aligned} &\delta+\beta+4\varepsilon = 0 \\ &\delta-\beta+8\varepsilon = 0 \end{aligned}\right\} \Rightarrow$$

1) $\delta = -6\varepsilon$

2) $\beta = 2\varepsilon$

3) $\varepsilon = -a/6h$

4) $\delta = a/h$; $\beta = -a/3h$

5) $\alpha = 1/\tau$

6) $\gamma = -\frac{a}{2h} - 1/\tau$

Т.о.

$$\frac{1}{\tau}(u_l^{n+1} - u_l^n) + \frac{a}{6h}(-3u_l^n - u_{l+2}^n + 6u_{l+1}^n - 2u_{l-1}^n) = f_l^n$$

Аппроксимация:

1) по τ – 1-ый порядок

2) $-3/2 - 4/2 + 6/2 - 2/2$ ✓

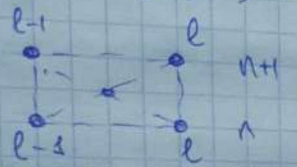
$-8/6 + 6/6 + 2/6$ ✓

$\Rightarrow$ 3-ий порядок по h.

VIII.1(г)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = f(x,t), \quad -\infty < x < \infty$$

$$u(x,0) = \psi(x), \quad 0 < t < T$$



Точка $(n+1/2;\ n+1/2)$

МНК: $\alpha u_l^{n+1} + \beta u_l^n + \gamma u_{l+1}^n + \delta u_{l+1}^{n+1} - f_{l+1/2}^{n+1/2} = \delta f^{(h)}$

$\alpha|$ $[u]_l^{n+1} = [u]_{l+1/2}^{n+1/2} + [\dot{u}_t]_{l+1/2}^{n+1/2}\tau/2 - [u'_x]_{l+1/2}^{n+1/2} h/2 +$
$+ [\ddot{u}]_{l+1/2}^{n+1/2}\tau^2/8 + [u''_{xx}]_{l+1/2}^{n+1/2} h^2/8 - [\dot{u}'_{tx}]_{l+1/2}^{n+1/2}\frac{h}{2}\frac{\tau}{2}$
$+ O(\tau^3 + h^3 + h^2\tau + \tau^2 h)$

$\beta|$ $[u]_l^n = [u]_{l+1/2}^{n+1/2} - [\dot{u}_t]_{l+1/2}^{n+1/2}\tau/2 - [u'_x]_{l+1/2}^{n+1/2} h/2 + [\dot{u}'_{tx}]\frac{h}{2}\frac{\tau}{2} +$
$+ [\ddot{u}]_{l+1/2}^{n+1/2}\tau^2/4 + [u''_{xx}]_{l+1/2}^{n+1/2} h^2/4$
$+ O(\dots)$

$\gamma|$ $[u]_{l+1}^n = [u]_{l+1/2}^{n+1/2} - [\dot{u}_t]_{l+1/2}^{n+1/2}\tau/2 +$
$h/2 + [\ddot{u}_{tt}]_{l+1/2}^{n+1/2}\tau^2/8 + \quad h\tau/4 +$
$+ [\dot{u}'_x]_{l+1/2}^{n+1/2} h^2/8 - [\dot{u}_{xt}]_{l+1/2}^{n+1/2}$
$+ [u''_{xx}]_{l+1/2}^{n+1/2}$
$+ O(\dots)$

$\delta|$ $[u]_{l+1}^{n+1} = [u]_{l+1/2}^{n+1/2} + [\dot{u}_t]_{l+1/2}^{n+1/2}\tau/2 + [\ddot{u}_{tt}]_{l+1/2}^{n+1/2}\tau^2/8 + [u''_{xx}]_{l+1/2}^{n+1/2} h^2/8$
$+ [u'_x]_{l+1/2}^{n+1/2} h/2$
$+ [\dot{u}'_{xt}]_{l+1/2}^{n+1/2} h\tau/4 + O(\dots)$

Группируем:

