c_i № 7.1.1.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 1/3 & 1/3 \\ \hline & 1/4 & 3/4 \end{array} \quad b_i$$

$$y_{l+1} = y_l + h(b_1 f_1 + b_2 f_2) = y_l + h\left(\frac{1}{4} f_1 + \frac{3}{4} f_2\right)$$

$$f_1 = f(x_l + c_1 h; y_l + h a_{11} f_1 + h a_{12} f_2) = f(x_l; y_l)$$

$$f_2 = f(x_l + c_2 h; y_l + h a_{21} f_1 + h a_{22} f_2) =$$

$$= f\left(x_l + \frac{2}{3} h; y_l + \frac{1}{3} h f_1 + \frac{1}{3} h f_2\right)$$

Порядок аппроксимации:

I пор: $\frac{1}{4} + \frac{3}{4} = 1$

II пор: $0 \times 1/4 + \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$

III пор: а) $\frac{3}{4} \cdot \left(\frac{2}{3}\right)^2 = 1/3$

б) $\sum b_i \sum a_{ij} c_j = \cancel{\frac{1}{4}\left(\frac{2}{9}\right)}^{0} + \frac{3}{4} \frac{1}{3}(0 + 2/3) = 1/6$

IV пор: а) $3/4 \left(2/3\right)^3 = 2/9$

$\Rightarrow$ сходимость $3^{\text{его}}$ порядка.

$$f_1 = J \Delta_l$$

$$f_2 = J\left(\Delta_l + \frac{1}{3} h J \Delta_l + \frac{1}{3} h f_2\right) \to f_2 = \frac{J\left(1 + \frac{1}{3} z\right) \Delta_l}{\left(1 - \frac{1}{3} z\right)},$$

где $z = Jh$

$$\Delta_{l+1} = \Delta_l + h \frac{1}{4} J \Delta_l + \frac{3}{4} h \frac{J\left(1 + \frac{1}{3} z\right) \Delta_l}{\left(1 - \frac{1}{3} z\right)} \Longrightarrow$$

$$\Rightarrow R(z) = \frac{z^2+4z+6}{2(3-z)}$$

Заметим, что z^2+4z+6 млад, как максимум, не яв-ся L-устойчивым, т.к. $\lim\limits_{z\to\infty} R(z) \neq 0$

$$|R(z)|^2 \leq 1.$$

$$\left|\frac{x^2+2xyi-y^2+4x+4iy+6}{6-6x-6iy}\right| \leq 1$$

$$\frac{(x^2-y^2+4x+6)^2+(2xy+4y)^2}{(6-6x)^2+6^2y^2} \leq 1.$$

в т. $(-10;-10)$ данное выражение >1

$\Rightarrow$ не яв-ся A-устойчивым.

№7.1.5.

c_i		
0	0	0
$3/4$	$1/4$	$1/2$
	$1/3$	$2/3$ b_i

$$y_{e+1} = y_e + h\left(\frac{1}{3}f_1 + \frac{2}{3}f_2\right)$$

$$f_1 = (x_e;\ y_e)$$

$$f_2 = \left(x_e + \frac{3}{4}h;\ y_e + h\frac{1}{4}f_1 + h\frac{1}{2}f_2\right)$$

I пор. $1/3 + 2/3 = 1$

II пор. $1/2$

III пор. $\frac{2}{3}\left(\frac{3}{4}\right)^2 = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{3}{4} = 3/8$

$\Rightarrow$ II-ой порядок.

$$f_1 = J\Delta e$$

$$f_2 = J\left(\Delta e + \frac{h}{3}J\Delta e + \frac{2h}{3}f_2\right) \Rightarrow$$

$$\Rightarrow f_2 = \frac{J\Delta e\left(1 + h/4\,J\right)}{1 - J\,h/2} = \frac{J\Delta e\left(1+z/4\right)}{1-z/2}.$$

$$\Delta e_{+1} = \Delta e + \frac{h}{3}J\Delta e + \frac{2Jh}{3}\left(\frac{1+z/4}{1-z/2}\right) =$$

$$= \Delta e\left(1 + \frac{z}{3} + \frac{2z}{3}\left(\frac{1+z/4}{1-z/2}\right)\right) =$$

$$= \Delta e\left(\frac{3+z}{3} + \frac{4z(1+z/4)}{3(2-z)}\right) = \Delta e\left(\frac{6-3z+2z-z^2+4z+z^2}{3(2-z)}\right.$$

$$= \Delta e\left(\frac{6+3z}{3(2-z)}\right)$$

$$\Rightarrow R(z) = \frac{1+z/2}{1-z/2}$$

$|R(z)|^2 < 1 \Rightarrow$ A-устойчив, т.к. $\mathrm{Re}\,z < 0$

$$\hookrightarrow \cancel{\frac{(-x^2-y^2+8x+24)^2+(2xy+8y)^2}{12^2(2-x)^2+12^2\cdot y^2}} \leq 1.$$

№7.1.8

c_i		
γ	γ	0
$1-\gamma$	$1-2\gamma$	γ
	$1/2$	$1/2$	b_i

При каких γ – 3-ий порядок уч?
При каких γ А-уч и L-уч?

