Probability

Counting Rules

Counting operations on *n* objects

- **Sort, order matters (perms)**
  - Distinct: $n!$
  - Some Distinct: $\dfrac{n!}{n_1!n_2!\ldots}$
- **Choose *k* (combinations)**
  - Distinct: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
- **Put in *r* buckets**
  - Distinct: r^n
  - None Distinct: $\dfrac{(n+r-1)!}{n!(r-1)!}$

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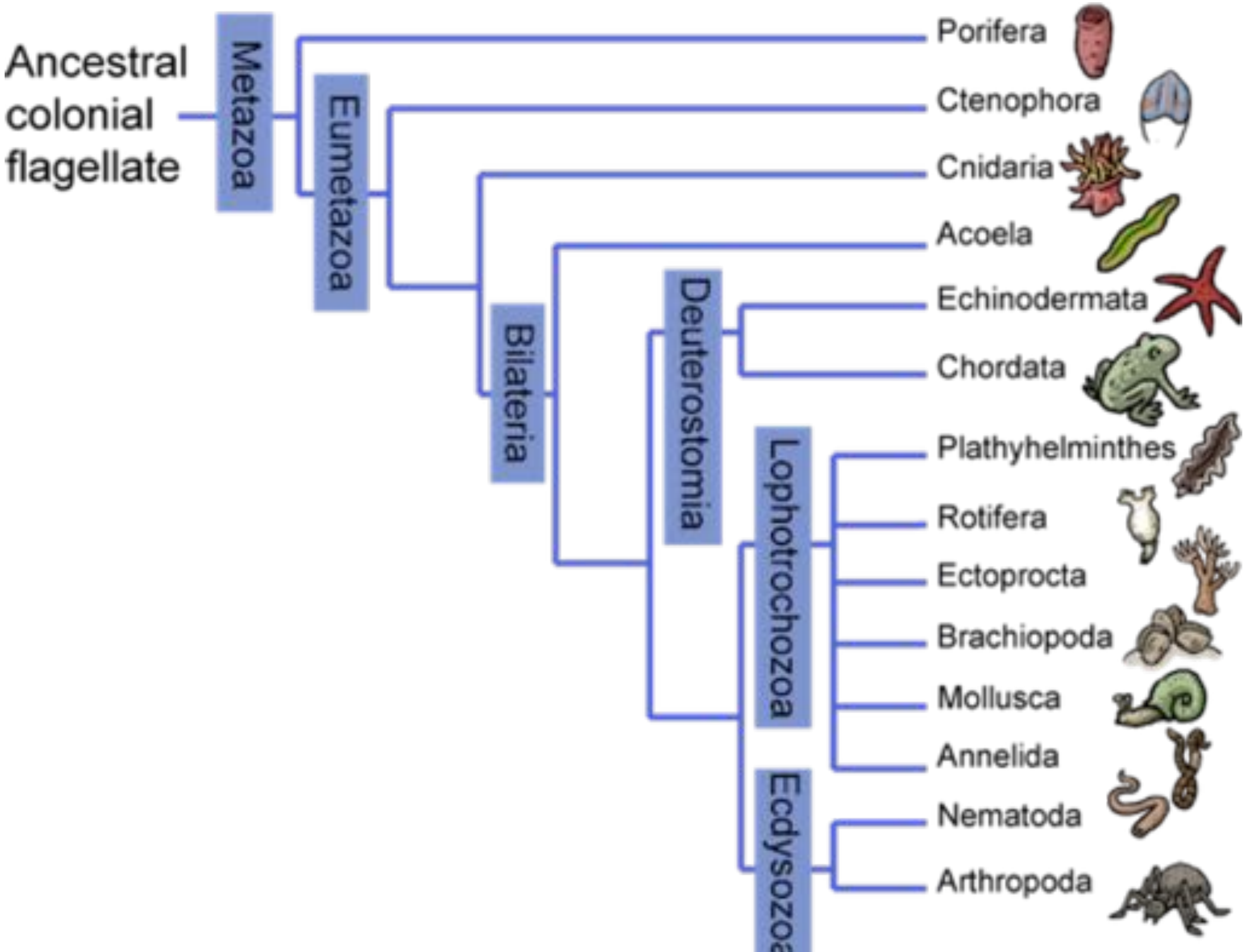
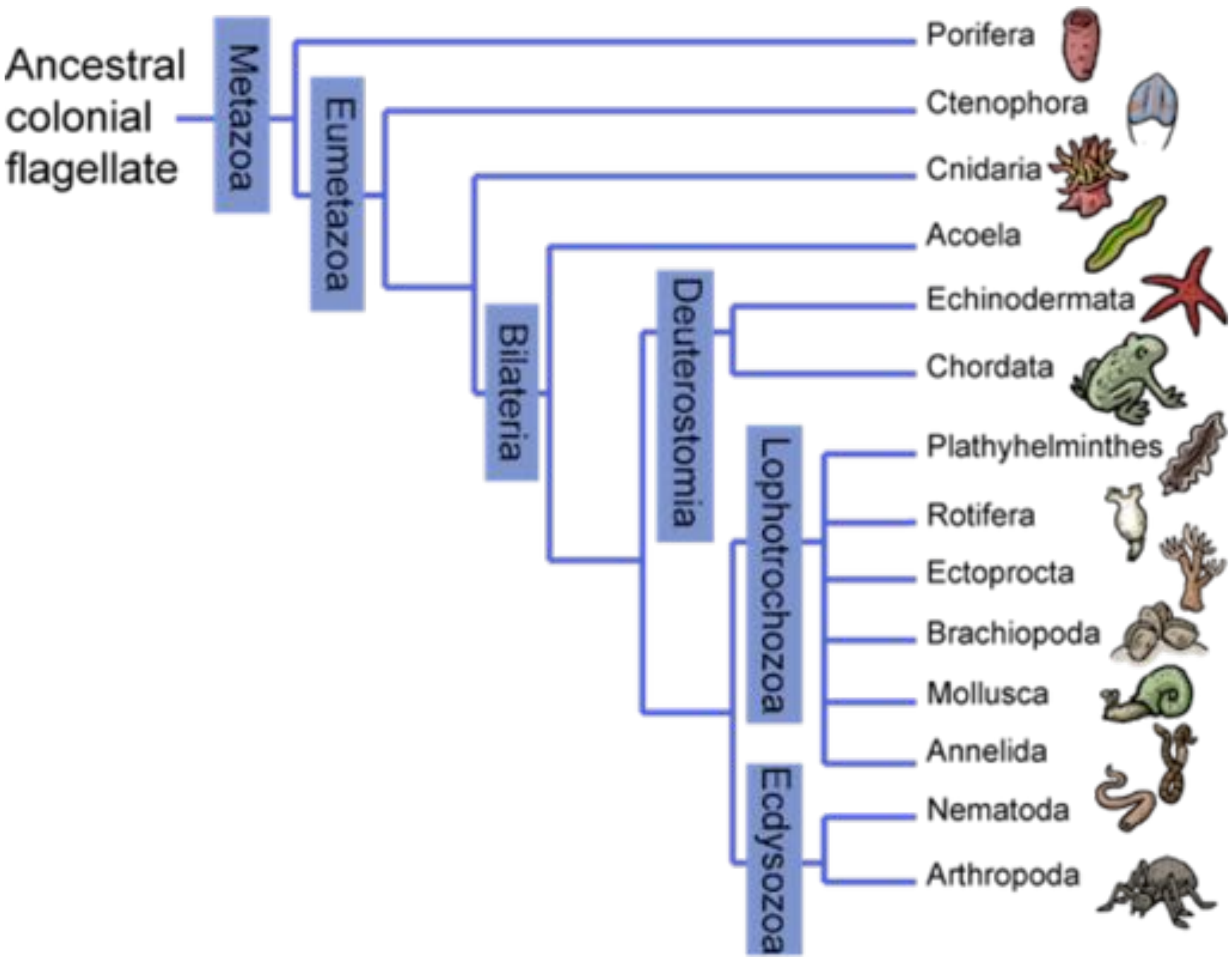
Counting Review

