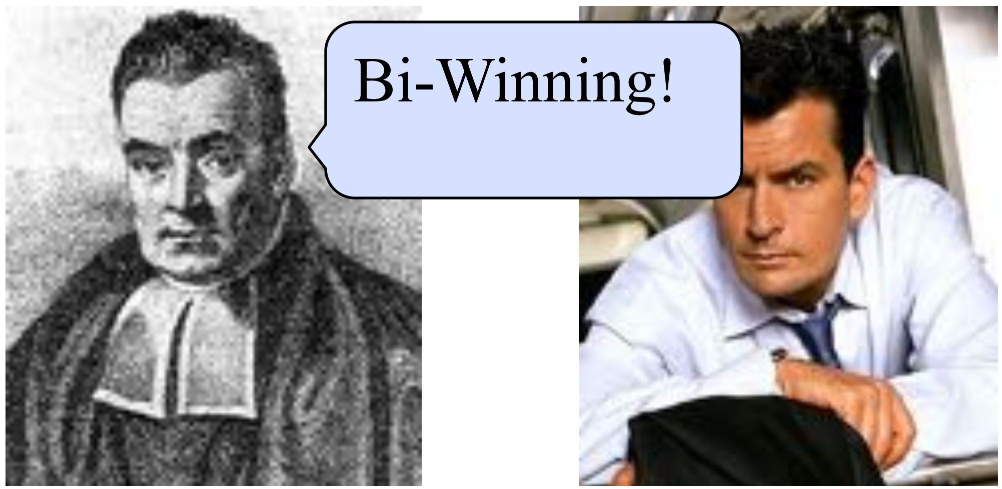
Thomas Bayes

- Rev. Thomas Bayes (1702 –1761) was a British mathematician and Presbyterian minister



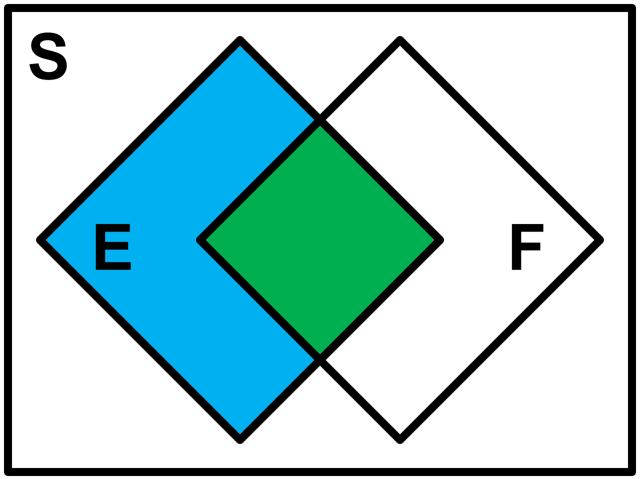
- He looked remarkably similar to Charlie Sheen
  - But that's not important right now...

But First!

