Combinatorics

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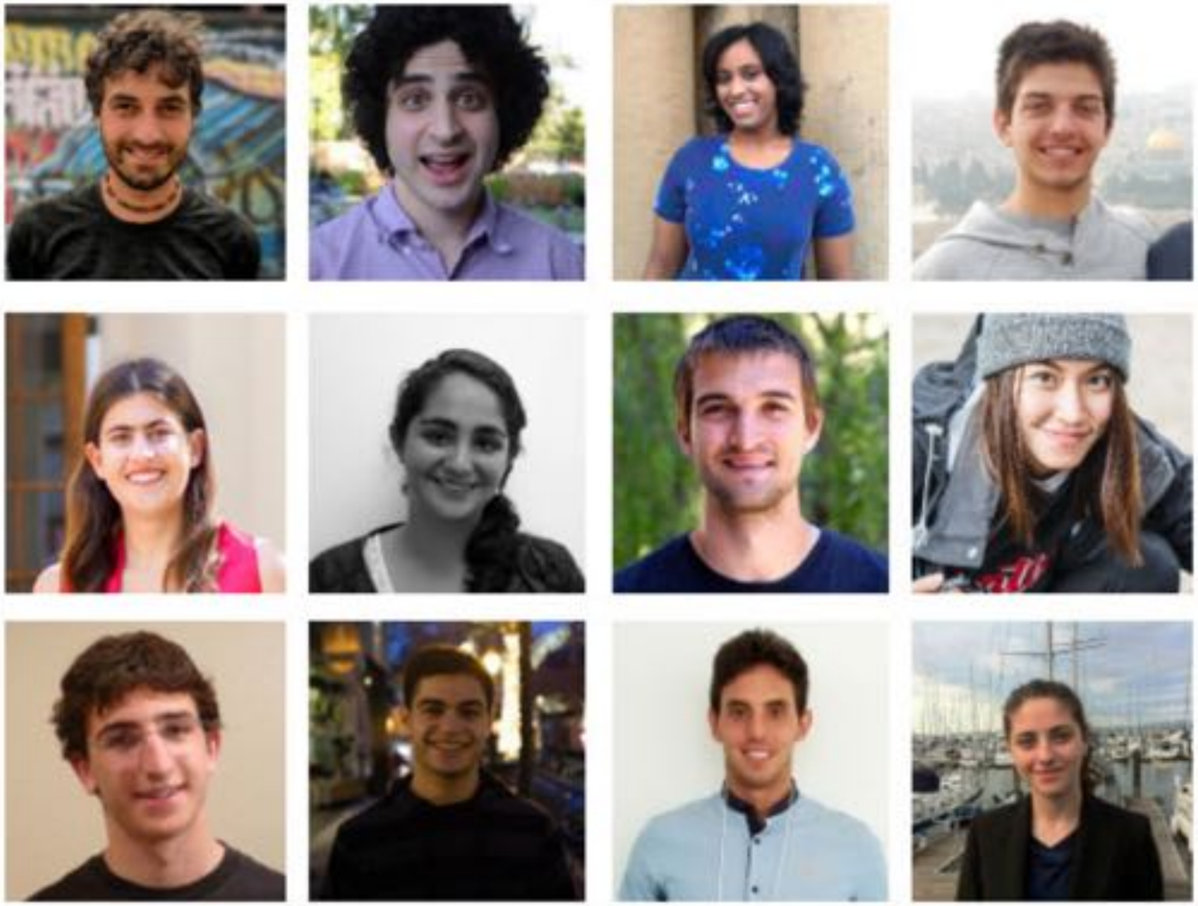
Review

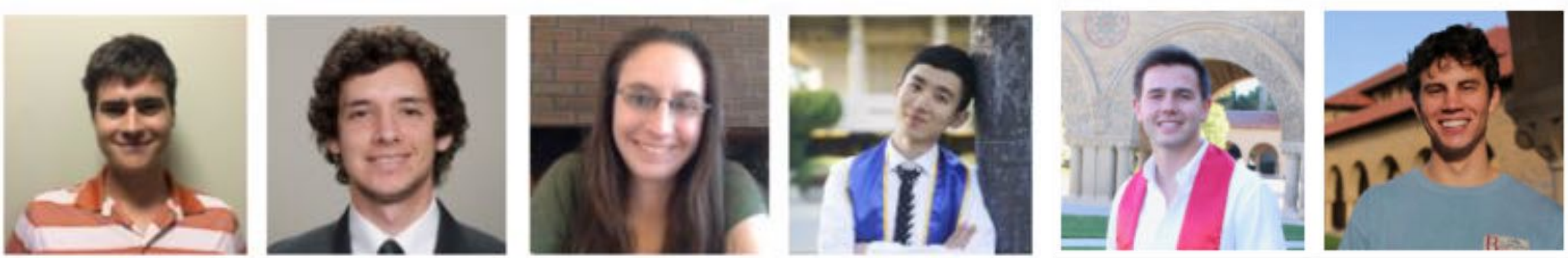
Essential Information

cs109.stanford.edu

CS109 Community



Dedicated, intelligent, hardworking teaching assistants



Counting

We are counting:

of **events**,

of **outcomes**,

of **objects**

Two Key Rules

Counting outcomes with or:

Inclusion Exclusion:

If outcomes can come from set A **or** set B, then the total number of outcomes is $|A| + |B| - |A \cap B|$.

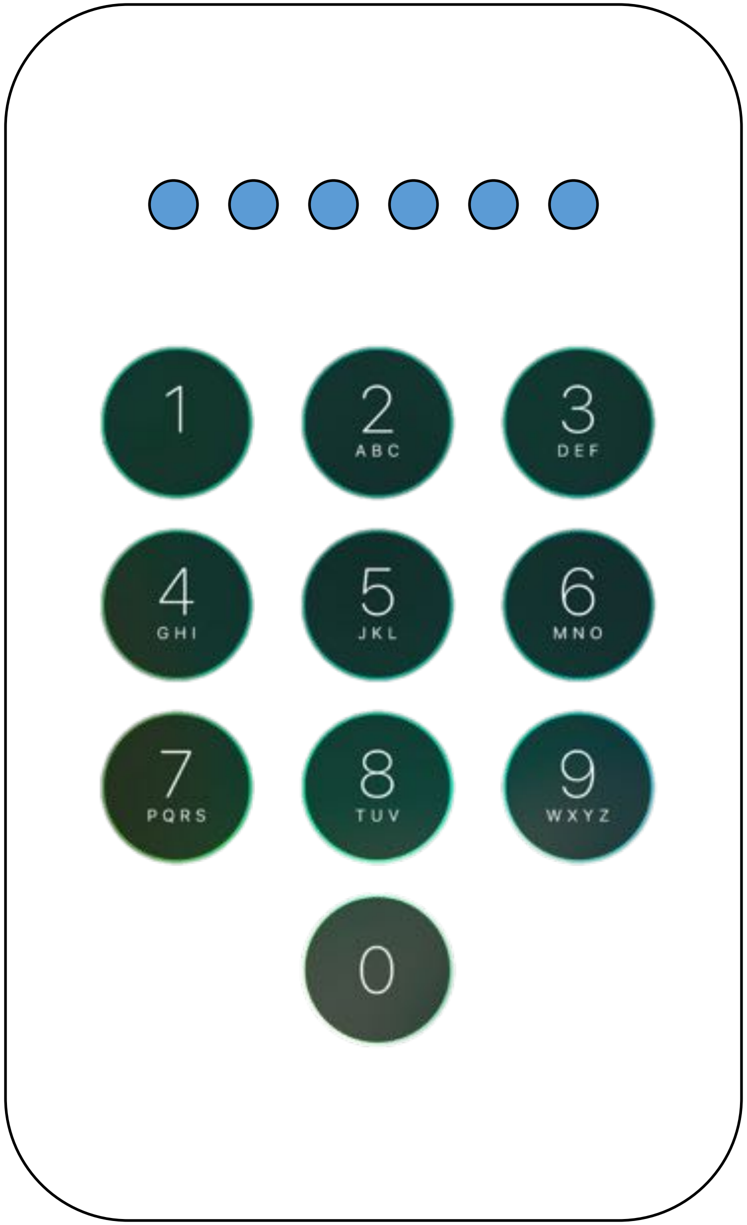
Counting outcomes with steps:

Product Rule of Counting:

If outcomes are generated via a process with r **steps**, where step i has n_i outcomes, then the total number of outcomes is:

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

How Many Unique 6 digit passcodes?



Approach: count by steps

Step 1: first digit in passcode
(10 outcomes)

Step 2: second digit in passcode
(10 outcomes)

...

Step 6: second digit in passcode
(10 outcomes)

$$\text{total} = n_1 \times n_2 \times \cdots \times n_r$$
$$= 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

End Review

Combinatorics

Counting tasks on *n* objects

- Sort objects (permutations)
- Choose *k* objects (combinations)
- Put objects in *r* buckets

Combinatorics

Counting tasks on *n* objects

- **Sort objects (permutations)**
  - Distinct: $n!$
  - Some Distinct: $\dfrac{n!}{n_1!n_2!\ldots}$
- **Choose *k* objects (combinations)**
  - Distinct: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
- **Put objects in *r* buckets**
  - Distinct: r^n
  - None Distinct: $\dfrac{(n+r-1)!}{n!(r-1)!}$

Combinatorics

Counting tasks on *n* objects

Sort objects (permutations)

Choose *k* objects (combinations)

Put objects in *r* buckets

Distinct

Sort *n* Distinct Objects

Ayesha Tim Irina Joey Waddie

Sort *n* Distinct Objects

Sort 5 distinct cans:

Step 1: Chose first can (5 options)

Irina

Sort *n* Distinct Objects

Sort 5 distinct cans:

Step 1: Chose first can (5 options)

Step 2: Chose second can (4 options)

Irina

Waddie

$$5 \times 4 \times 3 \times 2 \times 1 =$$

$\cdots$ **120 unique sorts**

Sort *n* Distinct Objects

Def Permutations:

A permutation is an ordered arrangement of distinct object.

n objects can be permuted in:

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

(Select 1st object out of n, then 2nd object out of n – 1, etc.)

Sort Distinct Objects

Ayesha Tim Irina Joey Waddie

$$= 120$$

Sort Semi-Distinct Objects

Coke Tim Coke Joey Waddie

$$= 120/2$$

Sort Semi-Distinct Objects

Making perms of distinct objects is a two step process

Step 1

Step 2

perms of distinct objects $=$ perms considering some objects are indistinct $\times$ perms of just the indistinct objects

Sort Semi-Distinct Objects

perms of distinct objects $=$ perms considering some objects are indistinct $\times$ perms of just the indistinct objects

Sort Semi-Distinct Objects

$$\frac{\text{perms of distinct objects}}{\text{perms of just the indistinct objects}} = \text{perms considering some objects are indistinct}$$

General Way to Count Permutations

Def: General Permutations:

When there are n objects
n_1 are the same (indistinguishable) and
n_2 are the same and

...

n_r are the same,

There are:

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Unique orderings ("permutations")

How many orderings?

Coke Coke0 Coke Coke0 Coke0

$$= 120/(3! \times 2!) = 10$$

How many orderings of letters?

MOO $= \frac{3!}{2!} = 3$

MISSISSIPPI

$$= \frac{11!}{1!4!4!2!} = 34,650$$

Combinatorics

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Sort objects (permutations)