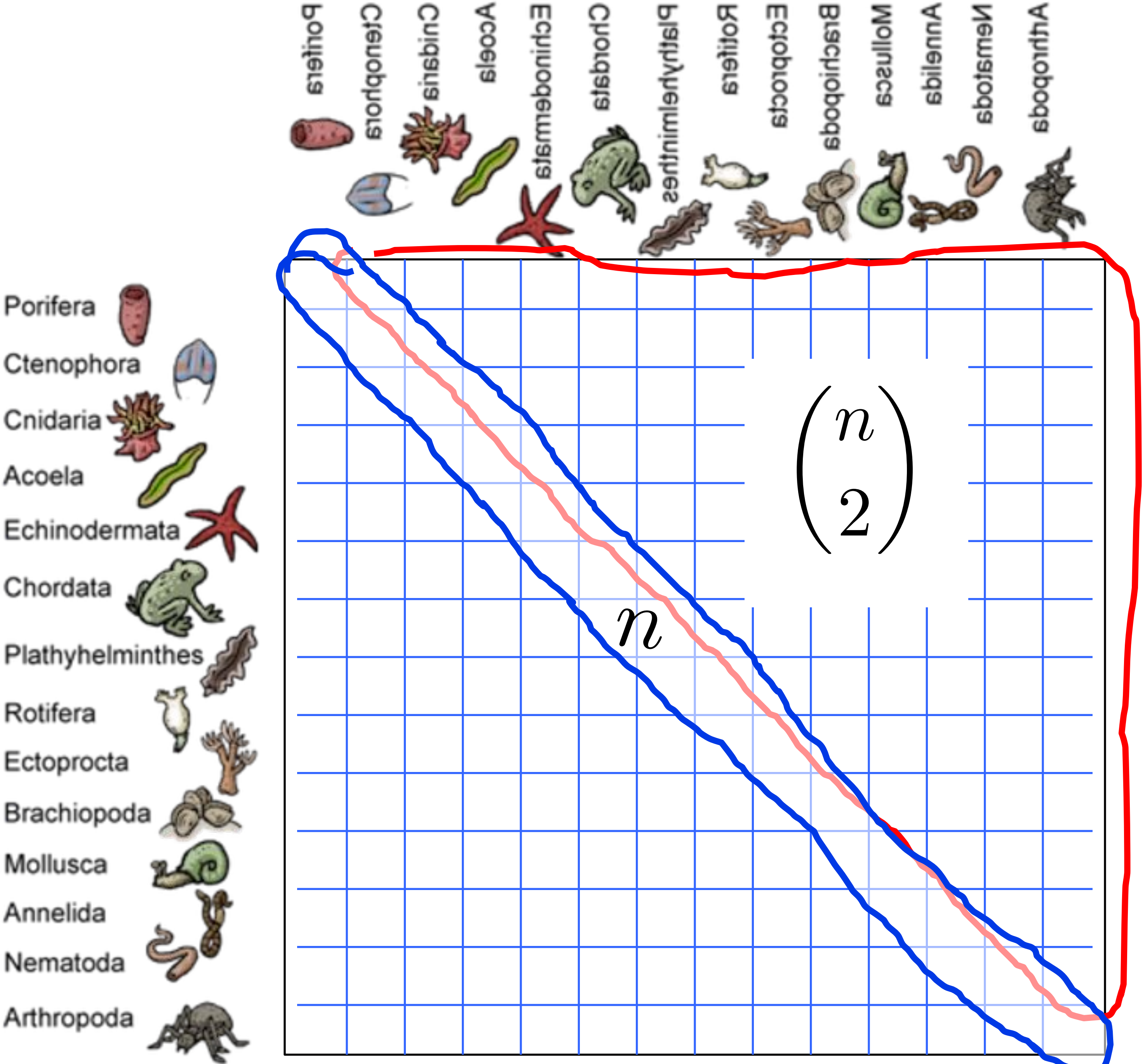
For a DNA tree we need to calculate the DNA distance between each pair of animals. How many calculations are needed?