$(\alpha+\beta+\gamma+\delta)[u]_{l+1/2}^{n+1/2} + (\alpha-\beta-\gamma+\delta)[\dot{u}_t]_{l+1/2}^{n+1/2}\tau/2$
$+ (-\alpha-\beta+\gamma+\delta)[u'_x]_{l+1/2}^{n+1/2} h/2 + (\alpha+\beta+\gamma+\delta)$
$\left([\ddot{u}_{tt}]_{l+1/2}^{n+1/2}\tau^2/8 + [u''_{xx}]_{l+1/2}^{n+1/2} h^2/8\right) +$
$+ (-\alpha+\beta-\gamma+\delta)[\dot{u}'_{xt}]\frac{h\tau}{4} + O(\dots) -$
$- f_{l+1/2}^{n+1/2}$

$(\alpha+\beta+\gamma+\delta) = 0$
$\alpha-\beta-\gamma+\delta = 2/\tau$
$-\alpha-\beta+\gamma+\delta = 2/h$
$-\alpha+\beta-\gamma+\delta = 0$

$\Rightarrow \beta = -\delta;\ \alpha = -\gamma;\ \beta = -2/\tau - \gamma$

$\beta = -\frac{1}{2\tau} - \frac{1}{2h};\ \alpha = \frac{1}{2\tau} - \frac{1}{2h};\ \gamma = \frac{1}{2h} - \frac{1}{2\tau};$
$\delta = \frac{1}{2\tau} + \frac{1}{2h}.$

$$\frac{1}{2}\left(\frac{u_{l+1}^{n+1}-u_{l+1}^{n}}{\tau}+\frac{u_{l}^{n+1}-u_{l}^{n}}{\tau}\right)+\frac{1}{2}\left(\frac{u_{l+1}^{n+1}-u_{l}^{n+1}}{h}+\frac{u_{l+1}^{n}-u_{l}^{n}}{h}\right)=f_{l+1/2}^{n+1/2}$$

$$u_l^n \to \lambda_\alpha^n e^{i\alpha l}$$

$$\frac{\lambda_\alpha^{n+1}e^{i\alpha(l+1)}-\lambda_\alpha^{n}e^{i\alpha(l+1)}+\lambda_\alpha^{n+1}-\lambda_\alpha^{n}e^{i\alpha l}}{\tau}+$$

$$+\frac{\lambda_\alpha^{n+1}e^{i\alpha(l+1)}-\lambda_\alpha^{n+1}e^{i\alpha l}+\lambda_\alpha^{n}e^{i\alpha(l+1)}-\lambda_\alpha^{n}e^{i\alpha l}}{h}=0$$

$$\frac{\lambda_\alpha e^{i\alpha}-e^{i\alpha}+\lambda_\alpha-1}{\tau}+\frac{\lambda_\alpha e^{i\alpha}-\lambda_\alpha+e^{i\alpha}-1}{h}=0$$

$$\Rightarrow \lambda_\alpha=\frac{1+\tau/h\left(\frac{1-e^{i\alpha}}{1+e^{i\alpha}}\right)}{1-\tau/h\left(\frac{1-e^{i\alpha}}{1+e^{i\alpha}}\right)}$$

$$\hookrightarrow \left|\frac{1+\tau/h+(1-\tau/h)e^{i\alpha}}{1-\tau/h+(1+\tau/h)e^{i\alpha}}\right|\le 1$$

$\Leftrightarrow |\lambda_\alpha|^2=1$ — безумовно устойчиве.