I пор. $\sum b_i = \frac{1}{2} + \frac{1}{2} = 1.$

II пор. $\sum b_i c_i = \frac{1}{2}\gamma + \frac{1}{2}(1-\gamma) = \frac{1}{2}$

III пор. а) $\sum b_i c_i^2 = \frac{1}{2}\gamma^2 + \frac{1}{2}(1-2\gamma+\gamma^2) =$

$= \gamma^2 - \gamma + \frac{1}{2} = \frac{1}{3}$

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\gamma_{1/2} = \frac{+1 \pm \sqrt{1/3}}{2} = \frac{3 \pm \sqrt{3}}{6}$ – при таких γ III пор.

Ф-ция устойчивости:

$f_1 = \mathcal{J}(\Delta_e + h\gamma f_1) \Rightarrow f_1 = \frac{\mathcal{J}\Delta_e}{1-\gamma z}$

$f_2 = \mathcal{J}\left(\Delta_e + h(1-2\gamma)\frac{\mathcal{J}\Delta_e}{1-\gamma z} + h\gamma f_2\right)$

$\Rightarrow f_2 = \mathcal{J}\Delta_e \frac{\left(1 + \frac{(1-2\gamma)z}{1-\gamma z}\right)}{1-\gamma z}$

П.о.

$\Delta_{e+1} = \Delta_e + h\left(\frac{1}{2}\frac{\mathcal{J}\Delta_e}{1-\gamma z} + \frac{1}{2}\frac{\mathcal{J}\Delta_e}{1-\gamma z} + \frac{\mathcal{J}\Delta_e(1-2\gamma)z}{2(1-\gamma z)^2}\right.$

$= \Delta_e\left(1 + \frac{z}{1-\gamma z} + \frac{(1-2\gamma)z^2}{2(1-\gamma z)^2}\right) =$

$= \Delta_e\left(\frac{2(1-2\gamma z+\gamma^2 z^2) + (z-\gamma z^2)_2 + z^2 - 2\gamma z^2}{2(1-\gamma z)^2}\right)$

$= \Delta_e\left(\frac{1 + (1-2\gamma)z + z^2(\gamma^2 + \frac{1}{2} - 2\gamma)}{(1-\gamma z)^2}\right)$

По Th о max. к.з. ф-ции на границе: $z \to iy$.

$|R(z)|^2 < 1 \Leftrightarrow$

$\frac{(1 - y^2(\gamma^2 + \frac{1}{2} - 2\gamma))^2 + (1-2\gamma)^2}{(1-y^2\gamma^2)^2 + 4y^2\gamma^2} < 1.$

$\frac{(1-y^2(\gamma^2+\frac{1}{2}-2\gamma))^2 + (1-2\gamma)^2 - 4y^2\gamma^2 - (1-y^2\gamma^2)^2}{(1-y^2\gamma^2)^2 + 4y^2\gamma^2 > 0} < 0$

$\Leftrightarrow (1-y^2(\gamma^2+\frac{1}{2}-2\gamma))^2 - 4\gamma + \gamma^2 \cdot 4 - 4y^2\gamma^2 + 2y^2\gamma^2 - y^4\gamma^4 < 0$

$1 - \cancel{2y^2\gamma^2} - y^2 + 4y^2\gamma + y^4(\gamma^2 + \frac{1}{2} - 2\gamma)^2 - 4\gamma - 4y^2\gamma^2 +$
$+ \cancel{2y^2\gamma^2} - y^4\gamma^4 < 0$

$1 + 4y^2\gamma + \frac{y^4}{4} + 4y^4\gamma^4 + y^4\gamma^2 - 4y^4\gamma^3 - 2\gamma y^4 - 4\gamma - 4y^2\gamma^2$

$-y^4 y^4 < 0 \hookrightarrow -4y^3+5y^2-2y+\frac{1}{4} \le 0$

$\Rightarrow$ метод A-устойчив при $y \ge 1/4$

L-устойчив при $y^2-2y+\frac{1}{2}=0$ $\left(\lim_{z\to\infty} R(z)=0\right)$

$y = \frac{2\pm\sqrt{4-2}}{2} = 1 \pm \frac{\sqrt{2}}{2}$.

№ 7.1.13.

0	0	0	0
1/2	1/2	0	0
1	−1	2	0
	1/6	2/3	1/6

$y' + 100y + 10001\sin x = 0$

$y(0) = 1,\ 0 \le x \le 1.$

$f_1 = J\Delta_l$

$f_2 = J(\Delta_l + \frac{h}{2} f_1)$

$f_3 = J(\Delta_l - hf_1 + 2hf_2)$

$y_{l+1} = y_l + h\left(\frac{1}{6}f_1 + \frac{2}{3}f_2 + \frac{1}{6}f_3\right)$

I пор: $\frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1$.

II пор: $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

III пор: а) $\frac{2}{3}\,\frac{1}{4} + \frac{1}{6}\cdot 1 = \frac{1}{3}$

б) $\frac{2}{3}\cdot 0 + \frac{1}{6}(1 = \frac{1}{6}$

IV пор а) $\frac{2}{3}\,\frac{1}{8} + \frac{1}{6} = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$

б) $= \frac{1}{6} \Rightarrow$ 3-ий порядок.

$f_2 = J(\Delta_l + \frac{z\Delta_l}{2}) = J$

$f_3 = J(\Delta_l - z\Delta_l + 2z\Delta_l + z^2\Delta_l)$

$\Delta_{l+1} = \Delta_l\left(1 + z\left(\frac{1}{6} + \frac{2}{3} + \frac{z}{3} + \frac{1}{6} + \frac{z}{6} + \frac{z^2}{6}\right)\right)$

$\Rightarrow R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}$ — не A и L устойчив

$|R(z)| < 1$.

