Asymptotic Behavior of Lévy Walks

BACHELOR THESIS

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by
Marius Bothe
born 23. February 1995

Supervising tutors:

1. *Prof. Dr. Igor Sokolov*
2. *Dr. Michael Zaks*

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1. Introduction

The Lévy walk is a space-time coupled continuous time random walk (CTRW) that was first presented in [4] as a way to model the anomalous diffusion found by Richardson [2]: When studying turbulent flow in the atmosphere he observed that a particles msd would diffuse with a cubic time dependence $\langle x^2 \rangle \propto t^3$, the so called Richardson regime.

The Lévy walk had two distinguishing properties

Continuous time random walks are frequently used to model and understand stochastic processes. Of particular interest for this thesis

2. Theoretical Background

2.1 The model

2.1.1 Lévy walks

The original motivation for the creation of the Lévy walk model goes back to the work of Richardson in 1926 [2], who studied the motion of particles in the turbulent flow of the atmosphere. Such a system contains jets and eddies that affect the behavior of the particle and lead to anomalous diffusion. In particular Richardson found that the mean squared displacement (MSD) of the particle scales with the third power of the time, i.e.

$$\langle \mathbf{x}^2 \rangle(t) \propto t^3, \tag{2.1}$$

which is known as the Richardson regime.

There were several attempts to find a random walk model that replicates this behavior. These attempts found that power-law models were particularly suitable for describing superdiffusion [1] which lead to the creation of the Lévy flight model: In this model the walker jumps instantaneously in a random direction with a jump length drawn from a distribution $g(|\,\mathbf{x}\,|)$. He now waits at the turning point for the duration of the waiting time, which is drawn from the distribution $\psi(t)$ and then performs a new jump in another direction, as can be seen in figure (2.1). Both the waiting time and the jump length distributions are power-laws, meaning for large arguments they take the form

$$\psi(t) \propto t^{-1-\gamma}, \qquad g(|\,\mathbf{x}\,|) \propto |\,\mathbf{x}\,|^{-1-\beta}, \qquad \gamma, \beta > 0. \tag{2.2}$$

However the Lévy flight model has a major drawback: Since the jumps happen instantaneously it has an infinite propagation speed, which causes its MSD and all higher moments to diverge [3].

Therefore the Lévy walk model was developed by Shlesinger, Klafter and West [4]. Here the walker no longer waits at the turning points, but his jumps now have a finite duration, turning them into steps. The step duration is coupled to the length of the step and prevents the infinite propagation speed that caused problems with the Lévy flights, which is illustrated in figure 2.1.

[1] meaning diffusion where $\langle \mathbf{x}^2 \rangle(t) \propto t^{1+\alpha}$, $\alpha > 0$

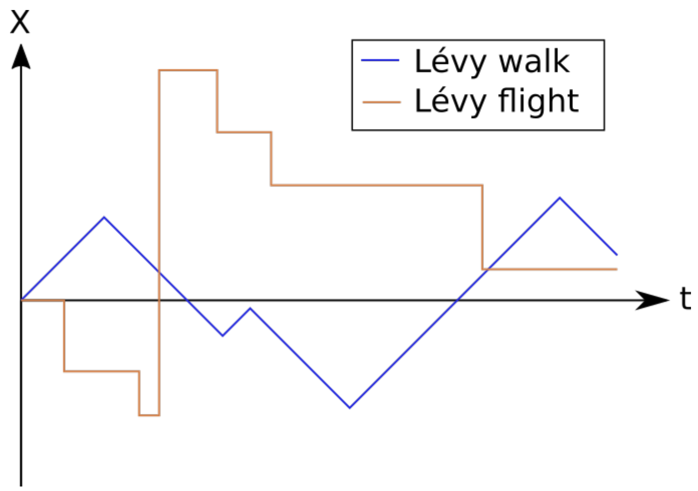

Figure 2.1: Comparison between the trajectories of the one dimensional Lévy flight and Lévy walk (for $\nu = 1$). Note that the jump length of the Lévy flight is independent of the waiting time.

The path of a walker in the new model is now described by a series of step durations $t_1, t_2, ...$ which are drawn from the power-law distribution

$$\psi(t_i) = \frac{\gamma}{t_0} \frac{1}{(1 + t_i/t_0)^{\gamma+1}}. \tag{2.3}$$

Here the parameter $\gamma > 0$ governs the width of the distribution and t_0 is the timescale of a step . These step durations are associated with their respective steps vectors $\mathbf{x}_1, \mathbf{x}_2, ...$, whose direction is chosen randomly. By partially summing up the step durations and the step lengths one obtains the turning times T_n and the turning points $\mathbf{X}_n$ respectively:

$$T_n = \sum_{j=1}^{n} t_j, \qquad \mathbf{X} = \sum_{j=1}^{n} \mathbf{x}_j \tag{2.4}$$

The walker is now being observed at the observation time t: Let the last turning time before t be $T_n = \max\{T_i | T_i \leq t\}$, then the distance covered from the last turning point is given by

$$|\,\mathbf{x}_{n+1}\,| = c(t_{n+1})^{\nu-1}(t - T_n), \tag{2.5}$$

where c is a constant with dimension $[st^{-\nu}]$. The speed

$$v = \frac{\partial}{\partial t} |\,\mathbf{x}_{n+1}\,| = c(t_{n+1})^{\nu-1} \tag{2.6}$$

is therefore constant during the entire step but depends on the step duration t_{n+1}, where the parameter $\nu > 0$ governs this dependence.
For any completed step we can now write down the joint probability to make a step of length $|\,\mathbf{x}\,|$ and duration t:

$$\psi(\mathbf{x},t) = \frac{\gamma}{t_0}\frac{1}{(1+t/t_0)^{\gamma+1}}\frac{\delta(|\,\mathbf{x}\,| - ct^{\nu})}{|\,\mathbf{x}\,|^{d-1}\,|\,S^{d-1}\,|}. \tag{2.7}$$

Here d is the spatial dimension of the process and $\left| S^{d-1} \right|$ is the surface area of a d-dimensional unit ball. Note that both the step duration distribution and the joint distribution are denoted by ψ, but their arguments are different. In conclusion we have a model that is governed by two parameters, ν and γ and can produce different kinds of anomalous diffusion.

Because of this versatility the Lévy walk model is used to describe a variety of systems: Besides the application in turbulent systems for which the model was originally invented it finds application in field like biology, where the special case of fixed velocities ($\nu = 1$) is used to approximate the motion of E. coli bacteria, who move with the help of microscopic flagella. These flagella either rotate in a synchronized manner, which leads to long stretches of relatively fast movement, or unsynchronized, which leads to a tumbling motion in which the bacterium changes its direction. The resulting motion was found to follow a power-law distribution with parameter $\gamma = 1.2$ [5]

more examples: light scattering, chaotic Hamiltonian systems

However it was found recently in [6] that the MSD of the model is actually divergent for certain values of its parameters, a fact that had previously gone unnoticed for the three decades of the models existence. The divergence can be seen directly when one writes down the contribution to the second moment of the distribution from the trajectories, that consist only of a single step longer than the observation, i.e. where the particle never stops:

$$\langle \mathbf{x}^2 \rangle (t) \geq \int_{\mathbb{R}^d}\int_t^{\infty} |\,\mathbf{x}\,|^2\,(t')\psi(\mathbf{x},t')dt'd^dx \tag{2.8}$$

$$= \frac{\gamma}{t_0}\int_0^{\infty}\int_t^{\infty} |\,\mathbf{x}\,|^2\,(t')\frac{1}{(1+t'/t_0)^{\gamma+1}}\delta(|\,\mathbf{x}\,| - c(t')^{\nu-1}t)dt'd\,|\,\mathbf{x}\,| \tag{2.9}$$

$$= \frac{\gamma t^2}{t_0}\int_t^{\infty} \frac{c^2(t')^{2\nu-2}}{(1+t'/t_0)^{\gamma+1}}dt'. \tag{2.10}$$

The integrand is proportional to $(t')^{2\nu-\gamma-3}$, therefore the integral will diverge at infinity whenever $2\nu \geq \gamma + 2$ holds. This includes the parameter region where the Richardson regime was expected, so the model that was essentially invented to cure the divergence in the description of the Richardson regime turns out to be divergent itself. In order to remedy this, a more general model model is necessary.

Say something about ballistic cone?

2.1.2 Generalized Lévy walks

Because the divergence of the second moment is caused by very long steps that result in arbitrarily high velocities throughout the entire step, a solution can be found by letting the particle start with a lower initial speed and compensating for the slower start by accelerating it throughout the step, so that it catches up with its constant velocity counterpart at the end of the step.
There are indeed some models that describe particles under acceleration [7, 8] that

are similar to the Drude model for solids. There it is shown that the MSD exists in the regime where one would expect the Richardson law.
Furthermore in the supplementary material of [6] a model is presented that introduces an additional parameter, η, that allows one to interpolate between the original Lévy model and the Drude like sheme. It is this model, which I will call generalized Lévy walk model, that I will investigate in this thesis.

The generalized model uses the same distribution of step durations as the previous model (**??**), but the position between two turning points is calculated differently. Instead of a linear time dependence we now have a dependence on the new parameter η for the displacement in the (n+1)th step:

$$|\,\mathbf{x}_{n+1}\,| = c(t_{n+1})^{\nu-\eta}(t - T_n)^{\eta}. \tag{2.11}$$

Therefore the particle moves in general with a non-constant speed

$$v = c\eta(t_{n+1})^{\nu-\eta}(t - T_n)^{\eta-1}. \tag{2.12}$$

We note, that this changes neither the turning points, nor the turning times and the distribution of completed steps is still given by

$$\psi(\mathbf{x}, t) = \frac{\gamma}{t_0}\frac{1}{(1 + t/t_0)^{\gamma+1}}\frac{\delta(|\,\mathbf{x}\,| - ct^{\nu})}{|\,\mathbf{x}\,|^{d-1}\,|\,S^{d-1}\,|}. \tag{2.13}$$

However the position in between the points now depends a on η, as illustrated in figure (2.2). This affects the last incomplete steps of the walk, which we have seen to be responsible for the divergence in the original model.
To summarize, the generalized model now depends on three parameters: ν determines how the step length of the walker depends on the step durations, η governs the acceleration in between the turning points and γ describes the width of the waiting time distribution. The value of γ has a major influence on the general properties of the model: It determines the power law of $\psi(t)$, such that for $\gamma < 2$ the mean squared step duration diverges and therefore the distribution of culminated duration is no longer subject to the central limit theorem, which states that the distribution of a sum of independent identically distributed random variables with finite variance converges towards a normal distribution.
Therefore the distribution of the sum of waiting times, and by extension the probability distribution function (PDF) of the walk, do not converge to a Gaussian. Instead it was shown by Paul Lévy that in this case the distribution of the sum of the variables converges to one of the so called Lévy alpha stable distributions [3], which is the statement of the generalized central limit theorem. It is this close relation to Lévy distributions that gave the Lévy flights and later the Lévy walks their name.

get first hand source(s) for this

A second important quality of the Lévy walk model in general and the generalized model in particular is that it is a semi-Markov process. A process is considered Markov if the behavior of the walker after a point in time t_0 only depends on its position and velocity at t_0, not on its history. For a CTRW, for which Lévy walks are an example of, this is usually not the case, as information about the last step,

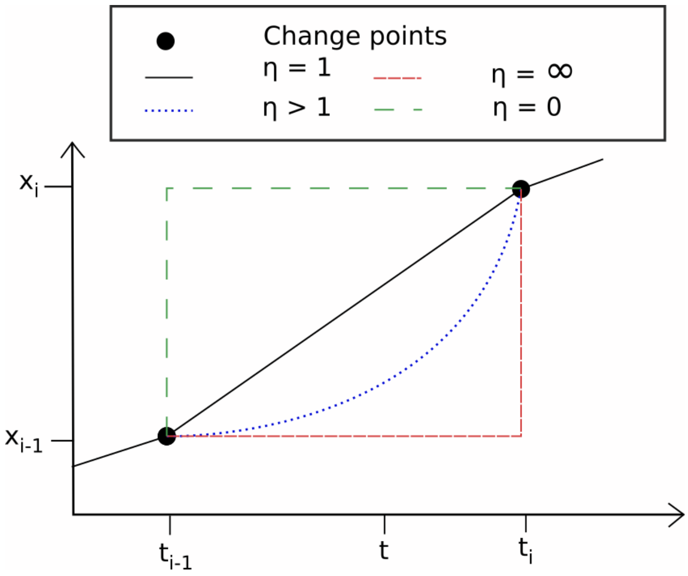

Figure 2.2: Comparison of the walkers motion for different values of η: The trajectories and turning points remain the same, but the position measured at time t varies. For $\eta = 1$ the walker moves with constant speed and has some non-linear time dependence for $\eta > 1$. In the limits $\eta = 0$ and $\eta = \infty$ we replicate the time-coupled Lévy flight, where the two limits correspond to the walker jumping first and the then waiting or waiting first and then jumping.

i.e. how long the walker is already moving, is important for predicting the future behavior (with the exception of Gaussian step distributions). However in our case this memory only extends to the last previous step, and the process is renewed at every turning point, therefore it is semi-Markov [3].
Since the path of the walker is dependent on its behavior prior to the beginning of observation it is of great interest to capture this dependence through suitable initial distributions. This was investigated for similar models in [9,10] and in this thesis the distribution of the first turning point $F(x,t|t_a)$ conditioned on the process aging for a time t_a is of major importance, as it is needed to calculate the MSD of the aged walk.