Пример:

$$\frac{\overline{w}_l^{n+1}-\overline{w}_l^{n}}{\tau}+\begin{pmatrix}4&10\\-6&-15\end{pmatrix}\frac{\overline{w}_{l+1}^{n+1}-\overline{w}_{l-1}^{n+1}}{2h}=f_l^{n+1}$$

Шаблон + спектр. устойчивость

$$\overline{w}_l^n \to \lambda_\alpha^n e^{i\alpha l}\overline{\Delta}_0$$

$$\left(\frac{\lambda_\alpha^{n+1}e^{i\alpha l}-\lambda_\alpha^{n}e^{i\alpha l}}{\tau}\right)\overline{\Delta}_0+\underbrace{\begin{pmatrix}4&10\\-6&-15\end{pmatrix}}_{A}\left(\frac{\lambda_\alpha^{n+1}e^{i\alpha(l+1)}-\lambda_\alpha^{n+1}e^{i\alpha(l-1)}}{2h}\right)$$

$$\cdot\overline{\Delta}_0=0$$

$$\lambda_\alpha^{n+1}/\lambda_\alpha^{n}=\lambda_\alpha$$

$$\frac{\lambda_\alpha-1}{\tau}\overline{\Delta}_0+A\lambda_\alpha\frac{e^{i\alpha}-e^{-i\alpha}}{2h}\overline{\Delta}_0=0$$

$$\left(\frac{\lambda_\alpha-1}{\tau}E+A\lambda_\alpha\frac{i\sin\alpha}{h}\right)\overline{\Delta}_0=0;\quad \forall\overline{\Delta}_0\Rightarrow \det(\ldots)=0$$

$$\begin{vmatrix}(\lambda_\alpha-1)/\tau+4\frac{\lambda_\alpha i\sin\alpha}{h} & 10\lambda_\alpha i\sin\alpha/h\\ -6\lambda_\alpha\frac{i\sin\alpha}{h} & \frac{\lambda_\alpha-1}{\tau}-15\frac{\lambda_\alpha i\sin\alpha}{h}\end{vmatrix}$$

$$=\left(\frac{\lambda_\alpha-1}{\tau}+4\frac{\lambda_\alpha i\sin\alpha}{h}\right)\left(\frac{\lambda_\alpha-1}{\tau}-15\frac{\lambda_\alpha i\sin\alpha}{h}\right)+$$

$$+60\frac{\lambda_\alpha^2(i\sin\alpha)^2}{h}=$$

$$=\left(\frac{\lambda_\alpha-1}{\tau}\right)^2-\frac{11\lambda_\alpha i\sin\alpha}{h}\frac{\lambda_\alpha-1}{\tau}=0$$

$$=\left(\frac{\lambda_\alpha-1}{\tau}\right)\left(\frac{\lambda_\alpha-1}{\tau}-\frac{11 i\lambda_\alpha\sin\alpha}{h}\right)=0$$

$\lambda_\alpha = 1$

$\lambda_\alpha - 1 - \frac{\tau}{h} 11 i \sin \alpha \cdot \lambda_\alpha = 0$

$$\Rightarrow \frac{1}{1 + 11^2 \frac{\tau^2}{h^2} \sin^2 \alpha} \le 1$$

$\Rightarrow \tau^2/h^2 \ge 0 \Rightarrow$ схема безусловно устойчива.

№ 11.2.5

$$\frac{\partial u}{\partial t} - 6 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = f(t,x)$$

$$\frac{\partial v}{\partial t} - 12 \frac{\partial v}{\partial x} - \frac{5 \partial u}{\partial x} = g(t,x)$$

$\overline{w} = \begin{pmatrix} u \\ v \end{pmatrix}$; $A = \begin{pmatrix} -6 & 1 \\ -5 & -12 \end{pmatrix}$; $\overline{F} = \begin{pmatrix} f \\ g \end{pmatrix}$

$\overline{w}_t + A \overline{w}_x = \overline{F}$; $A^T = \begin{pmatrix} -6 & -5 \\ 1 & -12 \end{pmatrix}$

С.з. матрицы A: $(-6-\lambda)(-1-\lambda) + 12 =$

$= (\cancel{6+\lambda})(1+\lambda) + 12 \quad \lambda^2 + 18\lambda + 77 = 0$

$\Rightarrow \lambda = -7$; $\lambda = -11$.