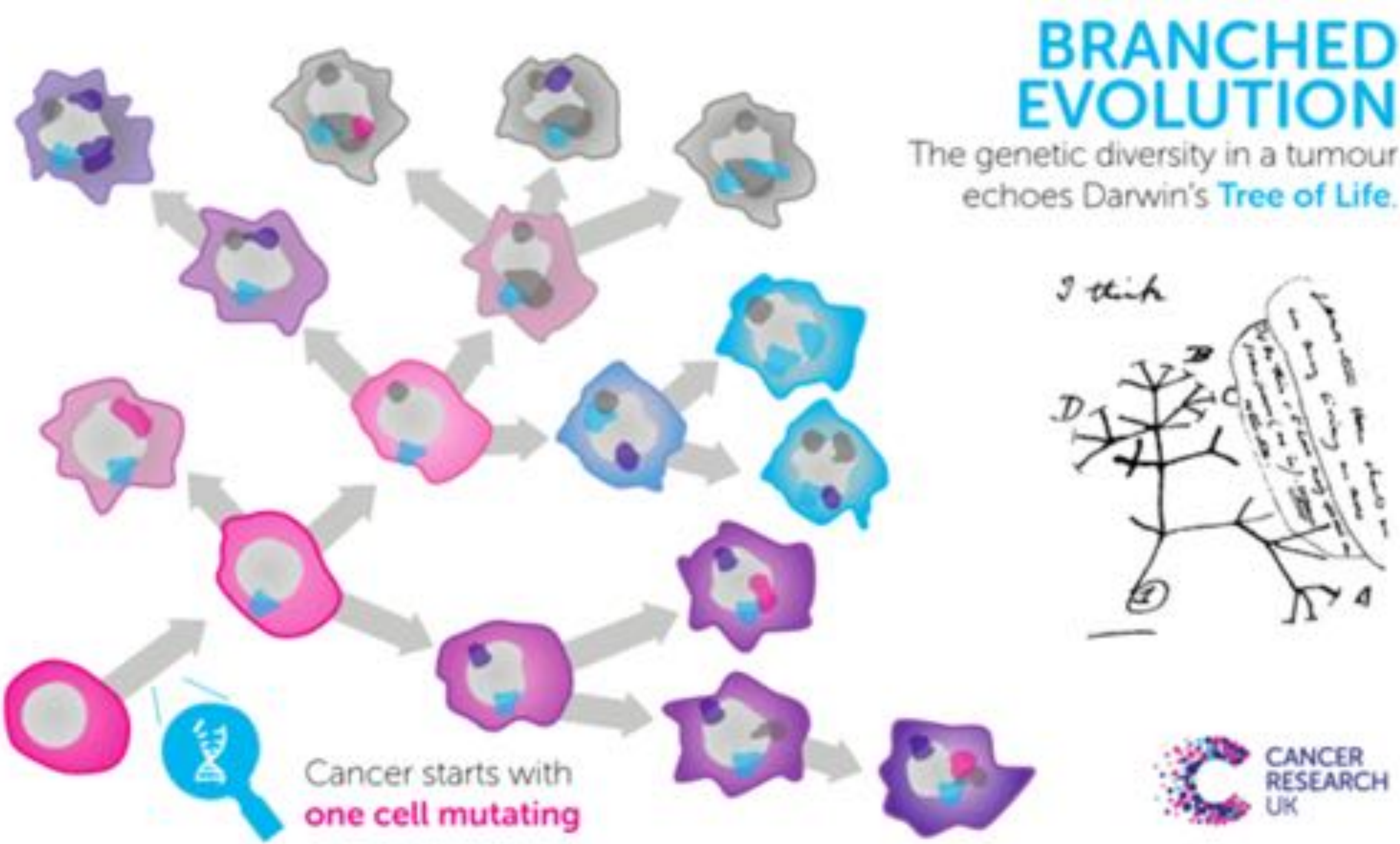
Counting Review

Q: There are *n* animals.
How many distinct pairs of animals are there?





Porifera
Ctenophora
Cnidaria
Acoela
Echinodermata
Chordata
Plathyhelminthes
Rotifera
Ectoprocta
Brachiopoda
Mollusca
Annelida
Nematoda
Arthropoda
n
$\binom{n}{2}$


BRANCHED
EVOLUTION
The genetic diversity in a tumour
echoes Darwin's Tree of Life.
I think
Cancer starts with
one cell mutating
CANCER
RESEARCH
UK

End Review

Sample Space

- **Sample space**, S, is set of all possible outcomes of an experiment
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
  - # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
  - YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Events

- **Event**, E, is some subset of S $(E \subseteq S)$
  - Coin flip is heads: $E = \{\text{Head}\}$
  - ≥ 1 head on 2 coin flips: $E = \{(H, H), (H, T), (T, H)\}$
  - Roll of die is 3 or less: $E = \{1, 2, 3\}$
  - # emails in a day ≤ 20: $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
  - Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Note: When Ross uses: $\subset$, he really means: $\subseteq$

What is a probability?

Number between 0 and 1

Ascribe Meaning

$$P(E)$$

* Our belief that an event E occurs

What is a Probability

What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

What is a Probability

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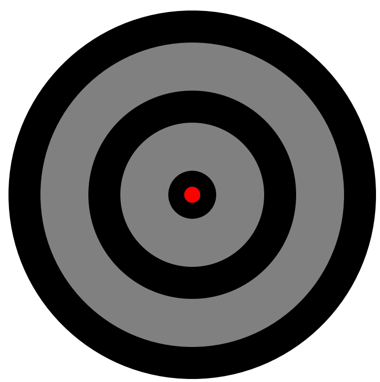
$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

Hit: 0

Thrown: 0



$P(E) \approx$

n is the number of trails

The "event" *E* is that you hit the target

What is a Probability

What is a probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

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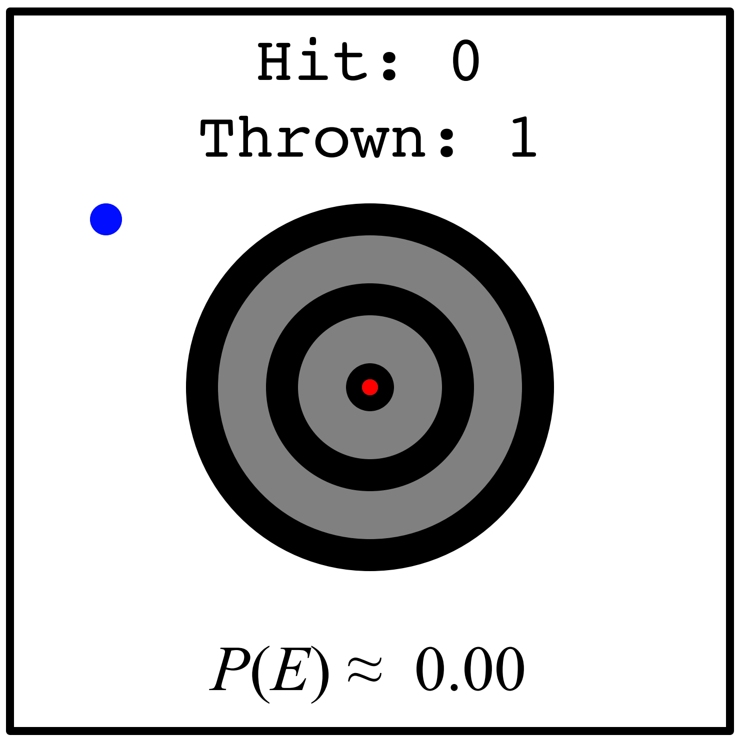

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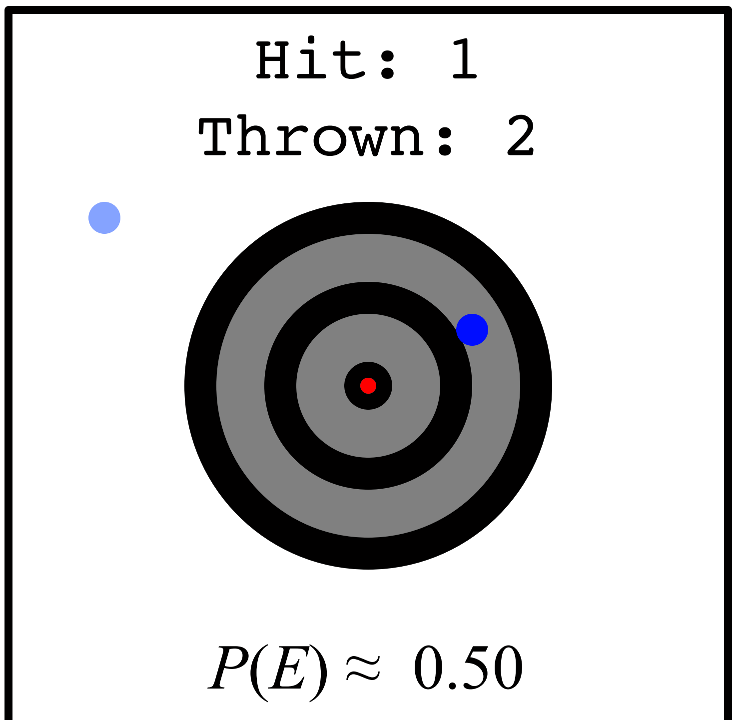