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Put objects in *r* buckets

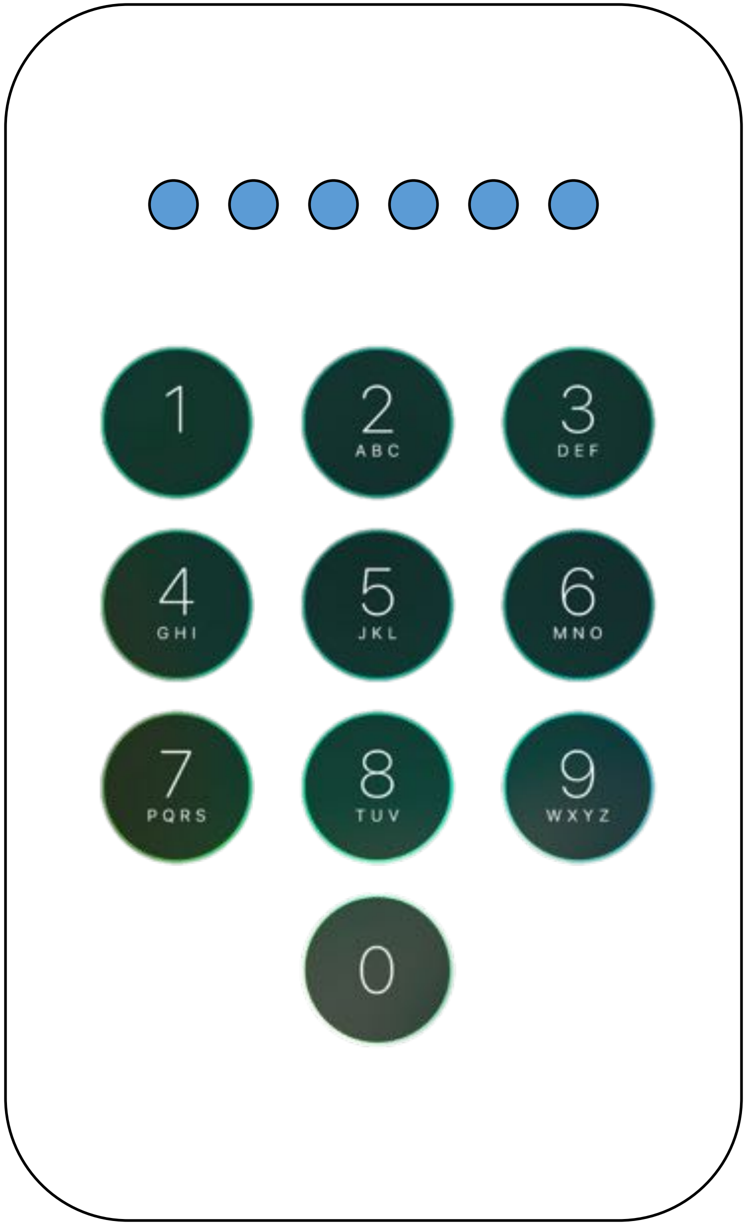
Distinct

$$n!$$

Some Distinct

$$\frac{n!}{n_1!n_2!\dots}$$

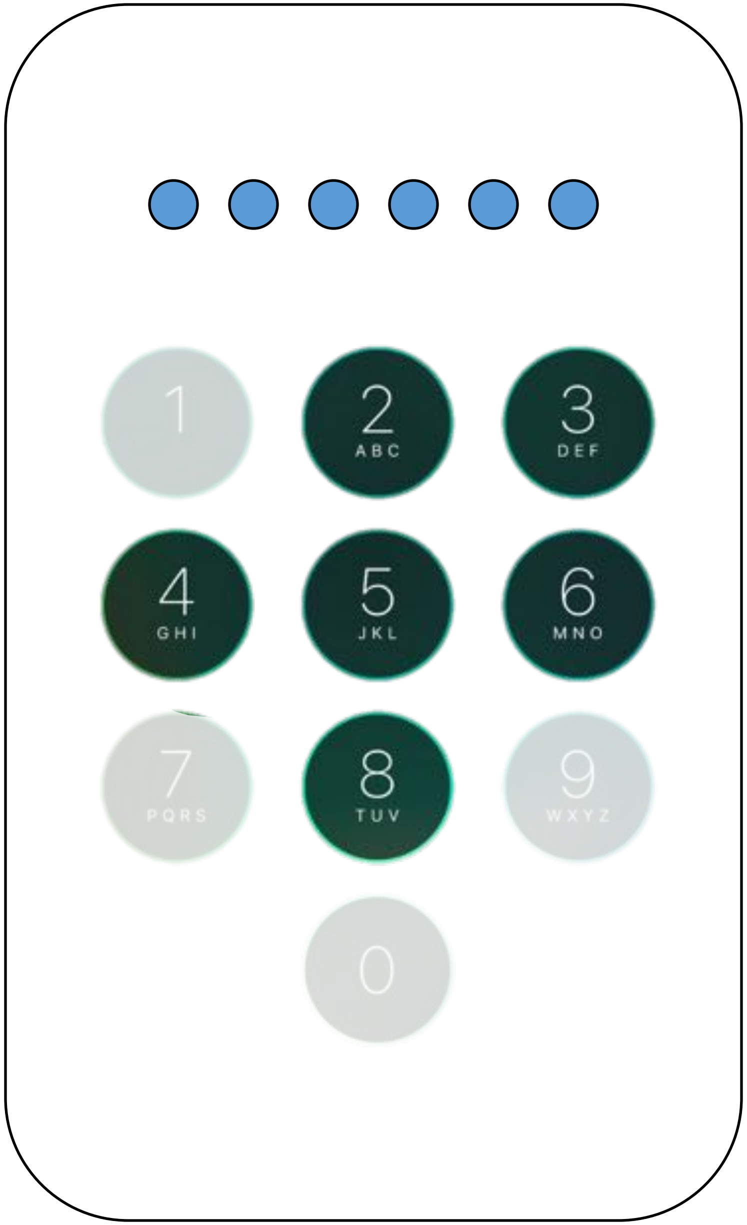
How Many Unique 6 digit passcodes?



How many unique 6 digit passcodes are there?

$$10^6 = 1,000,000$$

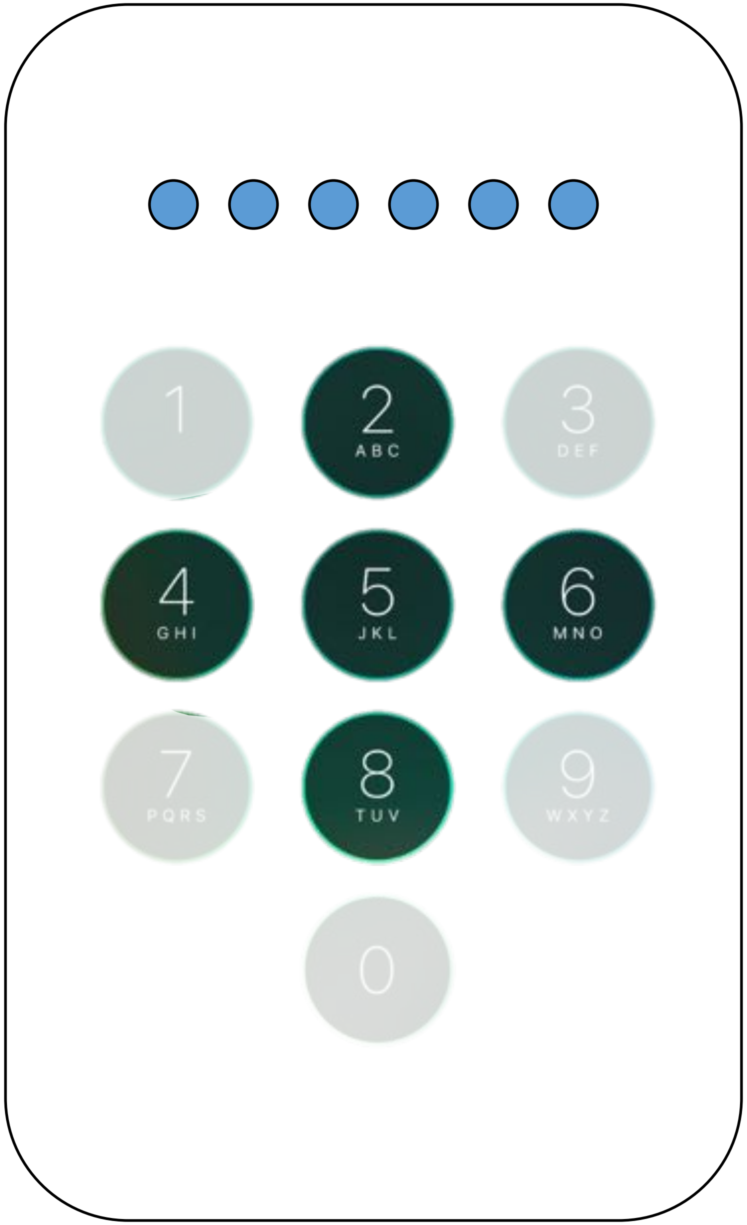
How many possible codes 6 smudges?



If a phone password uses each of **six** distinct numbers, how many unique six digit passcodes are there?

$$6! = 720$$

How many possible codes 5 smudges?



If a phone password uses each of **five** distinct numbers, how many unique six digit passcodes are there?

Five mutually exclusive cases:

2 was repeated

4 was repeated

5 was repeated

6 was repeated

8 was repeated

$$= 5 \times \frac{6!}{2!} = 1,800$$

Combinatorics

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Put objects in *r* buckets

Distinct

$$n!$$

Some Distinct

$$\frac{n!}{n_1!n_2!\ldots}$$

Distinct

Combinatorics

There are $n = 20$ people

How many ways can we **chose** $k = 5$ people to get cake?

Consider this
generative process

Step 1: Randomly order people

There are $n = 20$ people

How many ways can we **chose** $k = 5$ people to get cake?

$$\text{step 1 ways} = n!$$

Step 2: Draw a line at pos k

There are $n = 20$ people

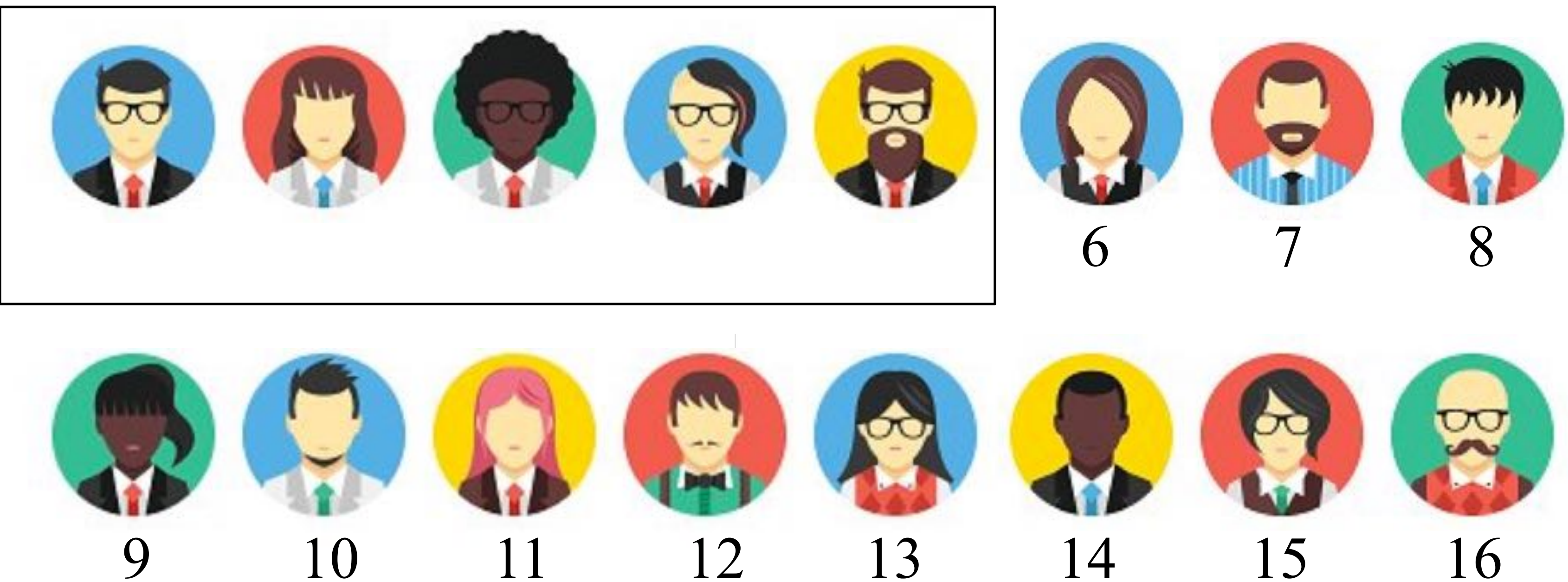
How many ways can we **chose** $k = 5$ people to get cake?

$$\text{step 2 ways} = 1$$

Step 3: Allow Cake Group to Mingle

There are $n = 20$ people

How many ways can we **chose** $k = 5$ people to get cake?

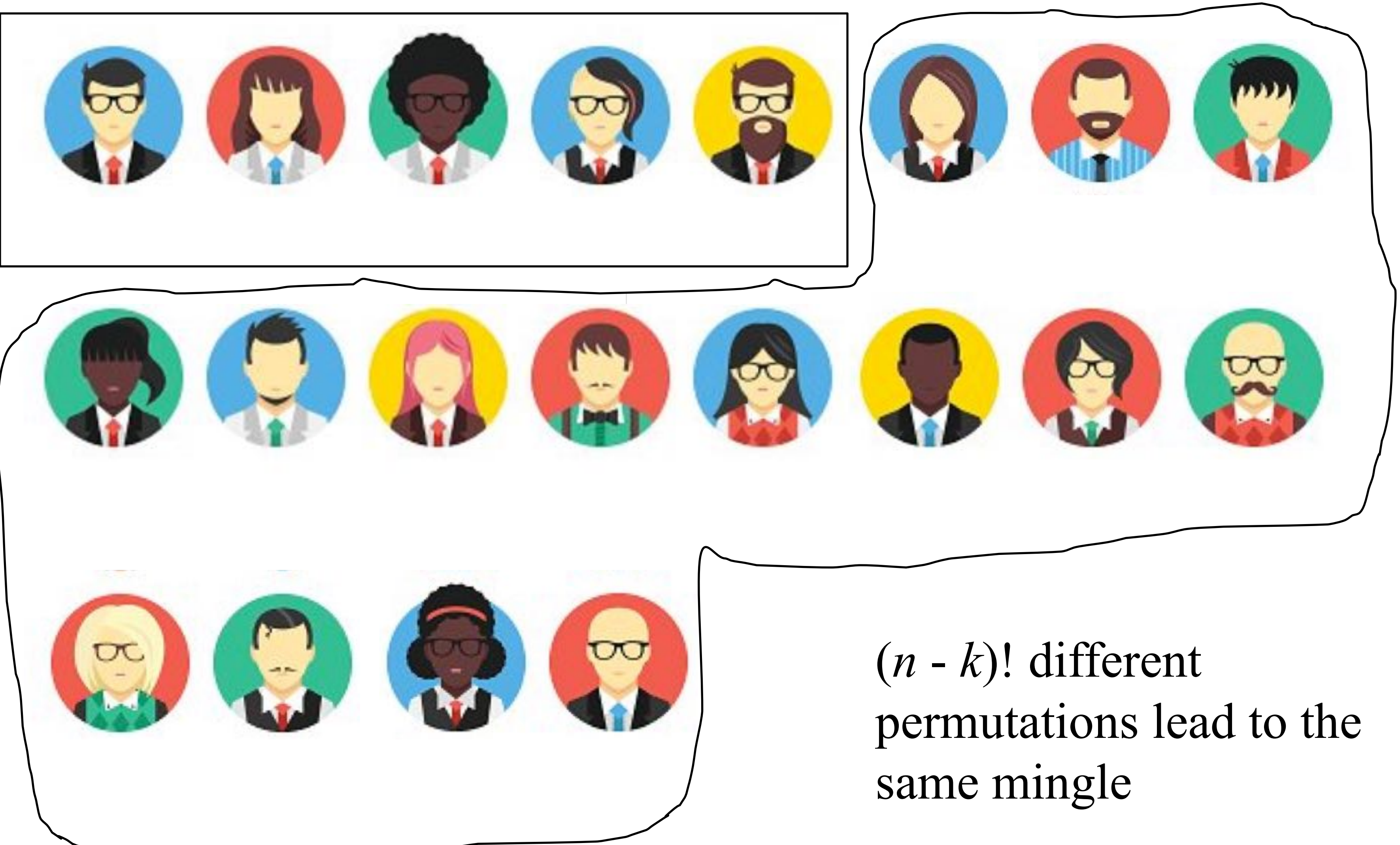


$k!$ different permutations lead to the same mingle

Step 4: Allow nonCake Group to Mingle

There are n = 20 people

How many ways can we **chose** k = 5 people to get cake?



