Bilgisayar Mühendisliğinde Matematik Uygulamaları Ders Notları

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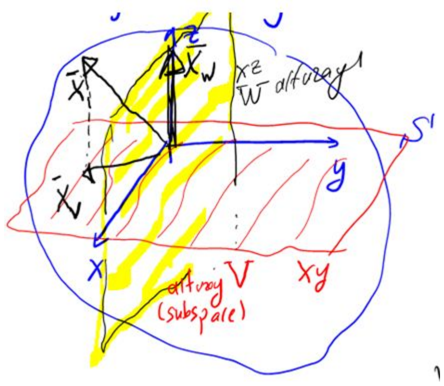
Bölüm 1

Bölüm1

1.1 Ortagonal Alt Uzaylar



Şekil 1.1: Ortagonal Alt Uzay

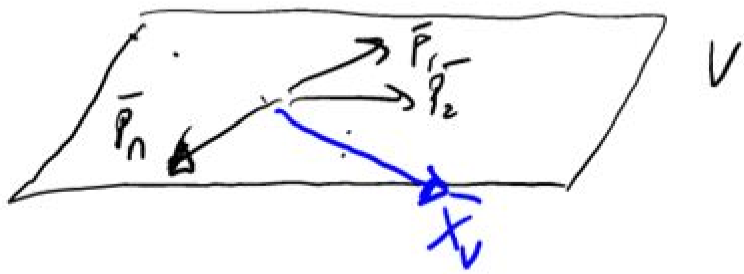
$$\overline{\mathbf{X}} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \quad \mathbf{S}$$

- $\mathbf{S} = R^3$
- $V \subset \mathbf{S}$
- $V^1 = W$
- $W^1 = V$

✓X ve W ortagonal altuzaylardır.
✓$\overline{X}$ vektoru S'de bir vektordur.
✓$\overline{X}_v$ V altuzayinda $\overline{X}'in$ izdüşüm vektörüdür.
✓$\overline{X}_w$ W altuzayinda $\overline{X}'in$ izdüşüm vektörüdür.

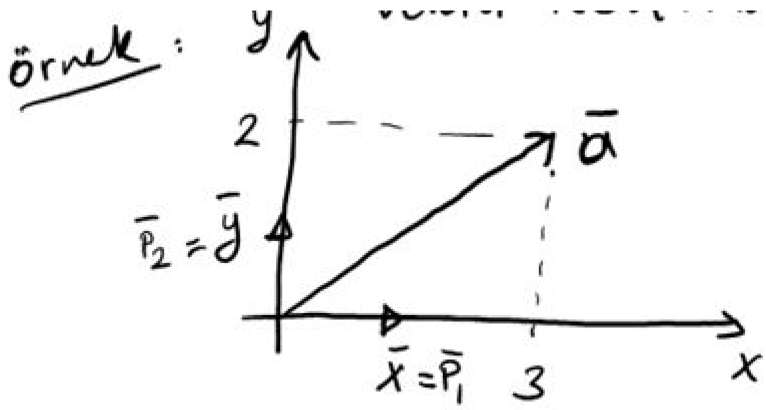
1.1.1 İzdüşüm Matrisi(Projection Matrix)

$\overline{P}_1, \overline{P}_2, \overline{P}_3 \cdots \overline{P}_n$ V altuzayını tarayan (span) vektörleri olsun



$\overline{X}_v = C_1\overline{P}_1 + C_2\overline{P}_2 + C_3\overline{P}_3 + \cdots + C_n\overline{P}_n$
$\mathbf{A_{m*n}} = \left[\overline{P}_1\overline{P}_2\overline{P}_3 \cdots \overline{P}_n\right]$
$\overline{P}_v = A_{m*n}(A^H_{m*n}A_{m*n})^{-1}A^H_{m*n}$ −> izdüşüm matrisi

1.1.2 Vektör İzdüşümü

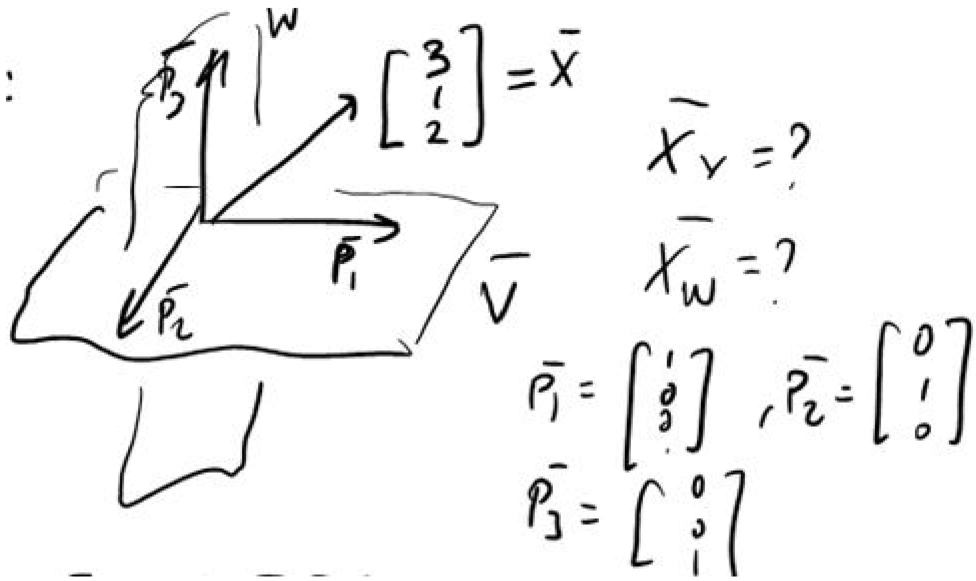


$\overline{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\overline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\overline{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\overline{a}_x =$ **?**
$< \overline{a}, \overline{x} >= \overline{a}^T * \overline{x} = \begin{bmatrix} 3 & 2 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3$

$\overline{a}_x = 3 * \overline{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
$\overline{a}_y = \textbf{?}$
$< \overline{a}, \overline{y} >= \overline{a}^T * \overline{y} = \begin{bmatrix} 3 & 2 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2$
$\overline{a}_y = 2 * \overline{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
$< \overline{a}, \overline{x} >= \|\overline{a}\| * \|\overline{x}\| * \cos(\Theta) = \|\overline{a}\| * \cos(\Theta)$ yandaki ifadede $\|\overline{x}\| = 1$ dir.

1.1.3 Ev Ödevi Sorusu

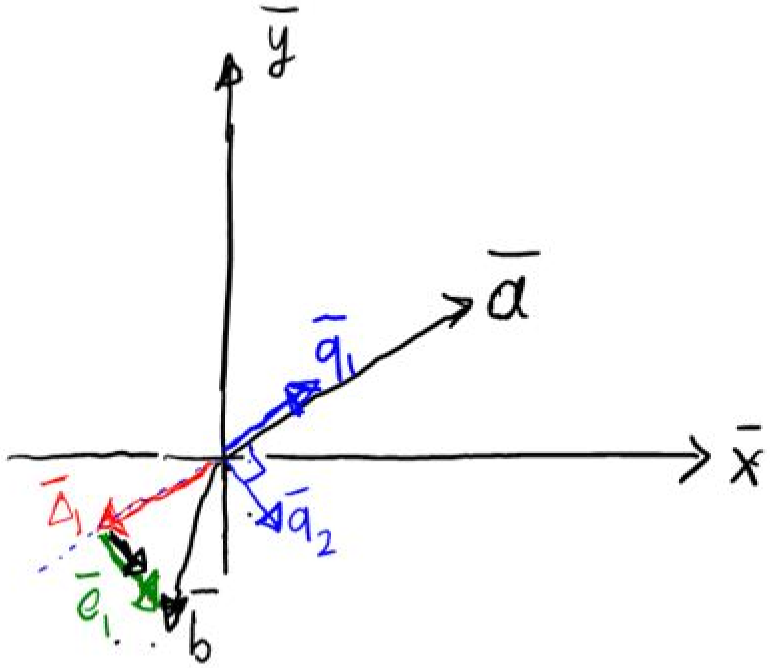
Ev ödevi sorusu aşağıdaki gibidir :



Şekil 1.2: Ev Ödevi Sorusu

$\mathbf{V} \longrightarrow \overline{A}_v = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \end{bmatrix} \longrightarrow \overline{P}_{A_v} = \cdots$
$\mathbf{W} \longrightarrow \overline{A}_w = \begin{bmatrix} \overline{P}_2 & \overline{P}_3 \end{bmatrix} \longrightarrow \overline{P}_{A_w} = \cdots$
$\overline{X}_v = \overline{P}_{A_v} \cdot \overline{X} \cdots$
$\overline{X}_w = \overline{P}_{A_w} \cdot \overline{X} \cdots$

1.2 Gram-Schmidt Dikleştirme Prosedürü



$\overline{a}$ ve $\overline{b}$'yi kullanarak birbirine dik vektörler nasıl oluşturulur?
$\| \overline{\mathbf{q}}_1 \|=1$, $\| \overline{\mathbf{q}}_2 \|=1$

1- $\overline{\triangle}_1 \longrightarrow \overline{b}'nin$ $\overline{\mathbf{q}}_1$ üzerindeki bileşeni

2- $\overline{\mathbf{e}}_1 \longrightarrow \overline{b}'nin$ $\overline{\triangle}_1$ 'den fark vektörü

3- $\overline{\mathbf{q}}_2 \longrightarrow \overline{\mathbf{e}}_1$'in normalize edilmiş hali

Algoritma:
$\mathbf{T} = \overline{P}_1, \overline{P}_2, \cdots, \overline{P}_n$ vektörleri için birbirine dik olan
$\mathbf{T}' = \overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2, \cdots, \overline{\mathbf{q}}_k \qquad k \leq n$

$< \overline{\mathbf{q}}_i, \overline{\mathbf{q}}_j >= \overline{\mathbf{q}}_i^T \cdot \overline{\mathbf{q}}_j = \delta_{ij}$

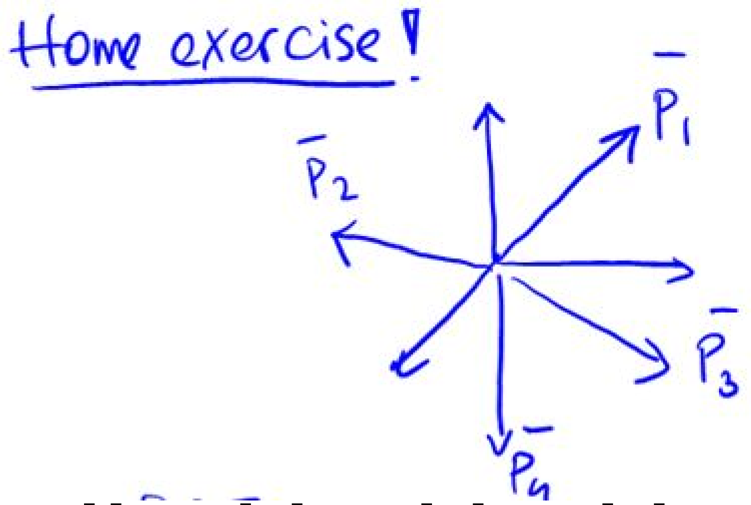
$$\delta_{ij} = \begin{cases} 1 & \text{i=j ise} \\ 0 & i \neq j \end{cases}$$

1) $\overline{\mathbf{q}_1} = \frac{\overline{\mathbf{p}_1}}{\|\overline{\mathbf{p}_1}\|}$
$\|\overline{\mathbf{p}_1}\| = (\overline{\mathbf{p}_1}^T \cdot \overline{\mathbf{p}_1})^{\frac{1}{2}}$

2) $\overline{\Delta_2} = <\overline{\mathbf{p}}_2, \overline{\mathbf{q}}_1> \overline{\mathbf{q}}_1$

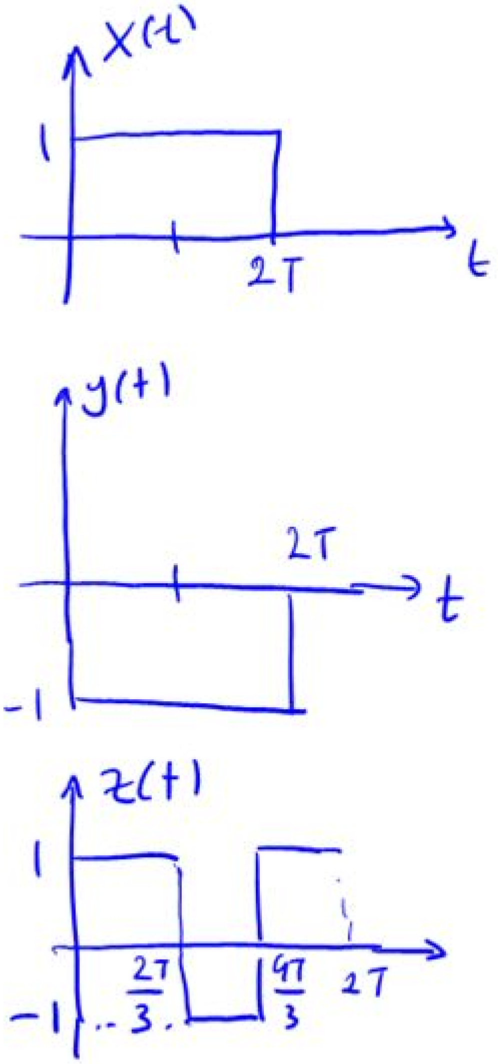
$\overline{\Delta_2} \quad \overline{\mathbf{p}}_2' nin \quad \overline{\mathbf{q}}_1$üzerindeki izdüşümü
$\overline{\mathbf{e}}_2 = \overline{\mathbf{p}}_2 - \overline{\Delta_2}$
$\overline{\mathbf{q}}_2 = \frac{\overline{\mathbf{e}}_2}{\|\overline{\mathbf{e}}_2\|}$
$\vdots$
k.adımda
$\overline{\mathbf{e}}_k = \overline{\mathbf{p}}_k - \sum_{i=1}^{k-1} <\overline{\mathbf{p}}_k, \overline{\mathbf{q}}_i> \overline{\mathbf{q}}_i$
$\overline{\mathbf{q}}_k = \frac{\overline{\mathbf{e}}_k}{\|\overline{\mathbf{e}}_k\|}$

1.2.1 Ev Ödevi Sorusu



$\overline{P}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \overline{P}_2 = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \overline{P}_3 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}, \overline{P}_4 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$
$\overline{P}_1$,$\overline{P}_2$,$\overline{P}_3$ ve $\overline{P}_4$'ü kullanarak Gram-Schmidt yöntemiyle birbirine dik olan vektörleri bulunuz ?