$\lambda = -7$:

$$\begin{pmatrix} 1 & -5 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \overline{w}_1$$

$$\overline{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Omega = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$; $\Omega^T = \Omega$; $\Lambda = \begin{pmatrix} -7 & 0 \\ 0 & -11 \end{pmatrix}$

$* \Omega^T$

$$\frac{\partial}{\partial t} \Omega^T \overline{w} + \overbrace{\Omega^T A}^{\Lambda \Omega^T} \frac{\partial}{\partial x} \overline{w} = \Omega^T F$$

$$\frac{\partial}{\partial t}(5u+v) - 7\frac{\partial}{\partial x}(5u+v) = 5f + g$$

$$\frac{\partial}{\partial t}(u+v) - 11\frac{\partial}{\partial x}(u+v) = f + g.$$

$$\frac{\partial \overline{J}}{\partial t} + \Lambda \frac{\partial \overline{J}}{\partial x} = \Omega^T F$$

I. Аппроксимация

$$\frac{\overline{J}_l^{n+1} - \overline{J}_l^n}{\tau} + \Lambda \frac{\overline{J}_{l+1}^n - \overline{J}_l^n}{h} = (\Omega^T F)_l^n$$

Раскладывая отн-но точки (n,l) получаем, что

$$\|\delta f^{(h)}\| = \left\| \frac{[J]_l^{n+1} - [J]_l^n}{\tau} + \Lambda \frac{[J]_{l+1}^n - [J]_l^n}{h} - (\Omega^T F)_l^n \right\|$$

$< C_1 \tau + C_2 h$

II. Устойчивость.

$\sigma = \tau/h$

$\overline{J}_n^l \to \lambda_\alpha^n e^{i\alpha l} \overline{\Delta}_0$

$$\left(\frac{\lambda_\alpha^{n+1} - \lambda_\alpha^n}{\tau}\right) \cancel{e^{i\alpha l}} \overline{\Delta}_0 + \Lambda \lambda_\alpha^n \cancel{e^{i\alpha l}} \frac{e^{i\alpha} - 1}{h} \overline{\Delta}_0 = 0$$

$$\left(\frac{\lambda_\alpha - 1}{\tau} E + \Lambda \frac{e^{i\alpha} - 1}{h}\right) \overline{\Delta}_0 = 0$$

$$[(\lambda_\alpha - 1)E + \Lambda(e^{i\alpha} - 1)\sigma]\Delta_0 = 0$$

$$\det[\ldots] = 0$$

$$\begin{vmatrix} (\lambda_\alpha - 1) - 7\sigma(e^{i\alpha} - 1) & 0 \\ 0 & (\lambda_\alpha - 1) - 11\sigma(e^{i\alpha} - 1) \end{vmatrix}$$

$$= (\lambda_\alpha - 1 - 7\sigma(e^{i\alpha} - 1))(\lambda_\alpha - 1 - 11\sigma(e^{i\alpha} - 1)) = 0$$

$$\lambda_{\alpha 1} = 1 + 7\sigma(e^{i\alpha} - 1) \qquad |\lambda_{\alpha_i}|^2 \le 1.$$

$$\lambda_{\alpha 2} = 1 + 11\sigma(e^{i\alpha} - 1)$$

$$\begin{cases} 7\sigma\cos\alpha + 7\sigma i\sin\alpha - 7\sigma + 1 \\ 11\sigma\cos\alpha + 11\sigma i\sin\alpha - 11\sigma + 1 \end{cases}$$

$$a)\ 1 - 14\sigma\sin^2\frac{\alpha}{2} + 7\sigma i\sin\alpha \,\Big|^2 \le 1$$

$$\cancel{1} - 28\sigma\sin^2\frac{\alpha}{2} + 196\sigma^2\sin^4\frac{\alpha}{2} + 49\sigma^2\sin^2\alpha \le \cancel{1}$$

$$196 \cdot 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}$$

$$-1 + 7\sigma\sin^2\frac{\alpha}{2} + 7\sigma\cos^2\frac{\alpha}{2} \le 0$$

$$7\sigma - 1 \le 0$$

$$\tau \le \frac{1}{7}h$$

$$\tau \le \frac{1}{11}h$$

№ 11.3.3.