(n - k)! different permutations lead to the same mingle

Step 4: Allow nonCake Group to Mingle

There are $n = 20$ people

How many ways can we chose $k = 5$ people to get cake?

Randomly order n objects

Designate the first k as chosen

$$\text{num ways} = n! \times 1 \times \frac{1}{k!(n-k)!}$$

Any ordering of chosen group is the same choice

Any ordering of non-chosen group is the same choice

Step 4: Allow nonCake Group to Mingle

There are $n = 20$ people

How many ways can we **chose** $k = 5$ people to get cake?

$$\text{num ways} = \binom{n}{k}$$

* Also called binomial coefficients $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

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Distinct

$$n!$$

Some Distinct

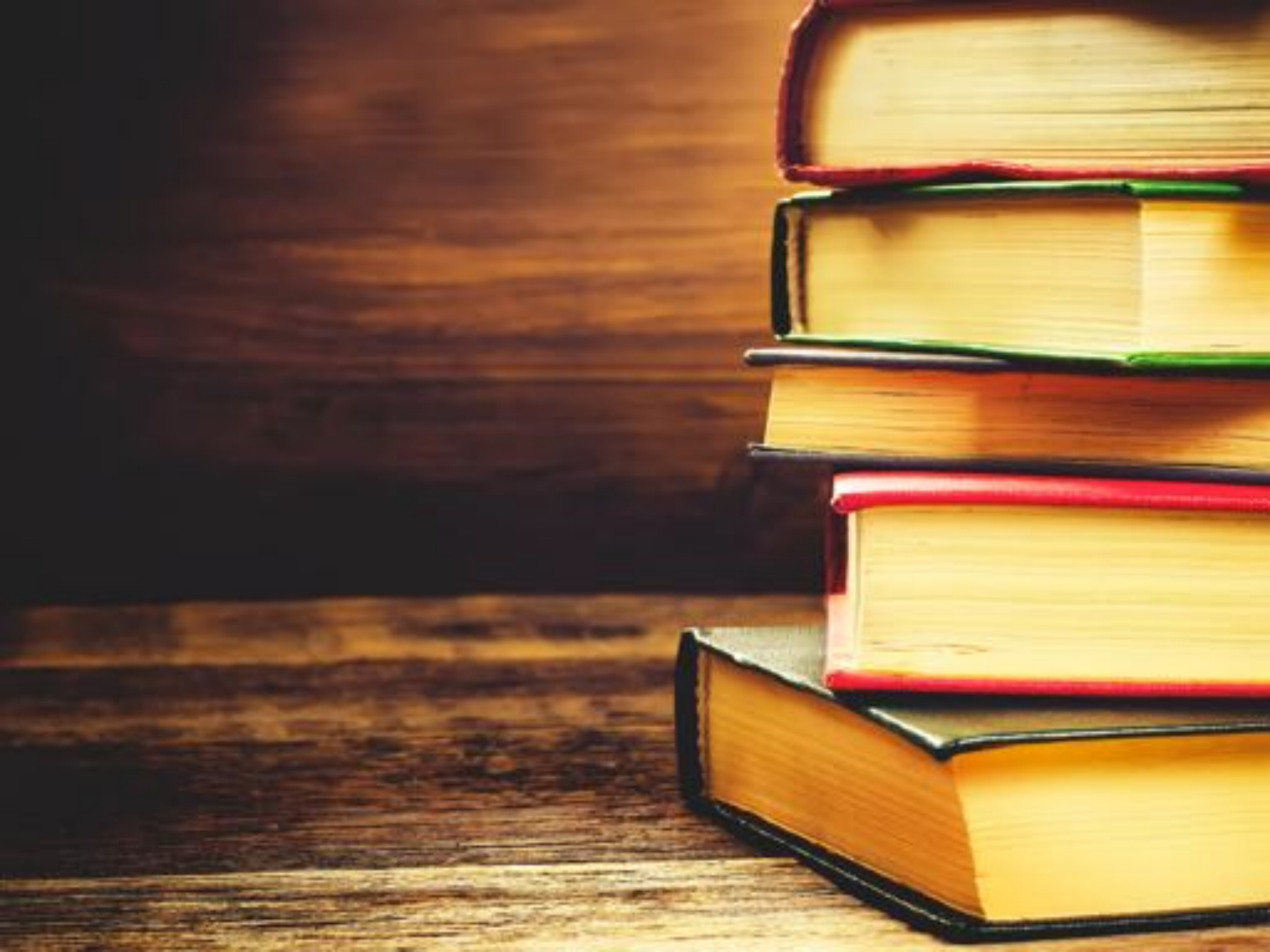
$$\frac{n!}{n_1!n_2!\dots}$$

Distinct

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

8,000 villagers.
How many distinct ways can you **chose** 2 to play a game?

$$= \frac{8000!}{7998!2!} = 31,996,000$$



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Choose *k* objects (combinations)

- Distinct: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Put objects in *r* buckets

- Distinct
- None Distinct

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Something is going on in the world of AI

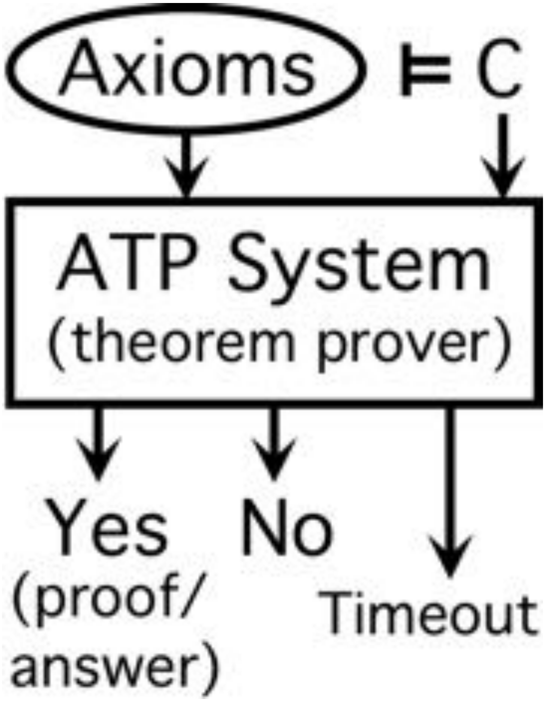
Modern AI
or, How we learned to combine probability and programming

Brief History

Early Optimism 1950

1952

1955



Early Optimism 1950

"Machines will be capable, within twenty years, of doing any work a man can do."
–Herbert Simon, 1952

Underwhelming Results 1950s to 1980s

The spirit is willing but the flesh is weak.

↓

(Russian)

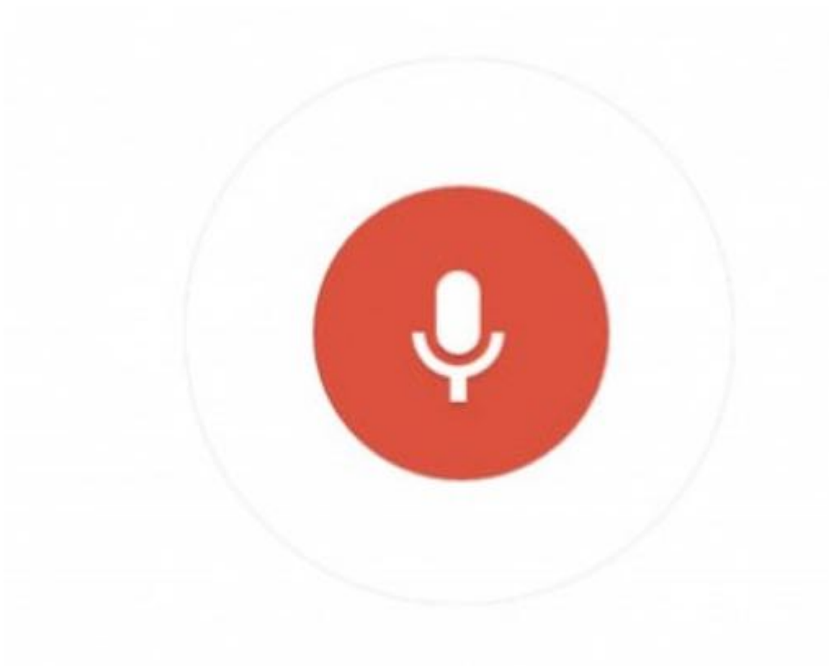
↓

The vodka is good but the meat is rotten.

The world is too complex


BRACE YOURSELVES
WINTER IS COMING

Told Speech Was 30 Years Out



Almost perfect...

What is going on?

[suspense]

Focus on one problem

Computer Vision

Classification

That is a picture
of a **one**

Classification

That is a picture of a **zero**

Classification

That is a picture of an **zero**

* It doesn’t have to be correct all of the time

Can you do it?

What number is this?

What number is this?

How about now?

What a computer sees

0	0	1	0	1	0	1	0	0	0	1	1	1	0	1
1	0	0	1	0	1	1	1	0	1	0	0	0	0	0
1	1	1	0	1	0	0	1	1	0	0	1	0	1	0
1	1	1	1	1	0	0	0	0	0	1	1	0	1	1
0	0	0	1	1	0	0	1	0	0					
1	0	0	1	1	0	0	0	1	0					
1	1	0	1	1	0	0	1	1	0					
1	0	1	0	0	1	0	0	1	0					
0	0	0	0	1	0	1	0	1	1					
0	1	1	0	0	0	0	0	1	1					
0	0	1	0	1	1	1	0	0	0					
0	1	1	1	0	1	0	0	1	0					
1	1	0	0	0	0	0	0	0	0					
0	0	0	0	0	0	0	0	1	1					
0	0	1	1	1	0	1	0	1	1					

What a human sees

Very hard to Program


```
public class HarryHat extends ConsoleProgram {

    public void run() {
        println(“Todo: Write program”);
    }

}
```

Two Great Ideas

1. Probability from Examples

2. Artificial Neurons

Two Great Ideas

1. Probability from Examples

2. Artificial Neurons

1. Probability From Examples

When Does the Magic Happen?

Lots of Data	+	Sound Probability

Machine Learning

Basically just a rebranding of statistics and probability.

Vision is Hard

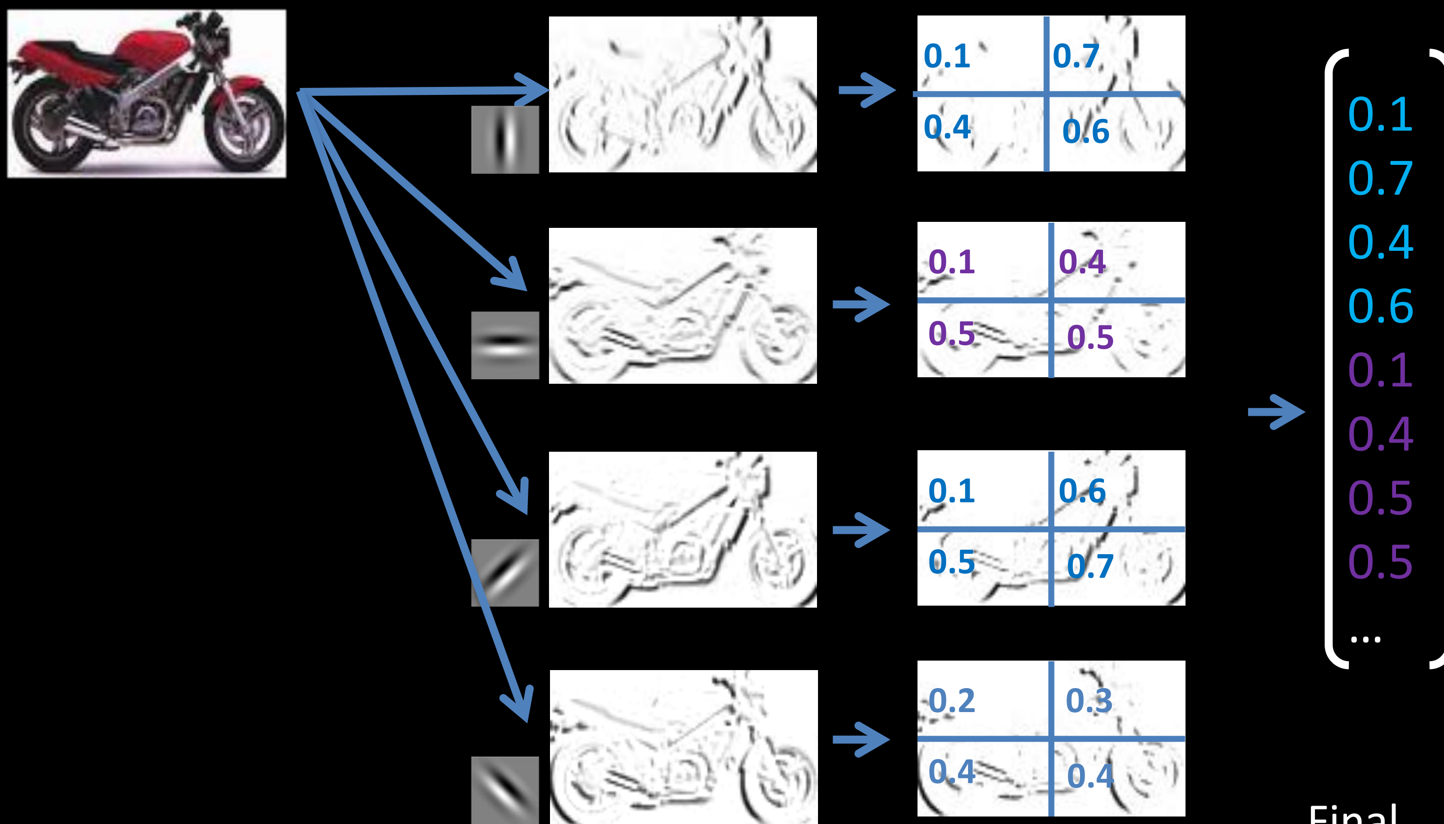
You see this:

But the camera sees this:

194	210	201	212	199	213	215	195	178	158	182	209
180	189	190	221	209	205	191	167	147	115	129	163
114	126	140	188	176	165	152	140	170	106	78	88
87	103	115	154	143	142	149	153	173	101	57	57
102	112	106	131	122	138	152	147	128	84	58	66
94	95	79	104	105	124	129	113	107	87	69	67
68	71	69	98	89	92	98	95	89	88	76	67
41	56	68	99	63	45	60	82	58	76	75	65
20	43	69	75	56	41	51	73	55	70	63	44
50	50	57	69	75	75	73	74	53	68	59	37
72	59	53	66	84	92	84	74	57	72	63	42
67	61	58	65	75	78	76	73	59	75	69	50

[Andrew Ng]

Human Designed Features



Find edges at four orientations

Sum up edge strength in each quadrant

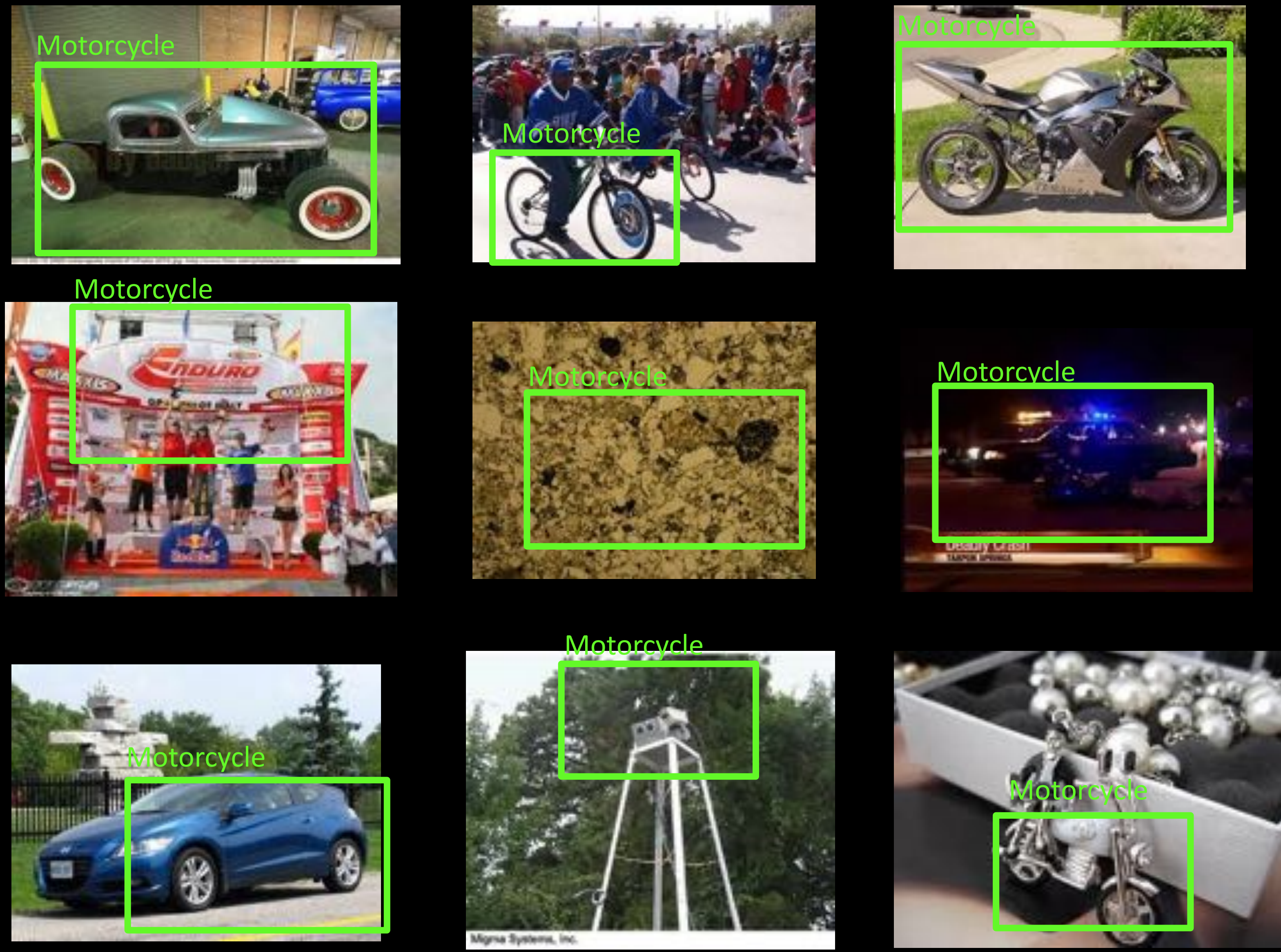
Final feature vector

[Andrew Ng]

Some Great Thinkers

Daphne Koller

Straight ML Not Perfect...



Two Great Ideas

1. Probability from Examples

2. Artificial Neurons

Two Great Ideas

1. Probability from Examples

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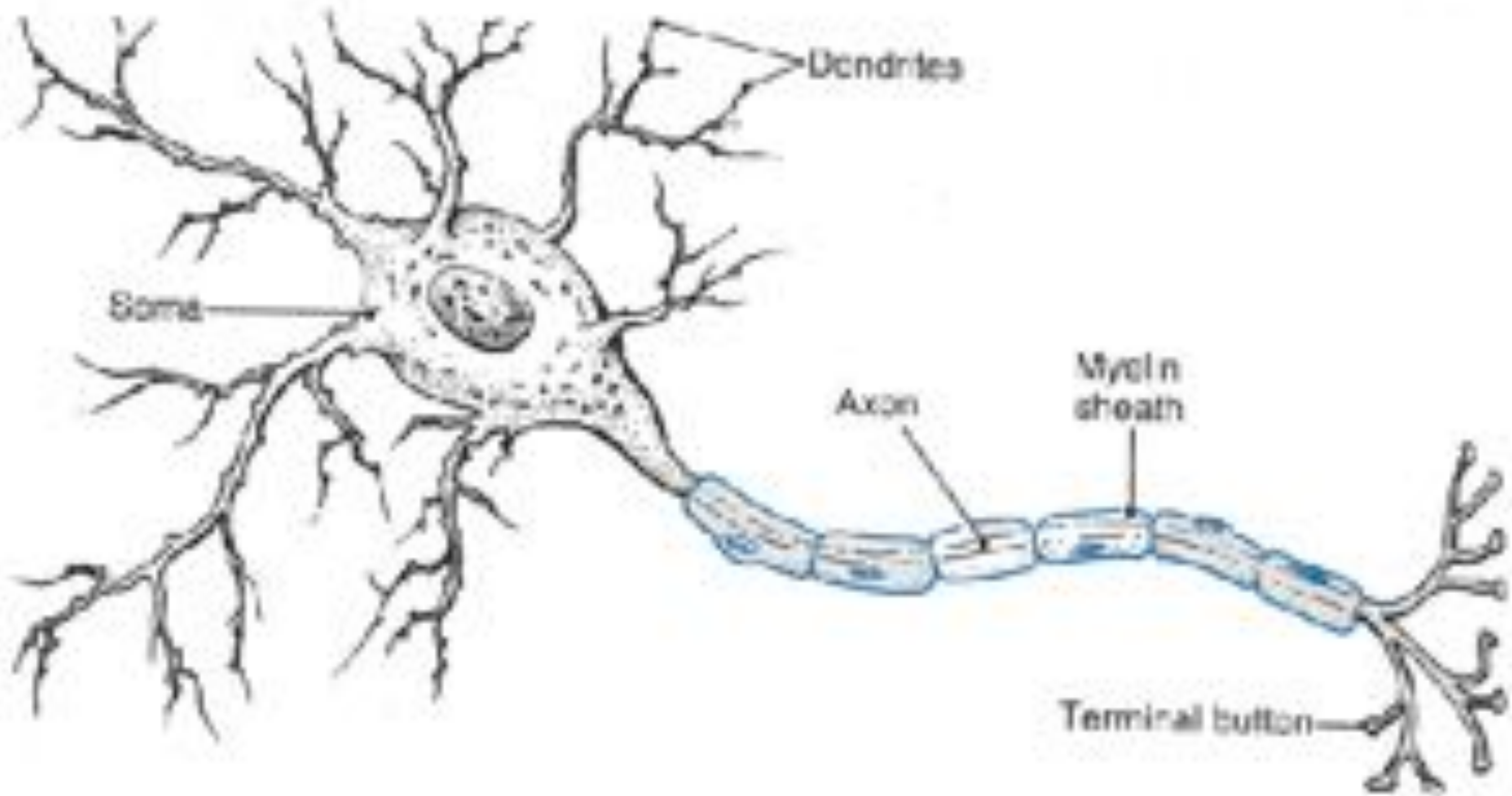
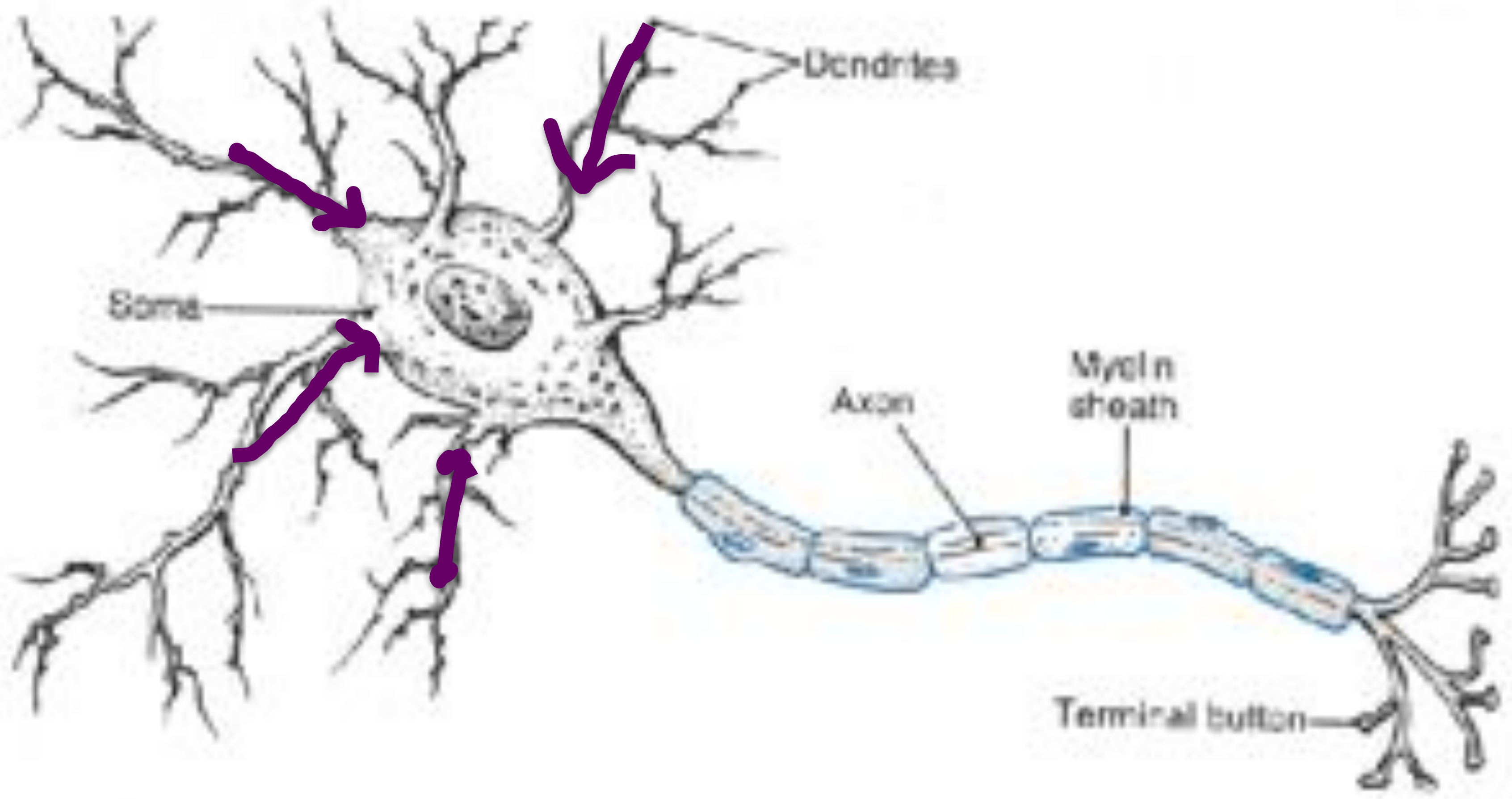
2. Artificial Neurons