Background Observation

- Say E and F are events in S

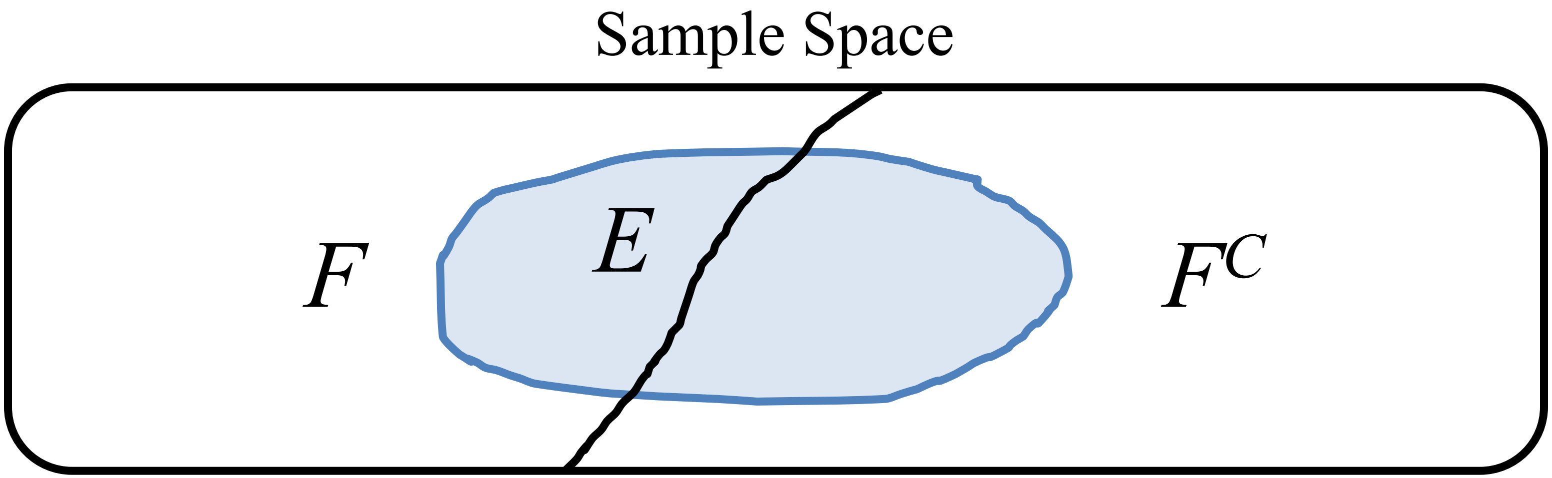
$$E = EF \cup EF^c$$



Note: $EF \cap EF^c = \emptyset$

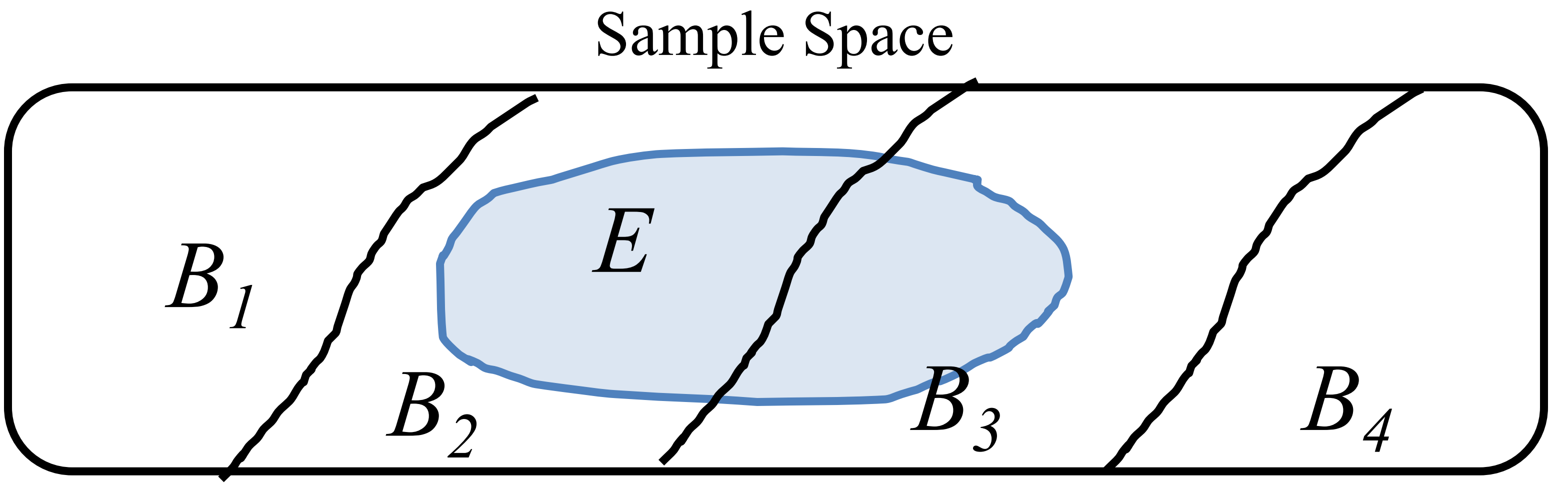
So, $P(E) = P(EF) + P(EF^c)$

Law of Total Probability



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability



$$P(E) = \sum_i P(B_i \cap E)$$
$$= \sum_i P(E|B_i)P(B_i)$$

Moment of Silence…

Bayes Theorem

$\mathrm{P}(\, F \mid E \,)$

I want to calculate
P(State of the world F | Observation E)
It seems so tricky!…

The other way around is easy
P(Observation E | State of the world F)
What options to I have, chief?

$\mathrm{P}(\, E \mid F \,)$

Bayes Theorem

Want $P(F \mid E)$. Know $P(E \mid F)$