1.2.2 Örnek



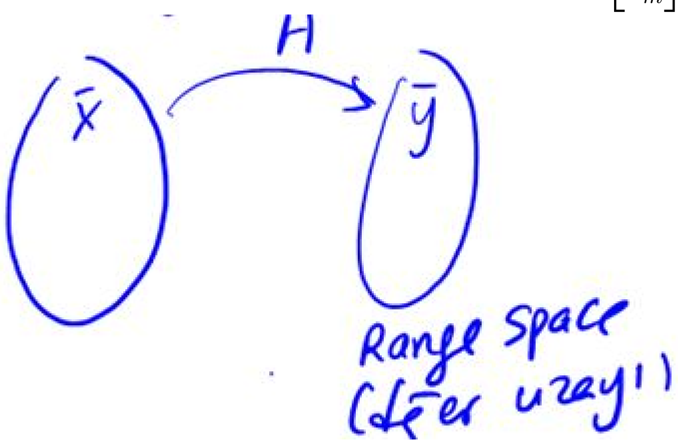
x(t), y(t) ve z(t)'yi birbirine dik (ortogonal) sinyaller şeklinde ifade ediniz?

$< x(t), y(t) >= \int_0^{2T} x(t)y(t)dt$

$\|x(t)\| = (< x(t), x(t) >)^{\frac{1}{2}}$

1.3 Range Space Of A Matrix(Bir Matrisin Değer Uzayı)

$$\mathbf{A} = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 & \cdots & \overline{P}_m \end{bmatrix} \qquad \mathbf{A} \cdot \overline{x} = \overline{y} \qquad \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbf{R}^m$$



$$\overline{y} = x_1\overline{P}_1 + x_2\overline{P}_2 + \cdots + x_m\overline{P}_m$$

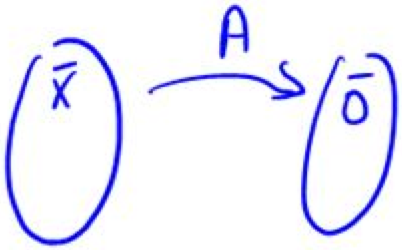
Örnek: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A \cdot \overline{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_1 + x_3 \end{bmatrix} = \overline{y}$$

$\overline{y}$ vektörü $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ vektörlerinin lineer birleşimidir.

R(A)= span{ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ } ,R(A)$\dashrightarrow$ değer uzayıdır .

1.3.1 Boş Uzay(NULL Space)



$$\overline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A$\overline{x} = \overline{0}$ işlemini sağlayan $\overline{x}$ vektörüne A'nın boş uzayı(null space) denir.N(A) ile gösterilir.
Örnek: Yukarıdaki A matrisi için :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0, x_1 + x_3 = 0 \rightarrow x_3 = 0 \quad x_2$ herhangi bir α değeridir

$$N(A) = span\{ \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} \}$$

1.4 Bazı Önemli Matris Ayrıştırmaları

A$\overline{x} = \overline{b}$ denklem sistemi için

1.4.1 A=mxm ise

Klasik yöntem:$\overline{x} = A^{-1} \cdot \overline{b}$
Bazı durumlarda A^{-1} almak oldukça zor ve karmaşık bir işlem.
$O(m^3)$ --→ polinom derecesi en fazla m^3 olan sayıda (+)ve (*) işlemi gerekir.

1.4.2 A->mxn matris ise m>n

A$\overline{x} = \overline{b}$
$(A^H_{n*m} \cdot A_{m*n})\overline{x}_{n*1} = A^H_{n*m}\overline{b}_{m*1}$
$\overline{x} = (A^H A)^{-1} A^H \cdot \overline{b}$
$(A^H A)^{-1} A^H \rightarrow A^\sharp$: A'nın pseduo-inxers

1.4.3 LU Matris ayrıştırması

A:m $\times$ m kare matris ,$A = L \cdot U$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & \vdots & \ddots & 0 \\ x & \dots & x & 1 \end{bmatrix}_{m*m} \qquad \Longrightarrow \text{Lower-Triangular(alt üçgensel)}$$

$$\mathbf{U} = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & \vdots & \ddots & x \\ 0 & \dots & 0 & x \end{bmatrix}_{m*m} \qquad \Longrightarrow \text{Upper-Triangular(üst üçgensel)}$$

Uygulaması:A$\overline{x} = \overline{b}$ denklem sistemi çözümünde kullanılır.(Eğer A:m*m ise)

1.4.4 Cholesky Matris ayrıştırması

A$=$ simetrik , pozitif-definite bir matris ise A$\rightarrow$ $m \times m$

$\overline{x}^T \cdot A\overline{x} > 0 \qquad \overline{x} \in R^n$ vektör ise A pozitif definite bir matris

$< \overline{x}, A\overline{x} >= \overline{x}^T \cdot A\overline{x} \qquad y = A\overline{x}$

$$A = LD\overline{L}^H \text{ ,D} \Rightarrow \text{köşegen} \quad D = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \dots & 0 & x \end{bmatrix}$$

Cholesky ayrıştırması aşağıdaki gibidir:

$$\begin{cases} A = UDU^H \\ A = LD\overline{L}^H \end{cases}$$

Uygulaması:Kestirim ve Kalman filtresi problemlerinin çözümünde

1.4.5 QR ayrıştırması

$A : m \times n$ kare olmayan bir matris ise

$A_{m\times n}\overline{x}_{n\times 1} = \overline{b}_{m\times 1} \qquad A = QR$

$$\text{Q unitary bir matris} \Rightarrow Q^H \cdot Q = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

$R =$ üst-üçgense bir matris ancak köşegendeki değerler ”1” değil·

$A\overline{x} = \overline{b}$

$\downarrow$

$Q \cdot R \cdot \overline{x} = \overline{b}$

$\underbrace{Q^H Q}_{I} \cdot R\overline{x} = \underbrace{Q^H \overline{b}}_{\overline{c}}$

$R \cdot \overline{x} = \overline{c} \longrightarrow$ çözümü $A\overline{x} = \overline{b}$ ' ye göre daha kolay $\cdot$

1.5 EigenValue Decomposition (EVD) Veya Singular Value Decomposition (SVD)

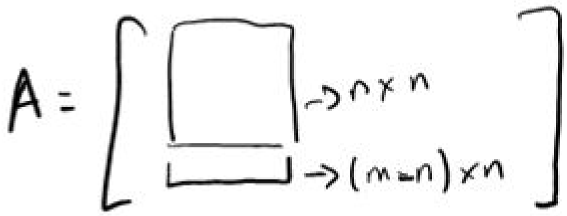
$A : m \times m$ kare $\Longrightarrow$EVD

$A : m \times n$ kare değil $\Longrightarrow$SVD

<u>SVD:</u> $A = U \sum V^H$

$$\sum = \begin{bmatrix} \Lambda_1 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \dots & 0 & \Lambda_n \end{bmatrix}_{n \times n}$$

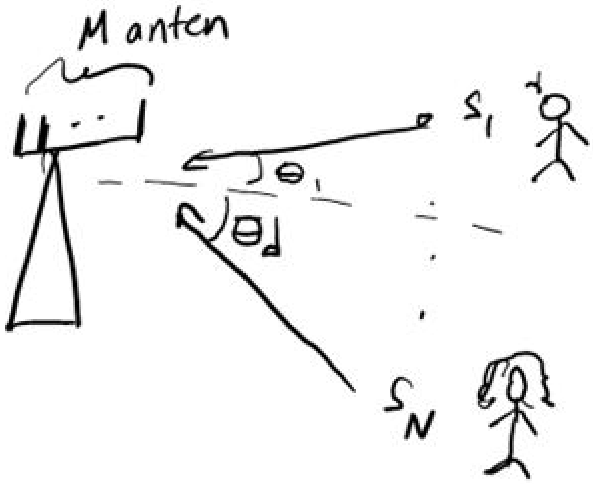
Λ_i : singular(tekil)değerler



$U = m \times n \Longrightarrow$ unitary $\quad U^H \cdot U = I_{n \times n}$

$V^H = m \times n \Longrightarrow$ unitary $\quad V \cdot V^H = I_{n \times n}$

Uygulama:Sinyal işleme , haberleşme , kestirim vs$\cdots$



Kullanıcı yönü $Q_i \quad i = 1, 2, \cdots, N$

1.6 LU Ayrıştırması Ve Denklem Sistemi Çözümü

$A_{m\times m}\overline{x}_{m\times 1} = \overline{b}_{m\times 1}$
$A = L \cdot U$
$\downarrow$
$LU\overline{x} = \overline{b}$
$U\overline{x} = \overline{y}$
$L\overline{y} = \overline{b} \Longrightarrow \overline{y} \Longrightarrow \overline{x}$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$y_1 = \dfrac{b_1}{l_{11}} \quad l_{11} = 1 \dashrightarrow y_1 = b_1$

$l_{21} \cdot y_1 + l_{22} \cdot y_2 = b_2 \dashrightarrow y_2 = \dfrac{b_2 - l21 \cdot y_1}{l_{22}} = b_2 - l21 \cdot y_1 \dashrightarrow l_{22} = 1$

$\downarrow$

$y_j = \left(b_j - \sum_{i=1}^{j-1} l_{ji} \cdot y_i\right)$

$j = 2, 3, 4, \cdots, m$ forward substitution (ileri yerine koyma)

$U\overline{x} = \overline{y}$
$\downarrow$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1m} \\ 0 & u_{22} & u_{23} & \cdots & u_{2m} \\ 0 & 0 & u_{33} & \cdots & u_{3m} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_m \end{bmatrix}$$

$x_m = \dfrac{1}{u_{mm}} \cdot y_m$

$x_{m-1} = \dfrac{1}{u_{m-1 \cdot m-1}} (y_{m-1} - x_m \cdot u_{m-1,m})$

$\vdots$

$x_j = \dfrac{1}{u_{jj}} \left(y_j - \sum_{k=j+1}^{m} (u_{jk} \cdot x_k) \right) \dashrightarrow$ Backward substitution(geriye yerine koyma)

$O(\dfrac{m^2}{2})$ işlem gerektirir ve $\overline{x} = \overline{A}^{-1} \cdot \bar{b}$ yöntemine göre daha az işlem yükü gerektirir.

<u>Zorluk:</u>

1-) u_{ii}'lerin "0" olması

2-) u_{ii}'ler pivot olarak adlandırılır .

Bölüm 2

Bölüm2

2.1 LU Ayrıştırması

$$\mathbf{A} \quad \longrightarrow \quad \mathbf{U} \quad \longrightarrow \quad \mathbf{L}$$

Gauss Elimination Satır işlemlerini saklayarak
(Birim Satır İşlemleri)

$$\mathbf{A} = \begin{bmatrix} \overline{a_1^T} \\ \overline{a_2^T} \\ \vdots \\ \overline{a_m^T} \end{bmatrix}_{m\times m} = \begin{bmatrix} \text{satır1} \\ \text{satır2} \\ \vdots \\ \text{satırm} \end{bmatrix}_{m\times m}$$

Birim Satır İşlemi(B.S.İ)= satır i $\leftarrow \alpha$ satır i $+\beta$ satır j $\quad \alpha, \beta \in$ R

<u>Örnek:</u>

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -5 \\ 6 & 8 & 1 \\ 4 & -8 & -3 \end{bmatrix} \longrightarrow \mathbf{A} = LU?, L =?, U =?$$

Amaç BSİ ile köşegen altındaki değerleri sıfırlamak

1)$S_2 \longleftarrow S_2 - 3S_1 \qquad 3 = \dfrac{a_{21}}{a_{11}}$

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 4 & -8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -5 \\ 6 & 8 & 1 \\ 4 & -8 & -3 \end{bmatrix} \longleftarrow \text{Orjinal A matrisi}$$

2)$S_3 \longleftarrow S_3 - 2S_1 \qquad 2 = \dfrac{a_{31}}{a_{11}}$

$$\mathbf{A}_2 = \begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 0 & -16 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 4 & -8 & -3 \end{bmatrix} \longleftarrow \text{updated(Bir önceki}$$

update edilmiş A)

$3) S_3 \longleftarrow S_3 - 4S_2 \qquad 4 = \dfrac{a_{32}}{a_{22}}$

$$\mathbf{A}_3 = \begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 0 & 0 & -57 \end{bmatrix} = \mathrm{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 0 & -16 & 7 \end{bmatrix} \longleftarrow \text{updated(Bir}$$
önceki update edilmiş A)

U matrisi oluşturuldu .