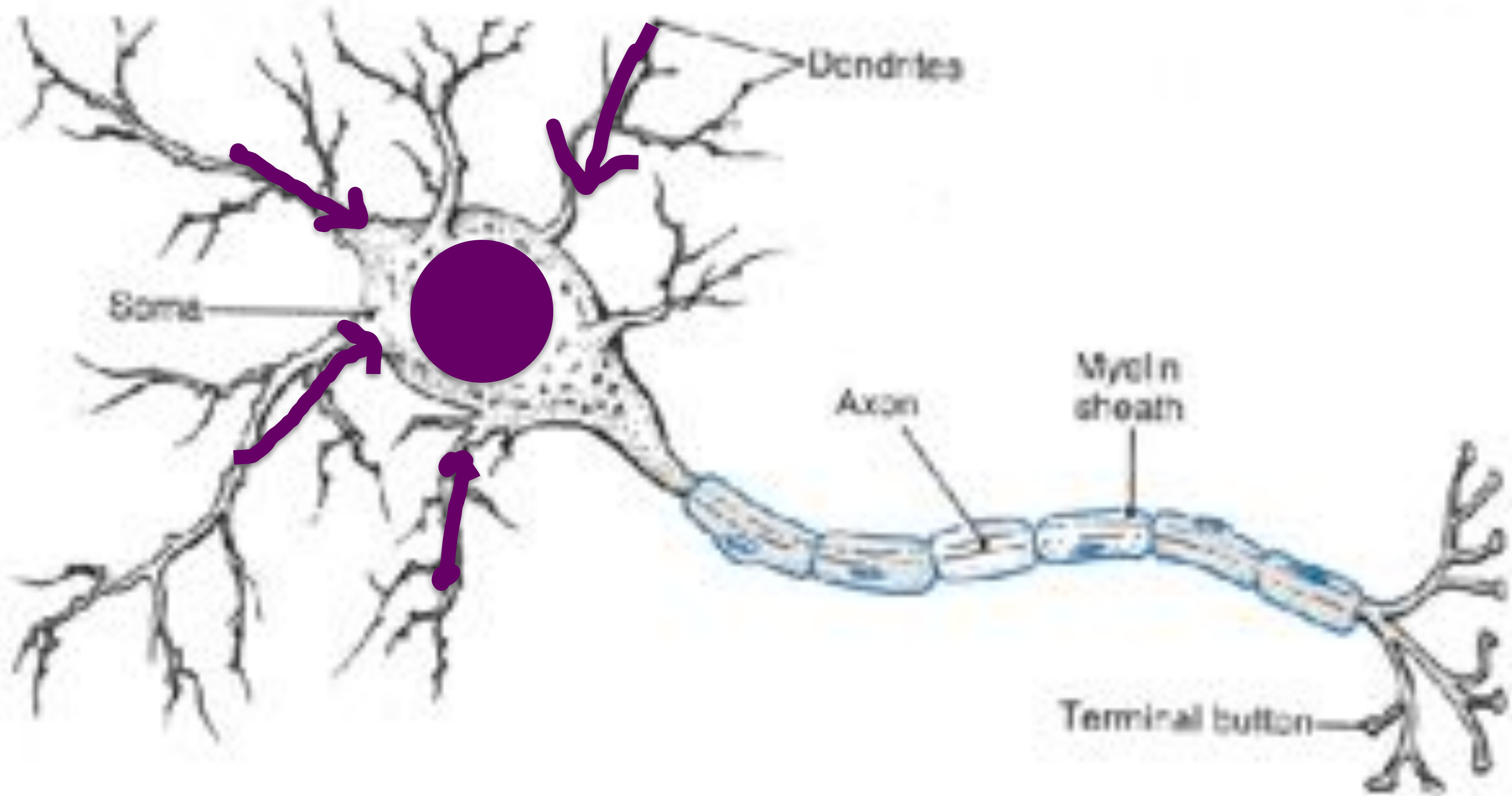
Neuron



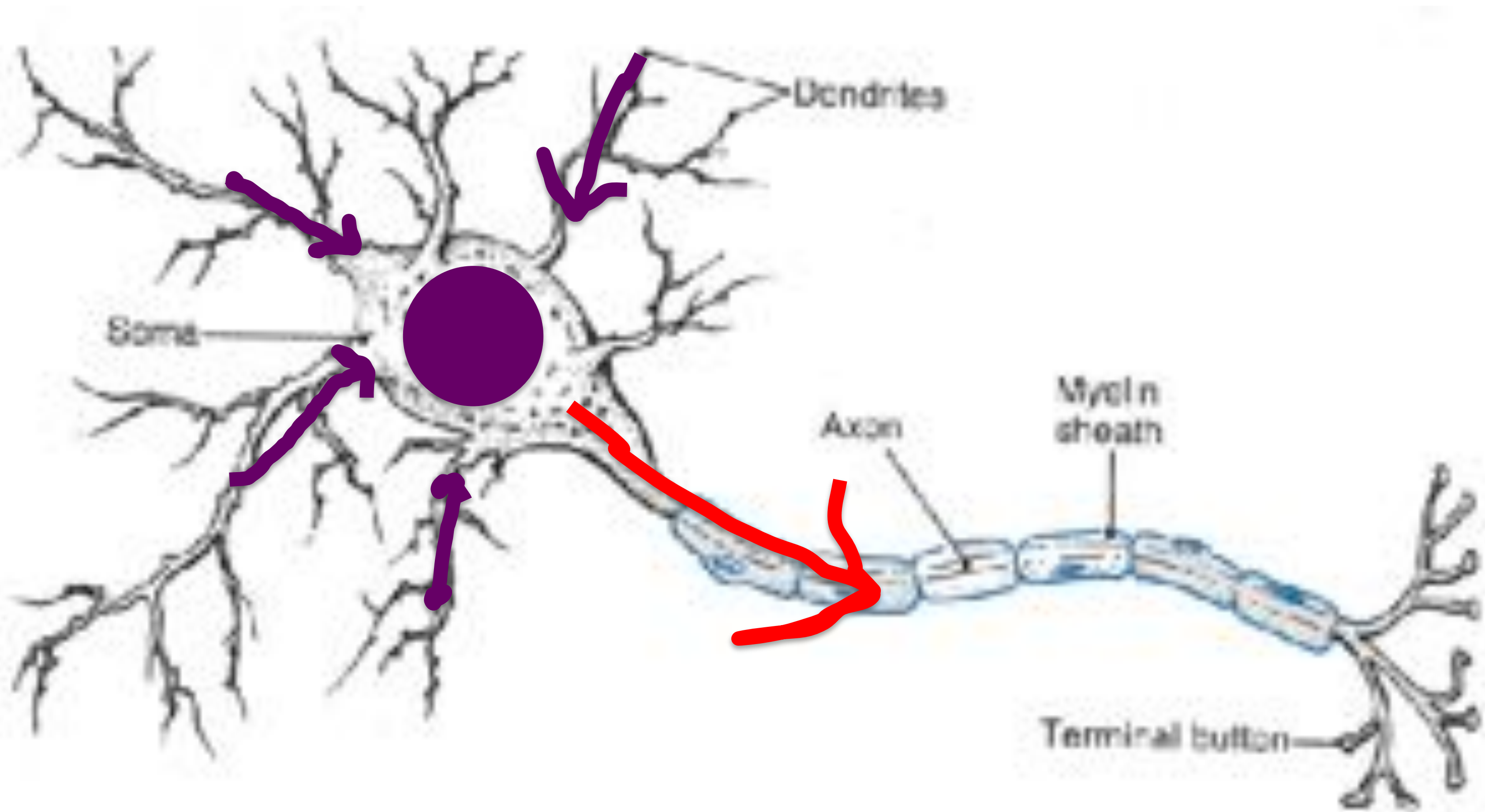
Neuron



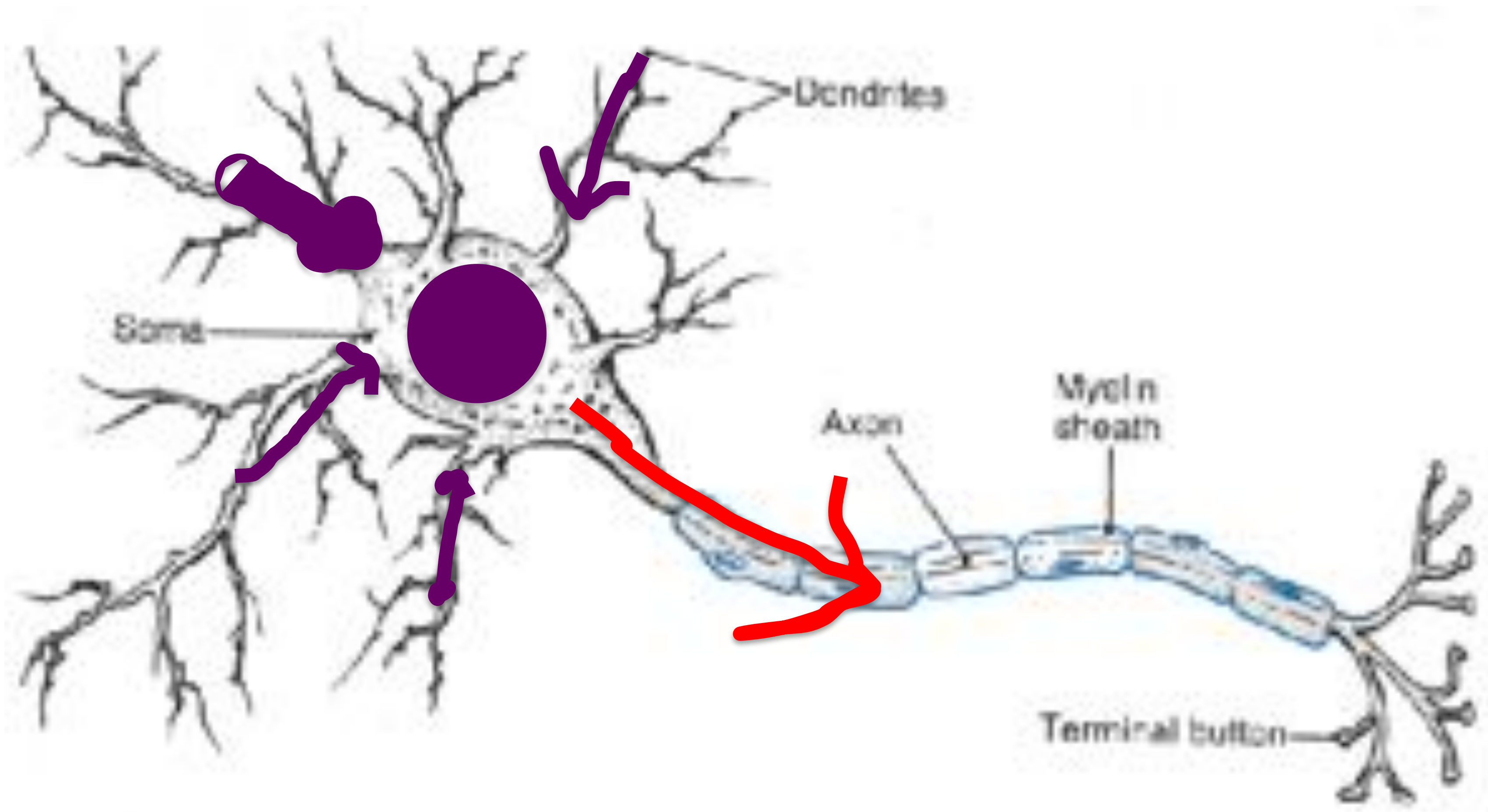
Neuron



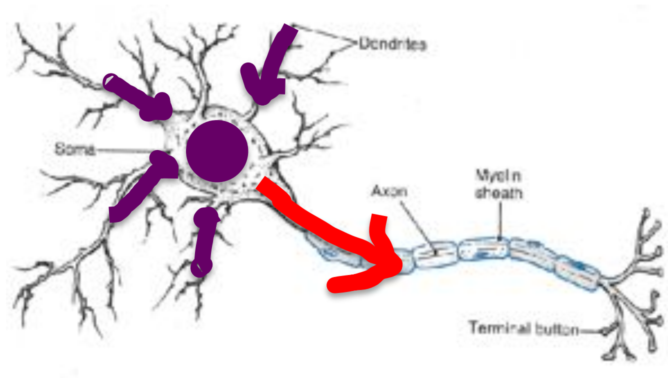
Neuron

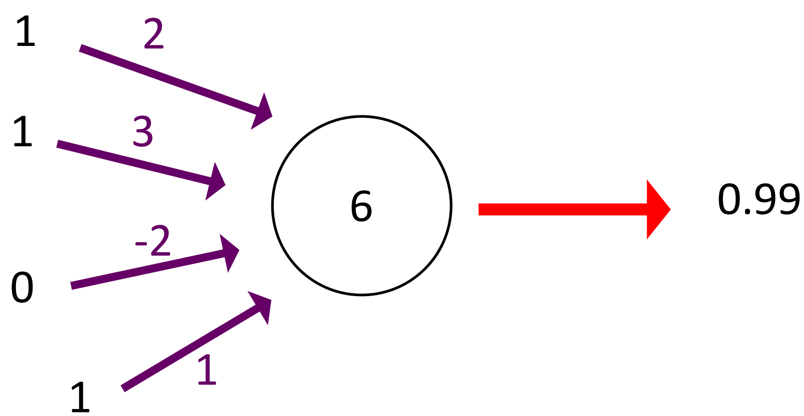


Some Inputs are More Important



Artificial Neuron



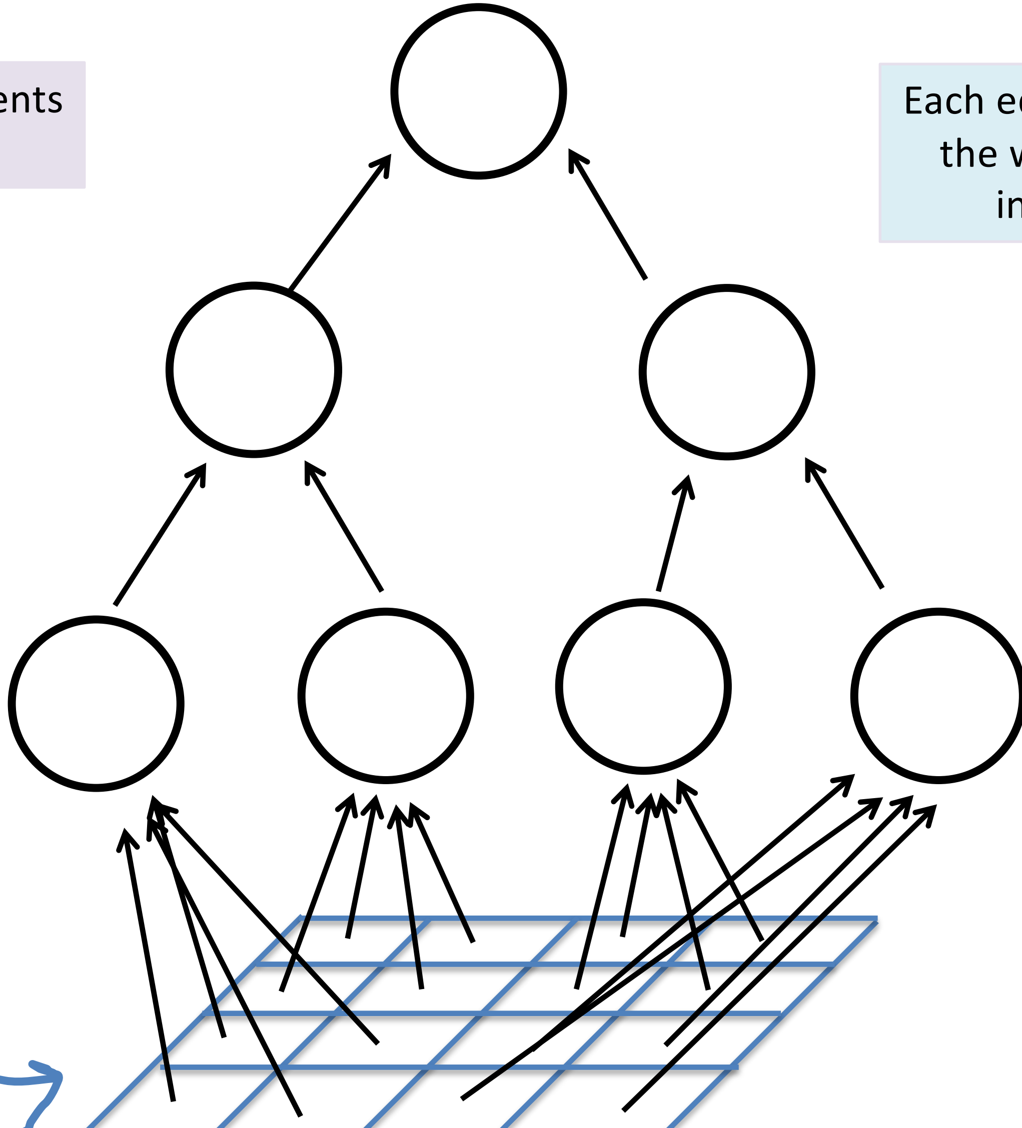


Neural Network

Each node represents a neuron