$$P(F|E) = \frac{P(EF)}{P(E)}$$ Def. of Conditional Prob.

A little while later…

$$= \frac{P(E|F)P(F)}{P(E)}$$ Chain Rule

Bayes Theorem

- Most common form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

- Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

H1N1 Testing

- A test is 98% effective at detecting H1N1
  - However, test has a "false positive" rate of 1%
  - 0.5% of US population has H1N1
  - Let E = you test positive for H1N1 with this test
  - Let F = you actually have H1N1
  - What is P(F | E)?
- Solution:

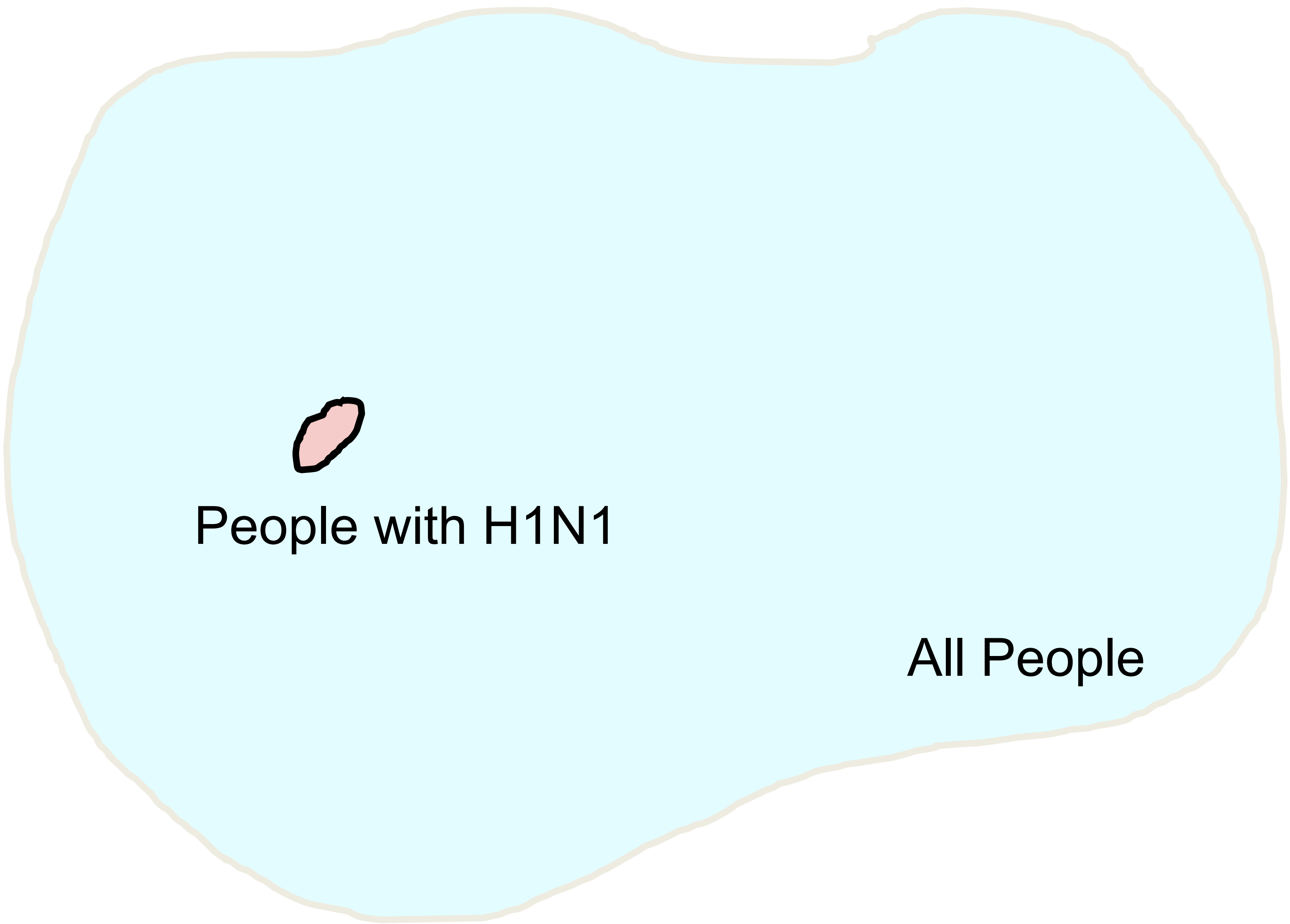
$$P(F \mid E) = \frac{P(E \mid F)\, P(F)}{P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)}$$