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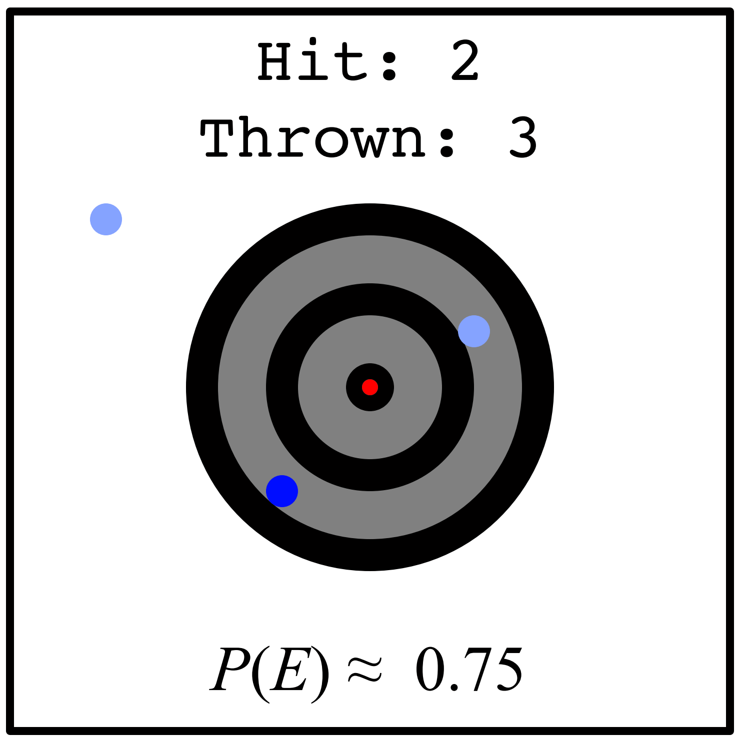

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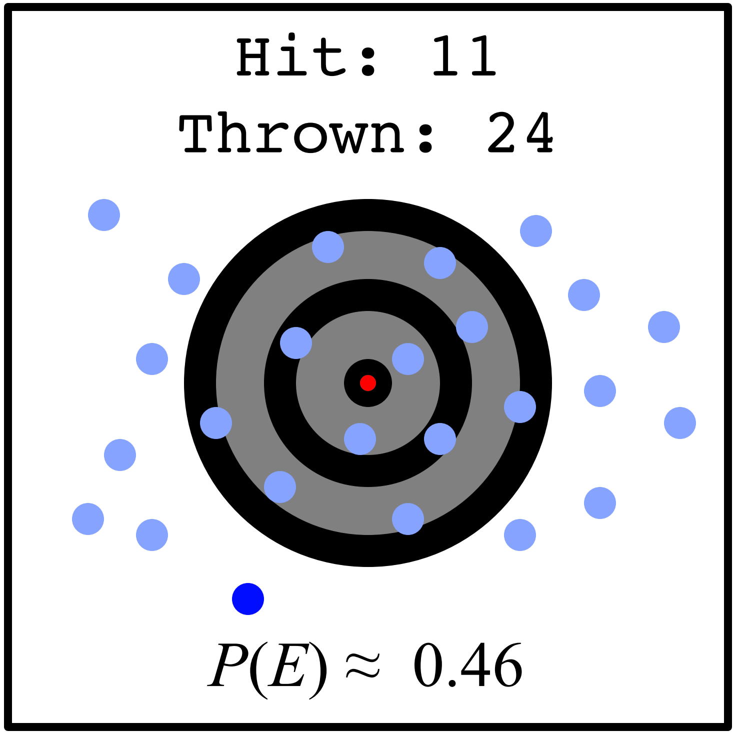
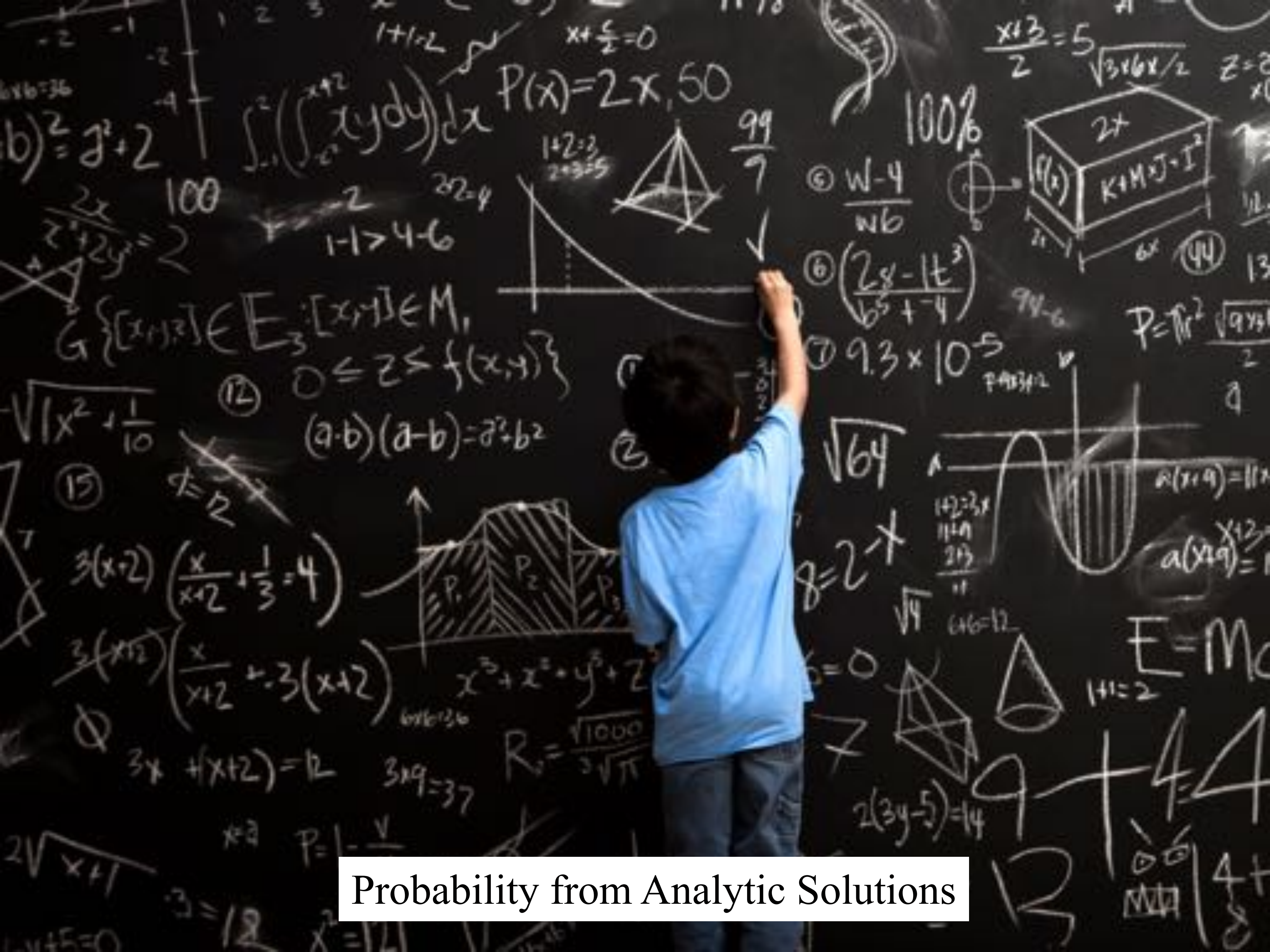