Each edge represents the weight of the interaction

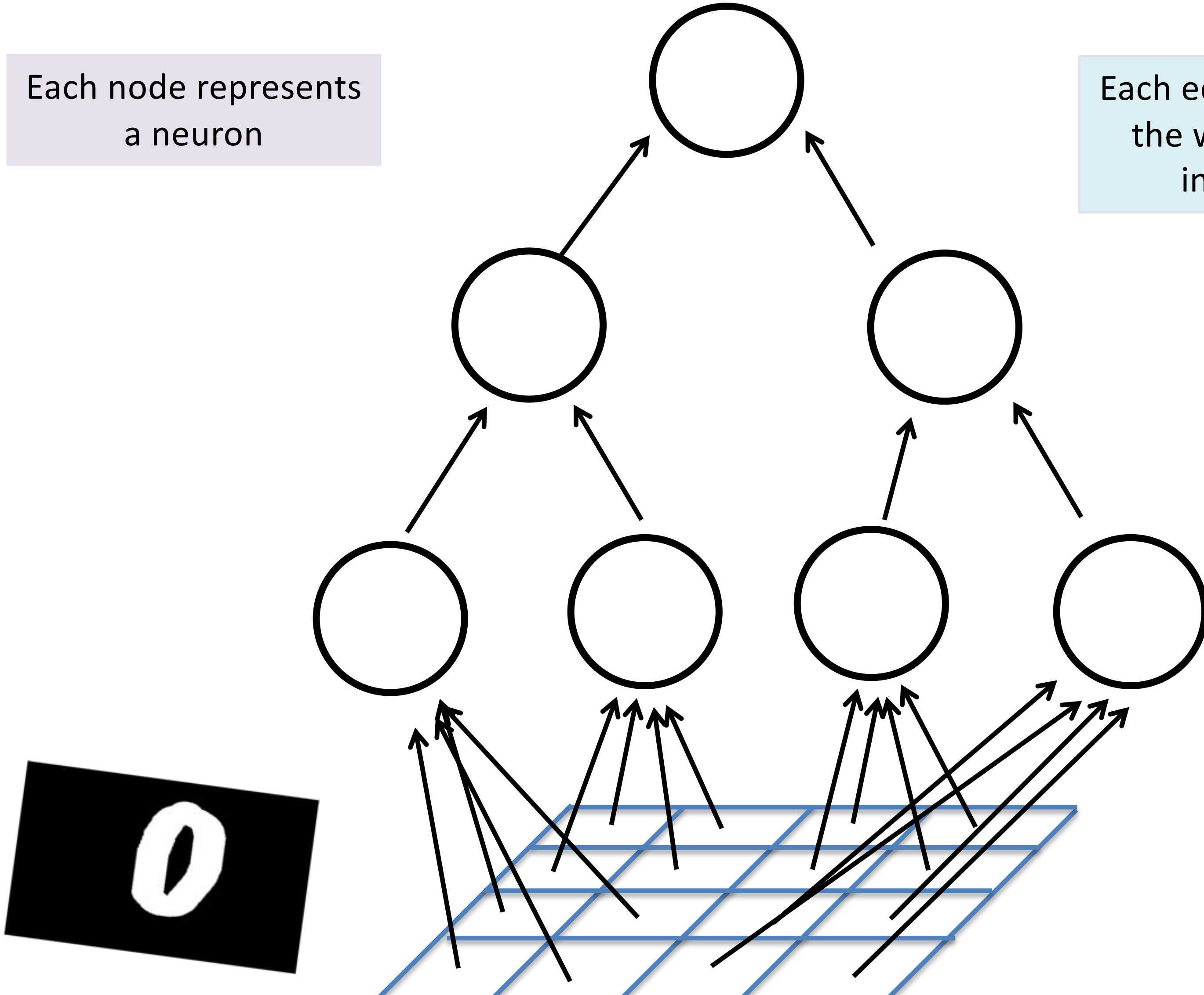
Pixels



Neural Network

Each node represents a neuron

Each edge represents the weight of the interaction



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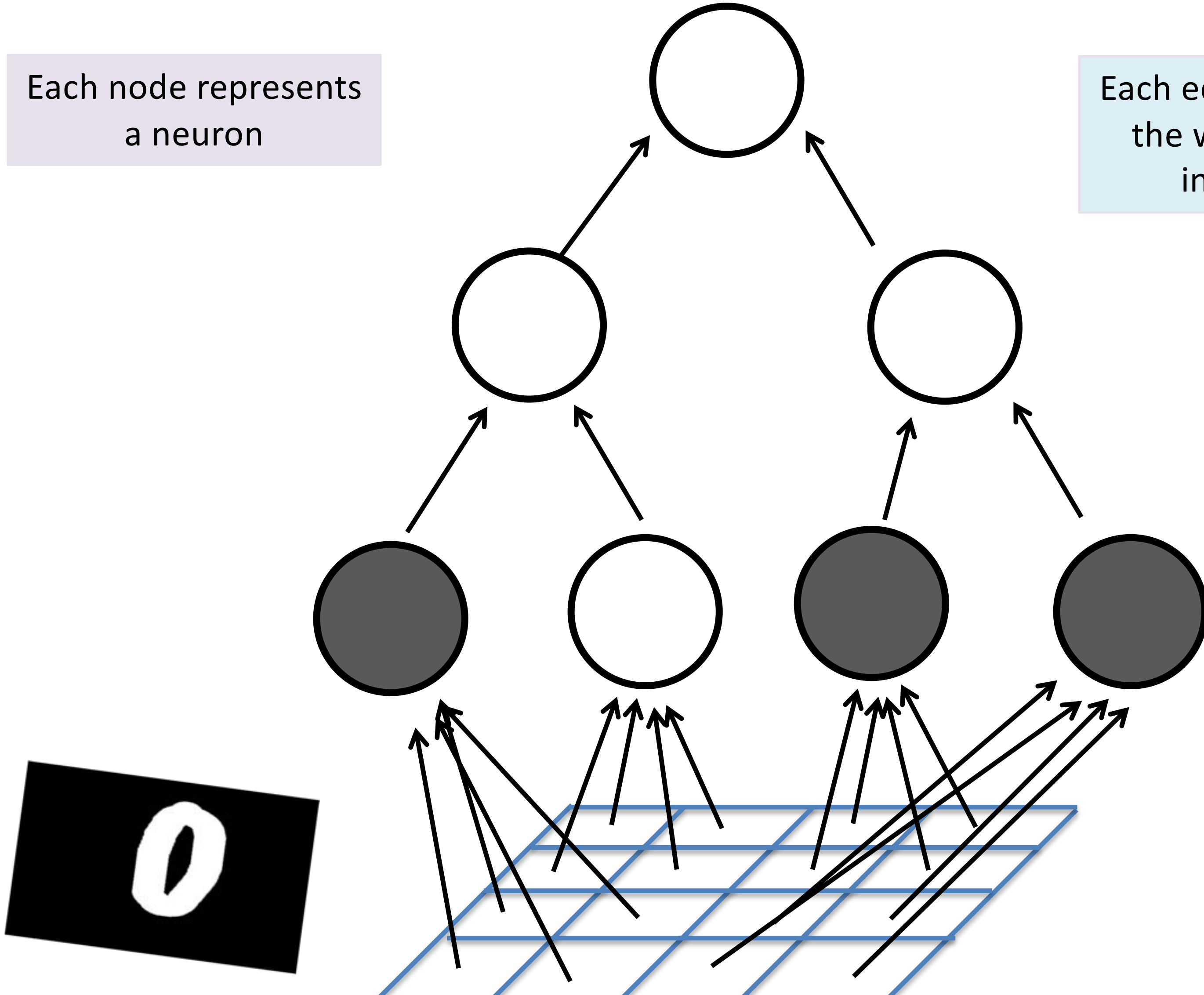
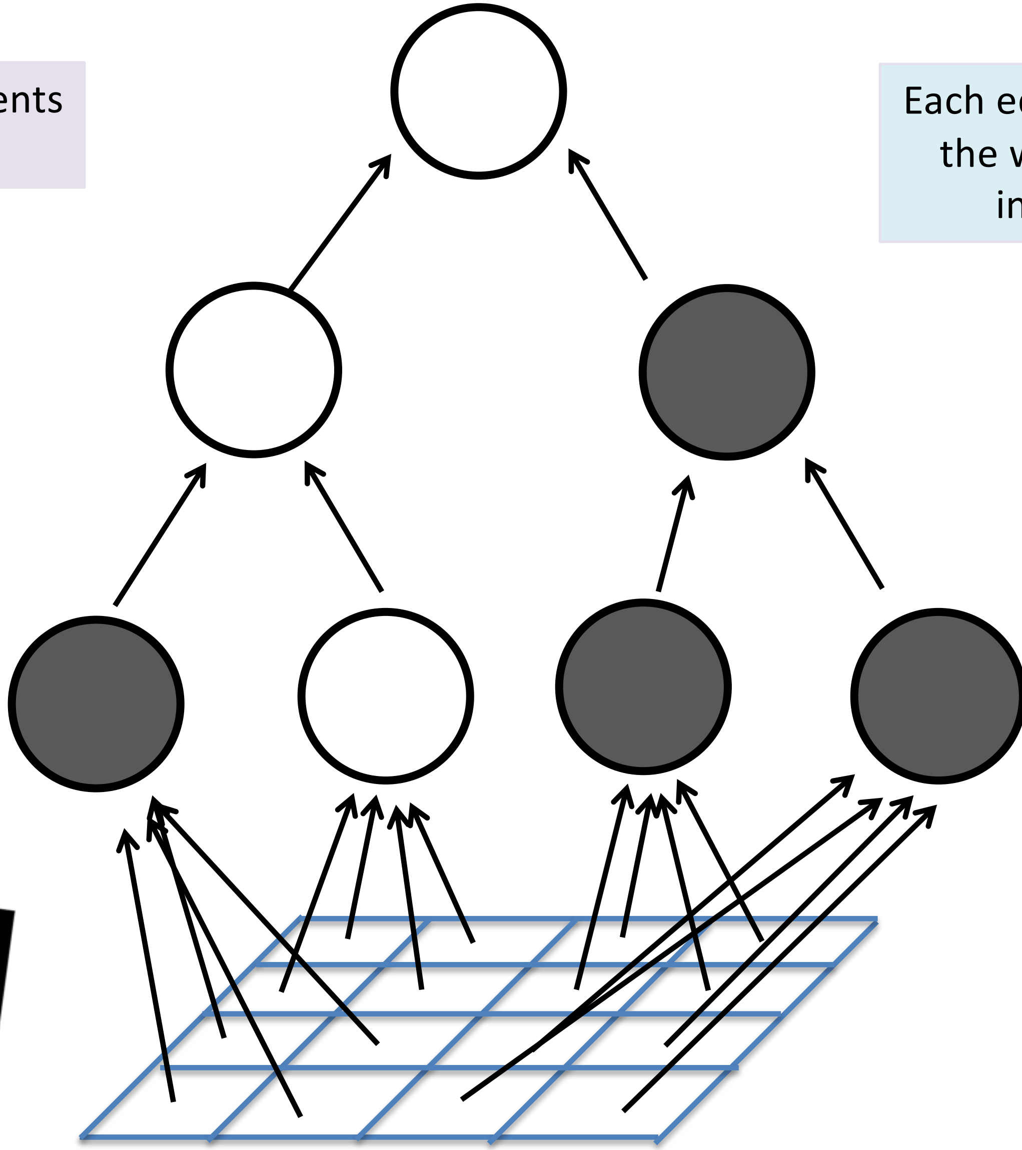
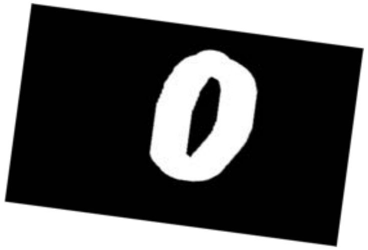