$f(x,y) = -100y - 10001\sin x$.

$J = \frac{\partial f}{\partial y} = -100$; $z = Jh = -100h = -\mu$

$-1 \le 1 - \mu + \mu^2/2 - \mu^3/6 \le 1$.

1) $\frac{\mu^3}{6} - \mu^2/2 + 1 \ge 0$ $\hookrightarrow$ вып-но всегда

2) $\mu^3 - 3\mu^2 + 6\mu - 12 \le 0$

$\mu \le \frac{5}{2} \Rightarrow h \le \frac{1}{40}$

№7.1.18 (продолжение)

Теперь исследуем на А-уя.

Линеаризуем задачу:

$$\Delta_{l+1} - \Delta_l = \frac{3}{2}\underbrace{hJ}_{z}\Delta_l - \frac{1}{2}\underbrace{hJ}_{z}\Delta_{l-1}$$

$$\Delta_l \to q^l$$

$$q^2 - q = \frac{3}{2}zq - \frac{1}{2}$$

$$q^2 - \left(\frac{3}{2}z + 1\right)q + \frac{1}{2} = 0$$

$$q^l = C_1 q_1^l + C_2 q_2^l$$

Условие сх-ти $|q_1| < 1$; $|q_2| < 1$.

$$D = \left(\frac{9}{4}z^2 + 3z + 1\right) - 2 = \frac{9}{4}z^2 + 3z - 1.$$

$$q_{1,2} = \frac{\frac{3}{2}z + 1 \pm \sqrt{\frac{9}{4}z^2 + 3z - 1}}{2 \cdot 1}$$

$$q_1 \underset{\text{при } z \to \infty}{=} \frac{\frac{3}{2}z - \left|\frac{3}{2}z\right|}{2} < -1$$

$\Rightarrow$ не будет А-уя.

о уе =

$\beta_0 = -\frac{1}{2}.$

$\frac{1}{2}\cdot 1 + \frac{3}{2}\cdot 1 + 0] = = 1$

$+0] = 3$

й порядок

№7.2.2.

$u' = -3u + 0\cdot v + 2w$, $u(0) = 1$

$v' = 3\cdot 10^{-2}u - 800v + 5\cdot 10^{-2}w$, $v(0) = 1.$

$w' = u + 0\cdot v - 4w$, $w(0) = 0$

1/10	1/10	0
9/10	8/10	1/10
	1/2	1/2

Вводим вектор неизвестных: $\bar{y} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$

Матрица А:

$$\begin{matrix} -3 & 0 & 2 \\ 0{,}03 & -800 & 0{,}05 \\ 1 & 0 & -4 \end{matrix}$$

$$\left.\begin{matrix} \bar{y}'_x = A\bar{y} \\ \bar{y}(0) = \bar{y} \end{matrix}\right\}; \quad J = \frac{\partial \bar{f}}{\partial y} = A$$

Найдем с.ч. и с.в. матрицы А:

$$\begin{pmatrix} -3-\lambda & 0 & 2 \\ 0{,}03 & -800-\lambda & 0{,}05 \\ 1 & 0 & -4-\lambda \end{pmatrix} = (-3-\lambda)(800+\lambda)(4+\lambda) + (800+\lambda)2 =$$

$$= (800+\lambda)(\lambda\lambda - 12 - 3\lambda - 4\lambda - \lambda^2) =$$

$$= -(800+\lambda)(\lambda^2 + 7\lambda + 10) = 0 =$$

$$= -(800+\lambda)(\lambda+5)(\lambda+2)$$

$\lambda_1 = -800$; $\lambda = -5$; $\lambda = -2$

Число жёсткости:

$$S = \frac{\max|\mathrm{Re}\,\lambda_i|}{\min|\mathrm{Re}\,\lambda_i|} = 400$$

№ 7.1.18

$$\frac{y_{l+1}-y_l}{h}=\frac{3}{2}f(x_l,y_l)-\frac{1}{2}f(x_{l-1};y_{l-1})$$

Ф-ла (7.1.12): $\alpha_k y_{l+k}+\alpha_{k-1}y_{l+k-1}+\ldots+\alpha_0 y_l=$
$=h(\beta_k f_{l+k}+\beta_{k-1} f_{l+k-1}+\ldots+\beta_0 f_l)$

$\alpha_2=1;\ \alpha_1=-1;\ \alpha_0=0;\ \beta_2=0;\ \beta_1=\frac{3}{2};\ \beta_0=-\frac{1}{2}$

Ф-ла: (7.1.13)

1) $\sum\limits_{j=0}^{k=2}\alpha_j=0=1-1+0=\underline{\underline{0}}$

2) $\sum\limits_{j=0}^{k=2}\alpha_j j^q=q\sum\limits_{j=0}^{k=2}\beta_j j^{q-1}$

При $q=1$: $(-1)(+1)^1+1(2)^1=\underline{\underline{1}}=1\left[-\frac{1}{2}\cdot1+\frac{3}{2}\cdot1+0\right]=\underline{\underline{1}}$

При $q=2$:

$(-1)(1)^2+1(2)^2=\underline{\underline{3}}=2\left[-\frac{1}{2}\cdot0+\frac{3}{2}1^1+0\right]=\underline{\underline{3}}$

$\Rightarrow$ Этот метод точно имеет 2-й порядок сходимости.

№ 7.2.2.