A third related property is weak ergodicity breaking in CTRWs. A process breaks ergodicity if its time averages and ensemble averages do not converge to the same values, usually because the trajectory of the solution observed in the time average can not reach the entire phase space, thus giving it only a partial sample. However in the case of weak ergodicity breaking the particle is able to reach the entire phase space, but the time to do this is on the same scale or larger then the total observation time, which means it does not converge reliably to the same value. Ergodicity breaking is of great interest for the theoretical as well as the experimental community as it determines what results we can expect from different kinds of measurements. Power law distributions as used in Lévy process are closely connected to weak ergodicity breaking, as their typical timescale for reaching a convergent average is divergent in subdiffusive regimes [11], and this behavior has been studied for example in [6,12].
While this thesis will not investigate the ergodicity breaking of the new model explicitly, this property is connected to the aged behavior of the walk, which is computed

for the MSD All averages are understood to be ensemble averages throughout the thesis.

mention further generalizations of the model

2.2 Theory of random walks

In this chapter I will briefly cover some of the main results of the theory of random walks that are used in this thesis. A more detailed description can be found in [13].

2.2.1 Continuous time random walks

Mean number of steps

The mean number of steps taken in a walk, $\langle n\rangle(t)$, gives an estimate for how fast a walker reaches the regime of asymptotic behavior that is calculated in this thesis and is therefore important for the comparison of analytical results with numerical simulations.
To derive a general expression for $\langle n\rangle(t)$ it is necessary to introduce three auxiliary quantities: The survival probability $\Psi(t)$ describes the chance of a time stretch to last longer than t and can be written as

$$\Psi(t) = \int_t^\infty \psi(t')dt', \tag{2.14}$$

where $\psi(t)$ is the distribution of time stretches. The Laplace transform of such an integral is known and results in

$$\Psi(s) = \frac{1-\psi(s)}{s}. \tag{2.15}$$

Additionally we need the probability of starting the n-th step at time t, denoted by $\psi_n(t)$. It obeys the recursion relation

$$\psi_n(t) = \int_0^t \psi_{n-1}(t')\psi(t-t')dt', \tag{2.16}$$

Using the convolution property of the Laplace transform (2.29) and induction we find in the Laplace domain:

$$\psi_n(s) = \psi^n(s). \tag{2.17}$$

These two quantities allow us to write down the probability of being in the n-th step at time t, $\chi(t)$, i.e. the probability of having started the n-th step at time t' and this step lasting longer than $t-t'$:

$$\chi_n(t) = \int_0^t \psi_n(t')\Psi(t-t')dt'. \tag{2.18}$$

Going into the Laplace domain and using results (2.15) and (2.17) we arrive at

$$\chi_n(s) = \psi_n(s)\Psi(s) = \psi^n(s)\frac{1-\psi(s)}{s}. \tag{2.19}$$

The desired mean number of steps can now be expressed as the sum over the χ_n:

$$\langle n\rangle(t) = \sum_{n=0}^\infty n\chi_n(t) \tag{2.20}$$

which yields a closed expression in the Laplace domain:

see how it de- pends on exis- tence of first

$$\langle n\rangle(s) = \frac{1-\psi(s)}{s}\sum_{n=0}^{\infty} n\psi^n(s) \tag{2.21}$$

$$= \frac{\psi(s)}{s(1-\psi(s))}. \tag{2.22}$$

2.2.2 Space-time coupled continuous time random walks

smooth in- tro- duc- tion from pre- vi- ous chap- ter

For a space-time coupled CTRW such as the Lévy walk a each step is determined by a joint distribution of both the step duration t and the step distance $\mathbf{x}$, where the coupling between space and time is introduced by one being conditioned on the other:

$$\psi(\mathbf{x}, t) = \psi(t) f(\mathbf{x}|t). \tag{2.23}$$

We are now interested in the distribution of completed steps $C(x,t)$, which is the probability density that a particle starting at $\mathbf{x} = 0$, $t = 0$ reaches a turning point at time t and position $\mathbf{x}$ after an arbitrary number of steps in between.
A transport equation can be written down for $C(x,t)$ [13], which reads

$$C(\mathbf{x}, t) = \int_{-\infty}^{\infty} d^d\mathbf{x}' \int_0^t dt' C(\mathbf{x}', t')\ \psi(\mathbf{x} - \mathbf{x}', t - t') + \delta(t)\delta(\mathbf{x}). \tag{2.24}$$

The two terms on the right hand side express two different contributions to the probability of finding a turning point at $(t, \mathbf{x})$: The first term is a convolution integral in $\mathbf{x}'$ and t'. It expresses the tautology that there will be a turning point at $(\mathbf{x}, t)$ exactly when there has been a turning point at the primed coordinates $(\mathbf{x}', t')$ and the particle performed a jump with displacement $\mathbf{x} - \mathbf{x}'$ and duration $t - t'$ from $(\mathbf{x}', t')$ to $(\mathbf{x}, t)$.
The second term just enforces that by definition of the density we will find the particle at the origin at the beginning of observation.
To evaluate this integral equation it is useful to go to the Fourier Laplace domain, where the transformations are defined as follows:

The Laplace transform of a function $f(t)$ is defined as the integral

$$\mathcal{L}\{f(t), s\} = f(s) = \int_0^{\infty} e^{-st} f(t) dt. \tag{2.25}$$

Note that the distinction between the function and its transform is only made in the argument of the function, which is either t or s.
The Laplace transform is unique up to a set of points with Lebesgue measure zero and can be inverted via the Bromwich integral

$$\mathcal{L}^{-1}\{f(s), t\} = f(t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{+i\infty+c} e^{st} f(s) ds, \tag{2.26}$$

where $c \in \mathbb{R}$ is chosen such that $f(s)$ exists on the contour.

move this to the first place

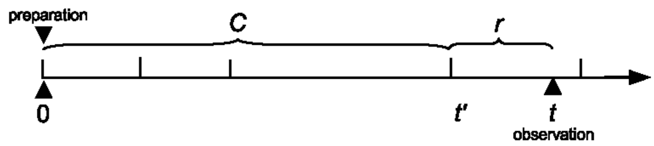

Figure 2.3: Illustration of the path of a Lévy walker on the time axis. Each tick on the line represents a turning time. The walker starts at $t = 0$ and is observed at time t during a final incomplete step described by the distribution $r(\mathbf{x}, t)$ after it has completed a series of steps, which is described by $C(\mathbf{x}, t)$.

For Fourier transforms I use the variables $\mathbf{x} \leftrightarrow \mathbf{k}$, with the distinction between the function and its transform being made only via the argument. It is defined as

$$\mathcal{F}\left\{f(\mathbf{x}), \mathbf{k}\right\} = f(\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d^d x, \tag{2.27}$$

with inverse

$$\mathcal{F}^{-1}\left\{f(\mathbf{k}), \mathbf{x}\right\} = f(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}) d^d k, \tag{2.28}$$

where the normalization factor $\frac{1}{2\pi}$ is kept in the inverse transform.

A useful property of the Fourier and the Laplace transform is that they turn convolutions of functions into simple products. In particular:

$$\mathcal{L}\left\{\int_0^t f(t')g(t-t')dt', s\right\} = f(s)g(s), \tag{2.29}$$

and

$$\mathcal{F}\left\{\int_{\mathbb{R}^d} f(\mathbf{x}')g(\mathbf{x}-\mathbf{x}')d^d x', \mathbf{k}\right\} = f(\mathbf{k})g(\mathbf{k}) \tag{2.30}$$

Applying this to the integral equation 2.24 we obtain

$$C(\mathbf{k}, s) = C(\mathbf{k}, s)\ \psi(\mathbf{k}, s) + 1 \tag{2.31}$$

and therefore

$$C(\mathbf{k}, s) = \frac{1}{1 - \psi(\mathbf{k}, s)} \tag{2.32}$$

Our next goal is to use this result to write down an expression for the probability distribution of the ordinary, meaning non-aged, Lévy walk $p(\mathbf{x}|t)$. It describes the probability of finding the walker at $\mathbf{x}$ given that it is observed at time t, where it can either be at a turning point or in motion, as illustrated in figure (2.3).

In order to write down a transport equation for $p(\mathbf{x}|t)$ we need to introduce the probability density for the rest of the walk after the last turning point, denoted by $r(\mathbf{x}|t)$ [2]. As shown in figure (2.3) $r(\mathbf{x}|t)$ describes the probability of the walker being

[2] Throughout the thesis I will use capital letters for probability densities that depend jointly on space and time, like $C(\mathbf{x}, t)$ and lower case letters for densities that depend only on space and are conditioned on time, like $r(\mathbf{x}|t)$. Note that the former have dimension $[L^{-d}t^{-1}]$, while the latter have dimension $[L_{-d}]$.

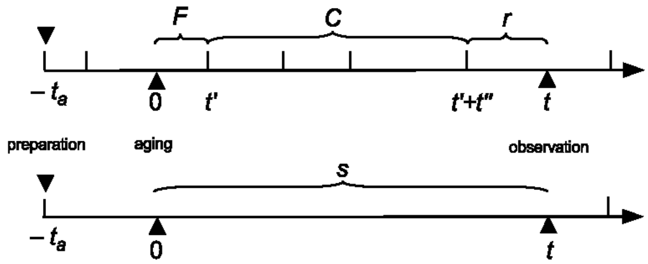

Figure 2.4: Illustration of an aged Lévy walk on the time axis. Each tick on the line represents a turning time. The walker starts at $-t_a$ and observation begins at $t = 0$. The upper picture shows the case where the first turning time t' is smaller than the observation time t, described by F. From here it performs a series of completed steps and a final incomplete step as in the ordinary case. The lower picture shows the case that the first turning point is after the end of observation, i.e. the walker never turns during observation. The probability density of this event is given by s.

displaced by the vector $\mathbf{x}$ after a given time t during a step whose total duration is larger or equal to t:

$$r(\mathbf{x}|t) = \delta(\mathbf{x} - \mathbf{x}'(t' = t)) \int_t^\infty \psi(\mathbf{x}', t')dt'. \tag{2.33}$$

With this expression for $r(\mathbf{x}|t)$ we can now write down $p(\mathbf{x}|t)$ as a convolution of $r(\mathbf{x}|t)$ and $C(\mathbf{x}, t)$

$$p(\mathbf{x}|t) = \int_{\mathbb{R}^d} \int_0^t C(\mathbf{x}', t')r(\mathbf{x} - \mathbf{x}'|t - t')dt'd^dx', \tag{2.34}$$

which describes the particle starting at the origin, performing a series of completed steps ending at $\mathbf{x}'$ and then performing a final, incomplete step that leaves it at position $\mathbf{x}$ at observation time t.
We now use the convolution properties of the Fourier and the Laplace transform, equations (2.30) and (2.29), to obtain

$$p(\mathbf{k}|s) = C(\mathbf{k}, s)r(\mathbf{k}|s). \tag{2.35}$$

Substituting for $C(\mathbf{k}, s)$ with formula (2.32) yields

$$p(\mathbf{k}|s) = \frac{r(\mathbf{k}|s)}{1 - \psi(\mathbf{k}, s)}, \tag{2.36}$$

which gives us an algebraic equation for the PDF in the Fourier Laplace domain, that depends only on the the transforms of the step probability density and its time integral. This is a key result for the treatment of space-time coupled CTRW and will be useful for the analytical investigation into the PDF later on.

A slightly more complicated approach is needed when aging effects are considered. In this case the walker has already moved for the duration of the aging time

t_a before the observation begins, which is illustrated in figure (2.4).
At the beginning of observation, which is set to $t = 0$ and $\mathbf{x} = 0$, the walker will already be in motion and has its first observed turning point some time after the beginning of observation, where we denote the probability of this turning point being at $\mathbf{x}'$ at time t' with $F(\mathbf{x}', t'|t_a)$.
Alternatively the walker can also perform a step so long that it stays in straight motion for the entire duration of observation. In this case no first turning point is observed and this event of a long, single step is instead described by $s(\mathbf{x}|t, t_a)$, which is the conditional probability that a walker that has aged for t_a performs a step of duration longer than t such that he is at $\mathbf{x}$ at time t.

The transport equation for the PDF therefore has two terms corresponding to these cases:

$$\begin{aligned} p(\mathbf{x}|t, t_a) = &\int_{\mathbb{R}^d} d^d x' \int_{\mathbb{R}^d} d^d x'' \int_0^t dt' \int_0^{t-t'} dt'' F(\mathbf{x}', t'|t_a) C(\mathbf{x}'', t'') r(\mathbf{x} - \mathbf{x}' - \mathbf{x}''|t - t' - t'') \\ &+ s(\mathbf{x}|t, t_a). \end{aligned} \tag{2.37}$$

The second term captures the contribution from the single step case while the double convolution describes a particle having its first turning point at t', then performing a series of completed steps for the duration t'' and then being found at observation time t in a final, incomplete step of duration $t - t' - t''$ or longer, as shown in the upper picture of figure (2.4).
Again using the convolution property of the Fourier and the Laplace transform, (2.30) and (2.30), we obtain the a closed expression for $p(\mathbf{k}|s, t_a)$:

$$p(\mathbf{k}|s, t_a) = F(\mathbf{k}, s|t_a) C(\mathbf{k}, s) r(\mathbf{k}|s) + s(\mathbf{k}|s, t_a). \tag{2.38}$$

Forward waiting time Consider a walker in a CTRW that has aged for a time t_a. When the observation begins the walker will in general not be at a turning point, but rather in some time interval that started before the beginning of observation. To describe this we use the forward waiting time $\psi_1(t|t_a)$, which gives the probability density that the first turning time after the beginning of observation is t when the aging time of the walk is t_a.