<u>L'yi bulmak için :</u>

$U = E_3 E_2 E_1 A \qquad \Longrightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$

$E_1 A = A_1 \quad E_2 E_1 A = A_2 \quad L = E_1^{-1} E_2^{-1} E_3^{-1}$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} = L$$

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}}_{\text{L}} \underbrace{\begin{bmatrix} 2 & 4 & -5 \\ 0 & -4 & 16 \\ 0 & 0 & -57 \end{bmatrix}}_{\text{U}}$$

2.2 Pivot Seçme İle LU Ayrıştırması

Amaç : Pivot(köşegen) elemanlarının o sütunda en büyük değer alacak şekilde satır yerlerinin değiştirilmesi (pivoting)

<u>Örnek:</u>

$$\begin{bmatrix} 2 & 4 & -5 \\ 6 & 8 & 1 \\ 4 & -8 & -3 \end{bmatrix}$$

$1-) a_{11}$ max olacak şekilde $\quad S_1 \longleftrightarrow S_2$

Bunun anlamı orjinal A matrisini yer değiştirme matrisi (P_{12}) ile soldan çarpmak demektir.

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yukarıdaki P_{12} matrisi birim matristeki 2.satırın yer değiştirilmiş halidir.

$$\mathbf{I_{3\times 3}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -5 \\ 6 & 8 & 1 \\ 4 & -8 & -3 \end{bmatrix}$$

\- $$= \begin{bmatrix} 6 & 8 & 1 \\ 2 & 4 & -5 \\ 4 & -8 & -3 \end{bmatrix}$$

\- $\downarrow$

2−)

a_{21}'i sıfırlamak için katsayı $\frac{-1}{3} = \frac{-a_{21}}{a_{11}} \rightarrow E_1$

a_{31}'i sıfırlamak için katsayı $\frac{-2}{3} = \frac{-a_{31}}{a_{11}} \rightarrow E_2$

$$A_2 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-2}{3} & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{A_1}_{P_{12}\cdot A}$$

\- $\uparrow$ $\uparrow$ $\downarrow$

\- E_2 E_1 $P_{12} \cdot A$

$$A_2 = \begin{bmatrix} 6 & 8 & 1 \\ 0 & \frac{4}{3} & \frac{-16}{3} \\ 0 & \frac{-40}{3} & \frac{-11}{3} \end{bmatrix} \longrightarrow \text{1. sütundaki } a'_{11} in \text{ altındaki elemanlar sıfırlandı .}$$

3−)

2.sütun için en son adımda güncellenmiş olan A_2 matrisinin a_{22} elemanı için pivot seçilir .

$$A_2 = \begin{bmatrix} 6 & 8 & 1 \\ 0 & 4/3 & -16/3 \\ 0 & -40/3 & -11/3 \end{bmatrix}$$

alt matris A_{21}

$|\frac{-40}{3}| > |\frac{4}{3}|$ olduğundan $S_2 \longleftrightarrow S_3 (A'_2 de)$ ya da $S_1 \longleftrightarrow S_2 (A'_{21} de)$

A'_2yi göz önüne alırsak :

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_3 = P_{23} \cdot A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 8 & 1 \\ 0 & \frac{4}{3} & \frac{-16}{3} \\ 0 & \frac{-40}{3} & \frac{-11}{3} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 6 & 8 & 1 \\ 0 & \frac{-40}{3} & \frac{-11}{3} \\ 0 & \frac{4}{3} & \frac{-16}{3} \end{bmatrix} \dashrightarrow a_{32}\text{'yi sıfırlamak için katsayı} = \frac{\frac{-4}{3}}{\frac{-40}{3}} = \frac{1}{10} \Rightarrow E_3$$

4−)

$A_4 = E_3 \cdot A_3 = E_3 \cdot P_{23} \cdot E_2 \cdot E_1 \cdot P_{12} \cdot A$

$A_1 = P_{12} \cdot A$

$A_2 = E_2 \cdot E_1 \cdot P_{12} \cdot A$

$A_3 = P_{23} \cdot E_2 \cdot E_1 \cdot P_{12} \cdot A$

$$A_4 = \underset{\substack{\uparrow \\ E_3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{10} & 1 \end{bmatrix}} \begin{bmatrix} 6 & 8 & 1 \\ 0 & \frac{-40}{3} & \frac{-11}{3} \\ 0 & \frac{4}{3} & \frac{-16}{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & 8 & 1 \\ 0 & \frac{-40}{3} & \frac{-11}{3} \\ 0 & 0 & \frac{-57}{10} \end{bmatrix}}_{U}$$

-

-

$U = E_3 P_{23} E_2 E_1 P_{12} A$

$\Longrightarrow A = \underbrace{P_{12}^{-1} E_1^{-1} E_2^{-1} P_{23}^{-1} E_3^{-1}}_{\mathbf{V}} U$

-

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{23}^{-1} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{10} & 1 \end{bmatrix} \rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{10} & 1 \end{bmatrix}$$

$$P_{23}^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{10} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-2}{3} & 0 & 1 \end{bmatrix} \rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} \cdot P_{23}^{-1} \cdot E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{10} & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{10} & 1 \\ \frac{2}{3} & 1 & 0 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} \cdot E_2^{-1} P_{23}^{-1} \cdot E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{10} & 1 \\ \frac{2}{3} & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{-1}{10} & 1 \\ \frac{2}{3} & 1 & 0 \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{12}^{-1}$$

$$P_{12}^{-1} \cdot E_1^{-1} E_2^{-1} P_{23}^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{-1}{10} & 1 \\ \frac{2}{3} & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-1}{10} & 1 \\ 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \end{bmatrix} = \mathbf{V}$$

$$A = \underbrace{P_{12}^{-1} \cdot E_1^{-1} E_2^{-1} P_{23}^{-1} E_3^{-1}}_{\mathbf{V}} \cdot \mathbf{U}$$

-

$$L = P_{23} P_{12} \cdot V = P_{23} \; P_{12} P_{12}^{-1} E_1^{-1} E_2^{-1} P_{23}^{-1} E_3^{-1}$$ [1]

-

$$= \underbrace{P_{12} P_{12}^{-1}}_{\mathbf{I}} E_1^{-1} E_2^{-1} \underbrace{P_{23} P_{23}^{-1}}_{\mathbf{I}} E_3^{-1}$$

-

-

$$= E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{-1}{10} & 1 \end{bmatrix}$$

Pivot seçme yönteminde

$$A = V \cdot U$$
$$P \cdot A = LU$$

[1] $P_{12} P_{12}^{-1}$ I birim matrisine eşittir

Bölüm 3

Bölüm3

3.1 EigenValue Decomposition(Özdeğer Ayrıştırması)

Fark denklemi (*):
$y_1(t+1) = -y_1(t) - 1.5y_2(t)$
$y_2(t+1) = 0.5y_1(t) + y_2(t)$
$\overline{y_2}(t+1) = \begin{bmatrix} y_1(t+1) \\ y_2(t+1) \end{bmatrix}$ $\overline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ =?

$$\Longrightarrow \overline{y}(t+1) = \underbrace{\begin{bmatrix} -1 & -1.5 \\ 0.5 & 1 \end{bmatrix}}_{\text{A}} \overline{y}(t) \Longrightarrow \overline{y}(t+1) = A\overline{y}(t)$$

-

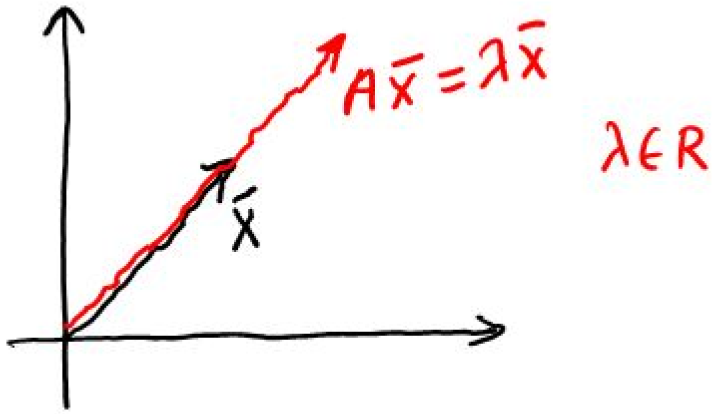
Çözüm: $y_1(t) = \lambda^t x_1 \qquad y_2(t) = \lambda^t x_2$
$\lambda^{t+1}x_1 = -\lambda^t x_1 - 1.5\lambda^t x_2$
$\lambda^{t+1}x_2 = 0.5\lambda^t x_1 + \lambda^t x_2$

$$\Longrightarrow A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\overline{x}} = \lambda \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\overline{x}}$$

-

$$\Longrightarrow A\overline{x} = \lambda\overline{x} \quad (**)$$

(**)denklem siteminin çözümü aynı zamanda (*) denklem sisteminin çözümüdür.
$\overline{x}$: eigenvector(özvektör)
λ:eigenvalue(özdeğer)



3.1.1 Özdeğer Ve ÖzVektör Nasıl Bulunur?

A$\overline{x} = \lambda\overline{x}$
Genel olarak :
$A : n \times n$
$\overline{x} : n \times 1$
$\lambda : 1 \times 1$

$\Longrightarrow A\overline{x} = \lambda\overline{x}$
$(A - \lambda I)\overline{x} = \overline{0}$
$\chi_A(\lambda) = det(A - \lambda I) = |A - \lambda I|$ karakteristik polinom
$\chi_A(\lambda) = det(A - \lambda I) = 0 \longrightarrow$ çözümü λ değerlerini verir.

<u>Örnek:</u>

$$A = \begin{bmatrix} -1 & -1.5 \\ 0.5 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -1 & -1.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & -1.5 \\ 0.5 & 1-\lambda \end{bmatrix}$$

$\chi_A(\lambda) = det(A - \lambda I) = (-1-\lambda)(1-\lambda) - (0.5)(-1.5)$
$\lambda^2 - 0.25 = 0$
$(\lambda - 0.5)(\lambda + 0.5) = 0$
$\downarrow$
$\lambda_1 = 0.5 \qquad \lambda_2 = -0.5$

bulduğumuz bu λ değerlerini $(A - \lambda I)\overline{x} = 0$'da yerine koyarsak :

$$\lambda_1 = 0.5 \longrightarrow \underbrace{\begin{bmatrix} -1-\lambda & -1.5 \\ 0.5 & 1-\lambda \end{bmatrix}}_{\text{B}} \overline{x}_1 = 0$$

-

$\overline{x}_1$ vektörü B'nin null space'ini oluşturur

$$\overline{x}_1 = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \qquad \alpha = 1 \longrightarrow \overline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|\overline{x}_1\| = \sqrt{\alpha^2 + \alpha^2} = \sqrt{2}\alpha \longrightarrow \overline{x}_1 = \frac{\overline{x}_1}{\|\overline{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = -0.5 \longrightarrow (A - \lambda_2 I)\overline{x}_2 = \underbrace{\begin{bmatrix} -0.5 & -1.5 \\ 0.5 & 1.5 \end{bmatrix}}_{\text{C}} \overline{x}_2 = \overline{0}$$

-

$\overline{x}_2$ C'nin null-space'idir.

$$\overline{x}_2 = \begin{bmatrix} -3\alpha \\ \alpha \end{bmatrix} \longrightarrow \|\overline{x}_2\| = \sqrt{9\alpha^2 + \alpha^2} = \sqrt{10}\alpha$$

$$\overline{x}_2 = \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

<u>Özdeğer</u> <u>ÖzVektör</u>

$$\lambda_1 = 0.5 \quad \longrightarrow \overline{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = -0.5 \quad \longrightarrow \overline{x}_2 = \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

Denklem sistemi çözümü :

$$\overline{y}_1(t) = \begin{bmatrix} \overline{y}_1(t) \\ \overline{y}_2(t) \end{bmatrix} = \lambda_1^t \overline{x}_1 = (0.5)^t \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\overline{y}_2(t) = \begin{bmatrix} \overline{y}_1(t) \\ \overline{y}_2(t) \end{bmatrix} = \lambda_2^t \overline{x}_2 = (-0.5)^t \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

Hem $\overline{y}_1(t)$ hem de $\overline{y}_2(t)$ orjinal denklem sist. çözümünü sağlıyor.
Sistem linear bir sistem olduğundan toplam çözüm:
$\overline{y}_t(t) = C_1 \lambda_1^t \overline{x}_1 + C_2 \lambda_2^t \overline{x}_2$

Denklem sisteminde başlangıç koşulları verilirse C_1 ve C_2 bulunabilir.

3.1.2 ÖzVektörlerin(EigenVectors)Lineer Bağımsızlığı?

$\overline{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \overline{x}_2 = \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$

$< \overline{x}_1, \overline{x}_2 >= \overline{x}_1^T \cdot \overline{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \frac{-3}{\sqrt{20}} - \frac{1}{\sqrt{20}}$

$\overline{x}_1 * \overline{x}_2 = \frac{-4}{\sqrt{20}} = \frac{-4}{2\sqrt{5}} = \frac{-2}{\sqrt{5}}$

Lema: $n \times n$ boyutlu A matrisinin tüm özdeğerleri birbirinden farklı ise A matrisinin özvektörleri brbirinden lineer bağımsızdır.