Neural Network

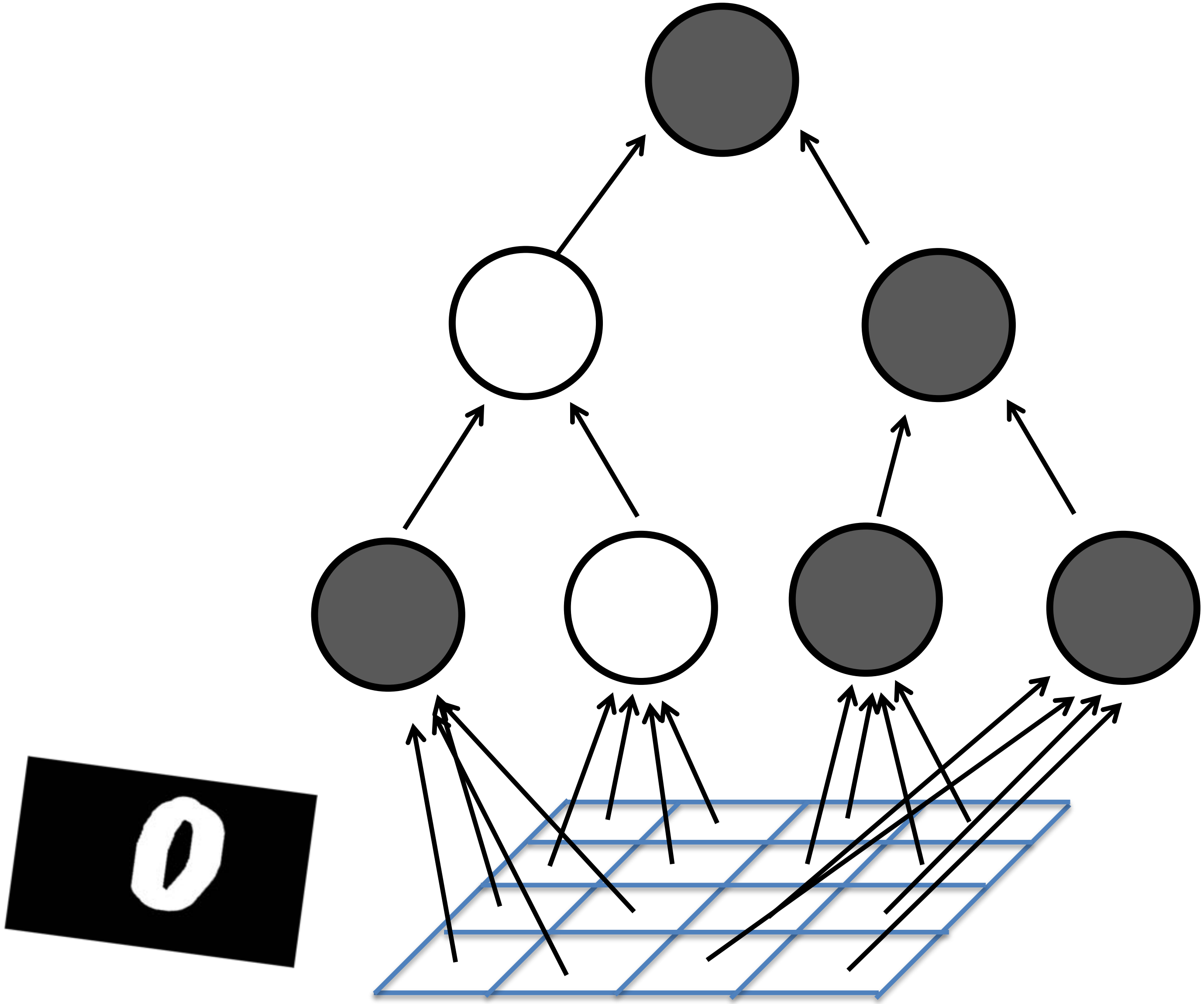
Each node represents a neuron

Each edge represents the weight of the interaction

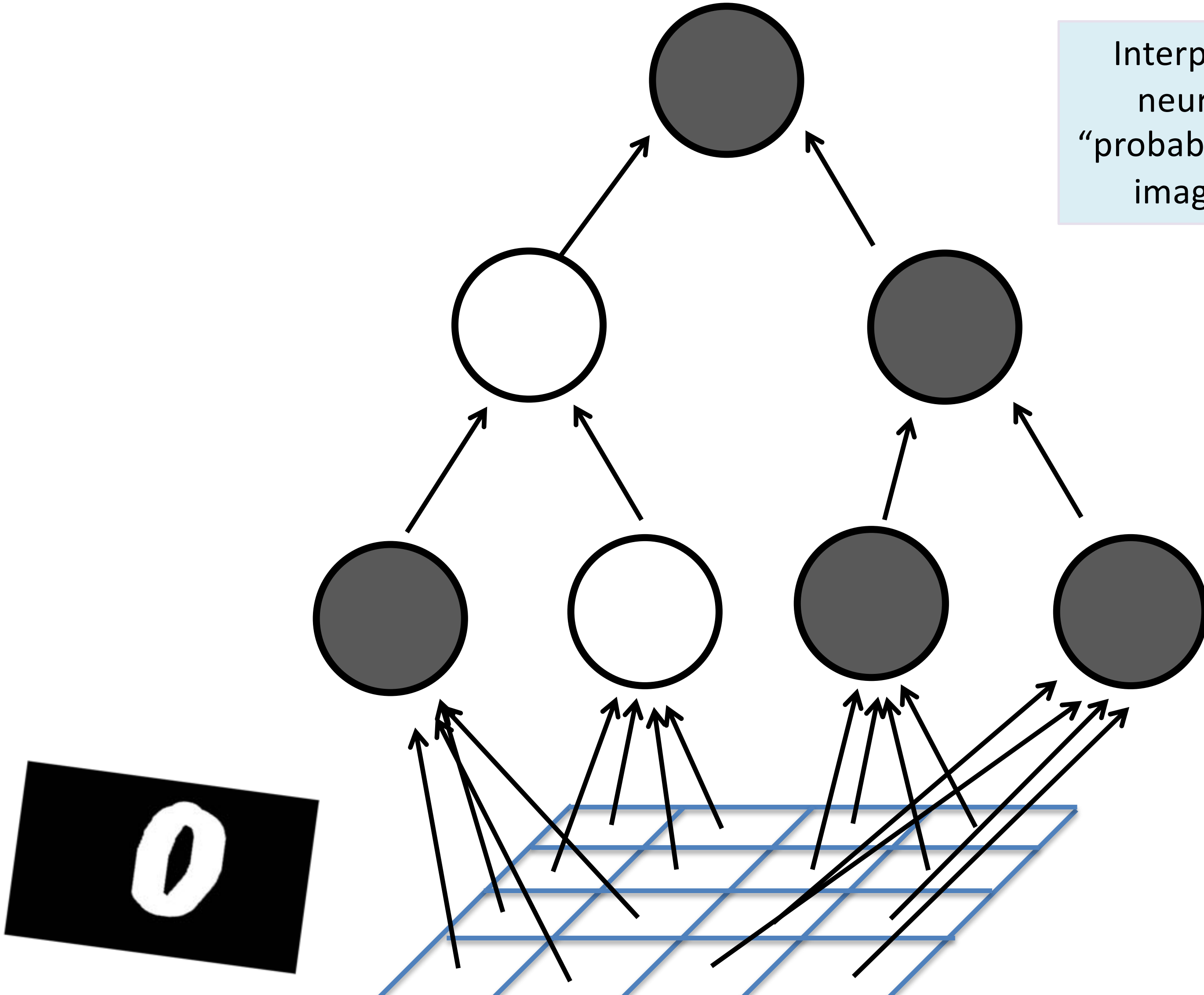




Neural Network

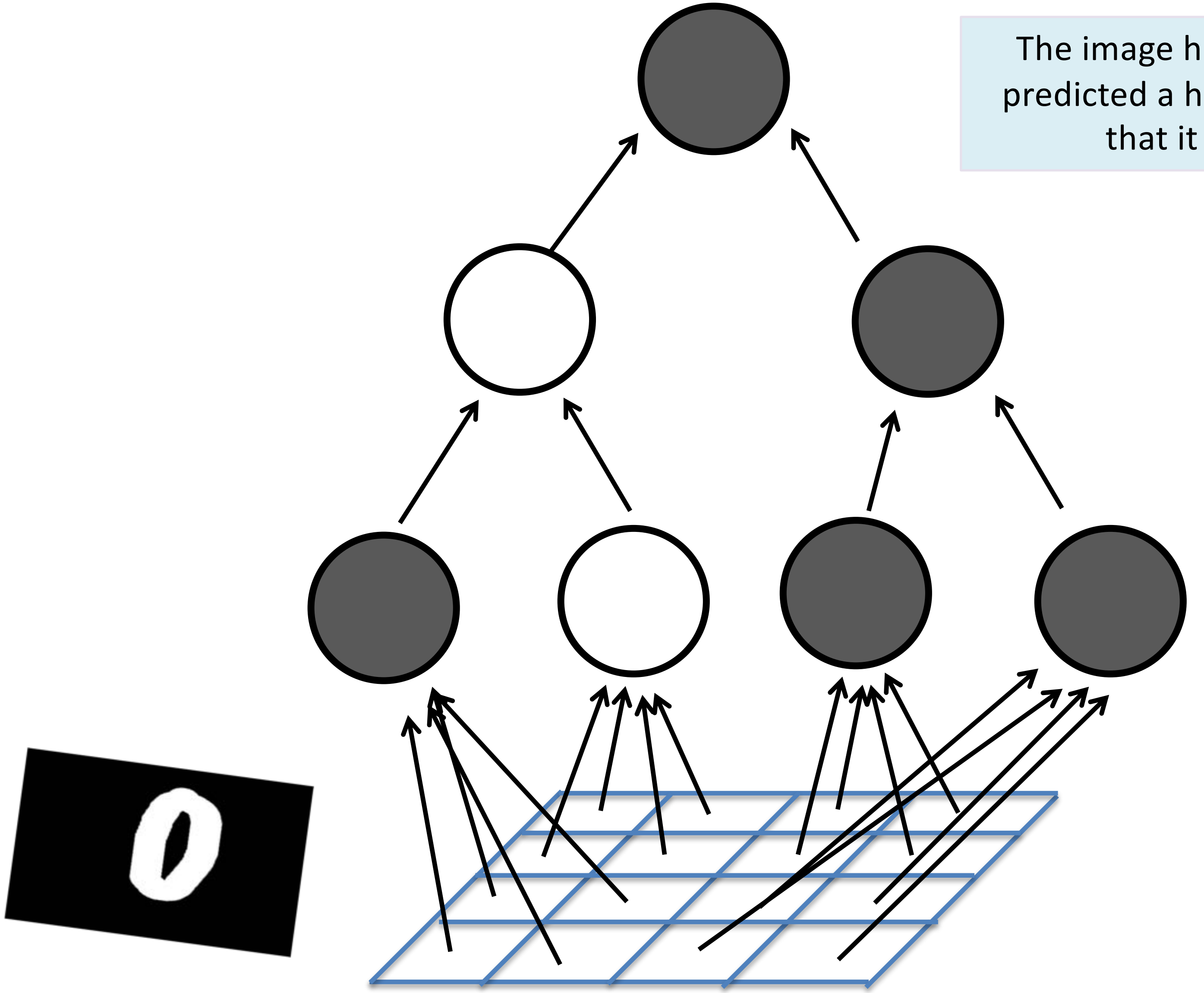


Neural Network



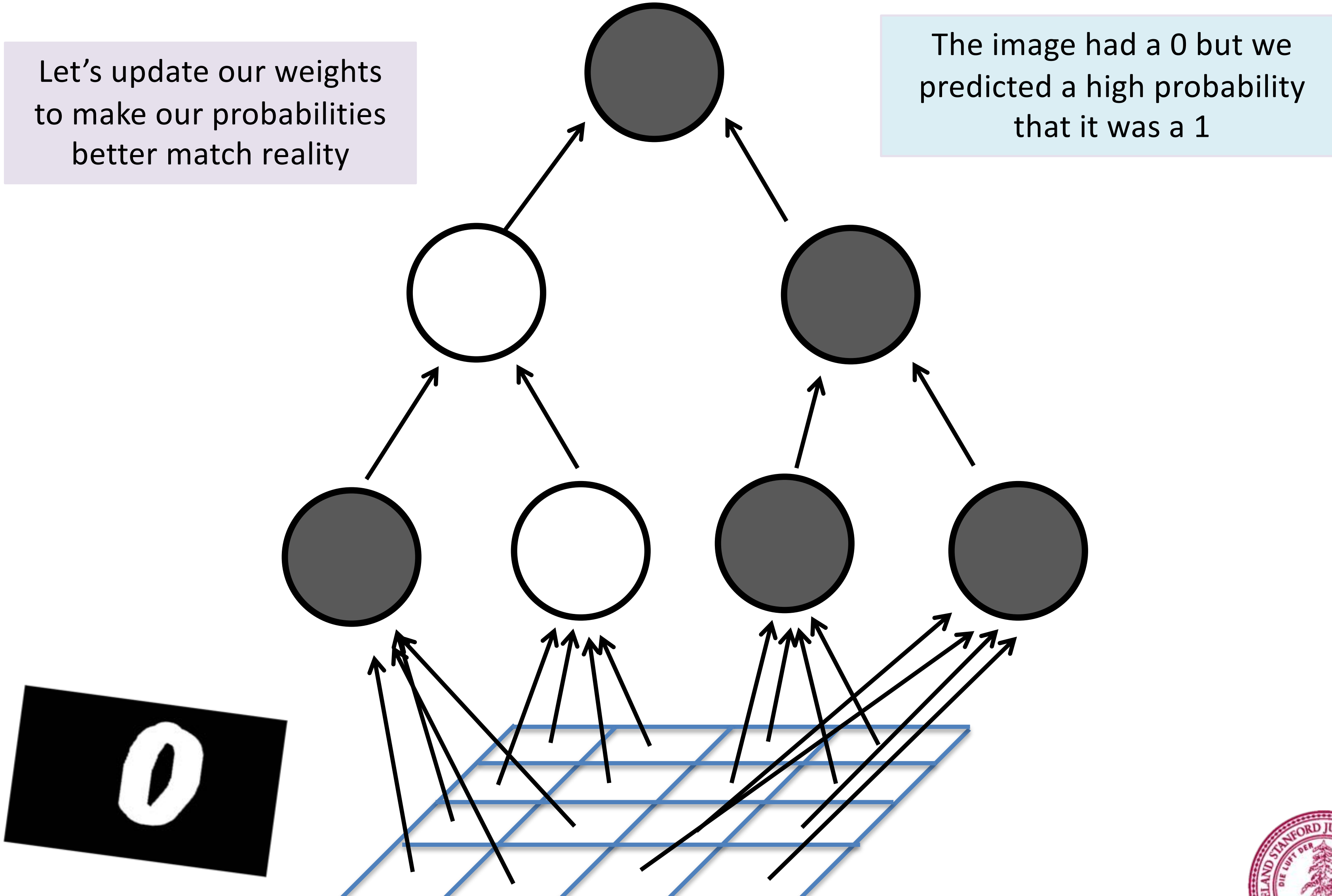
Interpret the last neuron as the "probability" that the image is of a 1