In order to derive a formula for $\psi_1(t|t_a)$ consider a walker starting at time zero, then performing exactly n steps that that end at $t' < t_a$ and then taking a final step of duration $t_a - t' + t$, i.e. a walker whose first turning point during the observation beginning at t_a is exactly at t. The probability density of this event is given by

$$\phi_n(t|t_a) = \int_0^{t_a} \psi_n(t') \psi(t_a - t' + t) dt'. \tag{2.39}$$

To obtain the forward waiting time we need to sum over all possible numbers of steps

$$\psi_1(t|t_a) = \sum_{n=0}^{\infty} \phi_n(t|t_a) = \int_0^{t_a} \left(\sum_{n=0}^{\infty} \psi_n(t') \right) \psi(t_a - t' + t) dt', \tag{2.40}$$

which can be expressed via the step rate

$$k(t) = \sum_{n=0}^{\infty} \psi_n(t'), \tag{2.41}$$

as

$$\psi_1(t|t_a) = \int_0^{t_a} k(t')\psi(t_a - t' + t)dt'. \tag{2.42}$$

For future calculations it is useful to note that $k(t)$ has a simple form in the Laplace domain,

$$k(s) = \frac{1}{1 - \psi(s)}, \tag{2.43}$$

which follows directly from the factorization of ψ_n in (2.17).

In the case of a power law time distribution that lacks the first moment, e.g. the distribution for the generalized Lévy walk (2.3) with $\gamma < 1$, the forward waiting time can be calculated exactly and reads [13]:

$$\psi_1 = \frac{\sin(\pi\gamma)}{\pi} \left(\frac{t_a}{t}\right)^{\gamma} \frac{1}{t + t_a}. \tag{2.44}$$

3. Methodology

3.1 Calculating the mean squared displacement

For the calculation of the MSD I will concentrate on the one-dimensional case, as this simplifies the calculations and generalizations to higher dimensions are clear, as the PDF of the process (2.7) is isotropic and the normalization takes care of the angular integral.

The one-dimensional MSD $\langle x^2\rangle(t)$ is defined via the integral

$$\langle x^2\rangle(t) = \int_{\mathbb{R}} x^2 \psi(x,t)dx, \tag{3.1}$$

which is closely related to the Fourier Laplace transform of the PDF for the process, as we can see when we expand it for small $\mathbf{k}$:

$$\begin{aligned} p(\mathbf{k}|s) &= \int_{\mathbb{R}} e^{ikx} p(x|s)dx & (3.2)\\ &= \int_{\mathbb{R}} p(x|s)dx + ik\int_{\mathbb{R}} xp(x|s)dx - \frac{k^2}{2}\int_{\mathbb{R}} x^2 p(x|s)dx & (3.3)\\ &= 1 - \frac{k^2}{2}\langle x^2\rangle(s) + \dots \quad , & (3.4)\end{aligned}$$

where I used that the PDF is normalized to one and that the first moment of an isotropic process vanishes. This implies

$$\langle x^2\rangle(s) = -\left[\frac{\partial^2}{\partial k^2} p(k|s)\right]_{k=0}, \tag{3.5}$$

which allows me to calculate the MSD directly without knowledge of the full PDF and then transforming it back into the time domain.

For the ordinary or non-aged case we can use expression (2.35) for the PDF in the Fourier Laplace domain:

$$p(k|s) = C(k,s)r(k|s). \tag{3.6}$$

We can expand C and r similarly to what we did for the PDF resulting in in

$$\begin{aligned} r(k|s) &= r_0(s) - \frac{1}{2}k^2 r_2(s) + o(k^2) & (3.7)\\ C(k,s) &= C_0(s) - \frac{1}{2}k^2 C_2(s) + o(k^2). & (3.8)\end{aligned}$$

Here we see that the first moments vanish again and I introduced the new notation $r_0(s) = r(k=0|s)$ and $r_2(s) = \left[\frac{\partial^2}{\partial k^2} r(k|s)\right]_{k=0}$.
Inserting these expression we find for the PDF

$$p(k|s) = C_0(s)r_0(s) - \frac{k^2}{2}\left[C_0(s)r_2(s) + C_2(s)r_0(s)\right] + o(k^2), \tag{3.9}$$

and therefore in the ordinary case the MSD is given by

$$\langle x^2\rangle(s) = C_0(s)r_2(s) + C_2(s)r_0(s). \tag{3.10}$$

For the aged case we start from the result found in (2.38),

$$p(\mathbf{k}|s,t_a) = F(\mathbf{k},s|t_a)C(\mathbf{k},s)r(\mathbf{k}|s) + s(\mathbf{k}|s,t_a), \tag{3.11}$$

and use similar expansions for the transforms of the single step density and the first step density:

$$s(k|s,t_a) = s_0(s,t_a) - \frac{1}{2}k^2 s_2(s,t_a) + o(k^2) \tag{3.12}$$

$$F(k,s|t_a) = F_0(s|t_a) - \frac{1}{2}k^2 F_2(s|t_a) + o(k^2). \tag{3.13}$$

Thus we find for the PDF

$$\begin{aligned} p(k|s) =& F_0(s|t_a)C_0(s)r_0(s) + s_0(s,t_a) - \frac{k^2}{2}s_2(s,t_a) \\ &- \frac{k^2}{2}\left[F_0(s|t_a)C_0(s)r_2(s) + F_0(s|t_a)C_2(s)r_0(s) + F_2(s|t_a)C_0(s)r_0(s)\right] + o(k^2). \end{aligned} \tag{3.14}$$

Therefore the MSD for the aged case reads

$$\langle x^2\rangle(s) = F_0(s|t_a)C_0(s)r_2(s) + F_0(s|t_a)C_2(s)r_0(s) + F_2(s|t_a)C_0(s)r_0(s) + s_2(s,t_a). \tag{3.15}$$

To extract the asymptotic results from these formulas we need to look at the $t \to \infty$ limit, which corresponds to the $s \to 0$ limit in the Laplace domain.
The general strategy is to find the expressions for s_2 directly in the time domain as it does not enter inside a product. The other quantities $C_0, F_0, r_0, C_2, F_2, r_2$ are individually calculated in the Laplace domain to leading order in s. They are then inserted the respective formula for the MSD and transformed back into the time domain using the Tauberian theorem.
The Tauberian theorem will be used frequently throughout this thesis as it gives the Laplace transform of a function $f(t)$ that behaves as a power law for large t through the formula

move Tauberian to first place it is used

$$f(t) \simeq t^{\rho-1}L(t) \quad \leftrightarrow \quad f(s) \simeq \Gamma(\rho)s^{-\rho}L\left(\frac{1}{s}\right), \tag{3.16}$$

if $\rho \geq 0$ and $L(t)$ is slowly varying, i.e.

$$\lim_{t\to\infty} \frac{L(C\,t)}{L(t)} = 1. \tag{3.17}$$

For general ρ the slightly more complicated formula

$$f(s) = \sum_{k=0}^{k_{\max}} \frac{(-1)^k}{k!} I_k^f s^k + L\Gamma(\rho) s^{-\rho}, \tag{3.18}$$

has to be used, which is derived in Sec. (A). Here $k_{\max}$ is the whole part of $-\rho$, and I_k^f is the moment integral

fix the Appendix reference!

$$I_k^f = \int_0^\infty t^k f(t) dt. \tag{3.19}$$

3.2 Finding the probability density function

3.3 Numerical simulation of the model

Numerical simulations can be used to supplement and support analytical computations by giving insight into the qualitative structure of the process, sharpening understanding of the model and giving a method of testing the results. Furthermore simulations allow investigation of regimes where analytical computations fail.
In general there are two possible approaches for the simulation of a Lévy walk model: On the one hand a direct simulation of the process, where I randomly generate step durations and directions for a large ensemble of walkers and record their positions; or on the other hand a description via a suitable Langevin equation, which has been shown to be equivalent to a Lévy walk. I decided to go with the former approach as it keeps closer to the model and avoids potential numerical instabilities that can appear during the integration of differential equation. It is also not entirely clear how the generalization of the Lévy walk studied in this paper could be reflected in a Langevin equation.

is it?

The two main quantities of interest in this thesis are the MSD and the PDF of the generalized Lévy walk, which can be obtained from an ensemble of simulated walkers via averaging and creating histograms respectively. The simulation was implemented in one dimension similarly to the analytical computation, as this captures most of the behavior in an isotropic walk.
When performing the simulation duration of the walk and the size of the ensemble have the biggest impact on computation times, where the second factor is of special importance for processes with power law distributions such as Lévy walks, because here the walk is often dominated by rare events which are only captured with sufficiently large ensembles.
To address this issue I use the independence of the different walkers to parallelize the computation and perform it on the available graphics cards (GPUs) using NVIDIAs C++ extension CUDA. The university computers are equipped with Quadro K4000 GPUs, that have 768 cores each. This is a far than greater number of cores than available on processor (CPU), which is usually less than ten, and thus allows for far greater parallelization, resulting in a considerable speedup of the simulation.

quantify this?

Possible hurdles to this approach are the latency in the data transfer between working memory and GPU, which can slow down performance, and the limited memory on the GPU (3GB). However by limiting the walker positions I save to selected measurement times and reducing communication between GPU and CPU to a minimum it was possible to simulate large ensembles of 10^9 particle in a few hours.

Another aspect that should be addressed is the generation of pseudo random numbers for the creation of the steps for the simulation. As these numbers are not truly random, i.e. not completely uncorrelated, they can, depending on the quality of the number generator, leave statistical artifacts that falsify the simulation results. To minimize this risk I use the cuRAND library, which implements a version of the Xorshift algorithm [14]. The documentation guarantees a period greater than 2^{190} for each independently seeded sequence of random numbers (i.e. each simulation), and each thread has an offset of 2^{67} in this sequence. At roughly 10^9 threads that each simulate a random walker with a step number lower than 10^7 I am more than ten orders of magnitude away from reaching the period of the random number sequence, which leaves little risk that statistical artifacts influenced the results.

4. Analytical Calculations

4.1 Asymptotic behavior of the ordinary MSD

Calculation of $\psi_0(s)$

All properties of the ordinary walk are derived from the waiting time density $\psi(t)$. The joint probability density of a displacement in a stretch and of the time of stretch is given by

$$\psi(x,t) = \frac{1}{2}\delta(|x| - ct^{\nu})\psi(t), \tag{4.1}$$

so that the Fourier transform of this function in x reads

$$\psi(k,t) = \psi(t) - \frac{k^2}{2}c^2 t^{2\nu}\psi(t) + o(k^2). \tag{4.2}$$

Then the Laplace transform in t should be performed. Although the exact Laplace transform of $\psi(t)$ as given by Eq.(2.3) is possible in quadratures, we are only interested in the asymptotic behavior, which can be found using the Tauberian theorem (3.18). The forms of the Laplace transform differ for different values of γ and ν. As explained above we are interested only in the lowest order terms of s-dependence.

$\gamma < 1$

For $0 < \gamma < 1$ the function $\psi_0(s) = \psi(s)$ belongs to an integrable class, and its Laplace representation reads

$$\psi_0(s) \simeq 1 + \gamma\Gamma(-\gamma)t_0^{\gamma}s^{\gamma} = 1 - \Gamma(1-\gamma)t_0^{\gamma}s^{\gamma}. \tag{4.3}$$

Keeping t_0 in all calculations and not putting it to unity is reasonable to be able to check the dimension of the ensuing results, especially in the aged case.

$\gamma > 1$

The Laplace transform of $\psi(t)$ now has an additional term, due to its first moment being finite:

$$\psi_0(s) \simeq 1 - \tau s - \Gamma(1-\gamma)t_0^{\gamma}s^{\gamma}. \tag{4.4}$$

Here τ is defined as

$$\tau = \frac{\gamma}{t_0}\int_0^{\infty}\frac{tdt}{(1+t/t_0)^{\gamma+1}} = \frac{t_0}{\gamma - 1}. \tag{4.5}$$

Calculation of $\psi_2(s)$

The marginal second moment of the step distribution is given by

$$\psi_2(t) = \int_{-\infty}^{\infty} x^2 \psi(x,t) dx = c^2 t^{2\nu} \psi(t). \tag{4.6}$$

The expressions for $\psi_2(s)$ depend on whether $2\nu < \gamma$ or $2\nu > \gamma$:

$2\nu < \gamma$

In this first case the function $\psi_2(t)$ is integrable, $\int_0^\infty \psi_2(t)dt < \infty$, and the expansion of its Laplace transform starts from a constant:

$$\psi_2(s) \simeq \gamma c^2 t_0^{2\nu} \int_0^\infty \frac{x^{2\nu}}{(1+x)^{\gamma+1}} dx + \gamma \Gamma(2\nu - \gamma) c^2 t_0^\gamma s^{\gamma - 2\nu} \tag{4.7}$$

where the integral is given by the dimensionless constant

$$\int_0^\infty \frac{x^{2\nu}}{(1+x)^{\gamma+1}} dx = \mathrm{B}(2\nu + 1, \gamma - 2\nu), \tag{4.8}$$

with $\mathrm{B}(a,b)$ being the Beta-function, which is defined as:

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{4.9}$$

$2\nu > \gamma$

In the second case $\psi_2(t)$ is non-integrable, the integral $\int_0^\infty \psi_2(t)dt$ diverges, and the asymptotics of its Laplace transform read

$$\psi_2(s) \simeq \gamma \Gamma(2\nu - \gamma) c^2 t_0^\gamma s^{\gamma - 2\nu}. \tag{4.10}$$

Calculation of $C_0(s)$ and $C_2(s)$

For complete steps our generalization does not differ from the original Lévy walk and we can use the general result for $C(x,t)$ in the Fourier-Laplace domain Eq. (2.32):

$$C(k,s) = \frac{1}{1 - \psi(k,s)}. \tag{4.11}$$

Expanding $C(k,s)$ for s small and in the limit $k \to 0$ we get

$$\begin{aligned} C(k,s) &\simeq \frac{1}{1 - \psi(s) + (k^2/2)\psi_2(s) + o(k^2)} \\ &\simeq \frac{1}{1-\psi(s)} - \frac{k^2}{2} \frac{\psi_2(s)}{[1-\psi(s)]^2} + o(k^2). \end{aligned} \tag{4.12}$$

Therefore

$$C_0(s) = \frac{1}{1-\psi(s)} \tag{4.13}$$

$$C_2(s) = \frac{\psi_2(s)}{[1-\psi(s)]^2} \ , \tag{4.14}$$

into which we now have to insert our results for ψ_0 and ψ_2:

$\gamma < 1$ and $2\nu < \gamma$

Here we find by using equations (4.3) and (4.7) :

$$C_0(s) \simeq \frac{1}{\Gamma(1-\gamma)} t_0^{-\gamma} s^{-\gamma}, \tag{4.15}$$

$$C_2(s) \simeq \gamma \frac{\mathrm{B}(2\nu+1, \gamma-2\nu)}{\Gamma^2(1-\gamma)} c^2 t_0^{2\nu-2\gamma} s^{-2\gamma}. \tag{4.16}$$

$\gamma < 1$ and $2\nu > \gamma$:

In this case we obtain for C_2:

$$C_2(s) \simeq \gamma \frac{\Gamma(2\nu-\gamma)}{\Gamma^2(1-\gamma)} c^2 t_0^{-\gamma} s^{-2\nu-\gamma}, \tag{4.17}$$

while C_0 is the same as in Eq.(4.15) as long as $\gamma < 1$.