n is the number of trails

The "event" *E* is that you hit the target


90
MLB 040



Probability from Analytic Solutions

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: $P(E^c) = 1 - P(E)$

Aside: axiom 3 is often stated as the probability of mutually exclusive events. We'll come back to that later in the lecture…

Special Case of
Analytic Probability

Equally Likely Outcomes

- Some sample spaces have equally likely outcomes
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
- P(Each outcome) = $\frac{1}{|S|}$
- In that case, P(E) = $\frac{\text{number of outcomes in E}}{\text{number of outcomes in S}} = \frac{|E|}{|S|}$

Rolling Two Dice

- Roll two 6-sided dice.
  - What is P(sum = 7)?

- S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
  (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
  (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
  (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
  (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
  (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}
- E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

- P(sum = 7) = |E|/|S| = 6/36 = 1/6

Not Equally Likely

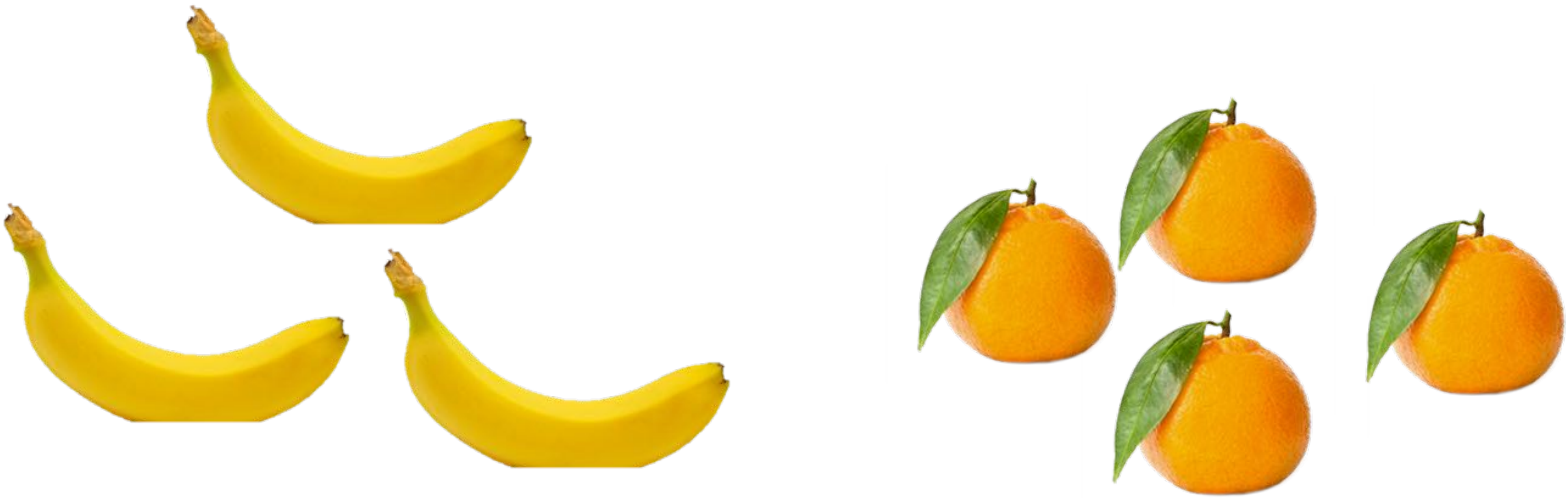
- Play lottery.
  - What is P(Win)?

- S = {Lose, Win}
- E = {Win}
- P(Win) = |E|/|S| = 1/2 = 50%

Mandarins and Bananas

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
  - What is P(1 Mandarin and 2 Bananas drawn)?

Equally likely sample space? Thought experiment



Mandarins and Grapefruit

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
  - What is P(1 Mandarin and 2 Bananas drawn)?
- Ordered:
  - Pick 3 ordered items: |S| = 7 * 6 * 5 = 210
  - Pick Mandarin as either 1st, 2nd, or 3rd item:
    |E| = (4 * 3 * 2) + (3 * 4 * 2) + (3 * 2 * 4) = 72
  - P(1 Mandarin, 2 Grapefruit) = 72/210 = 12/35
- Unordered:
  - $|S| = \binom{7}{3} = 35$
  - $|E| = \binom{4}{1}\binom{3}{2} = 12$
  - P(1 Mandarin, 2 Grapefruit) = 12/35

Make indistinct items **distinct** to get equally likely sample space outcomes

*You will need to use this “trick” with high probability

Any "Straight" Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - What is P(straight)?

What is an example of one outcome?