$$\begin{cases} \frac{\partial u}{\partial t} - 5\frac{\partial u}{\partial x} + 7\frac{\partial v}{\partial x} = f(t,x) \\ \frac{\partial v}{\partial t} + 0\cdot\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x} = g(t,x) \end{cases} \qquad 0 \le t \le 0,\ 0 \le x \le 1$$

Приводим к характ. виду

$$\frac{\partial \bar{w}}{\partial t} + A\frac{\partial \bar{w}}{\partial x} = \bar{F} \quad ; \quad A = \begin{pmatrix} -5 & 7 \\ 0 & 3 \end{pmatrix} \to A^T = \begin{pmatrix} -5 & 0 \\ 7 & 3 \end{pmatrix}$$

С.з. A^T: $(-5-\lambda)(3-\lambda) = 0$

$$\lambda_1 = -5; \quad \lambda_2 = 3$$

$$\lambda_1 = -5: \hookrightarrow \begin{matrix} 0 & 0 \\ 7 & -2 \end{matrix} \hookrightarrow 7w_1 = 2w_2 \hookrightarrow \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \bar{w}_1$$

$$\lambda_2 = 3: \begin{matrix} -8 & 0 \\ 7 & 0 \end{matrix} \hookrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{w}_2$$

$$\Omega = \begin{pmatrix} 2 & 0 \\ 7 & 1 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} -5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Omega^T = \begin{pmatrix} 2 & 7 \\ 0 & 1 \end{pmatrix}$$

$$* \Omega^T$$

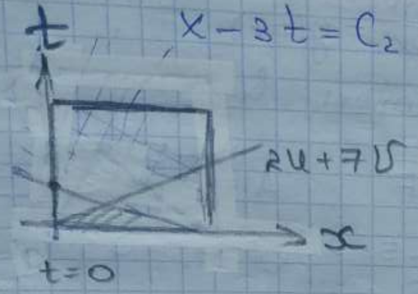
$$\frac{\partial(2u + 7v)}{\partial t} - 5\frac{\partial(2u + 7v)}{\partial x} = 2f + 7g$$

$$\frac{\partial(0u + 1v)}{\partial t} + 3\frac{\partial(0\cdot u + v)}{\partial x} = 0\cdot f + 1\cdot g$$

Характеристики:

$$\frac{dx}{dt} = -5 \Rightarrow x + 5t = C_1$$

$$\frac{dx}{dt} = 3 \qquad x - 3t = C_2$$



а) нет
б) нет
в) нет
г) да

№ 12.3.10

$$\left.\begin{aligned} &\Delta u = f(x,y) \\ &u|_\Gamma = \varphi(s) \\ &0 \le x, y \le 1 \end{aligned}\right\} \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u - f(x,y) \\ &u(0,x,y) = \psi(x,y) \\ &u|_\Gamma = \varphi(s) \end{aligned}$$

Сетка:

$$\frac{u^{n+1}_{m,l} - u^n_{l,m}}{\tau} = \frac{u^n_{l+1,m} - 2u^n_{l,m} + u^n_{l-1,m}}{h^2_x} + \frac{u^n_{l,m+1} - 2u^n_{l,m} + u^n_{l,m-1}}{h_y^2}; \quad u^n_{0,m} = \varphi_{0,m}$$

$$\varepsilon^n_{l,m} = u^n_{l,m} - u^{*n}_{l,m}$$

С-ма базисных ф-ций:

$$\psi^{(k,j)}_{l,m} = 2\sin\frac{k\pi l}{L}\sin\frac{j\pi m}{M}$$

$$\varepsilon^n_{l,m} = \sum_{k=1}^{L-1}\sum_{j=1}^{M-1} C^n_{k,j}\psi^{(k,j)}_{l,m}$$