$\gamma > 1$ and $2\nu < \gamma$

Now the first moment of $\psi(t)$ is finite, therefore

$$C_0(s) \simeq \frac{1}{\tau s} \tag{4.18}$$

$$C_2(s) \simeq \gamma \mathrm{B}(2\nu+1, \gamma-2\nu) c^2 \frac{t_0^{2\nu}}{\tau^2} s^{-2} \tag{4.19}$$

$\gamma > 1$ and $2\nu > \gamma$:

Again $C_0(s)$ doesn't change, and for C_2 we obtain

$$C_2 \simeq \gamma \Gamma(2\nu-\gamma) c^2 \frac{t_0^{\gamma}}{\tau^2} s^{\gamma-2\nu-2}. \tag{4.20}$$

Calculation of $r_0(s)$ and $r_2(s)$

The function $r(x|t)$ gives the distribution of the corresponding displacements conditioned on the fact that the total duration of a stretch is longer than t:

$$r(x|t) = \int_t^\infty \frac{1}{2} \delta(|x| - ct^\eta t'^{\nu-\eta}) \psi(t') dt' \ . \tag{4.21}$$

This is correctly normalized to the overall probability to stay within a single stretch for a time longer than t

$$\int r(x|t) dx = \int_t^\infty \psi(t') dt'. \tag{4.22}$$

Expanding the Fourier transform of $r(x,t)$ for small k,

$$\begin{aligned} r(k|t) = &\int_{-\infty}^{\infty} dx\ e^{ikx} \int_t^\infty \frac{1}{2} \delta(|x| - ct^\eta t'^{\nu-\eta}) \psi(t') dt' \\ = &r_0(t) - \frac{k^2}{2} r_2(t) + o(k^2), \end{aligned} \tag{4.23}$$

we find the marginal moments

$$r_0(t) = \frac{1}{(1+t/t_0)^\gamma} \tag{4.24}$$

$$r_2(t) = \gamma c^2 \frac{1}{t_0} t^{2\eta} \int_t^\infty \frac{t'^{2(\nu-\eta)}}{(1+t'/t_0)^{\gamma+1}} dt' \ , \tag{4.25}$$

whose Laplace transforms depend on the relationship between γ, ν and η:

$\gamma < 1$

In this case r_0 is non-integrable and we find

$$r_0(s) \simeq \Gamma(1-\gamma) t_0^\gamma s^{\gamma-1}. \tag{4.26}$$

For r_2 we can use the asymptotic form of $\psi(t)$, since we are interested in large t:

$$r_2(t) \simeq \gamma c^2 t_0^\gamma t^{2\eta} \int_t^\infty \tau^{2(\nu-\eta)-1-\gamma} d\tau \ , \tag{4.27}$$

which is only finite for $\gamma > 2(\nu - \eta)$ and diverges otherwise, meaning that no MSD exists. In the rest of the paper we will concentrate on the case when the second moment converges, where we find

$$r_2(t) \simeq \gamma \frac{1}{\gamma - 2(\nu-\eta)} c^2 t_0^\gamma t^{2\nu-\gamma}. \tag{4.28}$$

Since this expression is always in the non-integrable class we obtain by using Tauberian theorem (3.16)

$$r_2(s) \simeq \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma - 2(\nu-\eta)} c^2 t_0^\gamma s^{\gamma-2\nu-1}. \tag{4.29}$$

$\gamma > 1$ **and** $2\nu > \gamma - 1$

For $\gamma > 1$ $r_0(t)$ becomes integrable, therefore its Laplace transform reads

$$r_0 \simeq \tau + \Gamma(1-\gamma) t_0^\gamma s^{\gamma-1}. \tag{4.30}$$

The function $r_2(t)$ is still non-integrable and therefore identical to the previous case.

$\gamma > 1$ **and** $2\nu < \gamma - 1$

In this case r_0 does not change but $r_2(s)$ is now integrable and therefore its transform behaves as

$$r_2(s) \simeq I_0^{r_2} - \gamma \frac{\Gamma(2\nu+1-\gamma)}{2(\nu-\eta)-\gamma} c^2 t_0^\gamma s^{\gamma-2\nu-1} \ . \tag{4.31}$$

We evaluate the definite integral $I_0^{r_2}$ by manipulating the area of integration:

$$I_0^{r_2} = c^2 \int_0^\infty dt t^{2\eta} \int_t^\infty dt' \psi(t') (t')^{2(\nu-\eta)} \tag{4.32}$$

$$= c^2 \frac{1}{2\eta+1} \int_0^\infty \psi(t') t'^{2\nu+1} dt'$$

$$= \gamma \frac{1}{2\eta+1} \int_0^\infty \frac{x^{2\nu+1} dx}{(1+x)^{\gamma+1}} c^2 t_0^{2\nu+1}$$

$$= \gamma \frac{\mathrm{B}(2\nu+2, \gamma-2\nu-1)}{2\eta+1} c^2 t_0^{2\nu+1}, \tag{4.33}$$

meaning the r_2 is constant in the leading order. Therefore we find for $\gamma > 1$ and $2\nu < \gamma - 1$

$$r_2 \simeq \gamma \frac{\mathrm{B}(2\nu+2, \gamma-2\nu-1)}{2\eta+1} c^2 t_0^{2\nu+1} - \gamma \frac{\Gamma(2\nu+1-\gamma)}{2(\nu-\eta)-\gamma} c^2 t_0^{\gamma} s^{\gamma-2\nu-1}. \tag{4.34}$$

The results so far are summarized in Table 4.1.

$\gamma < 1$	$C_0(s) \simeq \frac{1}{\Gamma(1-\gamma)} t_0^{-\gamma} s^{-\gamma}$	$C_2(s) \simeq \begin{cases} \gamma \frac{\mathrm{B}(2\nu+1,\gamma-2\nu)}{\Gamma^2(1-\gamma)} c^2 t_0^{2\nu-2\gamma} s^{-2\gamma} & \text{for } 2\nu < \gamma \\ \gamma \frac{\Gamma(2\nu-\gamma)}{\Gamma^2(1-\gamma)} c^2 t_0^{-\gamma} s^{-2\nu-\gamma} & \text{for } 2\nu > \gamma \end{cases}$
$\gamma < 1$	$r_0(s) \simeq \Gamma(1-\gamma) t_0^{\gamma} s^{\gamma-1}$	$r_2(s) \simeq \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)} c^2 t_0^{\gamma} s^{\gamma-1-2\nu}$
$\gamma > 1$	$C_0(s) \simeq \frac{1}{\tau s}$	$C_2(s) \simeq \begin{cases} \gamma \mathrm{B}(2\nu+1, \gamma-2\nu) c^2 \frac{t_0^{2\nu}}{\tau^2} s^{-2} & \text{for } 2\nu < \gamma \\ \gamma \Gamma(2\nu-\gamma) c^2 \frac{t_0^{\gamma}}{\tau^2} s^{\gamma-2\nu-2} & \text{for } 2\nu > \gamma \end{cases}$
$\gamma > 1$	$r_0(s) \simeq \tau$	$r_2(s) \simeq \begin{cases} \gamma \frac{\mathrm{B}(2\nu+2,\gamma-2\nu-1)}{2\eta+1} c^2 t_0^{2\nu+1} & \text{for } 2\nu < \gamma - 1 \\ \gamma \frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)} c^2 t_0^{\gamma} s^{\gamma-2\nu-1} & \text{for } 2\nu > \gamma - 1 \end{cases}$

Table 4.1: Leading terms of the marginal moments of C and r in the Laplace domain for different parameter ranges.

Mean squared displacement

With these results we can now compute the MSD via the formula

$$\langle x^2(s) \rangle = C_0(s) r_2(s) + C_2(s) r_0(s) \; . \tag{4.35}$$

$\gamma < 1$ and $2\nu < \gamma$:

In the case $2\nu < \gamma$ we have

$$\begin{aligned} \langle x^2(s) \rangle \simeq & \gamma \left[\frac{\Gamma(2\nu+1-\gamma)}{\Gamma(1-\gamma)(\gamma-2(\nu-\eta))} c^2 s^{-2\nu-1} \right. \\ & \left. + \frac{\mathrm{B}(2\nu+1, \gamma-2\nu)}{\Gamma(1-\gamma)} c^2 t_0^{2\nu-\gamma} s^{-\gamma-1} \right], \end{aligned} \tag{4.36}$$

which translates to

$$\begin{aligned} \langle x^2(t) \rangle \simeq \gamma & \left[\frac{\Gamma(2\nu+1-\gamma)}{\Gamma(1-\gamma)(\gamma-2(\nu-\eta))\Gamma(2\nu+1)} c^2 t^{2\nu} \right. \\ & \left. + \frac{\mathrm{B}(2\nu+1, \gamma-2\nu)}{\Gamma(1-\gamma)\Gamma(1+\gamma)} c^2 t_0^{2\nu-\gamma} t^{\gamma} \right]. \end{aligned} \tag{4.37}$$

This is dominated by the second term since $2\nu < \gamma$, leading to $\langle x^2(t) \rangle \propto t^{\gamma}$ in the case. This means that for $\gamma < 1$ and $2\nu < \gamma$ the behavior of the walk merges with the one of a CTRW with a fixed step length.

$\gamma < 1$ **and** $2\nu > \gamma$

In this parameter regime we obtain

$$\begin{aligned}\langle x^2(s)\rangle \simeq&\gamma c^2 s^{-2\nu-1}\left[\frac{\Gamma(2\nu+1-\gamma)}{\Gamma(1-\gamma)(\gamma-2(\nu-\eta))}+\frac{\Gamma(2\nu-\gamma)}{\Gamma(1-\gamma)}\right]\\ =&\gamma\frac{\Gamma(2\nu-\gamma)}{\Gamma(1-\gamma)}\frac{2\eta}{\gamma-2(\nu-\eta)}c^2 s^{-2\nu-1},\end{aligned} \tag{4.38}$$

therefore we find in the time domain

$$\langle x^2(t)\rangle \simeq \gamma\frac{\Gamma(2\nu-\gamma)}{\Gamma(2\nu+1)\Gamma(1-\gamma)}\frac{2\eta}{2(\nu-\eta)-\gamma}c^2 t^{2\nu}. \tag{4.39}$$

$\gamma > 1$ **and** $2\nu < \gamma - 1$

In this case the MSD reads

$$\begin{aligned}\langle x^2(s)\rangle =&C_0(s)r_2(s)+C_2(s)r_0(s)\\ \simeq&\gamma\frac{\mathrm{B}(2\nu+2,\gamma-2\nu-1)}{2\eta+1}c^2\frac{t_0^{2\nu+1}}{\tau}\frac{1}{s}+\gamma\mathrm{B}(2\nu+1,\gamma-2\nu)c^2\frac{t_0^{2\nu}}{\tau}s^{-2},\end{aligned}$$

which is dominated by the second term. Therefore we find in the time domain in leading order:

$$\langle x^2(t)\rangle \simeq \gamma\mathrm{B}(2\nu+1,\gamma-2\nu)c^2\frac{t_0^{2\nu}}{\tau}t. \tag{4.40}$$

$\gamma > 1$ **and** $\gamma - 1 < 2\nu < \gamma$

Compared to the previous case only r_2 changes, therefore

$$\langle x^2(s)\rangle \simeq \quad \gamma\frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)}c^2\frac{t_0^{\gamma}}{\tau}s^{\gamma-2-2\nu}+\gamma\mathrm{B}(2\nu+1,\gamma-2\nu)c^2\frac{t_0^{2\nu}}{\tau}s^{-2}. \tag{4.41}$$

Since $\gamma - 2\nu > 0$ the term quadratic in s is again dominant, and the asymptotic behavior in the time domain is identical to the previous case:

$$\langle x^2(t)\rangle \simeq \gamma\mathrm{B}(2\nu+1,\gamma-2\nu)c^2\frac{t_0^{2\nu}}{\tau}t. \tag{4.42}$$

$\gamma > 1$ **and** $2\nu > \gamma$

Now C_2 is different, giving us

$$\langle x^2(s)\rangle \simeq\gamma\frac{\Gamma(2\nu+1-\gamma)}{\gamma-2(\nu-\eta)}c^2\frac{t_0^{\gamma}}{\tau}s^{\gamma-2\nu-2}+\gamma\Gamma(2\nu-\gamma)c^2\frac{t_0^{\gamma}}{\tau}s^{\gamma-2\nu-2}. \tag{4.43}$$

This leads to the time dependence

$$\langle x^2(t)\rangle \simeq \gamma\frac{2\eta}{\gamma-2(\nu-\eta)}\frac{\Gamma(2\nu-\gamma)}{\Gamma(2\nu+2-\gamma)}c^2\frac{t_0^{\gamma}}{\tau}t^{2\nu+1-\gamma}. \tag{4.44}$$

The results for the ordinary walk under the assumption that the convergence condition $\gamma > 2(\nu - \eta)$ is satisfied can be summarized as follows:

$$\langle x^2(t)\rangle \propto \begin{cases} t^{\gamma} & \text{for} \quad \gamma < 1,\ 2\nu < \gamma \\ t^{2\nu} & \text{for} \quad \gamma < 1,\ 2\nu > \gamma \\ t & \text{for} \quad \gamma > 1,\ 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for} \quad \gamma > 1,\ 2\nu > \gamma. \end{cases} \tag{4.45}$$

Thus, in the whole domain of γ there are four regimes with crossovers at $\gamma = 1$ and at $2\nu = \gamma$:

- For $2\nu < \gamma$ one has $\langle x^2(t)\rangle \propto t^{\gamma}$ for $\gamma < 1$ crossing over to a faster growth $\langle x^2(t)\rangle \propto t$ for $\gamma > 1$
- For $2\nu > \gamma$ one has universally $\langle x^2(t)\rangle \propto t^{2\nu}$ for $\gamma < 1$ crossing over to a slower growth $\langle x^2(t)\rangle \propto t^{2\nu+1-\gamma}$ for $\gamma > 1$.