Is each outcome equally likely?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \cdot \binom{4}{1}^5$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

Official “Straight” Poker Hand

- Consider 5 card poker hands.
  - “straight” is 5 consecutive rank cards of any suit
  - “straight flush” is 5 consecutive rank cards of same suit
  - What is P(straight, but not straight flush)?

$$|S| = \binom{52}{5}$$

$$|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \approx 0.00392$$

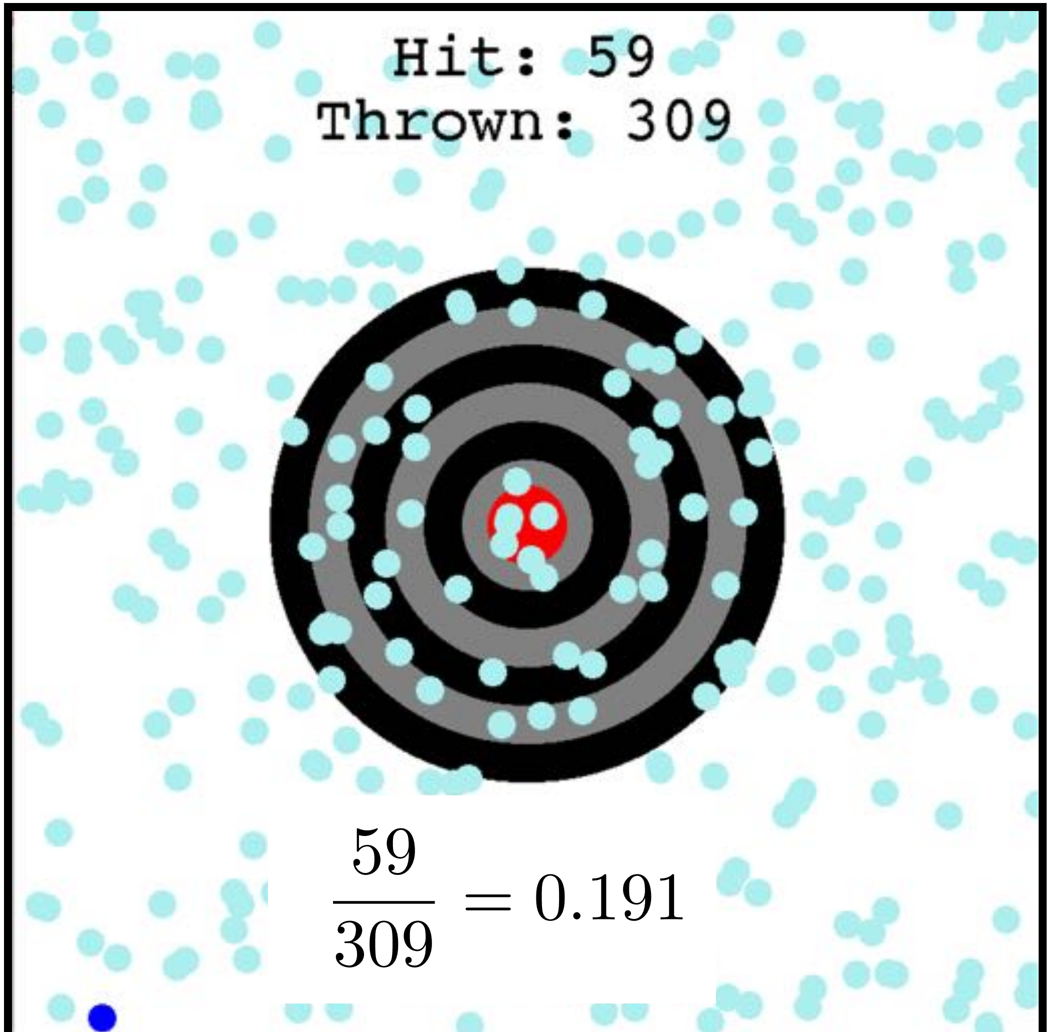
When approaching an "**equally likely probability**" problem, start by defining **sample spaces** and **event spaces**.

Chip Defect Detection

- n chips manufactured, 1 of which is defective.
- k chips randomly selected from n for testing.
  - What is P(defective chip is in k selected chips)?
- $|S| = \binom{n}{k}$
- $|E| = \binom{1}{1}\binom{n-1}{k-1}$
- P(defective chip is in k selected chips)

$$= \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{k}{n}$$

Target Revisited



$$\frac{59}{309} = 0.191$$

Screen size = 800x800
Radius of target = 200

The dart is equally likely to land anywhere on the screen.

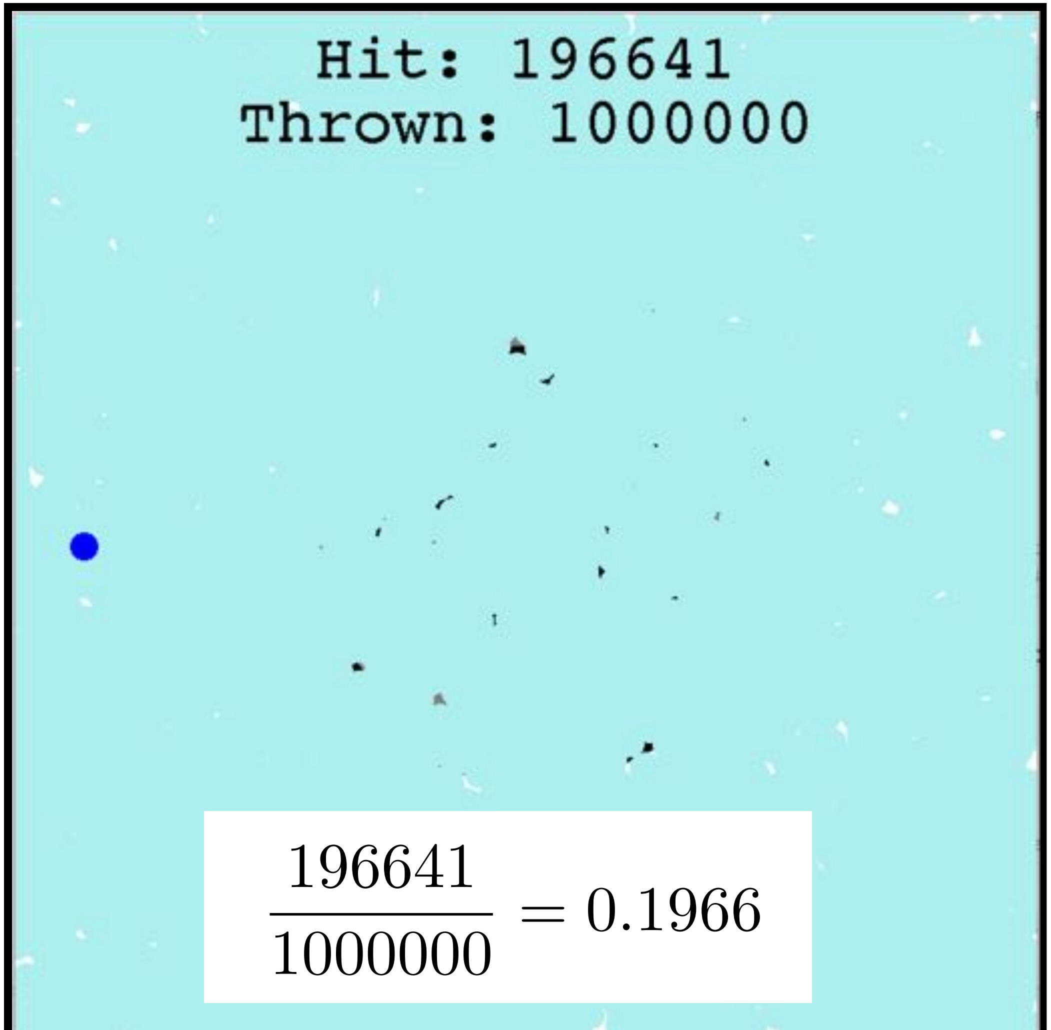
What is the probability of hitting the target?