$$P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx 0.330$$

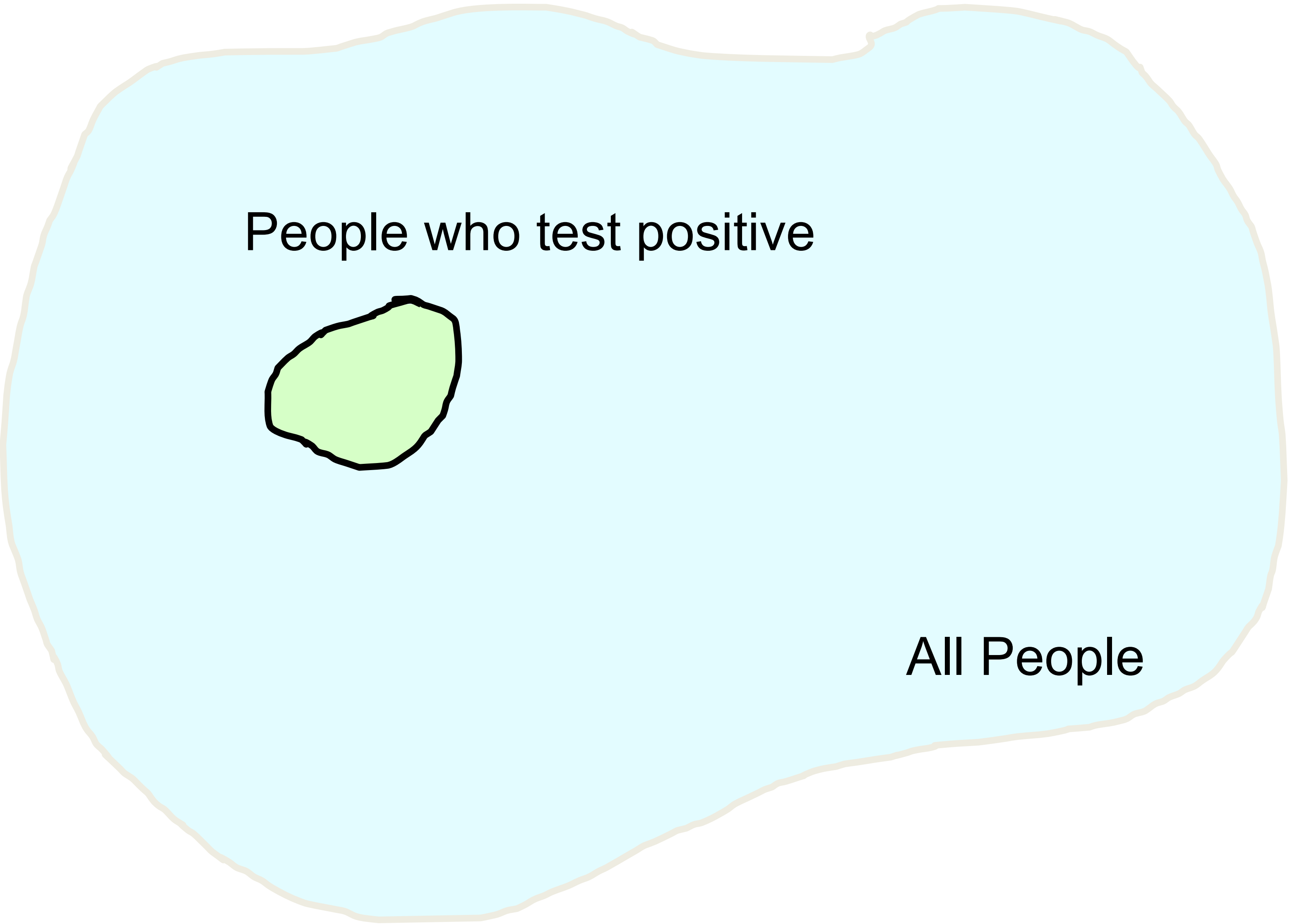
Intuition Time

Bayes Theorem Intuition

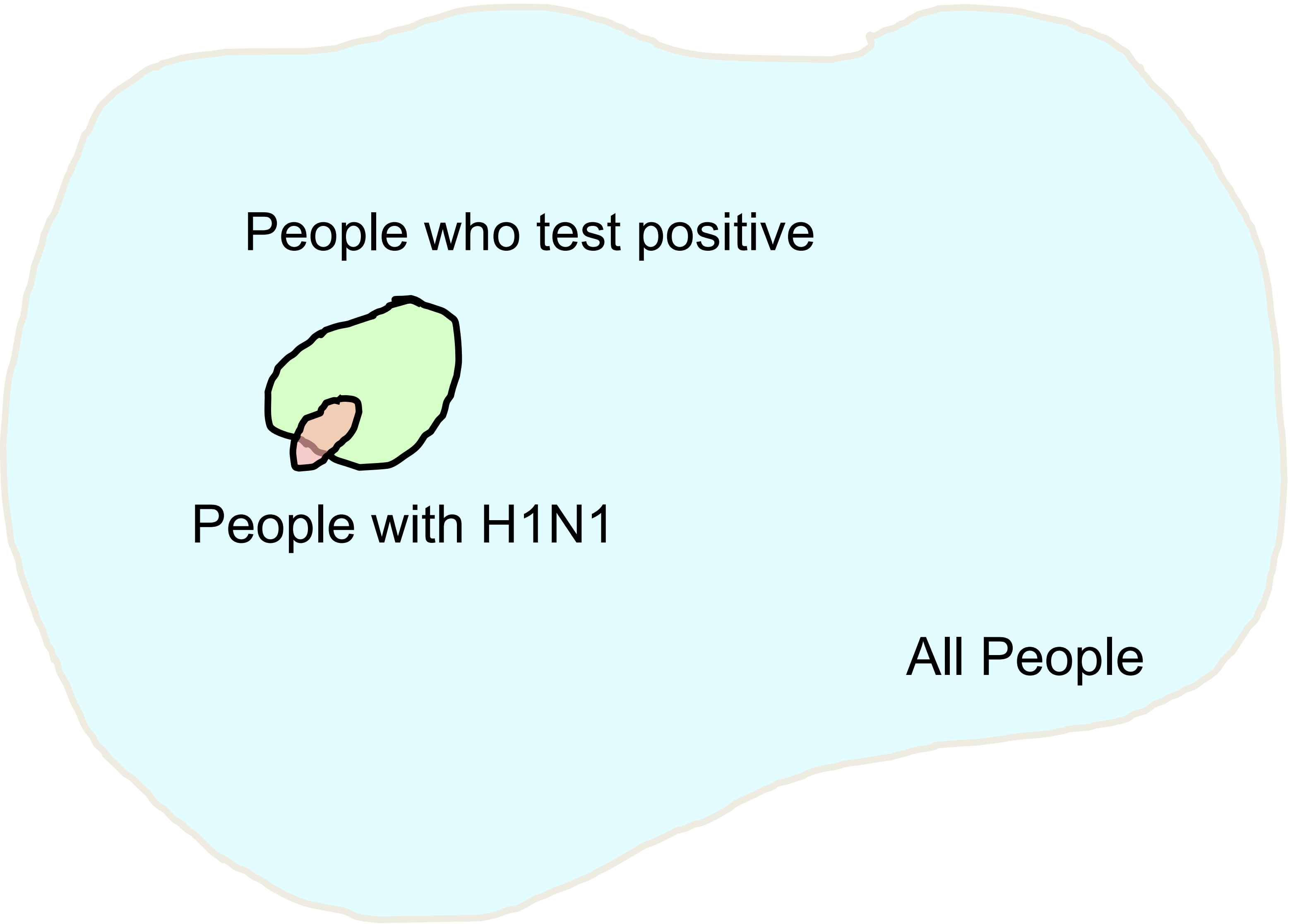
Bayes Theorem Intuition



Bayes Theorem Intuition

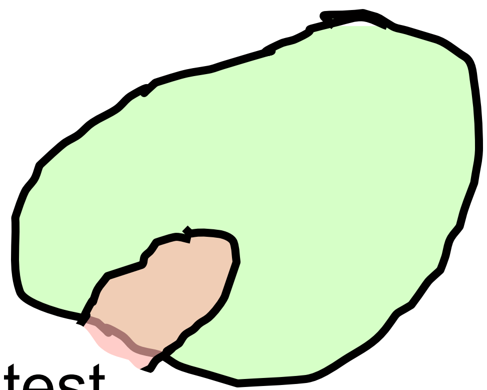


Bayes Theorem Intuition



Bayes Theorem Intuition

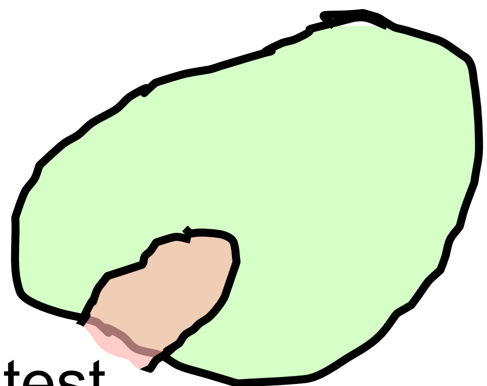
Conditioning on a positive result changes the sample space to this:



≈ 0.330

Bayes Theorem Intuition

Conditioning on a positive result changes the sample space to this:



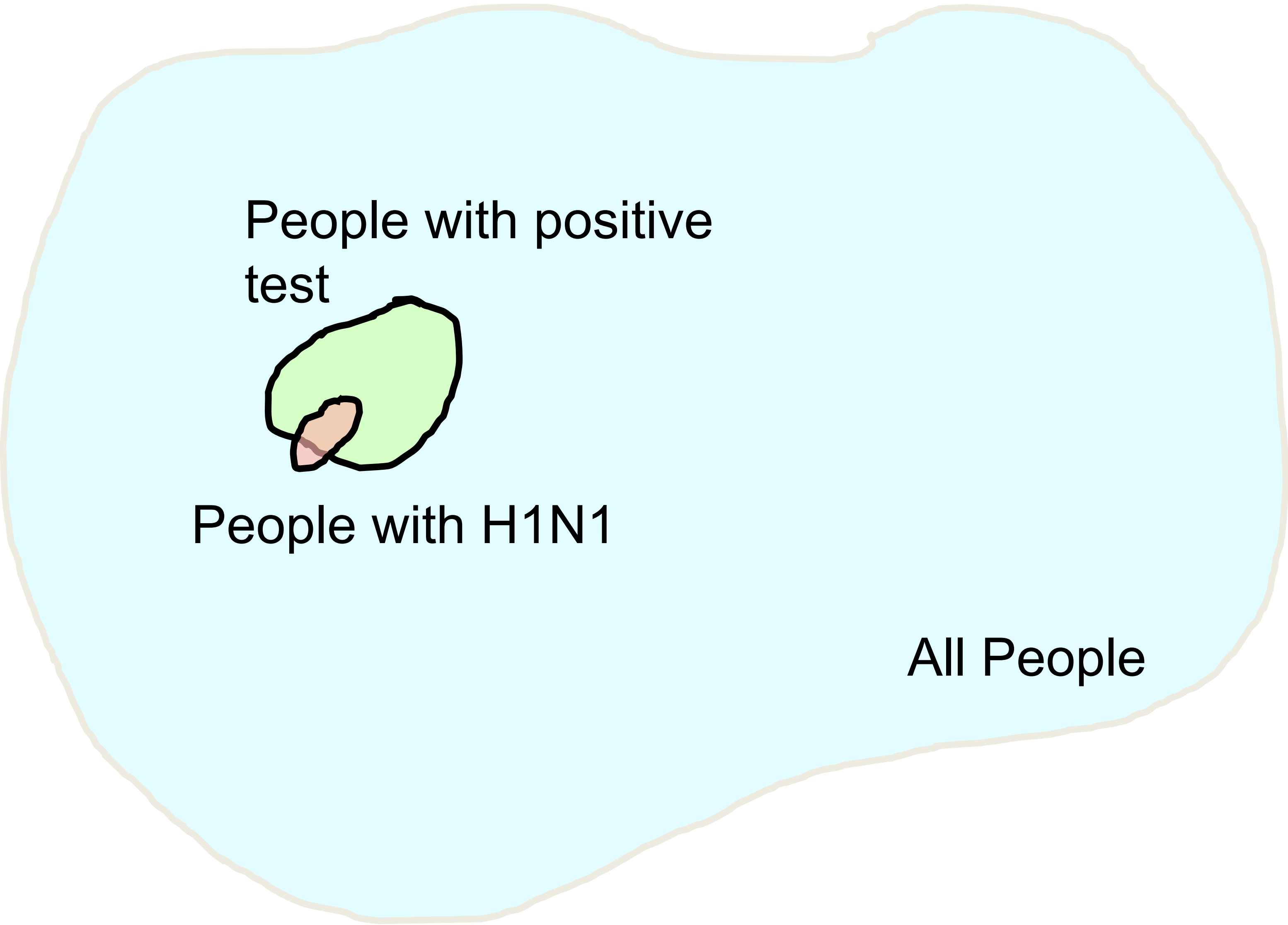
People who test positive

$$P(F)P(E|F) + P(F^c)P(E|F^c)$$

People who test positive and have H1N1 $P(F)P(E|F)$

≈ 0.330

Bayes Theorem Intuition



Bayes Theorem Intuition

Say we have 1000 people:

5 have H1N1 and test positive, 985 **do not** have H1N1 and test negative.
10 **do not** have H1N1 and test positive.

≈ 0.333

Why It's Still Good to get Tested

	H1N1 +	H1N1 –
Test +	$0.98 = P(E \mid F)$	$0.01 = P(E \mid F^c)$
Test –	$0.02 = P(E^c \mid F)$	$0.99 = P(E^c \mid F^c)$

- Let E^c = you test <u>negative</u> for H1N1 with this test
- Let F = you actually have H1N1
- What is $P(F \mid E^c)$?

$$P(F \mid E^c) = \frac{P(E^c \mid F)\,P(F)}{P(E^c \mid F)\,P(F) + P(E^c \mid F^c)\,P(F^c)}$$

$$P(F \mid E^c) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \approx 0.0001$$

Slicing Up Spam

In 2010 88% of email was spam

Simple Spam Detection

- Say 60% of all email is spam
  - 90% of spam has a forged header
  - 20% of non-spam has a forged header
  - Let E = message contains a forged header
  - Let F = message is spam
  - What is $P(F \mid E)$?
- Solution:

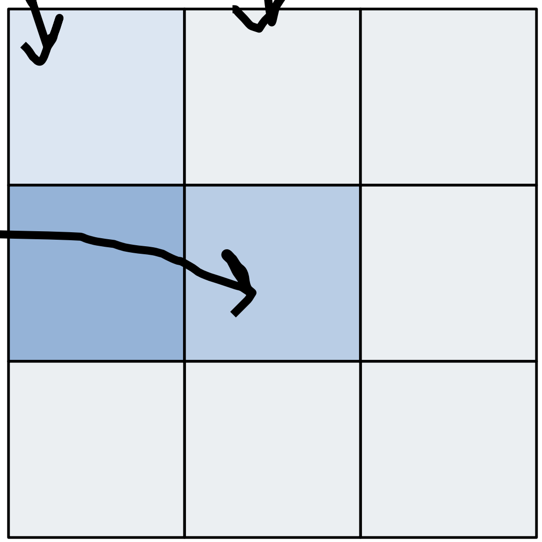
$$P(F \mid E) = \frac{P(E \mid F)\,\mathrm{P}(F)}{P(E \mid F)\,\mathrm{P}(F) + \mathrm{P}(E \mid F^c)\,P(F^c)}$$