Note that in an ordinary walk the Richardson regime is possible, and is achieved for $\nu = 3/2$ for $\gamma < 1$, and for $\nu = (\gamma + 2)/2$ for $\gamma > 1$, provided $\gamma > 2(\nu - \eta)$. The last restriction corresponds to $\eta > (3 - \gamma)/2$ for $\gamma < 1$, and to $\eta > 1$ for $\gamma > 1$. The original model with $\eta = 1$ indeed does not possess a Richardson regime.

4.2 Aged walk: General expressions

We now consider the functions F and s which are specific for aged walks. The general expression for F reads:

$$\begin{aligned} F(x,t|t_a) = &\int_0^{t_a} dt'\psi(t_a + t - t')k(t') \\ &\times \delta\left\{x - c[(t_a + t - t')^{\nu} - (t_a + t - t')^{\nu-\eta}(t_a - t')^{\eta}]\right\}, \end{aligned} \tag{4.46}$$

where $k(t) = C_0(t)$ is the time-dependent rate of steps. Note that the argument of the δ-function is shifted, due to the fact that the distance from the origin x is set to zero at the start of the measurement. The marginal normalization of $F(x,t|t_a)$ is

$$\begin{aligned} F_0(t|t_a) = &\int F(x,t|t_a)dx = \int_0^{t_a} \psi(t_a + t - t')k(t')dt' \\ = &\psi_1(t|t_a), \end{aligned} \tag{4.47}$$

where $\psi_1(t|t_a)$ is the forward waiting time PDF known from the theory of CTRW as discussed in section (2.2). The marginal second moment of F reads:

$$\begin{aligned} F_2(t|t_a) = &\int_0^{t_a} dt' c^2\psi(t_a + t - t')k(t') \\ &\times [(t_a + t - t')^{\nu} - (t_a + t - t')^{\nu-\eta}(t_a - t')^{\eta}]^2. \end{aligned} \tag{4.48}$$

Additionally we have to consider the term $s(x|t, t_a)$, which describes the case that both the aging time and the observation time belong to the same stretch:

$$\begin{aligned} s(x|t,t_a) = &\int_0^{t_a} dt'k(t') \int_{t_a+t-t'}^{\infty} dt''\psi(t'') \\ &\times \delta\left\{x - c[(t'')^{\nu-\eta}(t_a + t - t')^{\eta} - (t'')^{\nu-\eta}(t_a - t')^{\eta}]\right\}, \end{aligned} \tag{4.49}$$

where the inner integral gives the probability that no renewal took place during the time interval between t' and $t_a + t$. The zeroth order of this function,

$$s_0(t, t_a) = \int_0^{t_a} \Psi(t_a + t - t')k(t')dt', \tag{4.50}$$

where $\Psi(t) = \int_t^\infty \psi(t')dt'$ is the survival probability, is not necessary for what follows, and will not be calculated.
The second moment is given by

$$\begin{aligned} s_2(t, t_a) = c^2 \int_0^{t_a} k(t') \int_{t_a+t-t'}^{\infty} \psi(t'') \\ \times [(t'')^{\nu-\eta}(t_a + t - t')^\eta - (t'')^{\nu-\eta}(t_a - t')^\eta]^2 dt'' dt'. \end{aligned} \tag{4.51}$$

We note that the form of the integrals involved in F_2 and s_2 is very similar. In the following calculation we differentiate between two time regimes: The case of short aging times $t >> t_a >> t_0$ and the case of long aging times $t_a >> t >> t_0$, which will be discussed separately in the two following sections.

4.3 Aged walk: short aging times $t >> t_a$

Calculation of F_0 and F_2

We are interested in the Laplace transforms of the marginal moments of F_0 and F_2. Since both of them depend on $k(t) = C_0(t)$ whose form we found to be dependent on whether $\gamma < 1$ or $\gamma > 1$, we have to distinguish between these cases.

$\gamma < 1$

The function $F_0(t|t_a)$ is equal to the forward waiting time PDF $\psi_1(t|t_a)$. For $\gamma < 1$ we can use the result from Eq. (2.44)

$$F_0(t|t_a) = \psi_1(t|t_a) = \frac{\sin \pi\gamma}{\pi} \left(\frac{t_a}{t}\right)^\gamma \frac{1}{t + t_a}. \tag{4.52}$$

The expression is normalized to unity, and therefore in the Laplace domain the leading term in F_0 will be 1.

For $t >> t_a$ the expression in the square brackets in Eq.(4.48) can be approximated by $t^{2\nu}$ (since $\nu > 0$) and for F_2 we find in this limit

$$F_2(t|t_a) \simeq c^2 t^{2\nu} \int_0^{t_a} \psi(t_a + t - t')k(t')dt'. \tag{4.53}$$

The integral can again be expressed through the forward waiting time $\psi_1(t|t_a)$. We take the asymptotics of ψ_1 for t large, so that

$$F_2(t|t_a) \simeq \frac{\sin \pi\gamma}{\pi} c^2 t_a^\gamma t^{2\nu-\gamma-1}. \tag{4.54}$$

Here again two situations arise depending on the integrability:

$\gamma < 1$ **and** $2\nu > \gamma$

In this case F_2 is non-integrable, so that in the Laplace domain

$$F_2(s|t_a) \simeq \Gamma(2\nu - \gamma)\frac{\sin \pi\gamma}{\pi} c^2 t_a^{\gamma} s^{\gamma - 2\nu}. \tag{4.55}$$

$\gamma < 1$ **and** $2\nu < \gamma$

Now F_2 is integrable, and the lowest order in its Laplace transform tends to a constant:

$$F_2(s|t_a) \simeq \frac{\sin \pi\gamma}{\pi} c^2 t_a^{\gamma} \int_0^{\infty} \frac{t^{2\nu-\gamma}}{t + t_a} dt. \tag{4.56}$$

The corresponding integral is given by

$$\int_0^{\infty} \frac{t^{2\nu-\gamma}}{t + t_a} dt = \frac{\pi}{\sin(\pi(2\nu + 1 - \gamma))} t_a^{2\nu-\gamma}, \tag{4.57}$$

see Eq.(2.2.5.25) of Ref. [15], so that

$$F_2(s|t_a) \simeq \frac{\sin \pi\gamma}{\sin(\pi(2\nu + 1 - \gamma))} c^2 t_a^{2\nu}. \tag{4.58}$$

$\gamma > 1$

Now we consider the case $\gamma > 1$. From the previous section we know that $C_0(s) = \frac{1}{\tau s}$, therefore

$$k(t) = C_0(t) = \frac{1}{\tau}. \tag{4.59}$$

With this we can rewrite F_0 as

$$F_0(t|t_a) = \frac{1}{\tau} \int_0^{t_a} \psi(t + y) dy. \tag{4.60}$$

For $t \to \infty$ it decays as $t^{-\gamma-1}$ and therefore is of integrable type. To find its lowest order (constant) term of the Laplace transform we note that

$$\begin{aligned}
F_0(s|t_a) \simeq& \frac{1}{\tau} \int_0^{\infty} dt \int_0^{t_a} \psi(t + y) dy & (4.61)\\
=& \frac{1}{\tau} \int_0^{t_a} dy \int_0^{\infty} dt \psi(t + y) & (4.62)\\
=& \frac{1}{\tau} \int_0^{t_a} dy \int_y^{\infty} dt \psi(t) & (4.63)\\
=& \frac{1}{\tau} \int_0^{t_a} \Psi(y) dy, & (4.64)
\end{aligned}$$

which tends to unity since $t_a \gg t_0$ and since $\int_0^{\infty} \Psi(t')dt' = \tau$.

The term F_2 for $t \gg t_a$ can again be evaluated by approximating the expression in square brackets in Eq.(4.48) by t^{ν}:

$$\begin{aligned}
F_2(t|t_a) \simeq& c^2 \frac{1}{\tau} t^{2\nu} \int_0^{t_a} \psi(t + y) dy & (4.65)\\
=& c^2 \frac{t_0^{\gamma}}{\tau} [(t + t_0)^{-\gamma} - (t + t_a + t_0)^{-\gamma}] t^{2\nu}. & (4.66)
\end{aligned}$$

Since the power-law asymptotics of the expression in square brackets is $t^{-\gamma-1}$ the whole expression

$$F_2(t|t_a) \simeq \gamma c^2 \frac{t_0^\gamma}{\tau} t_a t^{2\nu-\gamma-1} \tag{4.67}$$

is of the non-integrable type for $2\nu > \gamma$ and of integrable type for $2\nu < \gamma$.

$\gamma > 1$ **and** $2\nu > \gamma$

In this first case $F_2(t|t_a)$ is non-integrable, therefore the Laplace transforms is

$$F_2(s|t_a) \simeq \gamma \Gamma(2\nu - \gamma) c^2 \frac{t_0^\gamma}{\tau} t_a s^{\gamma-2\nu}. \tag{4.68}$$

$\gamma > 1$ **and** $2\nu < \gamma$

In the second case the Laplace transform of the expression tends to a constant. To evaluate this we put down

$$F_2 \simeq \gamma c^2 \frac{t_0^\gamma}{\tau} \int_0^\infty dt\, t^{2\nu} \int_0^{t_a} \frac{1}{(t + t_0 + y)^{\gamma+1}} dy \;, \tag{4.69}$$

and interchange the sequence of integrations:

$$\begin{aligned}
F_2 =& \gamma c^2 \frac{t_0^\gamma}{\tau} \int_0^{t_a} dy \int_0^\infty \frac{t^{2\nu}}{(t + t_0 + y)^{\gamma+1}} dt \\
=& \gamma \mathrm{B}(2\nu + 1, \gamma - 2\nu) c^2 \frac{t_0^\gamma}{\tau} \int_0^{t_a} (t_0 + y)^{2\nu-\gamma} dy \\
=& \gamma \frac{\mathrm{B}(2\nu + 1, \gamma - 2\nu)}{2\nu + 1 - \gamma} c^2 \frac{t_0^\gamma}{\tau} [(t_0 + t_a)^{2\nu+1-\gamma} - t_0^{2\nu+1-\gamma}].
\end{aligned}$$

(in the transition to the second line the Eq.(2.2.5.24) of Ref. [15] is used). The corresponding expression is dominated by the first or by the second term in the square brackets, depending on whether $2\nu > \gamma - 1$ or $2\nu < \gamma - 1$.
We summarize our results for $\gamma > 1$ in the following formula:

$$F_2 = \begin{cases} \gamma \frac{\mathrm{B}(2\nu+1,\gamma-2\nu)}{\gamma-1-2\nu} c^2 \frac{t_0^{2\nu+1}}{\tau} & \text{for } 2\nu < \gamma - 1 \\ \gamma \frac{\mathrm{B}(2\nu+1,\gamma-2\nu)}{2\nu-\gamma+1} c^2 \frac{t_0^\gamma}{\tau} t_a^{2\nu+1-\gamma} & \text{for } \gamma - 1 < 2\nu < \gamma \\ \gamma \Gamma(2\nu - \gamma) c^2 \frac{t_0^\gamma}{\tau} t_a s^{\gamma-2\nu} & \text{for } 2\nu > \gamma \end{cases} \tag{4.70}$$

Calculation of s_2

We can calculate the second marginal moment of the single step PDF $s_2(t, t_a)$ directly in the time domain. For this we need the stepping rate $k(t) = C_0(t)$, whose behavior depends on whether $\gamma > 1$ or $\gamma < 1$.