$u'=-3u+0\cdot v+2w,\ u(0)=1$

$v'=3\cdot10^{-2}u-800v+5\cdot10^{-2}w,\ v(0$

$w'=u+0\cdot v-4w,\ w(0)$

Вводим вектор неизвестных:

Матрица A:
$$\begin{matrix}-3 & 0 & 2\\ 0{,}03 & -800 & 0{,}05\\ 1 & 0 & -4\end{matrix}$$

$$\left.\begin{matrix}\bar{y}'_x=A\bar{y}\\ \bar{y}(0)=\bar{y}\end{matrix}\right\};\ J=\frac{\partial f}{\partial y}=A$$

Найдем с.ч. и с.в. матрицы

$$\begin{pmatrix}-3-\lambda & 0 & 2\\ 0{,}03 & -800-\lambda & 0{,}05\\ 1 & 0 & -4-\lambda\end{pmatrix}=(-$$
$+(800$

$=(800+\lambda)(\lambda\ -12-3\lambda-$

$=-(800+\lambda)(\lambda^2+7\lambda+10$

$=-(800+\lambda)($

$\lambda_1=-800;\ \lambda=-5;\ \lambda=-$

Число жёсткости:

$$S=\frac{\max|Re\,\lambda_i|}{\min|Re\,\lambda_i|}=400$$

$$\bar{y}_{l+1} = \bar{y}_l + \frac{h}{2}(\bar{f}_1 + \bar{f}_2)$$

$$\bar{f}_1 = J\left(\bar{\Delta}_l + \frac{h}{10}\bar{f}_1\right) \Rightarrow \bar{f}_1 = \frac{J\bar{\Delta}_l}{1 - z/10}$$

$$\bar{f}_2 = J\left(\bar{\Delta}_l + \frac{8}{10}z\frac{\bar{\Delta}_l}{1 - z/10} + \frac{h f_2}{10}\right) \Rightarrow$$

$$\Rightarrow \bar{f}_2\left(1 - \frac{z}{10}\right) = J\bar{\Delta}_l \frac{10 + 7z}{10 - z}$$

$$\Rightarrow \bar{f}_2 = \frac{10 J\bar{\Delta}_l (10 + 7z)}{(10 - z)^2}$$

$$\bar{\Delta}_{l+1} = \bar{\Delta}_l\left(1 + \frac{5z}{10 - z} + \frac{5z(10 + 7z)}{(10 - z)^2}\right) =$$

$$= \left(\frac{100 - 20z + z^2 + 5z + 50z + 35z^2}{(10 - z)^2}\right)\bar{\Delta}_l$$

$$\Rightarrow R(z) = \frac{36z^2 + 35z + 100}{100 - 20z + z^2}$$

Условие устойчивости:

$$z \Rightarrow \lambda_i h = -\mu_i$$

$$-1 \le \frac{36\mu_i^2 - 35\mu_i + 100}{100 + 20\mu_i + \mu_i^2} \le 1$$

а) $35\mu_i \le 55 \Rightarrow \mu_i \le \frac{55}{35} = \frac{11}{7}$

1) $h \le \frac{11}{7 \cdot 800}$ 2) $h \le \frac{11}{35}$ 3) $h \le \frac{11}{14}$

$$\Rightarrow h \le \frac{11}{5600}$$

б) $D < 0$

№ 8.1.2.

$$y''_{xx} - \frac{x}{2}y'(x) + y(x) = 2, \quad 0 \le x \le 1$$

$$\Omega_h = \{x_l : x_l = lh;\ hL = 1,\ l = \overline{0, L},\ L = 2\}$$

$$y(0) = 1;\ y(1) = -1.$$

Общее решение:

$$y(x) = C_1\psi_1(x) + C_2\psi_2(x) + \Psi(x)$$

I. $(\psi_1)''_{xx} - \frac{x}{2}(\psi_1)'_x + \psi_1(x) = 0.$

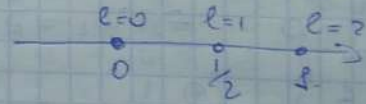
$\psi_1(0) = 0;\ (\psi_1)'_x(0) = -1.$

II. $(\psi_2)''_{xx} - \frac{x}{2}(\psi_2)'_x + \psi_2(x) = 0$

$\psi_2(0) = 1;\ \psi_2'(0) = 0$

III $(\Psi)''_{xx} - \frac{x}{2}\Psi'_x + \Psi = 2.$

$\Psi(0) = 0;\ \Psi'(0) = 0$



$\frac18 - \frac12 =$
$= \frac18 - \frac48 =$
$= -\frac38$

I. $\frac{(\psi_1)_2 - 2(\psi_1)_1 + (\psi_1)_0}{h^2} - \frac{x_1}{2}\frac{(\psi_1)_1 - (\psi_1)_0}{h} + (\psi_1)_1 = 0$

$(\psi_1)_0 = 0$; $\frac{(\psi_1)_1 - (\psi_1)_0}{h} = -1$.

$4((\psi_1)_2 + 1) + \frac12 \frac12 \cdot \frac12 - \frac12 = 0$.

$\Rightarrow (\psi_1)_2 = -\frac{3}{32} - 1 = -\frac{35}{32}$.

Т.о. $\psi_1 = \left(0; -\frac12; -\frac{35}{32}\right)^T$

II. $\frac{(\psi_2)_2 - 2(\psi_2)_1 + (\psi_2)_0}{h^2} - \frac{x_1}{2}\frac{(\psi_2)_1 - (\psi_2)_0}{h} + (\psi_2)_1 = 0$

$(\psi_2)_0 = 1$; $\frac{(\psi_2)_1 - (\psi_2)_0}{h} = 0$

$4((\psi_2)_2 - 1) - \frac12 \frac12 \cdot 0 + 1 = 0 \Rightarrow (\psi_2)_2 = \frac54$.