$$\sum_{k=1}^{L-1}\sum_{j=1}^{M-1} \frac{C^{n+1}_{k,j} - C^n_{k,j}}{\tau}\psi^{(k,j)}_{l,m} = \sum_{k=1}^{L-1}\sum_{j=1}^{M-1} 2C^n_{k,j}\left[\sin\frac{j\pi m}{M}\cdot\frac{\sin\frac{k\pi(l+1)}{L} - 2\sin\frac{k\pi l}{L} + \sin\frac{k\pi(l-1)}{L}}{h_x^2} + \sin\frac{k\pi l}{L}\frac{\dots}{h_y^2}\right]$$

$$= \sum_{k=1}^{L-1}\sum_{j=1}^{M-1} C^n_{k,j}\psi^{(k,j)}_{l,m}\Big[\underbrace{-\tfrac{4}{h_x^2}\sin^2\frac{k\pi}{2L} - \tfrac{4}{h_y^2}\sin^2\frac{j\pi}{2M}}_{\mu_{k,j}}\Big]$$

$$\Rightarrow C^{n+1}_{k,j} = C^n_{k,j}\left[1 - \tau_n\mu_{k,j}\right]$$

$$C^N_{k,j} = C^0_{k,j}\prod_{n=1}^{N}(1 - \tau_n\mu_{k,j})$$

$$\mu_m \approx 2\pi^2; \quad \mu_M \approx 4(L^2 + M^2)$$

$$\mu = \frac{\mu_m + \mu_M}{2} + \frac{\mu_M - \mu_m}{2}\vartheta, \quad \vartheta \in [-1;1]$$

$$T_N = \frac{1}{2^{N-1}}\cos(N\arccos\vartheta_n)$$

$$\vartheta_n = \cos\frac{\pi(2n-1)}{2N}$$

$$\tau_n = \frac{2}{(\mu_m + \mu_M) + (\mu_M - \mu_m)\cos\frac{\pi(2n+1)}{2N}}$$

№ 12.3.9

$$\frac{u^{n+1}_{l,m} - u^{n}_{l,m}}{\tau_n} = \frac{u^{n}_{l+1,m} - 2u^{n}_{l,m} + u^{n}_{l-1,m}}{h_x^2} + \frac{u^{n}_{l,m+1/2} - 2u^{n}_{l,m} + u^{n}_{l,m-1/2}}{h_y^2/4} -$$

$$- f_{l,m}$$

Замена: $\varepsilon^{n}_{l,m} = u^{*}_{l,m} - u^{n}_{l,m} = \sum_{k=1}^{L-1} \sum_{j=1}^{M-1} C^{n}_{k,j}\, 2\sin\frac{k\pi l}{L} \sin\frac{j\pi m}{M}$

П.Э.:

$$\frac{C^{n+1}_{k,j} - C^{n}_{k,j}}{\tau_n} \psi^{(k,j)}_{l,m} = C^{n}_{k,j} \psi^{(k,j)}_{l,m} \left[-\frac{4}{h_x^2} \sin^2\frac{k\pi l}{2L} - \right.$$

$$\left. - \frac{16}{h_y^2} \sin^2\frac{j\pi}{2M} \right], \quad l = \overline{1, L-1}$$

$$C^{n+1}_{k,j} = C^{n}_{k,j} (1 - \tau_n \mu_{k,j})$$

$$(\mu_{k,j})_{min} = \pi^2 + 4\pi^2 = 5\pi^2$$

$$(\mu_{k,j})_{max} = 4L^2 + 16M^2 = 4(L^2 + 4M^2)$$

$$\mu = \frac{\mu_m + \mu_M}{2} + \frac{\mu_M - \mu_m}{2} \nu, \quad \nu \in [-1; 1]$$

$$\overline{T}_N = \frac{1}{2^{N-1}} \cos(N \arccos \nu_n)$$

$$\nu_n = \frac{\pi(2n-1)}{2N}$$

$$\tau_n = \frac{2}{(\mu_m + \mu_M) + (\mu_M - \mu_m) \cos\frac{\pi(2n-1)}{2N}}$$