$\gamma < 1$

In this case we find by inverse transform of C_0 from table 4.1:

$$k(t) = \frac{1}{\Gamma(\gamma)\Gamma(1 - \gamma)} t_0^{-\gamma} t^{\gamma-1} = \frac{\sin \pi\gamma}{\pi} t_0^{-\gamma} t^{\gamma-1}, \tag{4.71}$$

where the Γ product formula was used for the second equality. Inserting this result into the general formula Eq. (4.51) we find

$$s_2(t,t_a) = \gamma\frac{\sin(\pi\gamma)}{\pi}c^2\int_0^{t_a}\int_{t_a+t-t'}^{\infty}[(t_a+t-t')^\eta-(t_a-t')^\eta]^2 \times (t')^{\gamma-1}(t'')^{2(\nu-\eta)}\frac{1}{(t_0+t'')^{\gamma+1}}dt''dt'. \tag{4.72}$$

Just like r_2 in the ordinary case, this integral only converges for $\gamma > 2(\nu-\eta)$, meaning that η again governs the existence of the second moment. In the limit $t_0 << t''$ we can write:

$$s_2(t,t_a) \simeq \gamma\frac{\sin(\pi\gamma)}{\pi}\frac{1}{\gamma-2(\nu-\eta)}c^2 \times \int_0^{t_a}(t_a+t-t')^{2(\nu-\eta)-\gamma}[(t_a+t-t')^\eta-(t_a-t')^\eta]^2(t')^{\gamma-1}dt'. \tag{4.73}$$

The expression in square brackets is again approximated by $t^{2\eta}$ and (t_a+t-t') by its value at t, so that

$$s_2(t,t_a) \propto c^2 t_a^\gamma t^{2\nu-\gamma}. \tag{4.74}$$

$\gamma > 1$

In this regime we can use $k(t) = \frac{1}{\tau}$ again. Substituting this into Eq. (4.51) and approximating the term in square brackets results in

$$s_2(t,t_a) \simeq \gamma\frac{1}{\gamma-2(\nu-\eta)}c^2\frac{t_0^\gamma}{\tau}t^{2\eta}\int_0^{t_a}(t_a+t-t')^{2(\nu-\eta)-\gamma}dt', \tag{4.75}$$

which gives us

$$s_2(t,t_a) \simeq \gamma\frac{1}{(\gamma-2(\nu-\eta))}c^2\frac{t_0^\gamma}{\tau}t_a t^{2\nu-\gamma}\ . \tag{4.76}$$

The results so far are summarized in Table 4.2. Here we used the fact that $\tau \propto t_0$, see Eq.(4.5).

$\gamma < 1$	$F_0(s\|t_a) \simeq 1$	$F_2(s\|t_a) \propto \begin{cases} c^2 t_a^{2\nu} & \text{for } 2\nu < \gamma \\ c^2 t_a^\gamma s^{\gamma-2\nu} & \text{for } 2\nu > \gamma \end{cases}$
$\gamma < 1$		$s_2(t,t_a) \propto c^2 t_a^\gamma t^{2\nu-\gamma}$
$\gamma > 1$	$F_0(s\|t_a) \simeq 1$	$F_2(s\|t_a) \propto \begin{cases} c^2 t_0^{2\nu} & \text{for } 2\nu < \gamma-1 \\ c^2 t_0^{\gamma-1} t_a^{2\nu+1-\gamma} & \text{for } \gamma-1 < 2\nu < \gamma \\ c^2 t_0^{\gamma-1} t_a s^{\gamma-2\nu} & \text{for } 2\nu > \gamma \end{cases}$
$\gamma > 1$		$s_2(t,t_a) \simeq c^2 t_0^{\gamma-1} t_a t^{2\nu-\gamma}$

Table 4.2: Results for F_0 and F_2 in the Laplace domain as well as s_2 in the time domain for different parameter ranges in the case of weak aging $t >> t_a$. Dimensionless prefactors are omitted.

Mean squared displacement

We are now ready to calculate the MSD in the weakly aged case. Recall our earlier result

$$\begin{aligned}\langle x^2\rangle(s|t_a) =& F_0(s|t_a)C_0(s)r_2(s) + F_0(s|t_a)C_2(s)r_0(s)\\ &+ F_2(s|t_a)C_0(s)r_0(s) + s_2(s, t_a).\end{aligned} \tag{4.77}$$

We can now write down the first three terms in the Laplace domain using the results from the tables 4.1 and 4.2, and transform them back into the time domain. The last term in the time domain is already known. The calculation results in different asymptotics depending on γ.

$\gamma < 1$

In this regime the terms $F_0C_0r_2$ and $F_0C_2r_0$ reproduce the result for the non-aged walks. The term $F_2C_0r_0(s) \propto t_a^\gamma s^{\gamma-2\nu-1}$ translates into $F_2C_0r_0(t) \propto c^2t^{2\nu}(t_a/t)^\gamma$, and is subdominant for $t \gg t_a$ for $2\nu > \gamma$. For $2\nu < \gamma$ this term tends to $const \cdot c^2 t_a^{2\nu} s^{-1}$, i.e. is a constant proportional to $c^2t_a^{2\nu}$ in the time domain, and is again subdominant with respect to the previous ones. The term s_2 has the same asymptotics as the previous one in the first case, $s_2 \propto c^2t^{2\nu}(t_a/t)^\gamma$ and therefore is also subdominant. The leading terms therefore behave as in the ordinary case.

$\gamma > 1$

For this regime the contributions $F_0C_0r_2$ and $F_0C_2r_0$ give the same behavior as in the non-aged case, $\propto t^{2\nu+1-\gamma}$ for $2\nu > \gamma$, or $\propto t$ in the opposite case. The contribution $F_2C_0r_0(s)$ either corresponds to $s^{\gamma-2\nu-1}$ and translates to $t^{2\nu-\gamma}$ for $2\nu > \gamma$, or to a constant for $2\nu < \gamma$, and is always subdominant. The contribution of s_2 is always subdominant as well.

In conclusion we find that the behavior for short aging times reproduces the behavior of the ordinary walk in leading order up to prefactors, as one might expect, and has the same range of convergence: The second moment exists for $\gamma > 2(\nu - \eta)$.

4.4 Aged walk: long aging times $t_a >> t$

Calculation of F_0 and F_2

Here again the cases $\gamma < 1$ and $\gamma > 1$ have to be distinguished.

$\gamma < 1$

In this domain we can reuse the previous result in Eq.(4.52), but we now expand it for $t_a >> t$:

$$F_0(t|t_a) = \frac{\sin\pi\gamma}{\pi}\left(\frac{t_a}{t}\right)^\gamma \frac{1}{t+t_a} \simeq \frac{\sin\pi\gamma}{\pi} t_a^{\gamma-1}t^{-\gamma}, \tag{4.78}$$

so that we get in the Laplace domain

$$F_0(s|t_a) \simeq \frac{\sin\pi\gamma}{\pi}\Gamma(1-\gamma)t_a^{\gamma-1}s^{\gamma-1}. \tag{4.79}$$

For $F_2(t|t_a)$ we can use our result for $k(t)$, Eq.(4.71), and insert it into (4.48):

$$F_2(t|t_a) = \gamma \frac{\sin(\pi\gamma)}{\pi} c^2 \int_0^{t_a} (t_a + t - t')^{2(\nu-\eta)} [(t_a + t - t')^\eta - (t_a - t')^\eta]^2 \times \frac{(t')^{\gamma-1}}{(t_0 + t_a + t - t')^{1+\gamma}} dt'. \quad (4.80)$$

Neglecting t_0 in the expression in the last line this simplifies to

$$F_2(t|t_a) \simeq \gamma \frac{\sin(\pi\gamma)}{\pi} c^2 t_a^{2\nu-2} \int_0^{t_a} \left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{2(\nu-\eta)-\gamma-1} \times \left[\left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^\eta - \left(1 - \frac{t'}{t_a}\right)^\eta\right]^2 \left(\frac{t'}{t_a}\right)^{\gamma-1} dt'. \quad (4.81)$$

We introduce the dimensionless variables $z = 1 - \frac{t'}{t_a}$ and $y = \frac{t}{t_a}$ and rewrite the integral as:

$$F_2(t|t_a) = \gamma \frac{\sin(\pi\gamma)}{\pi} c^2 t_a^{2\nu-1} \int_0^1 (z+y)^{2(\nu-\eta)-1-\gamma} [(z+y)^\eta - z^\eta]^2 (1-z)^{\gamma-1} dz. \quad (4.82)$$

Since we are going to encounter integrals of this type several times, we will calculate them separately. The general form

$$I_{a,b,c}(y) = \int_0^1 (z+y)^a [(z+y)^c - z^c]^2 (1-z)^b dz\ , \quad (4.83)$$

can be expressed in terms of Gauß hypergeometric functions, leading to the following asymptotic behavior for $y \to 0$

$$I_{a,b,c}(y) \simeq \begin{cases} C(a,c) y^{1+a+2c} & \text{for} \quad a + 2c < 1 \\ \mathrm{B}(1+b, a+2c-1) y^2 c^2 & \text{for} \quad a + 2c > 1. \end{cases} \quad (4.84)$$

A detailed derivation and the bounds for the constant $C(a,c)$ are given in Appendix B.

The behavior of F_2 follows with the substitutions $a = 2(\nu-\eta) - \gamma - 1$, $b = \gamma - 1$, $c = \eta$. Omitting dimensionless constants we obtain two distinct regimes in the limit $t_a >> t$, depending on the relation between ν and γ:

$$F_2(t|t_a) \propto \begin{cases} c^2 t_a^{2\nu-3} t^2 & \text{for } 2\nu > \gamma + 2 \\ c^2 t_a^{\gamma-1} t^{2\nu-\gamma} & \text{for } 2\nu < \gamma + 2. \end{cases} \quad (4.85)$$

Since both of these cases belong to the non-integrable class, we obtain in the Laplace-domain:

$$F_2(s|t_a) \propto \begin{cases} c^2 t_a^{2\nu-3} s^{-3} & \text{for} \quad 2\nu > \gamma + 2 \\ c^2 t_a^{\gamma-1} s^{\gamma-2\nu-1} & \text{for} \quad 2\nu < \gamma + 2. \end{cases} \quad (4.86)$$

$\gamma > 1$

By substituting $k(t') = \tau^{-1}$ one finds

$$F_0 = \int_0^{t_a} \psi(t_a + t - t') \frac{1}{\tau} dt' \to \frac{1}{\tau} \Psi(t) \simeq \frac{t_0^\gamma}{\tau} t^{-\gamma}. \quad (4.87)$$

Since $\gamma > 1$, the term F_0 is of the integrable type, and therefore

$$F_0(s|t_a) \simeq const. \tag{4.88}$$

Now we turn to F_2. Starting from equation (4.48) one finds

$$\begin{aligned} F_2(t|t_a) \simeq \gamma c^2 \frac{t_0^\gamma}{\tau} t_a^{2\nu-\gamma-1} \int_0^{t_a} \left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^{2(\nu-\eta)-\gamma-1} \\ \times \left[\left(1 + \frac{t}{t_a} - \frac{t'}{t_a}\right)^\eta - \left(1 - \frac{t'}{t_a}\right)^\eta\right]^2 dt' \, . \end{aligned} \tag{4.89}$$

The calculation is similar to the one in the case $\gamma < 1$. By applying Eq.(4.84) to $I_{2(\nu-\eta)-\gamma-1,0,\eta}(t/t_a)$ we obtain

$$F_2(t|t_a) \simeq \gamma c^2 \frac{t_0^\gamma}{\tau} \begin{cases} \eta^2 B(1, 2\nu - \gamma - 2) t_a^{2\nu-\gamma-2} t^2 & \text{for} \quad 2\nu > \gamma + 2 \\ C \cdot t^{2\nu-\gamma} & \text{for} \quad 2\nu < \gamma + 2. \end{cases} \, . \tag{4.90}$$

The case $2\nu > \gamma+2$ still belongs in the non-integrable class and therefore transforms into

$$F_2(s|t_a) \simeq 2\gamma\eta^2 B(1, 2\nu - \gamma - 2) c^2 \frac{t_0^\gamma}{\tau} t_a^{2\nu-\gamma-2} s^{-3}, \tag{4.91}$$

however for $2\nu < \gamma + 2$ we have to distinguish between $\gamma - 1 < 2\nu < \gamma + 2$, where the F_2 is non-integrable, and $2\nu < \gamma - 1$, where it is integrable. Therefore:

$$F_2(s|t_a) \simeq \gamma c^2 \frac{t_0^\gamma}{\tau} \begin{cases} \eta^2 \mathrm{B}(1, 2\nu - \gamma - 2) t_a^{2\nu-\gamma-2} s^{-3} & 2\nu > \gamma + 2 \\ C\, \Gamma(2\nu + 1 - \gamma) s^{\gamma-2\nu-1} & \gamma + 2 > 2\nu > \gamma - 1 \\ const\, t_0^{2\nu+1-\gamma} & 2\nu < \gamma - 1. \end{cases} \, . \tag{4.92}$$

Calculation of s_2

$\gamma < 1$

The calculations for s_2 from Eq. (4.51) are very similar to that for F_2 case and yield

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} \frac{\sin(\pi\gamma)}{\pi} c^2 \begin{cases} 2\eta^2 B(2\nu - \gamma - 1, \gamma) t_a^{2\nu-2} t^2 & \text{for} \quad 2\nu > \gamma + 1 \\ C\, t_a^{\gamma-1} t^{2\nu+1-\gamma} & \text{for} \quad 2\nu < \gamma + 1. \end{cases} \, . \tag{4.93}$$

$\gamma > 1$

In this case we have

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^\gamma}{\tau} \int_0^{t_a} \left[(t_a + t - t')^\eta - (t_a - t')^\eta\right]^2 (t_a + t - t')^{2(\nu-\eta)-\gamma} dt' \tag{4.94}$$

$$= \gamma \frac{1}{\gamma - 2(\nu - \eta)} I_{2(\nu-\eta)-\gamma,0,\eta} c^2 \frac{t_0^\gamma}{\tau} (t_a)^{2\nu+1-\gamma} \left(\frac{t}{t_a}\right). \tag{4.95}$$

Using Eq.(4.84) again we find

$$s_2(t, t_a) \simeq \gamma \frac{1}{\gamma - 2(\nu - \eta)} c^2 \frac{t_0^\gamma}{\tau} \begin{cases} \eta^2 B(2\nu - \gamma - 1, 1) t_a^{2\nu-\gamma-1} t^2 & \text{for} \quad 2\nu > \gamma + 1 \\ C t^{2\nu+1-\gamma} & \text{for} \quad 2\nu < \gamma + 1. \end{cases} \, . \tag{4.96}$$

The corresponding results for the case of long aging times are summarized in Table 4.3.