$k_1\overline{x}_1 + k_2\overline{x}_2 = \overline{0}$

$k_1 A\overline{x}_1 + k_2 A\overline{x}_2 = k_1\lambda_1\overline{x}_1 + k_2\lambda_2\overline{x}_2 = 0(*)$

$k_1\lambda_2\overline{x}_1 + k_2\lambda_2\overline{x}_2 = 0(**)$[1]

$(*) - (**) \longrightarrow k_1(\lambda_1 - \lambda_2)\overline{x}_1 = \overline{0}$

$\lambda_1 \neq \lambda_2, \overline{x}_1 \neq \overline{0} \longrightarrow k_1 = 0$ olmalı aynı şekilde $\lambda_1 \neq \lambda_2$ için $\longrightarrow k_1 = 0$ olmalı

.

3.2 Matrisin Ayrıştırılması

$A : n \times n$

Özvektörleri: $\overline{x}_1, \overline{x}_2, \overline{x}_3, \cdots, \overline{x}_n$

\- $\qquad \downarrow \ \downarrow \ \downarrow \qquad \downarrow$

Özdeğerleri: $\lambda_1, \lambda_2, \cdots, \lambda_n$

A$\overline{x}_i = \lambda_i\overline{x}_n \quad i = 1, 2 \cdots, n$

$\begin{bmatrix} A\overline{x}_1 & A\overline{x}_2 & A\overline{x}_3 & \cdots A\overline{x}_n \end{bmatrix}_{n\times n} = \begin{bmatrix} \lambda_1\overline{x}_1 & \lambda_2\overline{x}_2 & \lambda_3\overline{x}_3 & \cdots \lambda_n\overline{x}_n \end{bmatrix}_{n\times n}$

\- $\qquad \downarrow$

$$A_{n\times n} \underbrace{\begin{bmatrix} \overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \cdots \overline{x}_n \end{bmatrix}_{n\times n}}_{S} = \underbrace{\begin{bmatrix} \overline{x}_1 & \overline{x}_2 & \overline{x}_3 & \cdots \overline{x}_n \end{bmatrix}_{n\times n}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}_{n\times n}}_{\lambda}$$

\-

$\longrightarrow AS = S\lambda$

[1] $k_1\overline{x}_1 + k_2\overline{x}_2 = \overline{0}$ denklemi λ_2 ile çarpılırak elde edildi

Eğer eigenvector'ler lineer bağımsız iseler, S matrisi full rank ve tersi alınabilirler.

$\longrightarrow A\underbrace{SS^{-1}}_{\mathrm{I}} = S \wedge S^{-1}$

$\longrightarrow ASS^{-1} = S \wedge S^{-1} \quad veya \quad S^{-1}AS = \wedge$ Eigenvalue Decomposition(EVD)

A^n nasıl bulunur?

$ASS^{-1} = S \wedge S^{-1}$

$A^2 = S \wedge \underbrace{S^{-1}S}_{\mathrm{I}} \wedge S^{-1} = S \wedge^2 S^{-1}$

$A^3 = S \wedge^2 \underbrace{S^{-1}S}_{\mathrm{I}} \wedge S^{-1} = S \wedge^3 S^{-1}$

$\vdots$

$A^k = S \wedge^k S^{-1}$ eğer eigenvector'ler lineer bağımsız ise

$$\wedge = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\wedge^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

....

e^{At} nasıl bulunur?

$e^x = \sum_{i=0}^{\infty} \dfrac{x^i}{i!}$

$e^{At} = \sum_{i=0}^{\infty} \dfrac{A^i t^i}{i!}$

3.3 Özdeğer Ve Özvektörlerden Matris Oluşturma

$\lambda_i, \overline{x}_i \quad i = 1, 2, \cdots, n \qquad \overline{x}_i^H \overline{x}_j = 0$ olsun $(i \neq j)$

$n = 2$ ele alalım. $\qquad \|\overline{x}_i\| = 1$

$S = [\overline{x}_1 \quad \overline{x}_2] \qquad \bigwedge = \begin{bmatrix} \overline{x}_1 & 0 \\ 0 & \overline{x}_2 \end{bmatrix}$

$S^{-1}S = I$

$$\begin{bmatrix} - & \overline{x}_1^H & - \\ - & \overline{x}_2^H & - \end{bmatrix} \begin{bmatrix} \overline{x}_1 & \overline{x}_2 \end{bmatrix} = \begin{bmatrix} \|\overline{x}_1\|^2 & \overline{x}_1^H \overline{x}_2 \\ \overline{x}_2^H \overline{x}_1 & \|\overline{x}_2\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow S^{-1} = S^H$

$A = S \wedge S^{-1} = S \wedge S^H$

$$= \begin{bmatrix} \overline{x}_1 & \overline{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & \overline{x}_1^H & - \\ - & \overline{x}_2^H & - \end{bmatrix} = \begin{bmatrix} \lambda_1 \overline{x}_1 & \lambda_2 \overline{x}_2 \end{bmatrix}_{2\times 2} \begin{bmatrix} - & \overline{x}_1^H & - \\ - & \overline{x}_2^H & - \end{bmatrix}$$

$A = \lambda_1 \overline{x}_1 \overline{x}_1^H + \lambda_2 \overline{x}_2 \overline{x}_2^H$

Genel olarak ,özdeğerleri(λ_i) ve özvektörleri($\overline{x}_i$) verilen bir matris için , eğer $\overline{x}_i^H \cdot \overline{x}_j = 0$ ise bu matris şu şekilde oluşturulabilir:

$A = \lambda_1 \overline{x}_1 \overline{x}_1^H + \lambda_2 \overline{x}_2 \overline{x}_2^H + \cdots + \lambda_n \overline{x}_n \overline{x}_n^H$

$\Longrightarrow A = \sum_{i=1}^{n} \lambda_i \overline{x}_i \overline{x}_i^H \qquad P_i = \overline{x}_i \overline{x}_i^H$

<u>Örnek:</u>

$\lambda_1 = \lambda_2 = 1 \quad \lambda_3 = -1$

$$\overline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{x}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \overline{x}_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

A matrisini oluşturunuz?

<u>Cevap:</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{olmalı .}$$

<u>Örnek:</u>

$\lambda_1 = 5, \lambda_1 = 10$

$$\overline{x}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \ \overline{x}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$A = ?$

$$\overline{x}_1^H \overline{x}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = 0 \qquad \overline{x}_1 \perp \overline{x}_2$$

$$\overline{u}_1 = \frac{\overline{x}_1}{\|\overline{x}_1\|} = \tfrac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \overline{u}_2 = \tfrac{1}{5} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$P_1 = \overline{u}_1 \overline{u}_1^H = \tfrac{1}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \tfrac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$P_2 = \overline{u}_2\overline{u}_2^H = \frac{1}{25}\begin{bmatrix}-4\\3\end{bmatrix}\begin{bmatrix}-4 & 3\end{bmatrix} = \frac{1}{25}\begin{bmatrix}16 & -12\\-12 & 9\end{bmatrix}$
$A = \lambda_1 P_1 + \lambda_2 P_2 = 5 \cdot \frac{1}{25}\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix} + 10 \cdot \frac{1}{25}\begin{bmatrix}16 & -12\\-12 & 9\end{bmatrix}$
$= \frac{1}{5}\begin{bmatrix}9+32 & 12-24\\12-24 & 16+18\end{bmatrix}$
$A = \begin{bmatrix}\frac{41}{5} & \frac{-12}{5}\\ \\ \frac{-12}{5} & \frac{34}{5}\end{bmatrix}$

Not:A simetrik bir matris.

3.4 Self-Adjoint Matrislerin Köşegenleştirilmesi

Self$-$adjoint matris $\longrightarrow < A\overline{x}, \overline{x} >=< \overline{x}, A^H\overline{x} >$
- $\downarrow \qquad \downarrow$
- $\overline{x}^H A\overline{x} = \overline{x}^H Ax$

* A matrisinin elemanları reel değerler ise Self$-$adjoint matrise simetrik matris denir ve $A^T = A$

* A matrisinin elemanları kompleks değerler ise Self$-$adjoint matrise hermitian matris denir ve $A^T = A$

Özellikleri:

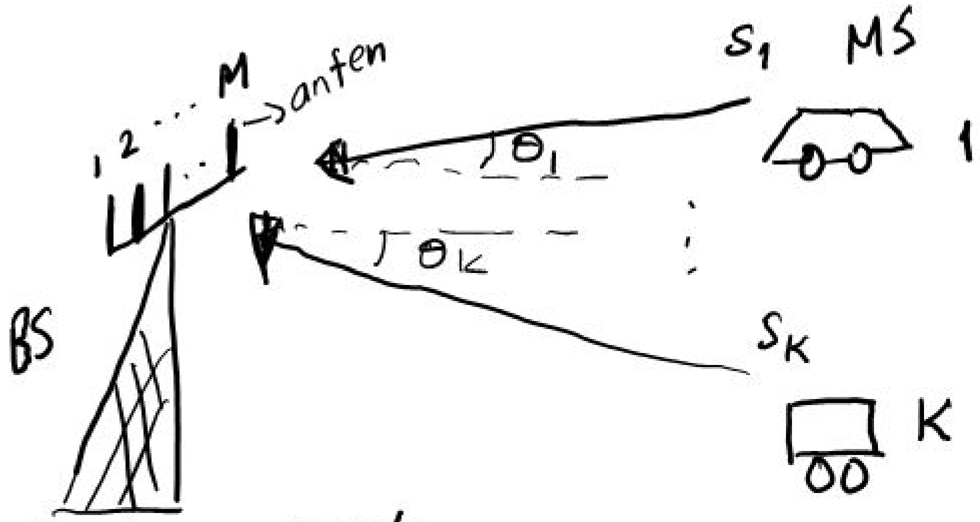
1- Self$-$adjoint matrisin ($A^H = A$ ise) eigenvalue'ları reel değerliklidir.
ispat:$< A\overline{x}, \overline{x} >=< \lambda\overline{x}, \overline{x} >= \lambda < \overline{x}, \overline{x} >= \lambda\overline{x}^H\overline{x}$
$< \overline{x}, A^H\overline{x} >= \lambda^* < \overline{x}, \overline{x} >= \lambda^*\overline{x}^H\overline{x}$
$\rightarrow \lambda\overline{x}^H\overline{x} = \lambda^*\overline{x}^H\overline{x} \rightarrow \lambda = \lambda^*$ için λ reel değerlikli olmalı.

2- Self$-$adjoint(simetrik veya hermitian)matrisler için birbirinden farklı eigenvalue'lara karşılık gelen eigenvektörler birbirine diktir.
ispat:$< A\overline{x}_1, \overline{x}_2 >=< \overline{x}_1, A^H\overline{x}_2 >=< \overline{x}_1, \lambda_2\overline{x}_2 >= \lambda_2 < \overline{x}_1, \overline{x}_2 >= \lambda_2\overline{x}_1^H\overline{x}_2$

aynı zamanda
$< A\overline{x}, \overline{x} >=< \lambda_1\overline{x}_1, \overline{x}_2 >= \lambda_1\overline{x}_1^H\overline{x}_2$
$\Rightarrow (\lambda_1\overline{x}_1^H\overline{x}_2 - \lambda_2\overline{x}_1^H\overline{x}_2) = 0$
$\Rightarrow (\lambda_1 - \lambda_2)(\overline{x}_1^H\overline{x}_2) = 0$

$\lambda_1 \neq \lambda_2$ ise $\overline{x}_1 \perp \overline{x}_2$ olmalı.
Teorem: $n \times n$ boyutu A matrisi hermitian bir matris ise ($A^H = A$ ise) A matrisinin EVD açılımı:
$A = U \bigwedge U^H = \sum_{i=1}^{n} \lambda_i \overline{U}_i \overline{U}_i^H$ şeklinde yazlır .
Bu arada U matrisi $U = [\overline{U}_1 \quad \overline{U}_2 \overline{U}_3 \cdots \overline{U}_n]$ unitary bir matristir.

3.5 EVD Uygulamaları



BS'de alınan sinyal·

$$\overline{x} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_M \end{bmatrix} \qquad \overline{a}(\Theta) = \begin{bmatrix} a_1(\Theta_i) \\ a_2(\Theta_i) \\ \vdots \\ a_M(\Theta_i) \end{bmatrix} \Longrightarrow \text{steering vector at } \theta_i$$

$\overline{x} = a(\theta_1)\alpha_1 S_1 + a(\theta_2)\alpha_2 S_2 + \cdots + a(\theta_k)\alpha_k S_k$

Korelasyon matrisi:R
$R = E\{\overline{x}\overline{x}^H\}$
M>K
$R \longrightarrow EVD \longrightarrow \lambda_i, \overline{U}_i$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \lambda_r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

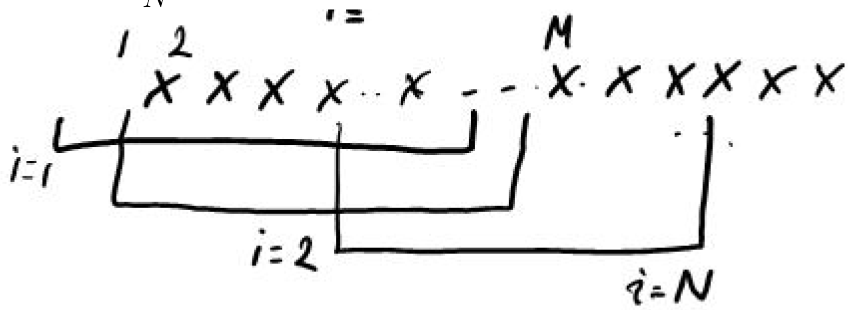
Dominant eigenvalue sayısı $\longrightarrow$ r , ortamda r adet kullanıcı vardır .
EVD $\longrightarrow$ MUSIC,ESPRIT,$\cdots$ vs high resolution algoritmalarla Q_i'ler bulunabilir .