Т.о. $\psi_2 = \left(1; 1; \frac54\right)^T$

III. $\frac{\psi_2 - 2\psi_1 + \psi_0}{h^2} - \frac{x_1}{2}\frac{\psi_1 - \psi_0}{h} + \psi_1 = 2$.

$\psi_0 = 0$; $\frac{\psi_1 - \psi_0}{h} = 0 \Rightarrow \psi_2 = \frac12$.

$$y(x) = C_1 \begin{pmatrix} 0 \\ -1/2 \\ -35/32 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ 5/4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$$

$C_2 = 1$; $C_1\left(-\frac12\right) + 1 + 0 = \emptyset \Rightarrow C_1 = 0$

$\Rightarrow \bar{y} = \begin{pmatrix} 1 \\ 1 \\ 7/4 \end{pmatrix}$

№ 8.3.2.

$y'' = y^{3/2}$, $y(0) = 1$, $y(2) = 1/2$, $0 \le x \le 2$

$D_h = \{x_l : x_l = lh;\ l = \overline{0, L},\ Lh = 2\}$, $\Delta = |y(2) - y_L| \le 0{,}2$

$l=0$ $l=1$ $l=2$
0 1 2

$|\frac72 \alpha + 2| \le 0{,}2$
$|\frac72 \alpha| \ge 1{,}8$
$-\frac72 \alpha \ge 1{,}8$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \frac32 y_1$

$y_0 = 1$; $\frac{y_1 - y_0}{h} = \alpha \Rightarrow y_1 = \alpha h + y_0/h = \alpha + 1$.

$y_2 - 2(\alpha + 1) + 1 = \frac32(\alpha + 1)$.

$y_2 = \frac32\alpha + \frac32 - 1 + 2\alpha + 2 = \frac72\alpha + \frac52$

Нужно подобрать α: $|y_2(\alpha) - \frac12| \le 0{,}2$

Возьмем, например, $\alpha_0 = -1 \Rightarrow$

$\Rightarrow -\frac72 + \frac52 = -1$ ↳ не подход.

$\alpha_1 = -\frac12$ ↳ не подх

$\alpha_2 = -\frac47$ ↳ не подх.

$\alpha_3 = -0{,}6$ ↳ подходит.

Т.о. $y_2(-0{,}6) = -0{,}1$

№ 8.5.7.

$$\frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] = f(x),\ x\in[0;1],\ y(0)=0,\ y(1)=0$$

$$(u,v)=\int_0^1 uv\,dx$$

Домножим обе части на -1:

$$Ly = -\frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right]$$

а) Самосопряженность:

$$(Lu,v) = (u,Lv)$$

$$(Ly,v) = -\int_0^1 \frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] v\,dx =$$

$$= -\int_0^1 v\,d\left[(x^2+1)\frac{dy}{dx}\right] =$$

$$= -v(x^2+1)\frac{dy}{dx}\Big|_0^1 + \int_0^1 (x^2+1)\frac{dy}{dx}\frac{dv}{dx}\,dx =$$

$$= \int_0^1 (x^2+1)\frac{dv}{dx}\,dy = \text{/по частям/}$$

$$= +(x^2+1)\frac{dv}{dx}y\Big|_0^1 - \int_0^1 \frac{d}{dx}\left[(x^2+1)\frac{dv}{dx}\right] y\,dx =$$

$$= (y, Lv)$$

б) Положительность:

$$(Ly,y)\geq 0;\ (Ly,y)=0 \Leftrightarrow y\equiv 0$$

$$(Ly;y) = -\int_0^1 \frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] y\,dx =$$

$$= \int_0^1 (x^2+1)\left(\frac{dy}{dx}\right)^2 dx \geq 0$$

$\Rightarrow$ решение краевой задачи доставляет min ф-ии

$$F(y) = (Ly,y) - 2(f,y)$$

№ 8.5.8

$$\frac{d}{dx}\left[(x^2-1)\,{}^{dy}\!/_{dx}\right] = f(x),\ x\in[0;1],\ u(0)=0,\ u(1)=0$$

с ск. пр-ием $(u,v)=\int_0^1 uv\,dx$

Отличия от задачи 8.5.7 только в (x^2-1)

$$\Rightarrow (Ly,y) = -\int_0^1 (x^2-1)\left(\frac{dy}{dx}\right)^2 dx$$

— не яв-ся положительно определенным.

№ 8.6.2.

$$D_h = \{x_l : x_l = -1+lh,\ l=\overline{0,2},\ h=1\}$$

$$\begin{cases} y''-5y'+6y=3 \\ y(-1)=1;\ y(1)=1,\ x\in[-1;1] \end{cases}$$

$$\varphi_1(x) = \begin{cases} 0, & |x|\geq 1 \\ x+1, & -1\leq x\leq 0 \\ 1-x, & 0\leq x\leq 1 \end{cases}$$

$$f(x) = y''_{xx} - 5y'_x + 6y - 3$$

$$\int_{-1}^{1} f(x)\varphi_n(x)\,dx = 0,\ n=1,$$

$$f(x)\equiv 0$$

$y = u(x) + 1$ ← любая ф-ция удовл. гр. ум.