$\gamma < 1$	$F_0(s\|t_a) \propto t_a^{\gamma-1} s^{\gamma-1}$	$F_2(s\|t_a) \propto c^2 \begin{cases} t_a^{2\nu-3} s^{-3} & 2\nu > \gamma+2 \\ t_a^{\gamma-1} s^{\gamma-2\nu-1} & 2\nu < \gamma+2 \end{cases}$
$\gamma < 1$		$s_2(t,t_a) \propto c^2 \begin{cases} t_a^{2\nu-2} t^2 & 2\nu > \gamma+1 \\ t_a^{\gamma-1} t^{2\nu+1-\gamma} & 2\nu < \gamma+1 \end{cases}$
$\gamma > 1$	$F_0(s\|t_a) \propto 1$	$F_2(s\|t_a) \propto c^2 \begin{cases} t_0^{\gamma-1} t_a^{2\nu-\gamma-2} s^{-3} & 2\nu > \gamma+2 \\ t_0^{\gamma-1} s^{\gamma-2\nu-1} & \gamma+2 > 2\nu > \gamma-1 \\ t_0^{2\nu} & 2\nu < \gamma-1 \end{cases}$
$\gamma > 1$		$s_2(t,t_a) \propto c^2 \begin{cases} t_0^{\gamma-1} t_a^{2\nu-\gamma-1} t^2 & 2\nu > \gamma+1 \\ t_0^{\gamma-1} t^{2\nu+1-\gamma} & 2\nu < \gamma+1 \end{cases}$

Table 4.3: Results for F_0 and F_2 in the Laplace domain as well as s_2 in the time domain for different parameter ranges in the case of long aging times $t_a >> t >> t_0$. Dimensionless prefactors are omitted.

Mean squared displacement

With these results we can now compute the MSD in the stronlgy aged case. Using Tables 4.1 and 4.3 we can write down the asymptotic behavior of the combinations $F_0C_0r_2$, $F_0C_2r_0$ and $F_2C_0r_0$ in the Laplace domain. The inverse transforms are then performed using Tauberian theorems. The corresponsing results for $\gamma < 1$ and for $\gamma > 1$ are summarized in Tables 4.4 and 4.5.

Therefore for considerably aged walks we have:

	$F_2C_0r_0$	$F_0C_0r_2$	$F_0C_2r_0$	s_2
$2\nu < \gamma$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_0^{2\nu-\gamma} t_a^{\gamma-1} t}$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$
$\gamma < 2\nu < 1+\gamma$	$\boldsymbol{c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}}$	$\boldsymbol{c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}}$	$\boldsymbol{c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}}$	$\boldsymbol{c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}}$
$1+\gamma < 2\nu < 2+\gamma$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_a^{2\nu-2} t^2}$
$2+\gamma < 2\nu$	$c^2 t_a^{2\nu-3} t^3$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$c^2 t_a^{\gamma-1} t^{2\nu+1-\gamma}$	$\boldsymbol{c^2 t_a^{2\nu-2} t^2}$

Table 4.4: Asymptotic behavior of the contributions to the MSD for $\gamma < 1$ in the limit $t_a >> t >> t_0$. All dimensionless prefactors are omitted. The dominant terms are highlighted in boldface.

$\gamma < 1$**:**

$$\langle x^2(t) \rangle \propto \begin{cases} t_a^{\gamma-1} t & \text{for} \quad 2\nu < \gamma \\ t_a^{\gamma-1} t^{2\nu+1-\gamma} & \text{for} \quad \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-2} t^2 & \text{for} \quad 1+\gamma < 2\nu. \end{cases} \tag{4.97}$$

Obviously, no Richardson superballistic regime $\langle x^2 \rangle \propto t^3$ is possible in the aged walk. This fact by itself does not make the model to be a poor candidate for description of turbulent dispersion, since the Richardson law in turbulence corresponds to starting the observation of the interparticle distance immediately after introducing the tracers into the stationary flow, i.e. to an ordinary, non-aged case.

	$F_2C_0r_0$	$F_0C_0r_2$	$F_0C_2r_0$	s_2
$2\nu < \gamma - 1$	$c^2t_0^{2\nu}$	$c^2t_0^{2\nu}$	$\mathbf{c^2t_0^{2\nu-1}t}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$
$\gamma - 1 < 2\nu < \gamma$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2t_0^{2\nu-1}t}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$
$\gamma < 2\nu < 1 + \gamma$	$\mathbf{c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}}$	$\mathbf{c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}}$
$1 + \gamma < 2\nu < 2 + \gamma$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2t_0^{\gamma-1}t_a^{2\nu-\gamma-1}t^2}$
$2 + \gamma < 2\nu$	$c^2t_0^{\gamma-1}t_a^{2\nu-\gamma-2}t^3$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$c^2t_0^{\gamma-1}t^{2\nu+1-\gamma}$	$\mathbf{c^2t_0^{\gamma-1}t_a^{2\nu-\gamma-1}t^2}$

Table 4.5: Asymptotic behavior of the contributions to the MSD for $\gamma > 1$ in the limit $t_a >> t >> t_0$. All dimensionless prefactors are omitted. The dominant terms are highlighted in boldface.

$\gamma > 1$

$$\langle x^2(t)\rangle \propto \begin{cases} t & \text{for} \quad 2\nu < \gamma \\ t^{2\nu+1-\gamma} & \text{for} \quad \gamma < 2\nu < 1+\gamma \\ t_a^{2\nu-\gamma-1}t^2 & \text{for} \quad 1+\gamma < 2\nu. \end{cases} \tag{4.98}$$

Therefore in the regime where the ordinary walk shows normal diffusion or enhanced diffusion there are no or weak changes due to aging (differing only by prefactor). In the regime where the ordinary walk is superballistic, it ages to a ballistic one. This finding complies with the fact that a ballistic walk with $\nu = 1$ shows only weak aging, i.e. again the ballistic aged behavior [16, 17].

Summarizing the findings for the case of considerably aged walks we state that for the parameter range where the ordinary walk showed subdiffusion we now find regular diffusion but with a prefactor that decays with growing aging times. There is no place for the Richardson regime in the considerably aged case. Again, the second moment extsts only for $\gamma > 2(\nu - \eta)$, and if it exists the value of η only enters the prefactors, but does not change the power-law dependences of the MSD on all times involved.

We moreover note that the double time-ensemble average $\langle\langle x^2(t)\rangle_T\rangle_E$, also discussed in Ref. [6], whose calculation involves an additional integration over the time,$\langle\langle x^2(t)\rangle_T\rangle_E \simeq (T-t)^{-1}\int_0^{T-t}\langle x^2(t|t_a)\rangle dt_a$ shows the same behavior as the aged walk if the measurement time t is associated with the time lag in the double average, and the aging time t_a is changed for the data acquisition time T.

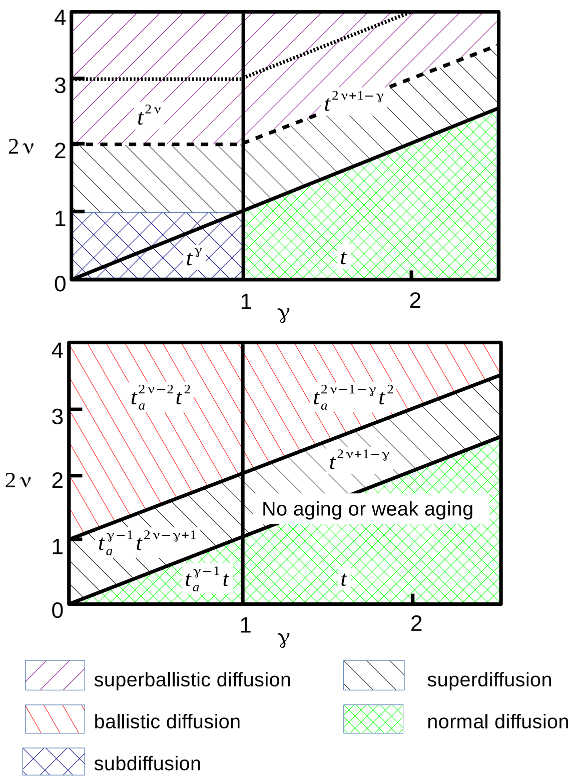

Figure 4.1: The upper panel shows the map of behaviors of the ensemble average $\langle x^2(t)\rangle$ in the ordinary Drude walk. The thick solid lines correspond to the changes in time-dependences while the hatchings represent the type of diffusion. The dashed line corresponds to the ballistic behavior, and the dotted one to the Richardson law. The lower panel shows the same for the considerably aged walk.

5. Results and Discussion

6. Conclusions

A. The Tauberian theorem

In this thesis I often consider Laplace transforms of functions asymptotically following power laws. The asymptotic form of Laplace transforms of functions which at long times behave as $f(t) = t^{\rho-1}L(t)$, where $L(t)$ is a slowly varying function, in the asymptotic regime $s \to 0$ as well as the corresponding inverse transforms may be obtained by use of the Tauberian theorem.

We assume that a Laplace transform $f(s) = \int_0^t f(t)e^{-st}dt$ exists, i.e. the function $f(t)$ does not possess a strong divergence at 0. All functions $f(t)$ appearing this thesis are non-negative. For such functions Laplace transforms are monotonically decaying functions of s. Depending on the behavior of $f(t)$ at infinity two cases should be considered:
The function $f(t)$ might be integrable on $[0, \infty)$, so that $\int_0^\infty f(t)dt = I_0^f < \infty$, or this integral may diverge. The first case corresponds to $\rho < 0$ and the second one to $\rho > 0$ (the case $\rho = 0$ may belong to the either class depending on the concrete form of $L(t)$).
In the second case the Tauberian theorem may be applied immediately, stating that if $f(t)$ is a regularly varying function, i.e. when its Laplace transform is given by

$$f(t) \simeq t^{\rho-1}L(t) \quad \leftrightarrow \quad f(s) \simeq \Gamma(\rho)s^{-\rho}L\left(\frac{1}{s}\right) \tag{A.1}$$

for $\rho \geq 0$. As in the main text, all slowly varying functions will be omitted (i.e. changed for constants L).

We note that for $\rho < 0$ equation, i.e. when $f(t)$ is integrable, equation (A.1) suggests $f(s)$ being a growing function of s and is therefore wrong. In this case let us consider the function

$$S(t) = \int_t^\infty f(t')dt'. \tag{A.2}$$

The integrability of $f(t)$ means that $S(t)$ is well-defined, and that $I_0^f = \int_0^\infty f(t')dt' = S(0)$ is finite. The function $S(t)$ has the power-law asymptotics

$$S(t) \simeq -\frac{Lt^\rho}{\rho}, \tag{A.3}$$

and, if this is no more integrable (i.e. for $\rho > -1$), can be transformed via the Tauberian theorem, so that

$$S(s) \simeq -L\frac{\Gamma(\rho+1)}{\rho}s^{-(\rho+1)} = -L\Gamma(\rho)s^{-(\rho+1)}, \tag{A.4}$$

where in the last equality the identity $\Gamma(x+1) = x\Gamma(x)$ was used. Noting that $f(t) = -\frac{d}{dt}S(t)$ and using the Laplace representation of the derivative, we get

$$f(s) = S(t=0) - sS(s) = I_0^f - L\Gamma(\rho)s^{-\rho}. \tag{A.5}$$

The direct application of the Tauberian theorem would give us a correct form of the second term (up to a sign), but omit the first one.

If $S(t)$ is still integrable, we consider the function $P(t) = \int_t^\infty S(t')dt'$, whose power-law asymptotics for $t \to \infty$ is

$$P(t) \simeq \frac{Lt^{\rho+1}}{\rho(\rho+1)}, \tag{A.6}$$

and whose connection to $f(t)$ is given by $f(t) = \frac{d^2}{dt^2}S(t)$. For $-2 < \rho$ the function $P(t)$ is not integrable, and the application of the Tauberian theorem gives

$$P(s) = L\frac{\Gamma(\rho+2)}{\rho(\rho+1)}s^{-\rho-2} = L\Gamma(\rho)s^{-\rho-2}. \tag{A.7}$$

Using the Laplace representation for the second derivative we get

$$f(s) = -sP(t=0) - P'(t=0) + s^2P(s). \tag{A.8}$$

The value of $P'(t=0)$ is $-S(t=0) = -I_0^f$. The value $P(t=0)$ is given by the integral

$$P(t=0) = \int_0^\infty dt \int_t^\infty f(t')dt'. \tag{A.9}$$

Changing the sequence of integrations in t and t' we get

$$P(t=0) = \int_0^\infty dt' f(t') \int_0^{t'} dt = \int_0^\infty t' f(t')dt'. \tag{A.10}$$

Since $f(t)$ decays with t faster than t^{-2}, the integral converges, and will be denoted by I_1^f. Therefore we have

$$f(s) = I_0^f - sI_1^f + L\Gamma(\rho)s^{-\rho}. \tag{A.11}$$

For $\rho < -2$ the procedure has to be repeated again for the function being the integral of $P(t)$, etc. The general result is

$$f(s) = \sum_{k=0}^{k_{\max}} \frac{(-1)^k}{k!} I_k^f s^k + L\Gamma(\rho)s^{-\rho} \tag{A.12}$$

with $k_{\max}$ being the whole part of $-\rho$, and I_k^f being the moment integral

$$I_k^f = \int_0^\infty t^k f(t)dt. \tag{A.13}$$

In the main text we never have to use more than first three terms of this expansion.

B. Estimates for the integral $I_{a,b,c}(y)$

We are interested in the integral

$$\begin{aligned} I_{a,b,c}(y) &= \int_0^1 (1-z)^b[(z+y)^c - z^c]^2(z+y)^a dz \\ &= \int_0^1 (1-z)^b \left[(z+y)^{a+2c} - 2(z+y)^{a+c}z^c + (z+y)^a z^{2c}\right] dz \end{aligned} \tag{B.1}$$

in the limit of small $y = \frac{t}{t_a} \ll 1$ for the parameter ranges $c > 0$, $b > -1$, $a \in \mathbb{R}$.