$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target Revisited



$$\frac{196641}{1000000} = 0.1966$$

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Let it find you.

SERENDIPITY

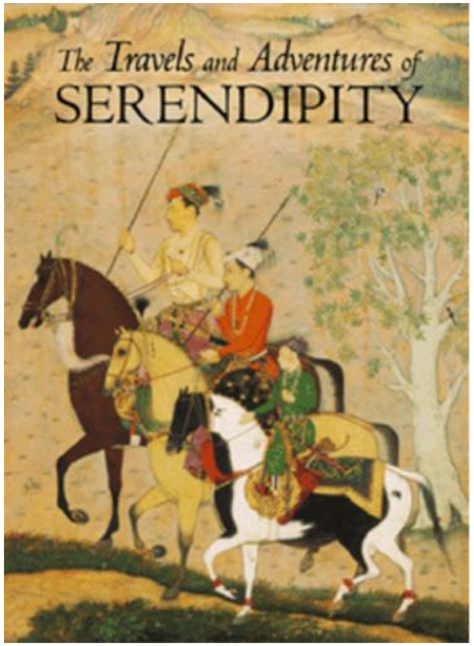
the effect by which one accidentally stumbles upon something truely wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND

Somewhere you didn't expect to.

Serendipity

- Say the population of Stanford is 17,000 people
  - You are friends with ?
  - Walk into a room, see 268 random people.
  - What is the probability that you see someone you know?
  - Assume you are equally likely to see each person at Stanford



Many times it is easier to calculate $P(E^C)$.

Back to Axiom 3

Axioms of Probability

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Aside: axiom 3 is often stated as the probability of mutually exclusive events. We'll come back to that later in the lecture…

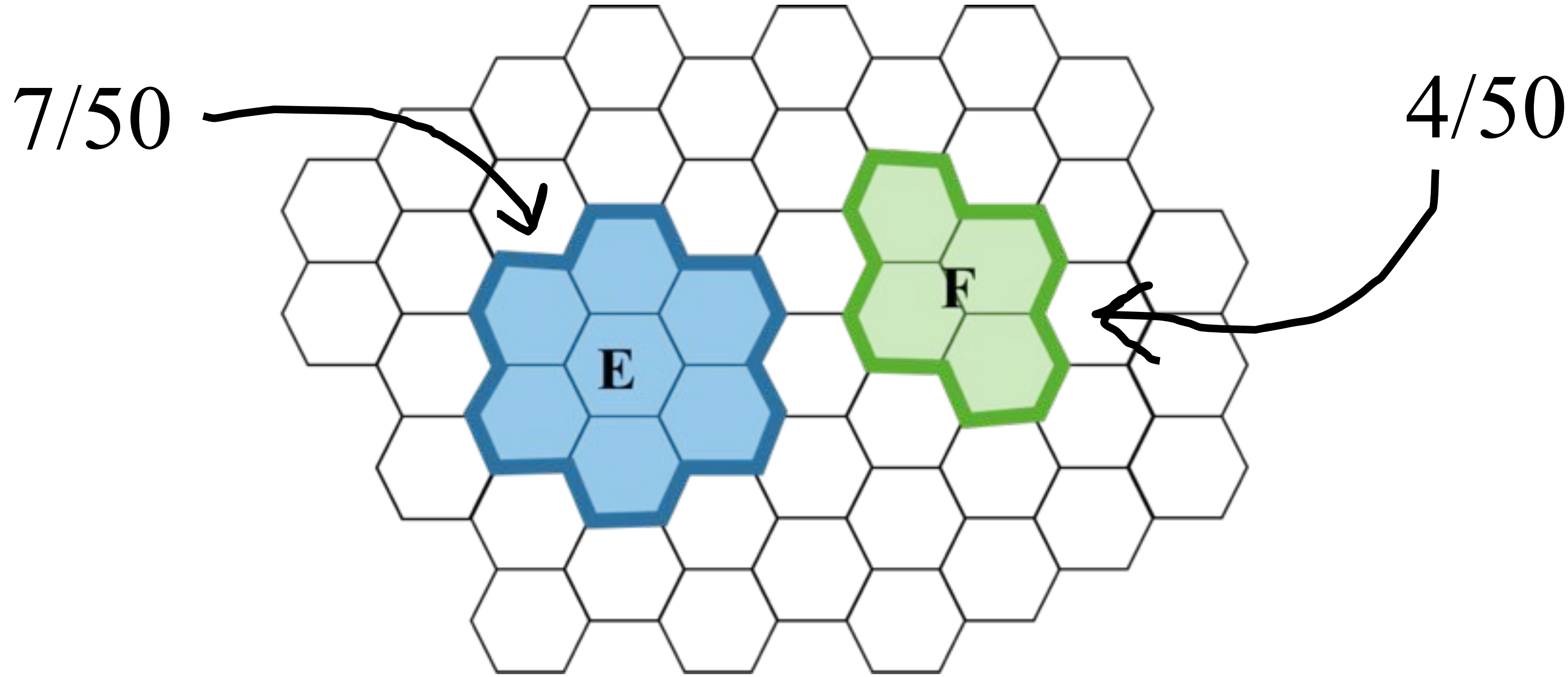
Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