3.6 Korelasyon Matrisi Özellikleri

1) R $\longrightarrow$ hermitian bir matris
<u>örnek:</u> $x(t) = e^{jwt} \longrightarrow$kompleks sinusoid
$t = 0, 1, \cdots, M-1$

$$\overline{\mathbf{x}} = \begin{bmatrix} 1 \\ e^{jw} \\ e^{j2w} \\ \vdots \\ e^{j(M-1)w} \end{bmatrix}$$

$R = E\{\overline{x}\overline{x}^H\} \cong \dfrac{1}{N} \sum_{i=}^{N} \overline{x}(i)\overline{x}^H(i)$



$$R = E\{ \begin{bmatrix} 1 \\ e^{jw} \\ e^{j2w} \\ \vdots \\ e^{j(M-1)w} \end{bmatrix} \begin{bmatrix} 1 & e^{jw} & e^{j2w} & \cdots & e^{j(M-1)w} \end{bmatrix} \}$$

$$= \mathrm{E}\{\begin{bmatrix} 1 & e^{-jw} & e^{-j2w} & \cdots & \cdots & e^{-j(M-1)w} \\ e^{jw} & 1 & e^{-jw} & e^{-j2w} & \cdots & e^{-j(M-1)w} \\ e^{j2w} & e^{jw} & 1 & \cdots & \cdots & e^{-j(M-3)w} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{j(M-1)w} & e^{j(M-1)w} & \cdots & \cdots & \cdots & 1 \end{bmatrix}_{m\times m} \}$$

R hermitian bir matris

2) R'nin eigenvalue'ları reel ve pozitif değerlerden oluşur .
$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m > 0$

3) R'nin eigenvektörleri biribirine diktir.($\overline{q}_i \perp \overline{q}_j$)
$\overline{q}_1, \overline{q}_2, \cdots, \overline{q}_m \rightarrow \overline{q}_i^H \cdot \overline{q}_j$

4) R^k matrisinin eigenvalue'ları
$\lambda_1^k, \lambda_2^k, \cdots, \lambda_m^k$ <u>ispatı ?</u> (ödev)

5) $Q = \begin{bmatrix} \overline{q}_1 & \overline{q}_2 & \cdots & \overline{q}_m \end{bmatrix} \; Q^H R Q = \bigwedge = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \lambda_r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \lambda m \end{bmatrix}$

Unitary− similarity transformation

6) $tr[R] = \sum_{i=1}^{M} \lambda_i$ <u>İspatı ?</u>

Bölüm 4

Bölüm4

4.1 QR Ayrıştırması

$A = QR$
Q:unitary matris
R:üst üçgensel matris

Unitary matris: $Q^H \cdot Q = I$
Eğer Q'nun sadece gerçek değerlikli elemanları varsa Q'ya ortogonal matris denir.
Lema1: $\overline{y} = Q\overline{x} \quad \|y\| = \|x\| \quad \overline{y} \in R^{m\times1}, \overline{x} \in R^{m\times1}$
Lema2: $Y = QX$, Y ve X birer matris

$$\|Y\|_F = \|X\|_F$$

Not: $\|\|_F \rightarrow$ matris Frobenius norm
$\|X\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|X_{ij}^2|)^{\frac{1}{2}} = (tr(X^HX))^{\frac{1}{2}}$
Örnek:
$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^H = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$
$X^HX = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$
$tr(X^HX) = a^2 + b^2 + c^2 + d^2 = (\sum_{i=1}^{2}\sum_{j=1}^{2}|X_{ij}^2|)$

4.2 Niçin QR ?

$A_{m\times n}\overline{x}_{n\times 1}=\overline{b}_{m\times 1}\longrightarrow$çözüm$\hat{x}=\underbrace{(A^HA)^{-1}A^H}\cdot\overline{b}$

- $\qquad A^{\sharp}$:pseduo-inverse

-bu çözüm $\|A\overline{x}-\overline{b}\|_2^2$ minimize eden bir çözümdür.

-aynı zamanda least−squares(en küçük kareler) çözümü olarakda adlandırılır.

$\|A\overline{x}-\overline{b}\|^2=(A\overline{x}-\overline{b})^H(A\overline{x}-\overline{b})=(\overline{x}^HA^H-\overline{b}^H)(A\overline{x}-\overline{b})$

$J(\overline{x})=\overline{x}^HA^HA\overline{x}-\overline{x}^HA^H\overline{b}-\overline{b}^HA\overline{x}+\overline{b}^Hb$

$\uparrow$

maliyet fonksiyonu

vektörel türev $\rightarrow\dfrac{\partial J(\overline{x})}{\partial\overline{x}}=A^HA\overline{x}+A^HA\overline{x}-A^H\overline{b}-A^Hb=0$

$$\Rightarrow 2A^HA\overline{x}-2A^H\overline{b}=0$$

$$\Rightarrow \overline{x}=(A^HA)^{-1}A^H\overline{b}\qquad \text{LS çözüm}$$

İpucu:

$\dfrac{\partial}{\partial\overline{x}}(x^HA)=A,\quad \dfrac{\partial}{\partial\overline{x}}(A\overline{x})=A^H$

Örnek:

$\overline{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\quad \overline{c}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

$\overline{x}^H\overline{c}=\begin{bmatrix}x_1c_1+x_2c_2+x_3c_3\end{bmatrix}$

$\dfrac{\partial}{\partial\overline{x}_1}\overline{x}^H\overline{c}=c_1\quad ,\dfrac{\partial}{\partial\overline{x}_2}\overline{x}^H\overline{c}=c_2\quad ,\dfrac{\partial}{\partial\overline{x}_3}\overline{x}^H\overline{c}=c_3$

$\dfrac{\partial}{\partial\overline{x}}=\begin{bmatrix}\dfrac{\partial}{\partial\overline{x}_1}\\\dfrac{\partial}{\partial\overline{x}_2}\\\dfrac{\partial}{\partial\overline{x}_3}\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\overline{c}$

$A_{m\times n}=Q_{m\times m}R_{m\times n}\qquad =Q\begin{bmatrix}R_1\\0\end{bmatrix}$

$m>n\qquad R_1:n\times n,\overline{0}:(m-n)\times n$

$$A_{m\times n}\overline{x}_{n\times 1}=b_{m\times 1}(*)$$

$\Rightarrow QR\overline{x} = \overline{b}$

$\underbrace{Q^H Q}_{\mathrm{I}} R\overline{x} = Q^H\overline{b}$

$\mathrm{R}\overline{x} = Q^H \cdot \overline{b}$

$Q^H_{m\times m} \cdot b_{m\times 1} = \begin{bmatrix} c \\ d \end{bmatrix} \rightarrow c : n \times 1 \qquad d : (m-n) \times 1$

$R\overline{x} = Q^H\overline{b}$

$\Rightarrow \begin{bmatrix} R_1 \\ \overline{0} \end{bmatrix} \overline{x} = \begin{bmatrix} c \\ d \end{bmatrix}$

$$R_{1n\times n}\overline{x}_{n\times 1} = \overline{c}_{n\times 1} \quad (**)$$

(*) ve (**) denklemlerinin çözümü aynı $\overline{x}$ vektörüdür.

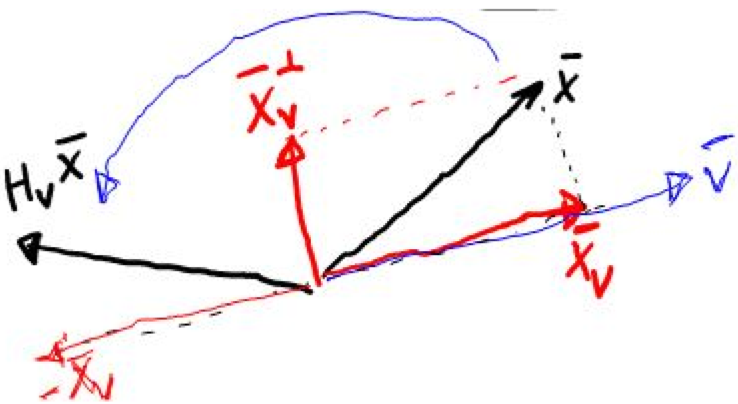
(**) kolaylıkla çözülebilir çünkü R_1 üst üçgensel bir matrisdir.

not: $m < n$ ise QR kullanılamaz!

QR Ayrıştırması:

Gram−Scmidt algoritması, Householder dönüşümü veya Givens rotation yöntemlerinden birisi kullanılarak yapılabilir.

4.3 HouseHolder Dönüşümü:



$\overline{x}_v$: $\overline{x}$ vektörünün $\overline{v}$ vektörü üzerine izdüşümü

$\overline{x}_v^{\perp}$: $(\overline{x} - \overline{x}_v)$ yani $\overline{x}$'in $\overline{x}_v$'ye dik olan bileşeni

$\overline{H}_v\overline{x}$: $\overline{x}$ vektörünün $\overline{x}_v^{\perp}$'ye göre yansıtılmışı(döndürülmüşü)

$$\overline{x}_v = P_v \cdot \overline{x}$$

$$P_v = \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})} \qquad , \overline{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\Rightarrow \overline{x}_v = \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})} \cdot \overline{x}$$

$$\overline{x}_v^\perp = \overline{P}_v^\perp \overline{x} \qquad , \overline{P}_v^\perp = I = \overline{P}_v = I - \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})}$$

$$\overline{x}_v^\perp = (I - \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})})\overline{x} = \overline{x} - \overline{x}_v$$

$$H_v = I - 2\frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})}$$

Householder dönüşüm matrisi $\overline{x}$'in yalnızca $\overline{V}$ vektörü üzerindeki bileşenin yönünü değiştirir.

$$H_v = \underbrace{I - \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})}} - \frac{\overline{V}\overline{V}^H}{(\overline{V}^H \overline{V})}$$

$$= P_v^\perp - P_v$$

$$H_v \overline{x} = P_v^\perp \overline{x} - P_v \overline{x} = \overline{x}_v^\perp - \overline{x}_v = \overline{x}_v^\perp + (-\overline{x}_v)$$

$$H_v^2 = I \quad H_v^H = H_v$$

Ödev: $H_v . H_v = I$ olduğunu gösteriniz.

$\overline{V}$ **Nasıl seçilir?**

$$\overline{\mathbf{X}} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_{n \times 1}$$

Amaç ,bu $\overline{X}$ vektörünü öyle bir H_v ile çarpmak ve sonuçta $x_2 = x_3 = \cdots = x_n = 0$ yapmak.

Amaç:

$$H_v \overline{x} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} = \alpha \overline{e}_1$$

$$\Rightarrow \left[I - 2\frac{\overline{V}\overline{V}^H}{(\overline{V}^H\overline{V})}\right]\overline{x} = \alpha\overline{e}_1$$

$$\overline{x} - 2\frac{(\overline{V}^H\overline{x})\overline{V}}{(\overline{V}^H\overline{V})} = \alpha\overline{e}_1$$

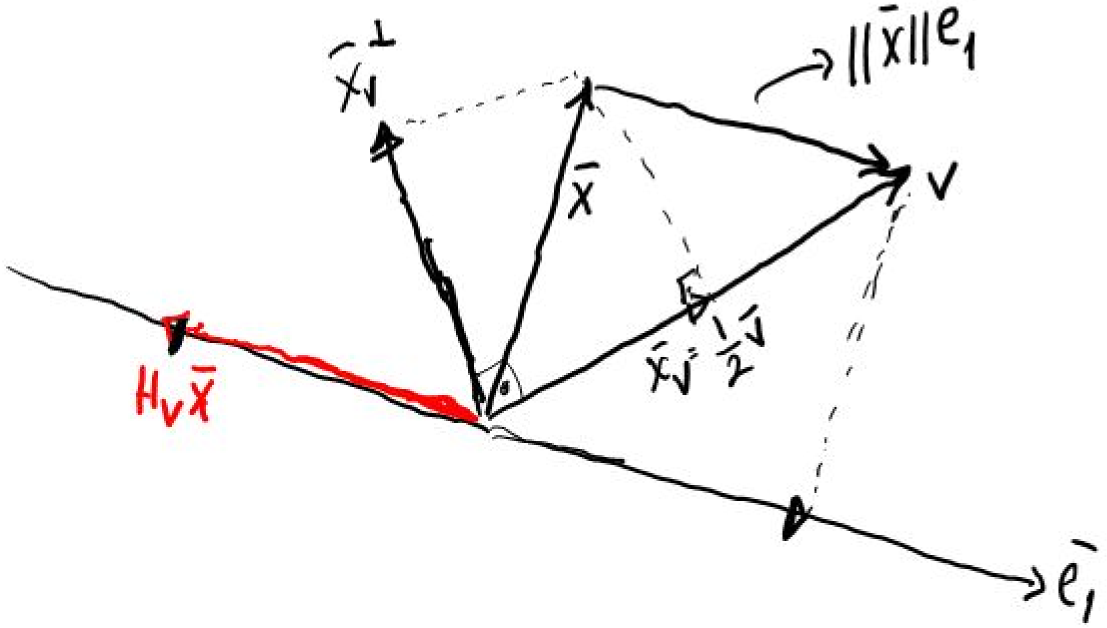
$$\Rightarrow \overline{x} - \alpha\overline{e}_1 = 2\frac{(\overline{V}^H\overline{x})}{(\overline{V}^H\overline{V})}\cdot\overline{V} \qquad (2\frac{(\overline{V}^H\overline{x})}{(\overline{V}^H\overline{V})} = 1 \text{kabul edersek})$$