$$u(x) = C_1\eta_1 = \sum_{n=1}^{\infty} C_n \varphi_n(x)$$

$$(u,v) = \int_a^b uv\,dx = \int_{-1}^{1} uv\,dx$$

$$\int_{-1}^{1} \{ (C_1\eta_1+1)''_{xx} - 5(C_1\eta_1+1)'_x + 6(C_1\eta_1+1) - 3 \} \eta_1(x)\,dx \equiv 0$$

$$= \int_{-1}^{1} \{ C_1[(\eta_1)''_{xx} - 5(\eta_1)'_x + 6\eta_1] + 6 - 3 \} \eta_1(x)\,dx = 0$$

На 2 ∫-ле, т.к. производная в нуле = 0

$$\int_{-1}^{0} (\eta_1)''_{xx}\eta_1\,dx = (\eta_1)'_x\eta_1 \Big|_{-1}^{0} - \int_{-1}^{0} \left(\frac{d\eta_1}{dx}\right)^2 dx = 0.$$

$$\int_{0}^{1} (\eta_1)''_{xx}\eta_1\,dx = (\eta_1)'_x\eta_1 \Big|_{0}^{1} - \int_{0}^{1} \left(\frac{d\eta_1}{dx}\right)^2 dx = 0$$

$$\int_{-1}^{1} 3\eta_1\,dx = \int_{-1}^{0} 3(x+1)\,dx + \int_{0}^{1} 3(1-x)\,dx =$$

$$= \frac{3(x+1)^2}{2}\Big|_{-1}^{0} - \frac{3(1-x)^2}{2}\Big|_{0}^{1} = \frac{3}{2} + \frac{3}{2} = 3$$

$$C_1\int_{-1}^{0} (-5+6+6x)(1+x)\,dx + C_1\int_{0}^{1} (5+6-6x)(1-x)\,dx + 3 = 0$$

$$= C_1\int_{-1}^{0} (1+7x+6x^2)\,dx + C_1\int_{0}^{1} (11-17x+6x^2)\,dx + 3 = C_1\left[1 - \frac{7}{2} + 2 + 11 - \frac{17}{2} + 2\right] + 3 =$$

$$= 4C_1 + 3 \Rightarrow C = -3/4$$

$$y = -\frac{3}{4}\eta_1(x) + 1$$

№ 8.6.4

$D_h - \{x_l : x_l = lh,\ l = \overline{0,2},\ h = 1/2\}$

$$\begin{cases} y'' - 2y' + 3y = 5x \\ y(0) = 1;\ y(1) = -3,\ x \in [0;1] \end{cases}$$

$$\eta_1(x) = \begin{cases} 0, & |x - \frac{1}{2}| \geq \frac{1}{2} \\ 8x^2(1-x) = 8x^2 - 8x^3, & |x - \frac{1}{2}| \leq \frac{1}{2} \Leftrightarrow -\frac{1}{2} \leq x - \frac{1}{2} \leq \frac{1}{2},\ 0 \leq x \leq 1 \end{cases}$$

$$f(x) = y'' - 2y' + 3y - 5x$$

$$\int_0^1 f(x)\varphi_n(x)\,dx = 0$$

$$f(x) \equiv 0$$

$$y = u(x) - 4x + 1$$

$$u(x) = C_1\eta_1 = \sum_{n=1}^{\infty} C_n\varphi_n(x) \quad ; \quad (u,v) = \int_0^1 uv\,dx$$

$$\int_0^1 \{ (C_1\eta_1 - 4x + 1)'' - 2(C_1\eta_1 - 4x + 1)' + 3(C_1\eta_1 - 4x + 1) - 5x \} \eta_1(x)\,dx$$

$$= \int_0^1 \{ C_1\eta_1'' - 2C_1\eta_1' + 8 + 3C_1\eta_1 - 12x + 3 - 5x \} \eta_1(x)\,dx$$

$$= \int_0^1 \{ C_1[\eta_1'' - 2\eta_1' + 3\eta_1] - 17x + 11 \} \eta_1(x)\,dx =$$

$$= \int_0^1 \{ C_1[16 - 48x - 16x + 24x^2 + 24x^2 - 24x^3] - 17x + 11 \}.$$

$\cdot \eta_1(x)\,dx =$

$8x^4 - 8x^3$

$$= \int_0^1 \{ C_1 [-24x^3 + 48x^2 - 16x + 16] - 17x + 11 \} \eta_1(x)\,dx =$$

$$= C_1 \left(-\frac{24 \cdot 8}{6} + \frac{24 \cdot 8}{7} \right) + C_1 \left(\frac{48 \cdot 8}{5} - \frac{48 \cdot 8}{6} \right) +$$

$$+ C_1 \left(-\frac{16 \cdot 8}{4} + \frac{16 \cdot 8}{5} \right) + C_1 \left(\frac{16 \cdot 8}{3} - \frac{16 \cdot 8}{4} \right) +$$

$$+ \left(-\frac{17 \cdot 8}{4} + \frac{17 \cdot 8}{5} \right) + \frac{11 \cdot 8}{3} - \frac{11 \cdot 8}{4} = 0.$$

32, -32, 64, 32, 22, $64 \cdot 8$

$$C_1 \left(-160 + \frac{24 \cdot 8}{7} + \frac{48 \cdot 8}{5} + \frac{16 \cdot 8}{5} + \frac{16 \cdot 8}{3} \right) =$$

$$= 56 - \frac{17 \cdot 8}{5} - \frac{11 \cdot 8}{3}.$$

$$C_1 \frac{1984}{105} = -\frac{8}{15} \Rightarrow C_1 = -\frac{7}{248}$$

$$\text{Т.о. } y(x) = -\frac{7}{248} \eta_1(x) - 4x + 1.$$

№ 2.2.6.