$$P(F \mid E) = \frac{(0.9)(0.6)}{(0.9)(0.6) + (0.2)(0.4)} \approx 0.871$$

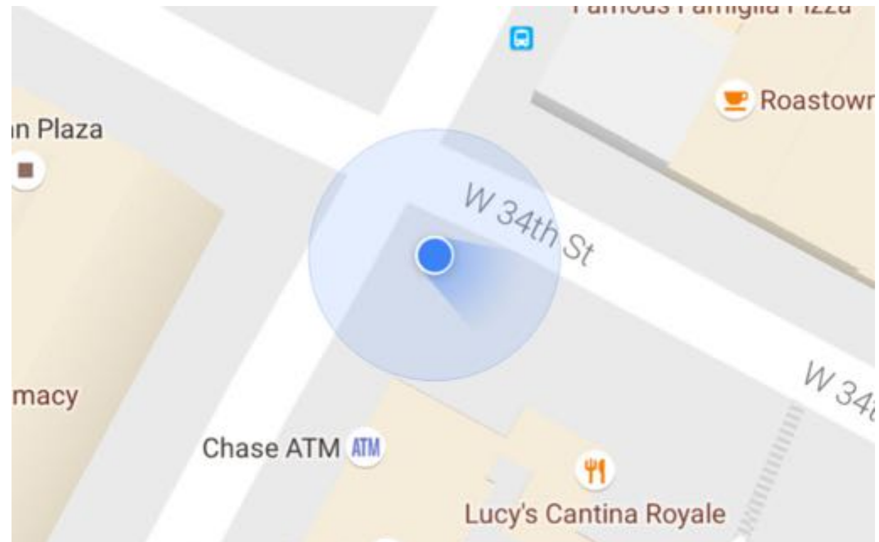
Update Belief

$P(L_1)$ $P(L_2)$

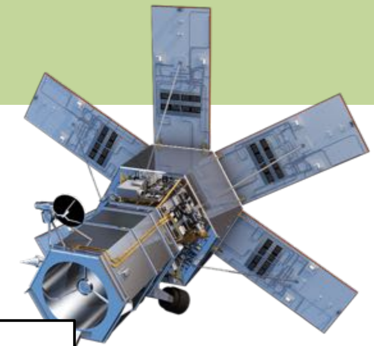
$P(L_5)$



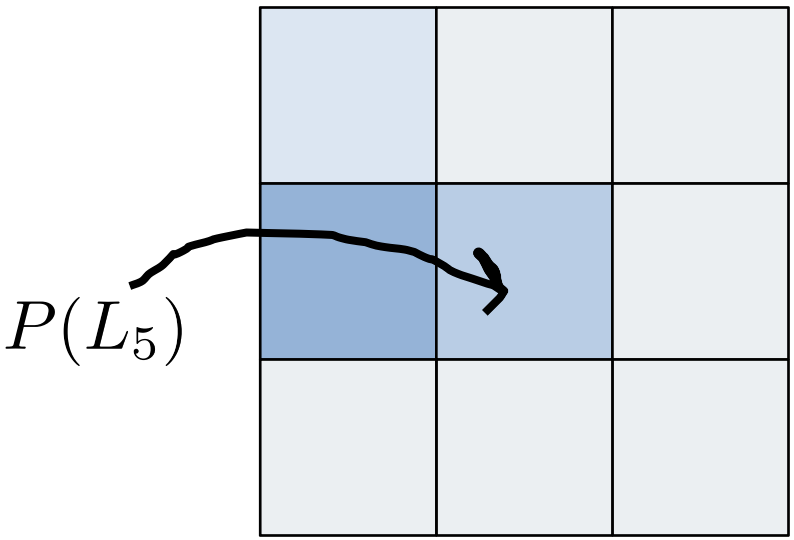
Before Observation



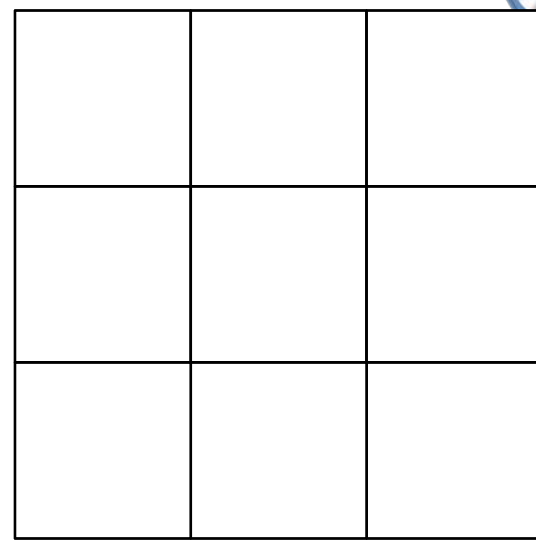
Update Belief



Know: $P(O|L_i)$

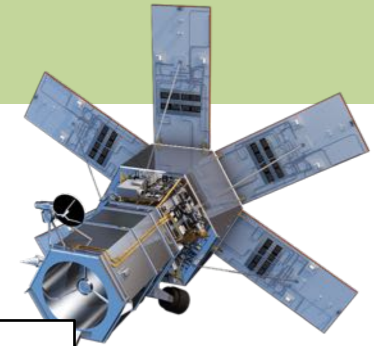


Before Observation



After Observation

Update Belief



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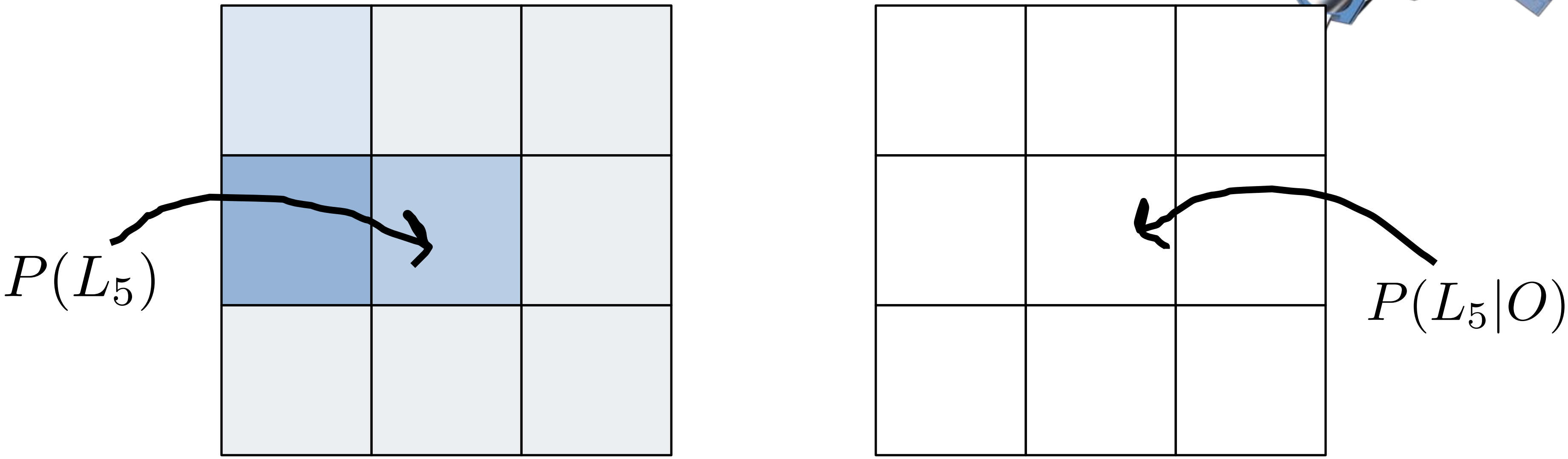
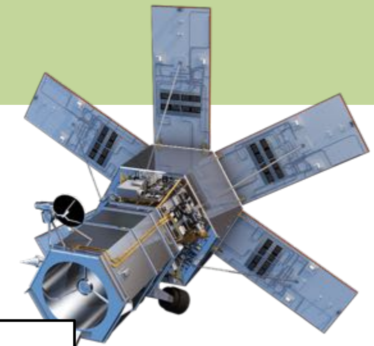