(Not:$H_v\overline{x} = \alpha\overline{e}_1 \qquad \|H_v\overline{x}\| = \|\alpha\overline{e}_1\| \qquad H_v$:ünitary. $\qquad \Rightarrow \|\overline{x}\| = \alpha$)

$\overline{x} - \alpha\overline{e}_1 = \overline{V}$

$\overline{x} \pm \|\overline{x}\|\overline{e}_1 = \overline{V}$

$\Rightarrow \overline{V} = \overline{x} + sign(x_1)\|x\|\overline{e}_1$



$$\mathbf{A} = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \qquad \text{genel olarak } A_{m\times n}, m > n$$

<u>Step1</u>

$$H_1A = \begin{bmatrix} \alpha_1 & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$$

$H_1 : 4 \times 4$

$H_1 = Q = I - 2\dfrac{v_1\overline{v}_1^H}{\|\overline{v}_1\|^2}$

$\overline{v}_1 = \overline{x} + sign(x_1)\|\overline{x}\|\overline{e}_1 \Rsh 4\times 1$

$\overline{e}_1 = [1000]^T$

$V_1 : 4\times 1$

<u>Step2</u>

$H_2 : 3\times 3$

$H_2 = I - 2\dfrac{\overline{v}_2\overline{v}_2^H}{\|\overline{v}_2\|^2}$

$\overline{v}_2 = \overline{y} + sign(y_1)\|\overline{y}\|e_1 \Rsh 3\times 1 \qquad V_2 : 3\times 1$

$$Q_2 = \begin{bmatrix} 1 & \overline{0}^T \\ \overline{0}^T & H_2 \end{bmatrix}_{2\times 2} \qquad \overline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad Q_2Q_1A = \begin{bmatrix} \alpha_1 & x & x \\ 0 & \alpha_2 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$$

<u>Step3:</u>

$H_3 : 2\times 2$

$H_3 = I - 2\dfrac{\overline{v}_3\overline{v}_3^H}{\|\overline{v}_3\|^2}$

$\overline{v}_3 = \overline{z} + sign(z_1)\|\overline{z}\|e_1 \Rsh 2\times 1 \qquad V_3 : 2\times 1$

$$Q_3 = \begin{bmatrix} 1 & 0 & \overline{0}^T \\ 0 & 1 & \overline{0}^T \\ \overline{0} & \overline{0} & H_3 \end{bmatrix}_{3\times 3} \qquad \overline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad Q_3Q_2Q_1A = \begin{bmatrix} \alpha_1 & x & x \\ 0 & \alpha_2 & x \\ 0 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Sonuçta:

$Q_3Q_2Q_1A = R$

$\underbrace{Q_3^HQ_3}_{\mathrm{I}} Q_2Q_1A = Q_3^HR$

$\underbrace{Q_2^HQ_2}_{\mathrm{I}} Q_1A = Q_2^HQ_3^HR$

$\underbrace{Q_1^HQ_1}_{\mathrm{I}} A = Q_1^HQ_2^HQ_3^HR$

$\Rightarrow A = \underbrace{Q_1^HQ_2^HQ_3^H}_{\mathrm{Q}} R$

-

$\Rightarrow A = QR$

$Q = Q_1^HQ_2^HQ_3^H \quad , Q^H = Q_3Q_2Q_1$

Denklem sistemi

$A\overline{x} = \overline{b}$

$QR\overline{x} = \overline{b}$

$R\overline{x} = Q^H b = \begin{bmatrix} \overline{c} \\ \overline{d} \end{bmatrix}$

$$\begin{matrix} n\,\{ \\ m-n\,\{ \end{matrix} \underbrace{\begin{bmatrix} R_1 \\ \hdashline \overline{0} \end{bmatrix}}_{n} \overline{X} = \begin{bmatrix} \overline{c} \\ \hdashline \overline{d} \end{bmatrix} \begin{matrix} \}\,n \\ \}\,m-n \end{matrix}$$

Problem $R_1\overline{X} = \overline{C}$ sistemi çözümüne indirgenmiştir.
R_1 üst üçgensel bir matris olduğundan çözüm geri yerine koyma yöntemiyle kolaylıkla bulunabilir.
<u>Not:</u>
2. ve 3. adımdaki Q_2 ve Q_3'ler

$$Q_{2_{4\times4}} = I - 2\frac{\tilde{V}_2\tilde{V}_2^H}{\|\tilde{V}_2\|^2} \qquad \tilde{V}_2 = \begin{bmatrix} 0 \\ \tilde{V}_2 \end{bmatrix}_{4\times1}$$

$$Q_{3_{4\times4}} = I - 2\frac{\tilde{V}_3\tilde{V}_3^H}{\|\tilde{V}_3\|^2} \qquad \tilde{V}_3 = \begin{bmatrix} 0 \\ 0 \\ \tilde{V}_3 \end{bmatrix}_{4\times1}$$

4.4 QR Yöntemi İle Denklem Sistemi Çözümü Matlab Örneği

$\gg A = [788; 862; 173; 073; 695]$

$$A = \begin{bmatrix} 788 \\ 862 \\ 173 \\ 073 \\ 695 \end{bmatrix}$$

$\gg b = [4726242339]'$

$$b = \begin{bmatrix} 47 \\ 26 \\ 24 \\ 23 \\ 39 \end{bmatrix}$$

$\gg [m, n] = size(A)$

$m =$

5

$n =$

3

−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−

function $v = makehouse(x)$

%

% Make the Householder vector v such that

Hx has zeros in

% all but the first component

%

% function v= makehouse(x)

%

% x= vector to be transformed

%

% v= Householder vector

% © 1999 by Todd K. Moon

$v = x(:);$

$nv = norm(v);$

$if(abs(x(1)) == nv)$

$v = 0 * v;$

$else$

$if(v(1))$

$v(1) = v(1) + sign(v(1)) * nv;$
$else$
$v(1) = v(1) + nv;$
end
end
$. >> v = makehouse(A(:, 1));$
$- -$

$$v = \begin{bmatrix} 19.2474 \\ 8.0000 \\ 0 \\ 6.0000 \end{bmatrix}$$

$\gg a = A(:, 1)$

$$a = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 0 \\ 6 \end{bmatrix}$$

$\gg na = (a' * a)^0.5$
$na =$
12.2474
$\gg e_1 = [10000]$
$e_1 =$
$[10000]$
$\gg e_1 = na * e_1$
$e_1 =$
$[12.2474 \quad 0000]$
$e_1 = e_1'$

$$e_1 = \begin{bmatrix} 12.2474 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\gg v = a + e_1$

$$v = \begin{bmatrix} 19.2474 \\ 8.0000 \\ 1.0000 \\ 0 \\ 6.0000 \end{bmatrix}$$

$\gg eye(5, 5)$

$$ans = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\gg H_1 = eye(5,5) - 2 * v * v'/(v' * v)$

$$H_1 = \begin{bmatrix} -0.5715 & -0.6532 & -0.0816 & 0 & -0.4899 \\ -0.6532 & 0.7285 & -0.0339 & 0 & -0.2036 \\ -0.0816 & -0.0339 & 0.9958 & 0 & -0.0225 \\ 0 & 0 & 0 & 1.0000 & 0 \\ -0.4899 & -0.2036 & -0.0255 & 0 & 0.8473 \end{bmatrix}$$

$\gg Q_1 = H_1$

$$Q_1 = \begin{bmatrix} -0.5715 & -0.6532 & -0.0816 & 0 & -0.4899 \\ -0.6532 & 0.7285 & -0.0339 & 0 & -0.2036 \\ -0.0816 & -0.0339 & 0.9958 & 0 & -0.0225 \\ 0 & 0 & 0 & 1.0000 & 0 \\ -0.4899 & -0.2036 & -0.0255 & 0 & 0.8473 \end{bmatrix}$$

$\gg A_1 = Q_1 * A$

$$A_1 = \begin{bmatrix} -12.2474 & -13.4722 & -8.5732 \\ -0.0000 & -2.9247 & -4.8885 \\ -0.0000 & 5.8844 & 2.1389 \\ 0 & 7.0000 & 3.0000 \\ -0.0000 & 2.3065 & -0.1664 \end{bmatrix}$$

$a_1 = A_1(2:5,2)$

$$a_1 = \begin{bmatrix} -2.9247 \\ 5.8844 \\ 7.0000 \\ 2.3065 \end{bmatrix}$$

$\gg v_2 = a_1 - (sqrt(a_1' * a_1)) * [1000]'$

$$v_2 = \begin{bmatrix} -12.7989 \\ 5.8844 \\ 7.0000 \\ 2.3065 \end{bmatrix}$$

$\gg H_2 = eye(4,4) - 2 * v_2 * v_2'/(v_2' * v_2)$

$$H_2 = \begin{bmatrix} -0.2962 & 0.5959 & 0.7089 & 0.2336 \\ 0.5959 & 0.7260 & -0.3259 & -0.1074 \\ 0.7089 & -0.3259 & 0.6123 & -0.1278 \\ 0.2336 & -0.1074 & -0.1278 & 0.9579 \end{bmatrix}$$

$Q_2 = zeros(5,5)$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\gg Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\gg Q_2(2:5, 2:5) = H_2$$

$$Q_2 = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & -0.2962 & 0.5959 & 0.7089 & 0.2336 \\ 0 & 0.5959 & 0.7260 & -0.3259 & -0.1074 \\ 0 & 0.7089 & -0.3259 & 0.6123 & -0.1278 \\ 0 & 0.2336 & -0.1074 & -0.1278 & 0.9579 \end{bmatrix}$$

$$\gg A_2 = Q_2 * A_1$$

$$A_2 = \begin{bmatrix} -12.2474 & -13.4722 & -8.5732 \\ 0.0000 & 9.8742 & 4.8105 \\ -0.0000 & -0.0000 & -2.3203 \\ -0.0000 & -0.0000 & -2.3046 \\ -0.0000 & 0 & -1.9142 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} -2.3203 \\ -2.3046 \\ -1.9142 \end{bmatrix}$$

$$\gg v_3 = a_3 - (sqrt(a_3' * a_3)) * [1 \quad 0 \quad 0]'$$

$$v_3 = \begin{bmatrix} -6.1096 \\ -2.3046 \\ -1.9142 \end{bmatrix}$$

$$\gg H_3 = eye(3,3) - 2 * v_3 * v_3' / (v_3' * v_3)$$

$$H_3 = \begin{bmatrix} -0.6123 & -0.6082 & -0.5052 \\ -0.6082 & 0.7706 & -0.1906 \\ -0.5052 & -0.1906 & 0.8417 \end{bmatrix}$$

$$\gg Q_3 = zeros(5,5)$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\gg Q3(1:2, 1:2) = eye(2,2)$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$\gg Q3(3:5, 3:5) = H_3$
$$Q_3 = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & -0.6123 & -0.6082 & -0.5052 \\ 0 & 0 & -0.6082 & 0.7706 & -0.1906 \\ 0 & 0 & -0.5052 & -0.1906 & 0.8417 \end{bmatrix}$$
$\gg A_3 = Q_3 * A_2$
$$A_3 = \begin{bmatrix} -12.2474 & -13.4722 & -8.5732 \\ 0.0000 & 9.8742 & 4.8105 \\ 0.0000 & 0.0000 & 3.7893 \\ 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0 \end{bmatrix}$$
$\gg U = A_3(1:3, :)$
$$U = \begin{bmatrix} -12.2474 & -13.4722 & -8.5732 \\ 0.0000 & 9.8742 & 4.8105 \\ 0.0000 & 0.0000 & 3.7893 \end{bmatrix}$$
$\gg Q_h = Q_3 * Q_2 * Q_1$
$$Q_h = \begin{bmatrix} -0.5715 & -0.6532 & -0.0816 & 0 & -0.4899 \\ 0.0304 & -0.2836 & 0.5975 & 0.7089 & 0.2431 \\ 0.7795 & -0.5901 & -0.1516 & -0.1083 & -0.0974 \\ 0.0695 & 0.1682 & -0.6686 & 0.6944 & -0.1940 \\ -0.2448 & -0.3413 & -0.4078 & -0.0596 & 0.8086 \end{bmatrix}$$
$\gg b$
$$b = \begin{bmatrix} 47 \\ 26 \\ 24 \\ 23 \\ 39 \end{bmatrix}$$
$\gg Q_h$
$$Q_h = \begin{bmatrix} -0.5715 & -0.6532 & -0.0816 & 0 & -0.4899 \\ 0.0304 & -0.2836 & 0.5975 & 0.7089 & 0.2431 \\ 0.7795 & -0.5901 & -0.1516 & -0.1083 & -0.0974 \\ 0.0695 & 0.1682 & -0.6686 & 0.6944 & -0.1940 \\ -0.2448 & -0.3413 & -0.4078 & -0.0596 & 0.8086 \end{bmatrix}$$
$\gg z = Q_h * b$

$$z = \begin{bmatrix} -64.9115 \\ 34.1800 \\ 11.3680 \\ 0.0000 \\ 0 \end{bmatrix}$$
$$c = \begin{bmatrix} -64.9115 \\ 34.1800 \\ 11.3680 \end{bmatrix}$$
$m = 3;$
$\gg x(3) = c(3)/U(3,3)$
$x = [0 \quad 0 \quad 3.0000]$

```
.........................................................
Geri yerine koyma Matlab kodu:
m=3;
x(m)=c(m)/U(m,m);
for j=m-1:-1:1;
tmp=0;
for k=j+1:m;
tmp=tmp+U(j,k)*x(k);
end
x(j)=(1/U(j,j))*(c(j)-tmp);
end
.........................................................
```

yukarıdaki kod kullanılabilir veya manual olarak tek tek şu şekilde hesaplanabilir:
$>> x(2) = (1/U(2,2)) * (c(2) - U(2,3) * x(3))$
$x =$