$D_h = \{ x_l : x_l = lh,\ hL = 1,\ l = \overline{0, L} \}$

Аппроксимация II порядка.

$$y''(x) + \frac{1+x}{1+x^2} y'(x) + \cos x \cdot y(x) = \frac{1}{1+x^2}$$

$$2y'(0) + y(0) = 3;\ y'_x(1) = 2;\quad 0 \le x \le 1.$$

Разностная задача.

$$\begin{cases} \dfrac{y_{l+1} - 2y_l + y_{l-1}}{h^2} + \dfrac{1+x_l}{1+x_l^2} \dfrac{y_{l+1} - y_{l-1}}{2h} + \cos x_l\, y_l = \dfrac{1}{1+x_l^2} \\ 2\dfrac{y_1 - y_0}{h} + y_0 - \{ 1 - \tfrac{3}{2}(y_0 - 1) \} h = 3 \\ \dfrac{y_L - y_{L-1}}{h} + \dfrac{h}{2} \{ y_L \cos 1 + \tfrac{3}{2} \} = 2 \end{cases}$$

Аппроксимация:

$$\delta f^{(h)} = L_h [y]_h - f^{(h)} =$$

$$\frac{[y]_{l+1} - 2[y]_l + [y]_{l-1}}{h^2} + \frac{1+x_l}{1+x_l^2} \frac{[y]_{l+1} - [y]_{l-1}}{2h} + \cos x_l \cdot [y]_l - \frac{1}{1+x_l^2}$$

$$2\frac{[y]_1 - [y]_0}{h} + [y]_0 - 3$$

$$\frac{[y]_L - [y]_{L-1}}{h} - 2$$

$$[y]_{l \pm 1} = [y]_l \pm [y'_x]_l + [y''_{xx}]_l \frac{h^2}{2} \pm [y'''_{xxx}] \frac{h^3}{3} + O(h^4)$$

$$[y]_1 = [y]_0 + [y'_x]_0 h + [y''_{xx}]_0 \frac{h^2}{2} + O(h^3)$$

$$[y]_{L-1} = [y]_L - [y'_x]_L h + [y''_{xx}]_L \frac{h^2}{2} + O(h^3)$$

$$\Leftrightarrow \begin{cases} \cancel{[y''_{xx}]_l + O(h^2) + \dfrac{1+x_l}{1+x_l^2}[y'_x]_l + \cos x_l\, y_l - \dfrac{1}{1+x_l^2}} \\ \cancel{2[y'_x]_0 + [y''_{xx}]_0 \dfrac{h}{2} + O(h^2) - 3 + [y]_0} \\ [y'_x]_L - [y''_{xx}]_L \dfrac{h}{2} + O(h^2) - 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} O(h^2) \\ [y''_{xx}]_0 h + O(h^2) \\ -[y''_{xx}]_L \dfrac{h}{2} + O(h^2) \end{cases}$$

Надо добавить с противоположными знаками

$$[y''_{xx}]_0 = 1 - [y]_0 - \overbrace{[y'_x]_0}^{\frac{[y]_0-3}{2}} = 1 - \frac{3}{2}(y_0 - 1)$$

$$[y''_{xx}]_L = \frac{1}{2} - [y]_L \cos 1 - [y'_x]_L = -[y]_L \cos 1 + \frac{3}{2}$$

№ 9.2.

$\lambda_1 = ?$

$$\frac{y_{l+1} - 2y_l + y_{l-1}}{h^2} = -\lambda y_l, \quad l = \overline{1, L-1}$$

$y_0 = 0$; $y_L = 0$

$y_l \Rightarrow q^l$

$$\frac{q^2 - 2q + 1}{h^2} = -\lambda q \Rightarrow q^2 - 2\left(1 - \frac{\lambda h^2}{2}\right)q + 1 = 0$$

– характеристич. ур-ние

$y_l = C_1 q_1^l + C_2 q_2^l$

$y_0 = 0$: $C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$

$y_L = 0$: $C_1(q_1^L - q_2^L) = 0 \Leftrightarrow$

По Th. Виета: $q_2 \cdot q_1 = 1 \Rightarrow q_2 = \dfrac{1}{q_1}$

$\Leftrightarrow C_1(q_1^L - q_1^{-L}) = 0 \Rightarrow q_1^{2L} = 1 = e^{i2\pi n}$

$\Rightarrow q_1 = e^{i\pi n/L}$; $q_2 = e^{-i\pi n/L}$

$$q_1 + q_2 = 2\left(1 - \frac{\lambda_n h^2}{2}\right) = 2\cos\frac{\pi n}{L}$$

$$\Rightarrow \lambda_n = \frac{4}{h^2}\sin^2\frac{\pi n}{2L}, \quad n = \overline{1, L-1}$$

Собств. ф-ции:

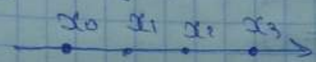
$$y_l = C_1\left(e^{i\frac{\pi n l}{L}} - e^{-i\frac{\pi n l}{L}}\right) = C_1' \sin\frac{\pi n l}{L}, \quad l = \overline{0, L}$$

№ 9.8.