Before Observation

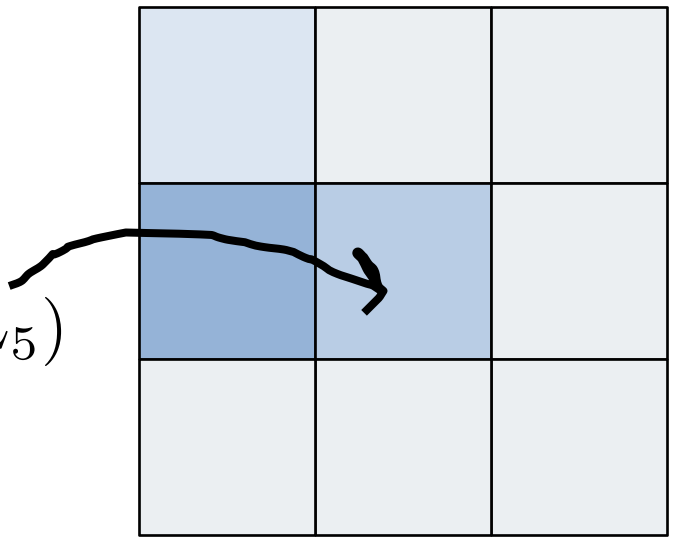
After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{P(O)}$$