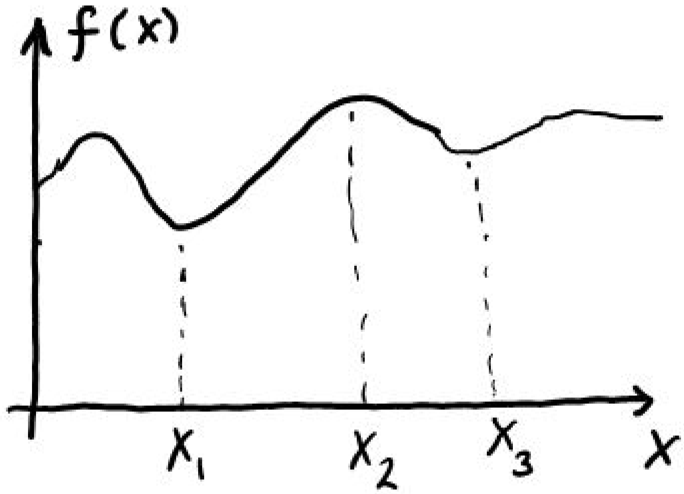
0 2.0000 3.0000
$>> x(1) = (1/U(1,1)) * (c(1) - [U(1,2) * x(2) + U(1,3) * x(3)])$
$x =$

1.0000 2.0000 3.0000

Bölüm 5

Bölüm5

5.1 Optimizasyon(En İyileme)



Tanım kümesi $\Omega : \{x : x \geq 0\}$

x_1 :kesin lokal minimum

x_2 : maksimum

x_3 : lokal minimum

$$\frac{\partial f(x)}{\partial x}|_{x=x_1} = 0 \quad , \frac{\partial f(x)}{\partial x}|_{x=x_2} = 0 \ , \quad \frac{\partial f(x)}{\partial x}|_{x=x_3} = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{\partial^2 f(x)}{\partial x^2}|_{x=x_1} > 0 \quad , \frac{\partial^2 f(x)}{\partial x^2}|_{x=x_2} < 0 \quad , \frac{\partial^2 f(x)}{\partial x^2}|_{x=x_3} > 0$$

<u>Tanım:</u> $f : R^n \to R \qquad \varepsilon > 0 \quad ve \quad x \in R$

$|x - x^*| < \varepsilon$ için $f(x) \geq f(x^*)$ sağlayan $x^* \in R$ noktasına <u>lokal minimum</u> noktası denir.

Eğer $f(x) > f(x^*)$ $\forall x$ için x^* noktasına <u>kesin lokal minimum</u> denir.

Tüm $x \in R$ kümesi içinde $f(x) \geq f(x^*)$ sağlayan $x \in R$ noktasına global minimum denir.

Gradyan Operatörü: (∇_x)

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\nabla_x f(\overline{x}) = \begin{bmatrix} \frac{\partial f(\overline{x})}{\partial x_1} \\ \frac{\partial f(\overline{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\overline{x})}{\partial x_n} \end{bmatrix}_{n \times 1} \qquad f(\overline{x}) : R^n \to R$$

Hessian Matrisi(H)

$$H = \nabla_x f(\overline{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}_{n \times n}$$

Minumum İçin Gerekli Koşul:

1. Eğer $\overline{x}^*$ f fonksiyonun Ω tanım kümesinde lokal minimum noktası ise ,

   $[\nabla f(\overline{x}^*)]^T \overline{d} \geq 0$

   $\overline{d}, \overline{x}^*$ noktasındaki uygulanabilir yön vektörü.

2. $\nabla f(\overline{x}^*) = \overline{o}$

3. Verilen $\overline{x}^*$ noktası için Hessian matrisi pozitif semidefinite bir matris olmalı

   $< \underbrace{\nabla^2 f(\overline{x}^*)}_{\text{H}} \overline{d}, \overline{d} >= \overline{d}^T \cdot \nabla^2 f(\overline{x}^*) \overline{d} \geq 0$

   \-

Örnek:

$\overline{x} = [x_1 x_2]^T$

$f(\overline{x}) = 3x_1^2 + 2x_1x_2 + 3x_1^2 - 20x_1 + 4x_2$
$x_1 \geq 0 \quad x_2 \geq 0$ için minimum noktası?

$$\frac{\partial f}{\partial x_1} = 6x_1 + 2x_2 - 20 \qquad \frac{\partial f}{\partial x_2} = 2x_1 + 6x_2 + 4$$

$$\nabla f(\overline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 6x_1 + 2x_2 - 20 \\ 2x_1 + 6x_2 + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Çözüm: $\overline{x}^* = \begin{bmatrix} \overline{x}_1^* \\ \overline{x}_2^* \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

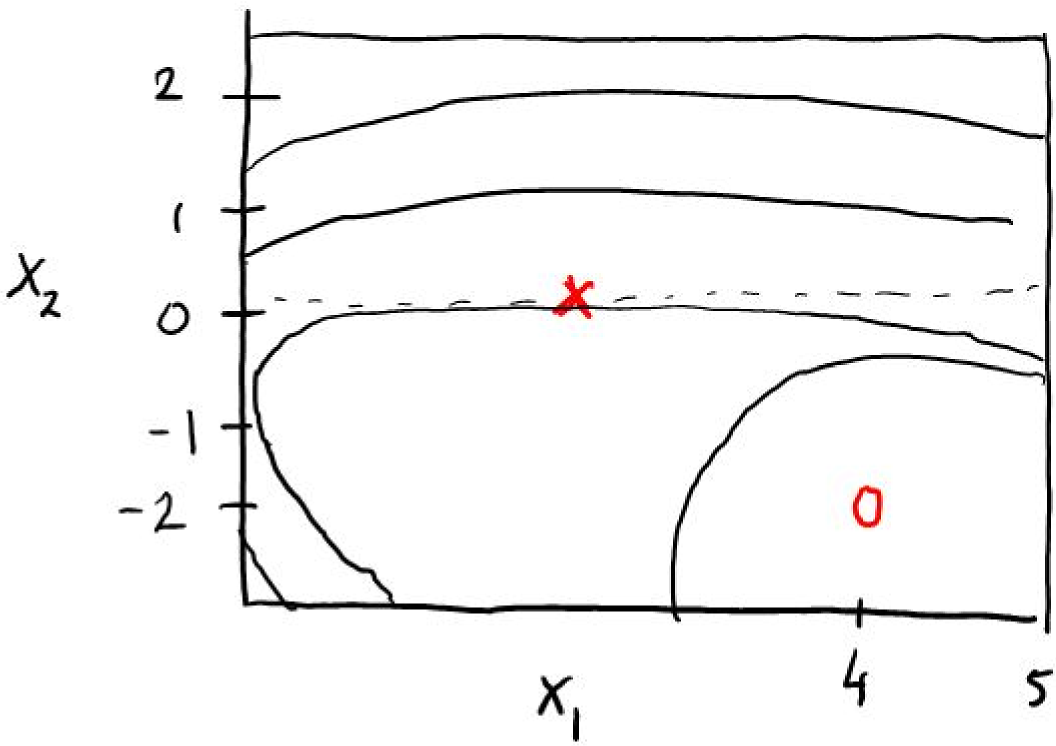
$f(\overline{x}^*) = f(4, -2) = -44 \quad \overline{x}^* \geq 0$ koşulunu sağlamıyor.

$\overline{x}_1^* \geq 0 \quad \overline{x}_2^* \geq 0 \rightarrow$ sınırlandırmaları(constraint) gözönüne alırsak:

minimum nokta $\rightarrow \begin{bmatrix} \overline{x}_1^* \\ \overline{x}_2^* \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ 0 \end{bmatrix}$

fonksiyonun lokal minimumu $\rightarrow f(\frac{10}{3}, 0) = -33,33$

$\overline{x}^* = \begin{bmatrix} \frac{10}{3} \\ 0 \end{bmatrix}$ için $\rightarrow \nabla f(\overline{x}^*) = \begin{bmatrix} 6.\frac{10}{3} + 2.0 - 20 \\ 2.\frac{10}{3} + 6.0 + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{32}{3} \end{bmatrix}$



o:global minimum
x:sınırlandırılmış(constraint)minimum

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Örnek:

$f(x,y) = x^2y$
$g(x,y) = 3y^2x$
$h(x,y) = x-2y$

$$F(x,y) = f(g(x,y),h(x,y)) = (3y^2x)^2(x-2y)$$

$$\frac{\partial F}{\partial x} =? \quad \frac{\partial F}{\partial y} =?$$

$$v = g(x,y)\ ,w = h(x,y) \longrightarrow u = f(v,w) = v^2w$$

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} =$$

$$\frac{\partial u}{\partial x} = 2vw = 2g(x,y).h(x,y) = 2(3y^2x)(x-2y) = 6x^2y^2 - 12xy^3$$

$$\frac{\partial v}{\partial x} = 3y^2$$

$$\frac{\partial u}{\partial w} = v^2 = (3y^2x)^2 = 9x^2y^4$$

$$\frac{\partial w}{\partial x} = 1$$

$$\frac{\partial F}{\partial x} = (6x^2y^2-12xy^3)3y^2 + 9x^2y^4 = 18x^2y^4 - 36xy^5 + 9x^2y^4 = 27x^2y^4 - 36xy^5 //$$

Genel olarak $x_1, x_2, \cdots, x_n$ bağımsız değişkenler
$g_1(x_1,x_2,\cdots,x_n), g_2(x_1,x_2,\cdots,x_n), \cdots, g_m(x_1,x_2,\cdots,x_n)$

$$F(x_1,x_2,\cdots,x_n) = f(g_1,g_2,\cdots,g_n)$$

$$\frac{\partial F}{\partial x_j} = \sum_{i=1}^{m} D_i f g_i \qquad D_i : i. \text{ argümanın türevi}$$

$$= \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial x_j} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial x_j} + \cdots + \frac{\partial f}{\partial g_m}\cdot\frac{\partial g_m}{\partial x_j}$$

Örnek:

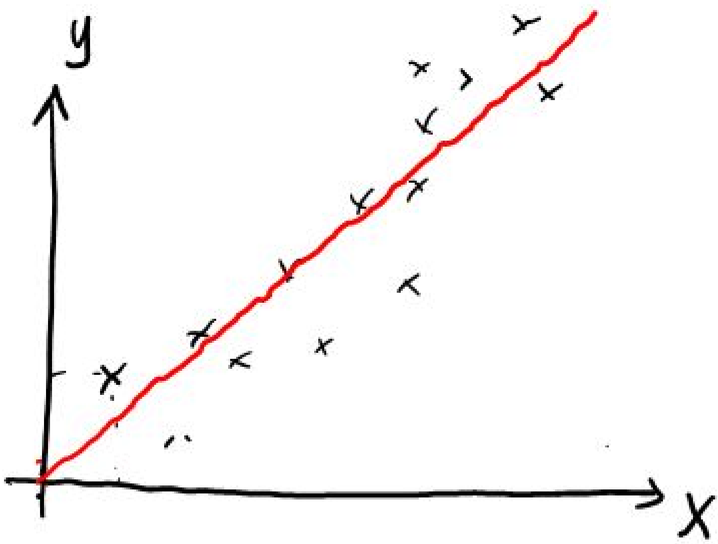
$f(x_1,x_2) = 3x_1^3x_2^3 - 2x_1^2x_2 + 5 \qquad \nabla_x f :? \qquad H =? \quad x_1 = 1, x_2 = 1$

$$\nabla_x f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 6x_1x_2^3 - 4x_1x_2 \\ \\ 9x_1^2x_2^2 - 2x_1 \end{bmatrix} |_{x_1=1,x_2=1} = \begin{bmatrix} 2 \\ \\ 7 \end{bmatrix}$$

Hessian matrisi

$$H = \begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}f & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}f \\ \\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}f & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}f \end{bmatrix} = \begin{bmatrix} 6x_2^3 - 4x_2 & 18x_1x_2^2 - 4x_1 \\ \\ 18x_1x_2^2 - 4x_1 & 18x_1^2x_2 \end{bmatrix}_{x_1=1,x_2=1} = \begin{bmatrix} 2 & 14 \\ \\ 14 & 18 \end{bmatrix}$$

Örnek:



Ölçüm değerleri : $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ $\quad y = ax + b \quad a =?, b =?$
least− squares çözümü a ve b'yi verir?

model:$y = ax + b$

ölçüm x_i, y_i

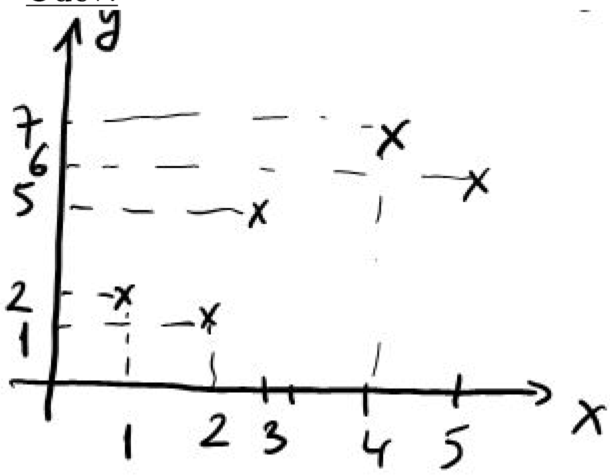
1.ölçüm:$e_1 = y - y_1 = (ax_1 + b) - y_1$
2.ölçüm:$e_2 = y - y_2 = (ax_2 + b) - y_2$
$\vdots$
n.ölçüm:$e_n = y - y_n = (ax_n + b) - y_n$

Amaç: hataların karelerinin ortalamasını minimize eden a ve b değerlerini bulmak .