$D_h = \{x_l : x_l = lh;\ l = \overline{0,3};\ h = 1\}$

$\lambda_m = ?$

$$\begin{cases} y'' + \lambda(1+x)y = 0 \\ y'(0) = y(3) = 0, \quad 0 \le x \le 3 \end{cases}$$



$x=x_1$: $\frac{y_2-2y_1+y_0}{h^2}+\lambda(1+x_1)y_1=0$

$x=x_2$: $\frac{y_3-2y_2+y_1}{h^2}+\lambda(1+x_2)y_2=0$

$y_3=0$; $\frac{y_1-y_0}{h}=0 \Rightarrow y_1=y_0$.

$y_2-y_1+2\lambda y_1=0$

$(3\lambda-2)y_2+y_1=0$

$$\begin{vmatrix} 2\lambda-1 & 1 \\ 1 & 3\lambda-2 \end{vmatrix}=6\lambda^2-7\lambda+1=0$$

$\lambda_{1,2}=\frac{7\pm5}{2\cdot6}$

$\lambda_m=\frac{1}{6}$; $\lambda_M=1$

№9.1.

$\frac{y_{l+1}-2y_l+y_{l-1}}{h^2}=-\lambda y_l$, $y_0=y_1$; $y_N=0$, $l=\overline{1,N-1}$

$y_l \Rightarrow q^l$:

$\frac{q^2-2q+1}{h^2}=-\lambda q$

$q^2-2\left(1-\frac{\lambda h^2}{2}\right)q+1=0$

$y_l=C_1q_1^l+C_2q_2^l$

$y_0=y_1$: $C_1+C_2=C_1q_1+C_2q_2$

$y_N=0$: $C_1q_1^N+C_2q_2^N=0$

$C_1(1-q_1)=C_2(q_2-1) \Rightarrow C_1/C_2=\frac{q_2-1}{1-q_1}$

$\underline{q_1\cdot q_2=1}$,

$\frac{C_1}{C_2}\frac{q_1^N}{q_2^N}+1=\frac{q_2-1}{1-q_1}q_1^{2N}+1=0$

$\frac{1/q_1-1}{1-q_1}q_1^{2N}+1 \Rightarrow q_1^{2N-1}=-1=e^{i\pi n}$

$\Rightarrow q_1=e^{i\frac{\pi n}{2N-1}}$; $q_2=e^{-i\frac{\pi n}{2N-1}}$

$q_1+q_2=e^{i\frac{\pi n}{2N-1}}+e^{-i\frac{\pi n}{2N-1}}=2\left(1-\frac{\lambda h^2}{2}\right)=$

$=2\cos\frac{\pi n}{2N-1}$.

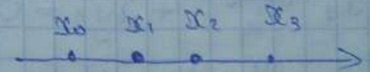
$\Rightarrow \lambda_n=\frac{4}{h^2}\sin^2\frac{\pi n}{2(2N-1)}$

№9.10.

$D_h=\{x_l:\ x_l=lh;\ l=\overline{0,3},\ h=1\}$

$\lambda_m=?$

$$\begin{cases} y''+\lambda xy=0 \\ y'(0)=y(3)=0,\ 0<x\le3 \end{cases}$$



$x = x_1$: $\dfrac{y_2 - 2y_1 + y_0}{h^2} + \lambda x_1 y_1 = 0$

$x = x_2$: $\dfrac{y_3 - 2y_2 + y_1}{h^2} + \lambda x_2 y_2 = 0$

$y_3 = 0$; $\dfrac{y_1 - y_0}{h} = 0 \Rightarrow y_1 = y_0$:

$y_2 - y_1 + \lambda y_1 = 0$

$-2y_2 + y_1 + 2\lambda y_2 = 0$

$$\begin{vmatrix} \lambda - 1 & 1 \\ 1 & 2\lambda - 2 \end{vmatrix} = 2\lambda^2 - 2\lambda - 2\lambda + 2 - 2 =$$

$$= 2\lambda^2 - 4\lambda = 2\lambda(\lambda - 2) = 0$$

$$\underline{\underline{\lambda = 2}}$$

№ 9.18.

① $D_h = \{x_i : x_i = ih;\ i = \overline{0,3},\ h = 1\}$

$\lambda_m = ?$

$y'' - 2xy' + \lambda y = 0$

$y(0) = y'(3) = 0$, $0 \le x \le 3$

$x_0 \quad x_1 \quad x_2 \quad x_3$

$x = x_1$: $\dfrac{y_2 - 2y_1 + y_0}{h^2} - 2x_1 \dfrac{y_1 - y_0}{h} + \lambda y_1 = 0$

$x = x_2$: $\dfrac{y_3 - 2y_2 + y_1}{h^2} - 2x_2 \dfrac{y_2 - y_1}{h} + \lambda y_2 = 0$

$y_0 = 0$; $y_3 = y_2$

$y_2 - 2y_1 - 2y_1 + \lambda y_1 = 0$

$-y_2 + y_1 - 4y_2 + 4y_1 + \lambda y_2 = 0$

$$\begin{vmatrix} \lambda - 4 & 1 \\ 5 & \lambda - 5 \end{vmatrix} = \lambda^2 - 5\lambda - 4\lambda + 20 - 5 =$$

$$= \lambda^2 - 9\lambda + 15 = 0$$

$$\lambda = \frac{9 \pm \sqrt{81 - 60}}{2} = \frac{9 \pm \sqrt{21}}{2}$$

$$\Rightarrow \lambda_{min} = \frac{9 - \sqrt{21}}{2}$$

$\exists\, y(x)$ – точное решение диф. ур-ния