Neural Network



The image had a 0 but we predicted a high probability that it was a 1

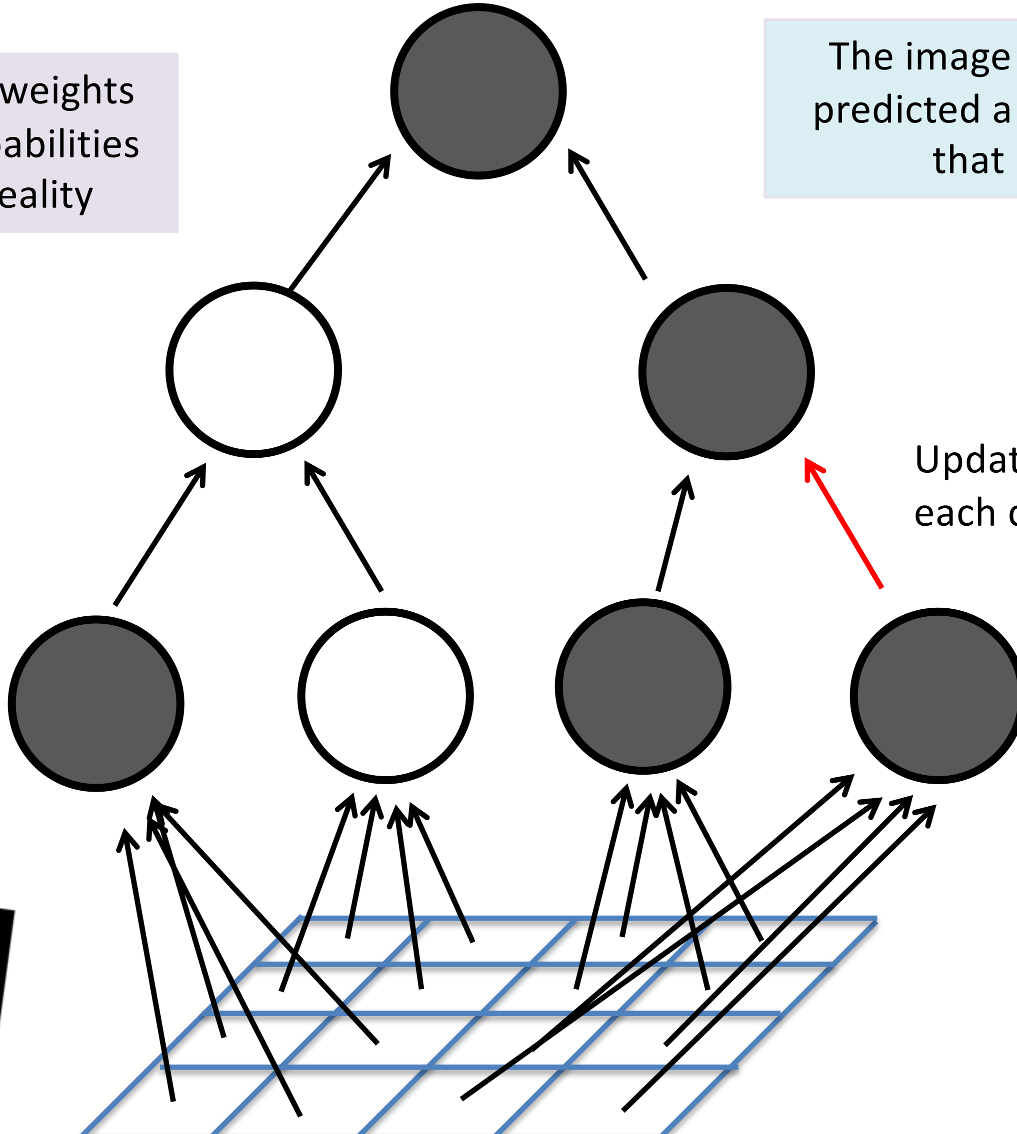
Neural Network



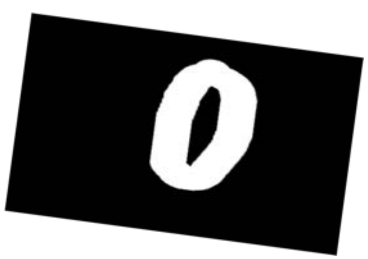
Neural Network

Let's update our weights to make our probabilities better match reality

The image had a 0 but we predicted a high probability that it was a 1



Update the parameter at each connection



Gradient of output layer params

$$\frac{\partial L}{\partial \theta_i^{(\hat{y})}} = \frac{\partial L}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$

$$\hat{y} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)$$

$$\frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)\left[1 - \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)\right] \cdot \frac{\partial}{\partial \theta_i^{(\hat{y})}} \sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}$$

$$= \hat{y}[1 - \hat{y}] \cdot \frac{\partial}{\partial \theta_i^{(\hat{y})}} \sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}$$

$$= \hat{y}[1 - \hat{y}] \cdot h_i$$

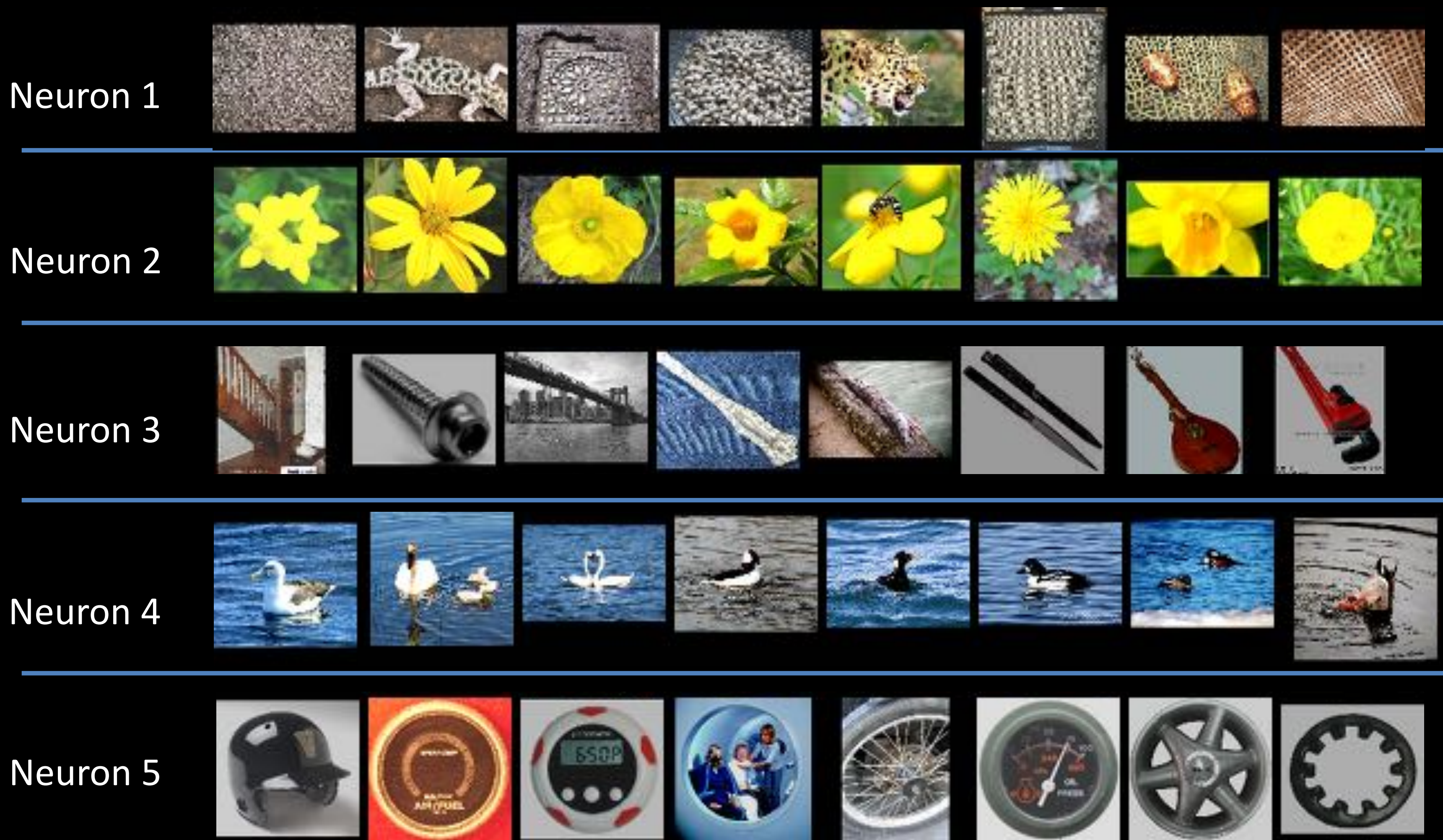
You will be able to do this.

Google Brain

Google Brain

1 Trillion Artificial Neurons

Other Neurons



Le, et al., *Building high-level features using large-scale unsupervised learning*. ICML 2012

A Neuron That Fires When It Sees Cats

Top stimuli from the test set

Optimal stimulus
by numerical optimization

Le, et al., *Building high-level features using large-scale unsupervised learning*. ICML 2012

ImageNet Classification

…
smoothhound, smoothhound shark, Mustelus mustelus
American smooth dogfish, Mustelus canis
Florida smoothhound, Mustelus norrisi
whitetip shark, reef whitetip shark, Triaenodon obseus
Atlantic spiny dogfish, Squalus acanthias
Pacific spiny dogfish, Squalus suckleyi
hammerhead, hammerhead shark
smooth hammerhead, Sphyrna zygaena
smalleye hammerhead, Sphyrna tudes
shovelhead, bonnethead, bonnet shark, Sphyrna tiburo
angel shark, angelfish, Squatina squatina, monkfish
electric ray, crampfish, numbfish, torpedo
smalltooth sawfish, Pristis pectinatus
guitarfish
roughtail stingray, Dasyatis centroura
butterfly ray
eagle ray
spotted eagle ray, spotted ray, Aetobatus narinari
cownose ray, cow-nosed ray, Rhinoptera bonasus
manta, manta ray, devilfish
Atlantic manta, Manta birostris
devil ray, Mobula hypostoma
grey skate, gray skate, Raja batis
little skate, Raja erinacea
…

Stingray

Mantaray

ImageNet Classification

0.005%	1.5%	?
Random guess	Pre Neural Networks	GoogLeNet

Le, et al., *Building high-level features using large-scale unsupervised learning*. ICML 2012

ImageNet Classification

0.005%	1.5%	43.9%
Random guess	Pre Neural Networks	GoogLeNet

Szegedy et al, Going Deeper With Convolutions, CVPR 2015

ImageNet Classification

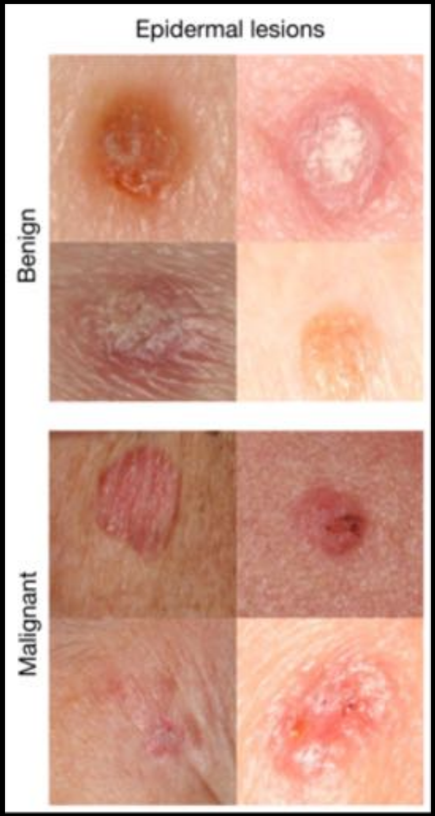
0.005%
Random guess

1.5%
Pre Neural Networks

82.7%
NASNet

https://arxiv.org/pdf/1707.07012.pdf

Where is this useful?



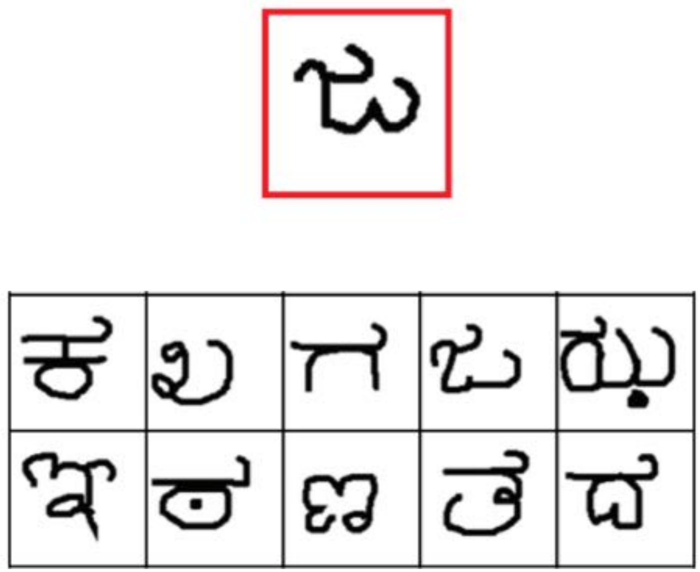
A machine learning algorithm performs **better than** the best dermatologists.

Developed this year, at Stanford.

Esteva, Andre, et al. "Dermatologist-level classification of skin cancer with deep neural networks." *Nature* 542.7639 (2017): 115-118.

Open Problem: One Shot Learning

B Lake, R Salakhutdinov, J Tenenbaum. Science 2015.
Human-level concept learning through probabilistic program induction.



Current deep learning methods are not enough to move the needle as far as we want, **especially on socially relevant problems** that often do not have the benefit of massive public datasets. The best new ideas are coming from probability theory

Prediction: The person who solves one shot learning problem will use core probability

Closest thing to magic you can learn.
Now is the time, Stanford is the place.