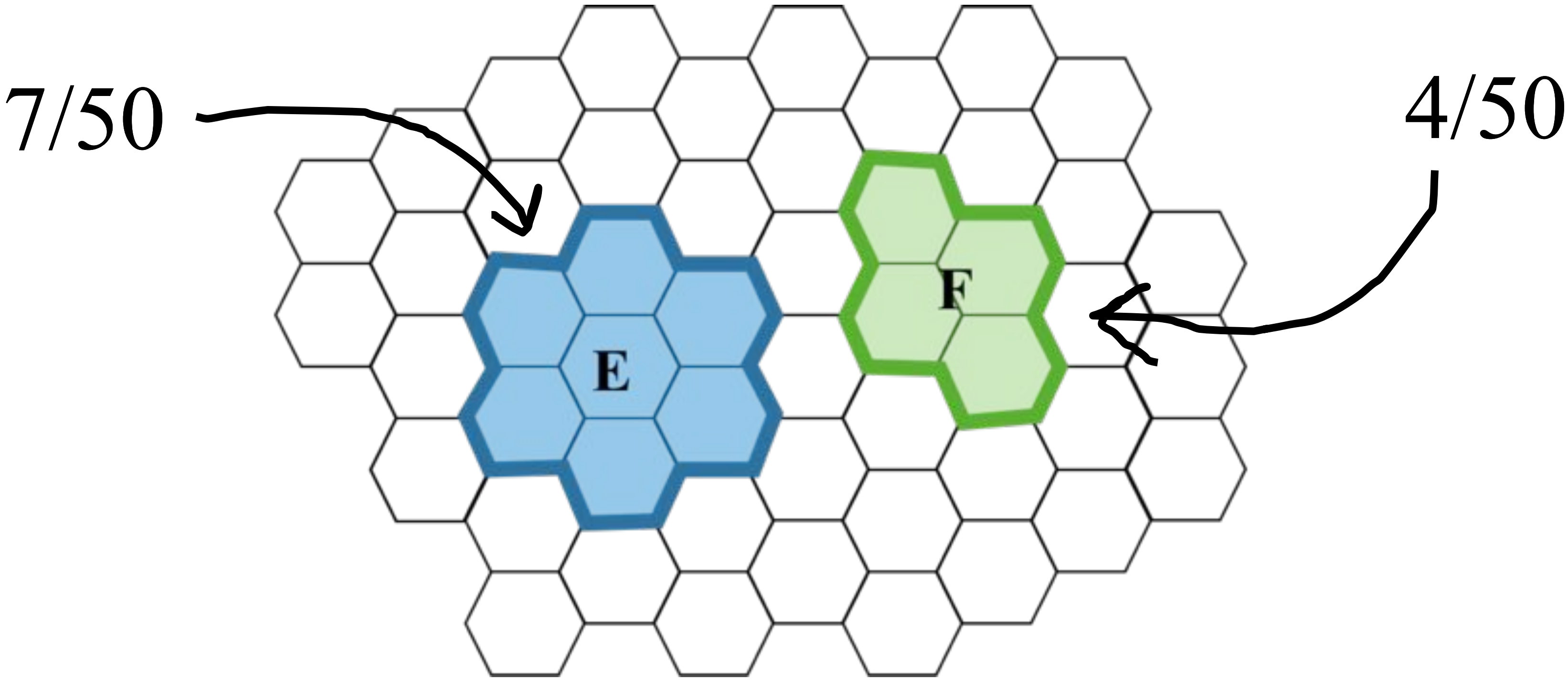
Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

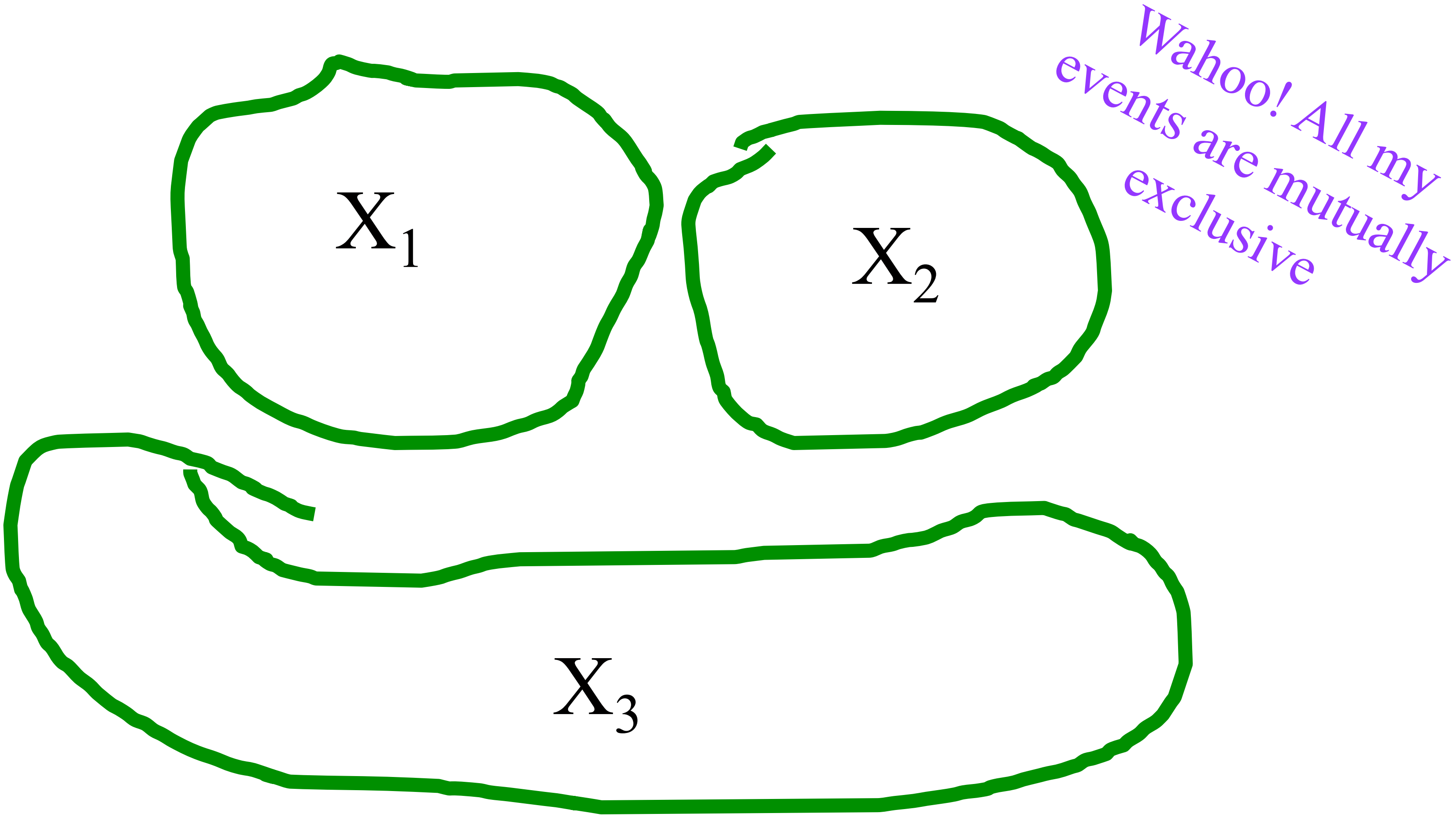
Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

OR with Many Mutually Exclusive Events



$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^{n} P(X_i)$$

If events are *mutually exclusive* probability of OR is easy!

$$P(E^c) = 1 - P(E)?$$

$$P(E \cup E^c) = P(E) + P(E^c)$$

Since E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since everything must either be in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2

$$P(E^c) = 1 - P(E)$$

Rearrange

Trailing the dovetail shuffle to it’s lair – Persi Diaconosis

Making History

- What is the probability that in the *n* shuffles seen since the start of time, yours is unique?
  - $|S| = (52!)^n$
  - $|E| = (52! - 1)^n$
  - P(no deck matching yours) = $(52!-1)^n/(52!)^n$

- For $n = 10^{20}$,
  - P(deck matching yours) < 0.000000001

* Assume 7 billion people have been shuffling cards once a second since cards were invented