To evaluate it we use Euler's integral representation for the Gauß hypergeometric function for $\Re\, c' > \Re\, b' > 0$

$$_2F_1(a', b'; c'; x) = \frac{1}{\mathrm{B}(b', c'-b')} \int_0^1 z^{b'-1}(1-z)^{c'-b'-1}(1-zx)^{-a'}. \tag{B.2}$$

As the existence condition $1 + b > 0$ is always satisfied for all three terms in (B.1) we can write the integral as

$$\begin{aligned} I_{a,b,c}(y) = y^a \Bigg[& y^{2c}\mathrm{B}(1, 1+b)\,_2F_1\left(-a-2c, 1; 2+b; -\frac{1}{y}\right) \\ & -2y^c\mathrm{B}(1+c, 1+b)\,_2F_1\left(-a-c, 1+c; 2+b+c; -\frac{1}{y}\right) \\ & +\mathrm{B}(1+2c, 1+b)\,_2F_1\left(-a, 1+2c; 2+b+2c; -\frac{1}{y}\right)\Bigg] \end{aligned} \tag{B.3}$$

with $\mathrm{B}(x, y)$ being the Beta function. Athough the integral can be expressed in terms of three Gauß hypergeometric functions, its investigation is somewhat tricky, since the asymptotic regimes appear as a subleading terms in a sum of three large contributions whose leading terms cancel. First, to avoid evaluating hypergeometric functions at $-\infty$ we make use of the Pfaff transformations:

$$_2F_1(a', b'; c'; z) = (1-z)^{-b'}\ _2F_1\left(b', c'-a'; c; \frac{z}{z-1}\right) \tag{B.4}$$

$$_2F_1(a', b'; c'; z) = (1-z)^{-a'}\ _2F_1\left(a', c'-b'; c; \frac{z}{z-1}\right). \tag{B.5}$$

These two forms will be applicable in different domains of parameters. Under the transformations the argument of the corresponding functions on the r.h.s., equal to $\frac{1}{1+y}$, will tend to 1. Applying the Pfaff transformation Eq.(B.4) to the integrals in

Eq.(B.3) we find:

$$I_{a,b,c}(y) = y^{1+a+2c}\Bigg[(1+y)^{-1}\mathrm{B}(1,1+b)\ {}_2F_1\left(1,2+a+b+2c;2+b;\frac{1}{1+y}\right)$$
$$-\,2(1+y)^{-1-c}\mathrm{B}(1+c,1+b){}_2F_1\left(1+c,2+a+b+2c;2+b+c;\frac{1}{1+y}\right)$$
$$+\,(1+y)^{-1-2c}\mathrm{B}(1+2c,1+b)\ {}_2F_1\left(1+2c,2+a+b+2c;2+b+2c;\frac{1}{1+y}\right)\Bigg].$$

We now use the Euler integral representation (B.2) again, but exchange the roles of a' and b':

$$ {}_2F_1(a',b';c';x) = \frac{1}{\mathrm{B}(a',c'-a')}\int_0^1 z^{a'-1}(1-z)^{c'-a'-1}(1-zx)^{-b'}$$

for $\Re\, c' > \Re\, a' > 0$. Note that the existence condition for the integrals is the same as before, $b+1>0$, which is satisfied for all cases relevant in this thesis, so we can write:

$$I_{a,b,c}(y) = y^{1+a+2c} \int_0^1 \Bigg[(1+y)^{-1}(1-z)^b\left(1-\frac{z}{1+y}\right)^{-2-a-b-2c}$$
$$-2(1+y)^{-1-c}z^c(1-z)^b\left(1-\frac{z}{1+y}\right)^{-2-a-b-2c}$$
$$+(1+y)^{-1-2c}z^{2c}(1-z)^b\left(1-\frac{z}{1+y}\right)^{-2-a-b-2c}\Bigg]\,dz.$$

The integrals of each of three contributions in square brackets would diverge for $y\to 0$, but the integral of whole sum is convergent for $a+2c<1$ since for $y\to 0$ the integrand tends to

$$(1-2z^c+z^{2c})(1-z)^{-2-a-2c} = (1-z^c)^2(1-z)^{-2-a-2c},$$

and the integral

$$C(a,c) = \int_0^1 (1-z^c)^2(1-z)^{-2-a-2c}dz$$

of this expression converges in the range $a+2c<1$ (to prove the convergence it is enough to expand the first term in vicinity of $z=1$). This integral cannot be expressed in terms of "simple" functions, but the (loose) bounds for it follow easily.

Let us find two constants $B>A>0$ such that for all $0<z<1$

$$A(1-z) < 1-z^c < B(1-z).$$

To do so consider the function

$$f(z) = \frac{1-z^c}{1-z},$$

with $f(0)=1$ and with its limiting value at $z\to 1$ given by the l'Hôpital's rule $\lim_{z\to 1} = c$. Therefore the limit of the function at 1 is larger than its value at 0 when $c>1$ and smaller than this value when $c<1$. For $c=1$ this function equals to unity identically.

Now we consider $c \neq 1$ and proseed to show that the function $f(z)$ is monotonically growing for $c > 1$ and monotonically decaying for $c < 1$. To show this it is enough to show that its derivative on $[0, 1]$ does not vanish. The derivative of the corresponding function is

$$f'(z) = \frac{1 - z^c + cz^c - cz^{c-1}}{(1-z)^2},$$

and can only vanish when the numerator, $g(z) = 1 - z^c + cz^c - cz^{c-1}$, vanishes somewhere at $0 \leq z < 1$. Vanishing of the numerator at $z = 1$ does not pose a problem since $f'(z)$ diverges and tends to $(c-1)(1-z)^{-2}$ for $z = 1$, being positive in vicinity of $z = 1$ for $c > 1$ and negative for for $c < 1$ due to the fact that the denominator vanishes even faster. Now we show that this function never changes its sign on $0 \leq z < 1$. Calculating the derivative

$$g'(z) = -c(c-1)z^{c-2} + c(c-1)z^{c-1} = -c(c-1)z^{c-2}(1-z)$$

we see that it is strictly positive for all $z < 1$ for $c < 1$ and strictly negative for $c > 1$. Therefore the bounds for the function $f(z)$ are given by its limiting values of 1 at $z = 0$ and c at $z = 1$. Therefore we have $A = \min(1, c^2)$ and $B = \max(1, c^2)$. Since for $1 > a + 2c$

$$\int_0^1 (1-z)^{-a-2c} dz = \frac{1}{1-a-2c}$$

we get

$$\frac{\min(1, c^2)}{1-a-2c} \leq C \leq \frac{\max(1, c^2)}{1-a-2c}. \tag{B.6}$$

Therefore, for $a + 2c < 1$ we have, for y small,

$$I_{a,b,c}(y) \simeq C y^{1+a+2c} \tag{B.7}$$

where the bounds for the constant C are given by Eq.(B.6).

For the opposite case $a + 2c > 1$ we have to use the other Pfaff transformation, Eq.(B.5), resulting in:

$$\begin{aligned} I_{a,b,c}(y) = (1+y)^a \Big[&(1+y)^{2c} \mathrm{B}(1, 1+b) {}_2F_1\left(-a-2c, 1+b; 2+b; 1/(1+y)\right) \\ &- 2(1+y)^c \mathrm{B}(1+c, 1+b) {}_2F_1\left(-a-c, 1+b; 2+b+c; 1/(1+y)\right) \\ &+ \mathrm{B}(1+2c, 1+b) {}_2F_1\left(-a, 1+b; 2+b+2c; 1/(1+y)\right) \Big]. \end{aligned} \tag{B.8}$$

Using the integral representation Eq.(B.2) again we find

$$\begin{aligned} I_{a,b,c}(y) = (1+y)^a \int_0^1 \Bigg[&(1+y)^{2c} z^b \left(1 - \frac{z}{1+y}\right)^{a+2c} \\ &- 2(1+y)^c z^b (1-z)^c \left(1 - \frac{z}{1+y}\right)^{a+c} \\ &+ z^b (1-z)^{2c} \left(1 - \frac{z}{1+y}\right)^a \Bigg] dz. \end{aligned} \tag{B.9}$$

Now we expand the expression in each term of the integrand up to second order in y. Using the fact that

$$(1+y)^\alpha \simeq 1+\alpha y+\frac{\alpha(\alpha-1)}{2}y^2,$$

and

$$\left(1-\frac{z}{1+y}\right)^\alpha \simeq (1-z)^\alpha+\alpha(1-z)^{\alpha-1}zy$$
$$+\frac{1}{2}\left[\alpha(\alpha-1)(1-z)^{\alpha-2}z^2-2\alpha(1-z)^{\alpha-1}z\right]y^2,$$

as well as the definition of the Beta function

$$\mathrm{B}(a,b)=\int_0^1 z^{a-1}(1-z)^{b-1}dz,$$

we find:

$$I_{a,b,c}(y)\simeq c^2y^2 \quad [\mathrm{B}(1+b,1+a+2c)$$
$$+2\mathrm{B}(2+b,a+2c)+\mathrm{B}(3+b,a+2c-1)].$$

The first two orders in y have canceled, so the leading term goes as y^2. From the argument of the last Beta function it is clear, that the result only holds for $a+2c>1$, i.e. exactly in the parameter range where Eq.(B.7) ceases to be applicable, and that the exponents are continuous at $a+2c=1$. Rewriting the Beta functions as $\mathrm{B}(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ and (repeatedly) using the identity $\Gamma(x+1)=x\Gamma(x)$ we find a compact representation of the sum of the three beta functions, namely

$$I_{a,b,c}(y)\simeq c^2y^2\mathrm{B}(1+b,a+2c-1). \tag{B.10}$$

In conclusion we have:

$$I_{a,b,c}(y)\simeq\begin{cases} C(a,c)y^{1+a+2c} & \text{for} \quad a+2c<1\\ c^2\mathrm{B}(1+b,a+2c-1)y^2 & \text{for} \quad a+2c>1,\end{cases}$$

which is the Eq.(4.84) of the main text, with the bounds on a constant $C(a,c)$ given by Eq.(B.6).

Bibliography

[1] Pearson, K: The Problem of the Random Walk. In: *letter, Nature* 72 (1905), S. 294

[2] Richardson, Lewis F.: Atmospheric diffusion shown on a distance-neighbour graph. In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 110 (1926), Nr. 756, S. 709–737

[3] Zaburdaev, V ; Denisov, S ; Klafter, J: Lévy walks. In: *Reviews of Modern Physics* 87 (2015), Nr. 2, S. 483

[4] Shlesinger, MF ; West, BJ ; Klafter, Joseph: Lévy dynamics of enhanced diffusion: Application to turbulence. In: *Physical Review Letters* 58 (1987), Nr. 11, S. 1100

[5] Korobkova, Ekaterina ; Emonet, Thierry ; Vilar, Jose M. ; Shimizu, Thomas S. ; Cluzel, Philippe: From molecular noise to behavioural variability in a single bacterium. In: *Nature* 428 (2004), Nr. 6982, S. 574

[6] Albers, Tony ; Radons, Günter: Exact results for the nonergodicity of d-dimensional generalized levy walks. In: *Physical review letters* 120 (2018), Nr. 10, S. 104501

[7] Schulz-Baldes, Hermann: Anomalous Drude Model. In: *Physical review letters* 78 (1997), Nr. 11, S. 2176

[8] Benkadda, Sadruddin ; Zaslavsky, George M.: *Chaos, Kinetics and Nonlinear Dynamics in Fluids and Plasmas: Proceedings of a Workshop Held in Carry-Le Rouet, France, 16–21 June 1997.* Bd. 511. Springer Science & Business Media, 1998

[9] Barkai, Eli: Aging in Subdiffusion Generated by a Deterministic Dynamical System. In: *Phys. Rev. Lett.* 90 (2003), Mar, 104101. `http://dx.doi.org/10.1103/PhysRevLett.90.104101`. – DOI 10.1103/PhysRevLett.90.104101

[10] Barkai, Eli ; Cheng, Yuan-Chung: Aging continuous time random walks. In: *The Journal of chemical physics* 118 (2003), Nr. 14, S. 6167–6178

[11] Klages, Rainer ; Radons, Günter ; Sokolov, Igor M.: *Anomalous transport.* Wiley Online Library, 2008

[12] Brokmann, X. ; Hermier, J.-P. ; Messin, G. ; Desbiolles, P. ; Bouchaud, J.-P. ; Dahan, M.: Statistical Aging and Nonergodicity in the Fluorescence of Single Nanocrystals. In: *Phys. Rev. Lett.* 90 (2003),

Mar, 120601. http://dx.doi.org/10.1103/PhysRevLett.90.120601. – DOI 10.1103/PhysRevLett.90.120601

[13] Klafter, Joseph ; Sokolov, Igor M.: *First steps in random walks: from tools to applications*. Oxford University Press, 2011

[14] Marsaglia, George u. a.: Xorshift rngs. In: *Journal of Statistical Software* 8 (2003), Nr. 14, S. 1–6

[15] Prudnikov, AP: *Yu. A. Brychkov, and OI Marichev. 1986. Integrals and Series, Vol. 1. Elementary Functions*. 1986

[16] Magdziarz, Marcin ; Zorawik, Tomasz: Aging ballistic Lévy walks. In: *Physical Review E* 95 (2017), Nr. 2, S. 022126

[17] Froemberg, Daniela ; Barkai, Eli: Time-averaged Einstein relation and fluctuating diffusivities for the Lévy walk. In: *Physical Review E* 87 (2013), Nr. 3, S. 030104

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Ort, Datum

Unterschrift