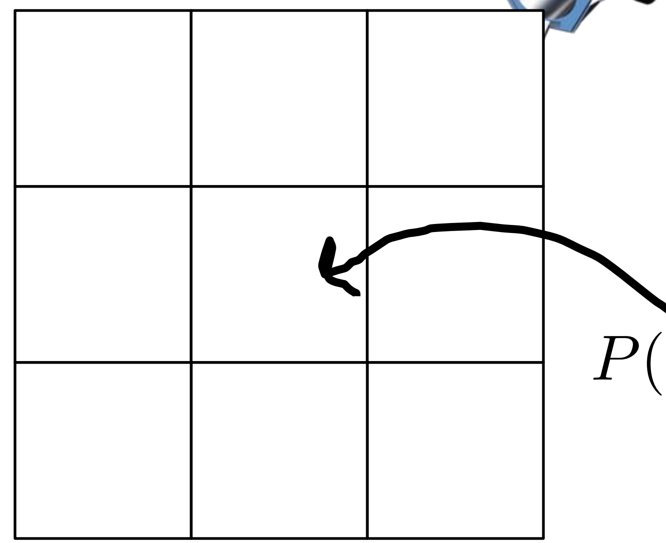
Update Belief



$P(L_5)$



Before Observation

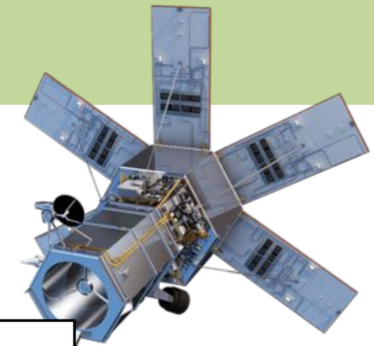


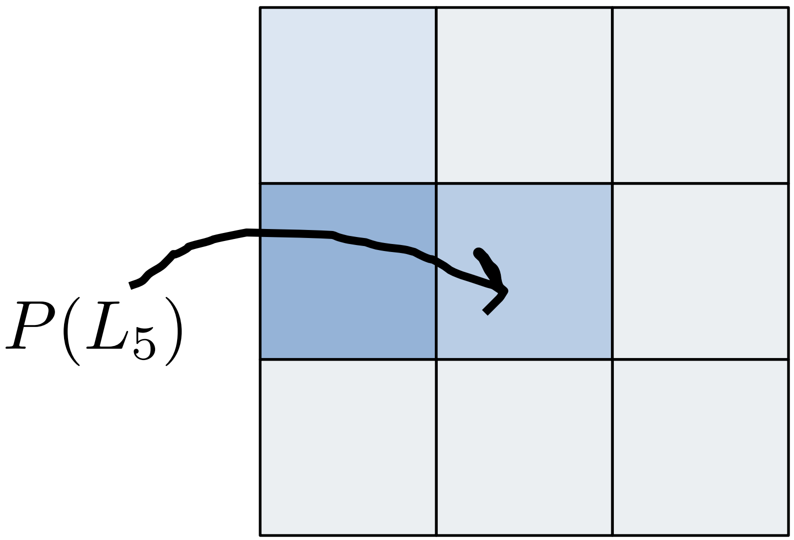
$P(L_5|O)$

After Observation

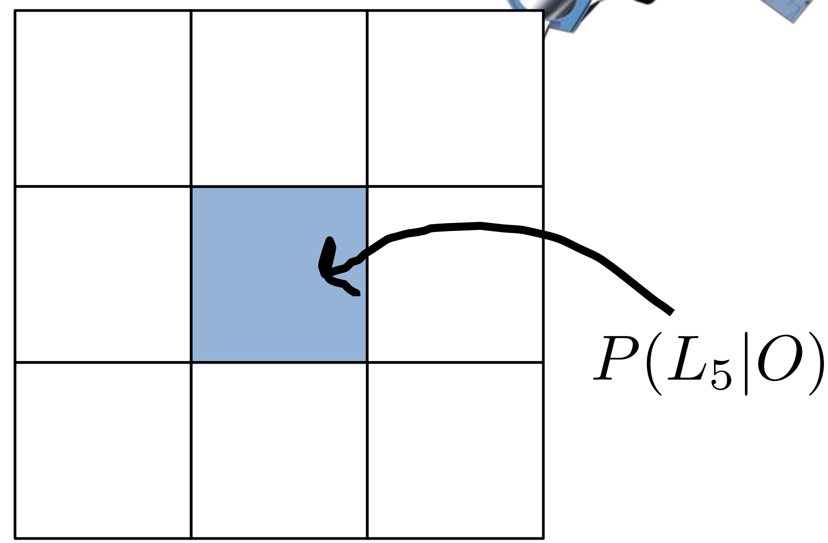
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Update Belief





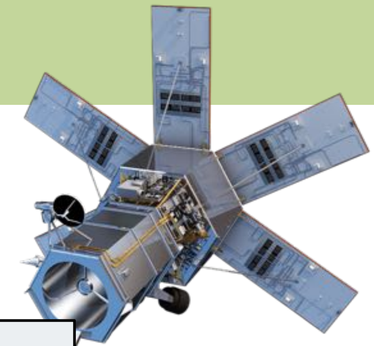
Before Observation

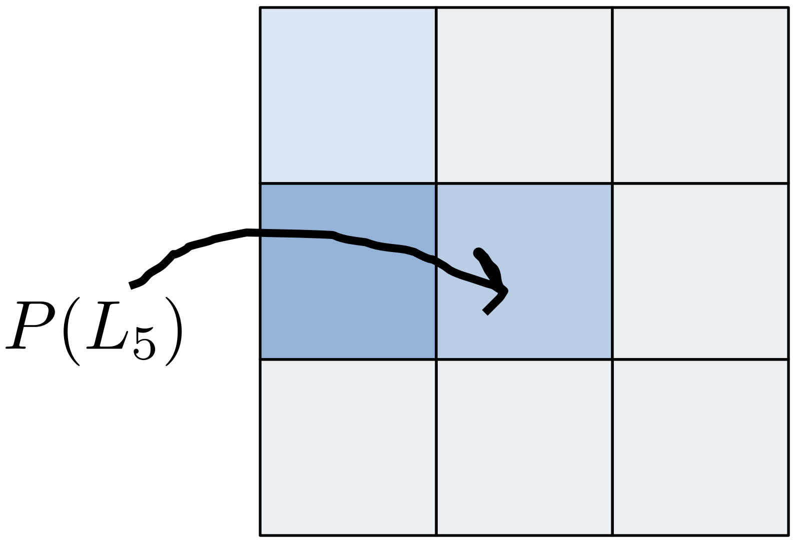


After Observation

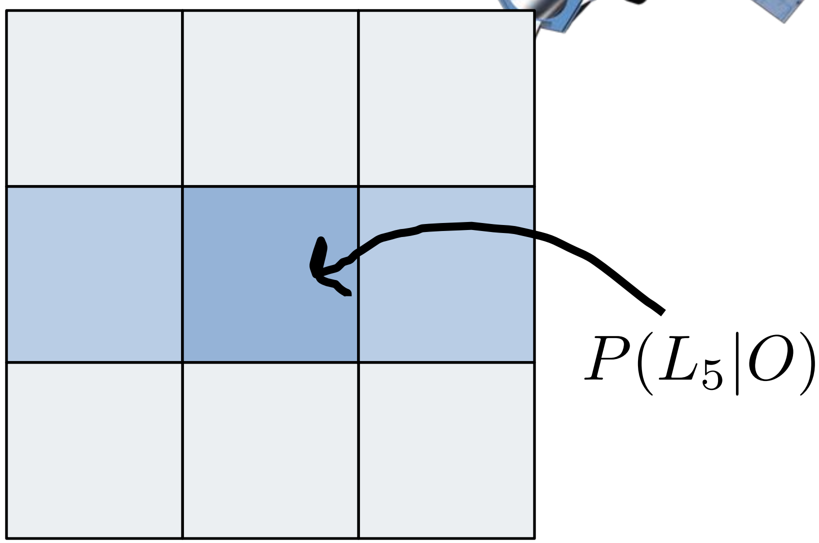
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Update Belief





Before Observation



After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Monty Hall

Let's Make a Deal

- Game show with 3 doors: A, B, and C

  - Behind one door is prize (equally likely to be any door)
  - Behind other two doors is nothing
  - We choose a door
  - Then host opens 1 of other 2 doors, revealing nothing
  - We are given option to change to other door
- Should we?
  - Note: If we don't switch, P(win) = 1/3   (random)

Let's Make a Deal

- Without loss of generality, say we pick A
  - P(A is winner) = 1/3
    - Host opens either B or C, we always lose by switching
    - P(win | A is winner, picked A, switched) = 0
  - P(B is winner) = 1/3
    - Host must open C (can't open A and can't reveal prize in B)
    - So, by switching, we switch to B and always win
    - P(win | B is winner, picked A, switched) = 1
  - P(C is winner) = 1/3
    - Host must open B (can't open A and can't reveal prize in C)
    - So, by switching, we switch to C and always win
    - P(win | C is winner, picked A, switched) = 1
  - Should always switch!
    - P(win | picked A, switched) = (1/3*0) + (1/3*1) + (1/3*1) = 2/3

Slight Variant to Clarify

- Start with 1,000 envelopes, of which 1 is winner
  - You get to choose 1 envelope
    - Probability of choosing winner = 1/1000
  - Consider remaining 999 envelopes
    - Probability one of them is the winner = 999/1000
  - I open 998 of remaining 999 (showing they are empty)
    - Probability the last remaining envelope being winner = 999/1000
  - Should you switch?
    - Probability winning without switch = $\dfrac{1}{\text{original \# envelopes}}$
    - Probability winning with switch = $\dfrac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$