$J(a, b) = \frac{1}{n}\sum_{i=1}^{n} e_i^2 \longrightarrow$ hataların karelerinin ortalaması(mean squared error-MSE)

$\bigtriangledown J = \begin{bmatrix} \frac{\partial J}{\partial a} \\ \\ \frac{\partial J}{\partial b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ MSE'yi minimum yapan değer Minimum Mean Squared Error(MMSE) -Least Squares(LS)

Ödev:



$y = ax + b \longrightarrow J(a, b)$
$y = ax^2 + bx + c \longrightarrow J(a, b, c)$

5.2 Koşullu (Constraint) Optimizasyon

Problem tanımı:$minf(\overline{x})$
Koşul

$$\left.\begin{array}{l} h_1(\overline{x}) = 0 \\ h_2(\overline{x}) = 0 \\ \vdots \\ h_m(\overline{x}) = 0 \end{array}\right\} \text{eşitlik koşulları} \quad \left.\begin{array}{l} g_1(\overline{x}) \leq 0 \\ g_2(\overline{x}) \leq 0 \\ \vdots \\ g_p(\overline{x}) \leq 0 \end{array}\right\} \text{eşitsizlik koşulları}$$

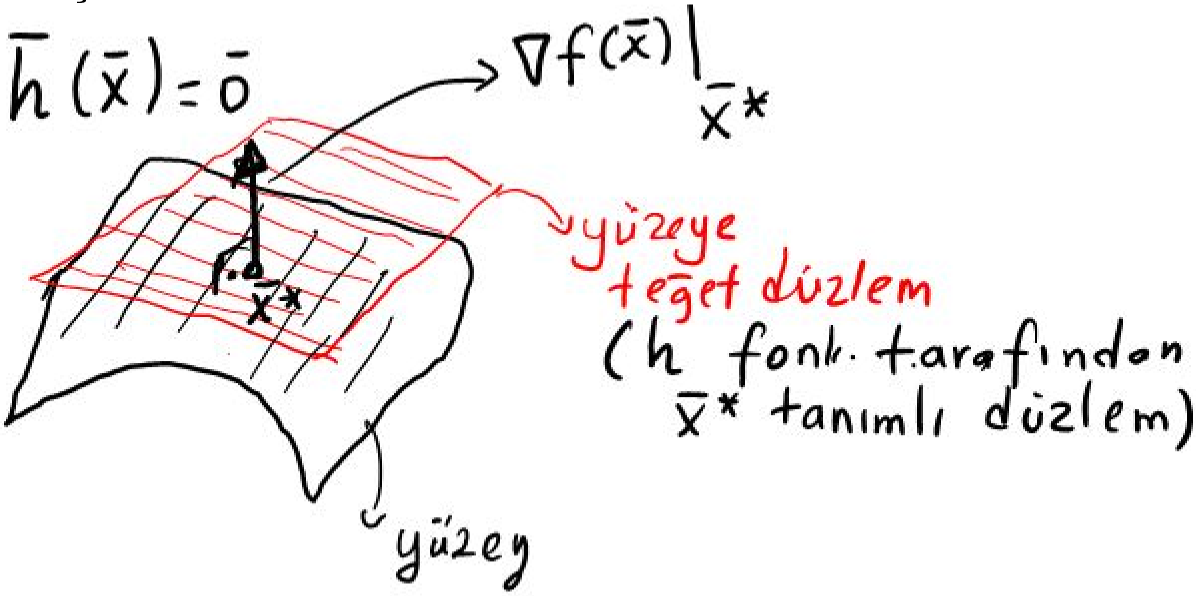
$\bar{h} = (h_1, h_2, \cdots, h_m)$

$\bar{g} = (g_1, g_2, \cdots, g_p)$

x$\in \Omega \quad , \Omega \in R^n$

Yukarıdaki problem tekrar yazılırsa
$minf(\bar{x})$
$\bar{h}(\bar{x}) = \bar{0}$
$\bar{g}(\bar{x}) \leq 0 \quad \bar{x} \in R^n$

Eşitlik koşulu:



Teorem: (Eşitlik sınırlaması için gerekli koşul)
$\bar{x}^*$ noktası f fonksiyonunun $\bar{h}(\bar{x}) = 0$ koşulu altında lokal ekstremum noktası için aşağıdaki koşulu sağlaması gerekir.
$\nabla f(\bar{x}^*) + \nabla \bar{h}(\bar{x}^*).\Lambda = 0$
$\Lambda \in R^m$ sayısı lagrange çarpanı olarak adlandırılır.
$\nabla f(\bar{x}^*) + \sum_{i=1}^{m} \nabla h_i(\bar{x}^*).\Lambda_i = 0$

Lagrange fonksiyonu:
$L(\bar{x}, \bar{\Lambda}) = f(\bar{x}) + \bar{h}(\bar{x})^T.\bar{\Lambda}$
$\nabla_x L(\bar{x}, \bar{\Lambda}) = 0$
$\nabla_\Lambda L(\bar{x}, \bar{\Lambda})$
min veya max olduğunu bulmak için 2.derece koşullar:
$F_{n\times n}(\bar{x}, \bar{\Lambda}) = \sum_{k=1}^{m} \dfrac{\partial^2 \bar{h}_k(\bar{x})}{\partial x_i \partial x_j} x_k \leftarrow$ H matrisinin (i,j). elemanı
$H_{n\times n}(\bar{x}) = \dfrac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j} \leftarrow$ F matrisinin (i,j). elemanı
$L(\bar{x}^*) = H(\bar{x}^*) + F(\bar{x}^*, \Lambda)$
$L(\bar{x}^*)$ pozitif semidefinite bir matris ise bulunan $\bar{x}^*$ noktası minimum noktasıdır.

Örnek:
$f(x_1, x_2) = 3x_1^2 + 4x_2^2 + 6x_1x_2 - 8x_2 - 6x_1$
Sınırlama:$h_1(x_1, x_2) = x_1 + x_2 - 9 = 0$

verilen koşulda f'yi minimize eden değeri($x_1^* =?, x_2^* =?$)
çözüm: $\bar{x} = [x_1 \quad x_2]^T, \quad$ Lagrange çarpanı Λ_1
$L(\bar{x}_1\Lambda_1) = f(\bar{x}) + \Lambda_1 h_1(\bar{x})$
$\nabla_x L = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \end{bmatrix}$

$$L(\bar{x}, \Lambda_1) = (3x_1^2 + 4x_2^2 + 6x_1x_2 - 8x_2 - 6x_1) + \Lambda_1(x_1 + x_2 - 9)$$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x_1} = 6x_1 + 6x_2 - 6 + \Lambda_1 (*) \\ \frac{\partial L}{\partial x_2} = 8x_2 + 6x_1 - 8 + \Lambda_1 (**) \\ \frac{\partial L}{\partial \Lambda_1} = x_1 + x_2 - 9 (***) \end{array}\right\} \text{3 bilinmeyenli 3 denklem .}$$

$$(**) - (*) \rightarrow 2x_2 - 2 = 0 \Rightarrow \boxed{x_2^* = 1}$$

$$(***) \rightarrow \boxed{x_1^* = 8}$$

(*)'de x_1^* ve x_2^* yerine yazılırsa $\rightarrow \boxed{\Lambda_1 = -48}$
$\bar{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

Örnek:
Bir önceki örnekte $h_1(\bar{x}) = x_1^2 + x_2^2 - 9 = 0$ verilirse $\qquad x_1^*, x_2^* =?$

Örnek:
üstteki örnekte
$$\left.\begin{array}{l} h_1(x_1, x_2) = x_1^2 + x_2^2 - 9 = 0 \\ h_1(x_1, x_2) = 2x_1 - x_2 - 4 = 0 \end{array}\right\} x_1^* =? \quad x_2^* =?$$
$L(\bar{x}, \bar{\Lambda}) = f(\bar{x}) + \Lambda_1 h_1(\bar{x}) + \Lambda_2 h_2(\bar{x})$

5.3 Ara Sınav Çözümleri

1) a)

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{l} \left[\begin{array}{cccc} 10 & 15 & 20 & 30 \\ 9 & 10 & 15 & 20 \\ 20 & 25 & 30 & 40 \end{array}\right] \begin{array}{l} \to \text{elma} \\ \to \text{portakal} \\ \to \text{nar} \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ MA & MB & MC & MD \end{array} \end{array}$$

$$B = \begin{array}{l} \left[\begin{array}{ccc} 20 & 30 & 40 \\ 0.21 & 0.11 & 0.31 \end{array}\right] \begin{array}{l} \to \text{fiyat (KR)} \\ \to \text{ağırlık (kg)} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{elma} & \text{port.} & \text{nar} \end{array} \end{array}$$

$$C = BA = \begin{array}{l} \left[\begin{array}{cccc} 1270 & 1600 & 2050 & 2800 \\ 9.3 & 12 & 15.2 & 20.9 \end{array}\right] \begin{array}{l} \to \text{fiyat} \\ \to \text{ağırlık} \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ MA & MB & MC & MD \end{array} \end{array}$$

b)MATLAB

$$A(2:3,1:2:4)=\begin{bmatrix}9 & 15\\20 & 30\end{bmatrix}$$

c)B(1,:)*A(:,1)
d)B(2,:)*A(:,3)

2) $\bar{x}_1=\begin{bmatrix}\alpha\\-1\\2\end{bmatrix} \qquad \bar{x}_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$

a) $<\ \bar{x}_1,\bar{x}_2\ >=\ \bar{x}_1^T.\bar{x}_2\ =\ [\alpha \quad -1 \quad 2]\begin{bmatrix}1\\-1\\1\end{bmatrix}\ =\ \alpha+1+2\ =\ 0\ \Rightarrow$

$\boxed{\alpha=-3}$

b) $\bar{x}_1 \perp \bar{x}_2$ olduğundan $\bar{e}_1$ $\quad ve \quad$ $\bar{e}_2$'yi bulurken normlarına bölmemiz yeterli

$$\|\bar{x}_1\|=\sqrt{\bar{x}_1^T.\bar{x}_1}=\left(\ [-3 \quad -1 \quad 2]\ \begin{bmatrix}-3\\-1\\2\end{bmatrix}\right)^{\frac{1}{2}}=\sqrt{14}$$

$$\bar{e}_1=\tfrac{\bar{x}_1}{\|\bar{x}_1\|}=\tfrac{1}{\sqrt{14}}\begin{bmatrix}-3\\-1\\2\end{bmatrix}$$

$$\|\bar{x}_2\|=\sqrt{\bar{x}_2^T.\bar{x}_2}=\left(\ [1 \quad -1 \quad 1]\ \begin{bmatrix}1\\-1\\1\end{bmatrix}\right)^{\frac{1}{2}}=\sqrt{3}$$

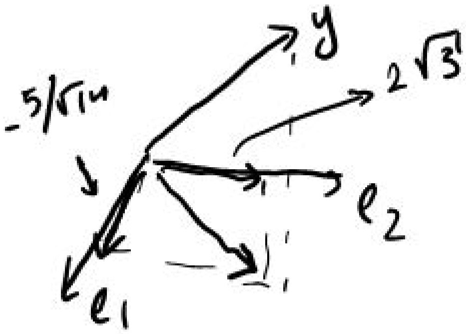
$$\bar{e}_2=\tfrac{\bar{x}_2}{\|\bar{x}_2\|}=\tfrac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

c) $\bar{y}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$

$\bar{y}_1 \qquad \bar{y}_2=?$

$$<\bar{y},\bar{e}_1>=[3 \quad -2 \quad 1]\begin{bmatrix}\frac{-3}{\sqrt{14}}\\\frac{-1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\end{bmatrix}=\tfrac{-5}{\sqrt{14}}$$

$$< \bar{y}, \bar{e}_2 >= [3 \quad -2 \quad 1] \begin{bmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \end{bmatrix} = \tfrac{6}{\sqrt{3}} = 2\sqrt{3}$$



$$\bar{y} = \tfrac{-5}{\sqrt{14}}\bar{e}_1 + 2\sqrt{3}\bar{e}_2$$

3) $A = \begin{bmatrix} 2 & 5 & 9 \\ 1 & 4 & 7 \\ 3 & 2 & 1 \end{bmatrix}$

a) without pivoting $\longrightarrow L = ?$

$\dfrac{a_{21}}{a_{11}} = \frac{1}{2} \quad , \quad \dfrac{a_{31}}{a_{11}} = \frac{3}{2}$

$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-3}{2} & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 5 & 9 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{-11}{2} & \frac{-25}{2} \end{bmatrix} \longleftarrow \dfrac{a_{32}}{a_{22}} = \dfrac{\frac{-11}{2}}{\frac{3}{2}} = \frac{-11}{3}$

$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{3} & 1 \end{bmatrix} A_1 = \begin{bmatrix} 2 & 5 & 9 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{-10}{3} \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & \frac{-11}{3} & 1 \end{bmatrix} \checkmark$

b) pivoting $\longrightarrow E_2 = ?$

$| a_{31} | > | a_{11} | \longrightarrow S_3 \longleftrightarrow S_1$

$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A_1 = P_{13}A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} \frac{a_{31}}{a_{11}} = \frac{2}{3} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-2}{3} & 0 & 1 \end{bmatrix} \checkmark$