On para-Kähler Lie algebroids and contravariant pseudo-Hessian structures

Mohamed Boucetta
m.boucetta@uca.ac.ma
Joint work with Saïd Benayadi

Cadi-Ayyad University
Faculty of Sciences and Technology Marrakesh Morocco

Séminaire Algèbre, Géométrie, Topologie et Applications
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Motivations

Example.

1. *Let* $M = \mathbb{R}^n$ *and* $\phi(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2$. *Then* $(g_{ij}) = \left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)$ *is a Riemannian metric on* M.

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2. *Let* $M = \{x \in \mathbb{R}^n, x_1 > 0, \ldots, x_n > 0\}$ *and* $\phi(x) = \sum_{i=1}^n x_i \ln x_i$. *Then* $(g_{ij}) = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$ *is a Riemannian metric on* M.

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3. *Let* $M = \left\{x \in \mathbb{R}^n, x_n > \frac{1}{2}\sum_{i=1}^{n-1} x_i^2\right\}$ *and* $\phi(x) = -\ln\left(x_n - \frac{1}{2}\sum_{i=1}^{n-1} x_i^2\right)$. *Then* $(g_{ij}) = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$ *is a Riemannian metric on* M.

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4. *Let* $M = \{(x, y) \in \mathbb{R}^2, y > 0\}$ *and* $\phi(x, y) = \frac{1}{2}x \ln(x^2 + y^2) + y \arctan\left(\frac{x}{y}\right)$. *Then* $(g_{ij}) = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)$ *is a Lorentzian metric on* M.

Theorem.

Let M be a manifold of dimension n. Then the following assertions are equivalent:

1. *There exists on M a flat and torsionless connection ∇.*
2. *There exists on M an atlas $((U_i, \phi_i)_{i \in A})$ such that, for any $i, j \in A$, the change of charts $\phi_{i,j} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is given by*

$$\phi_{i,j}(x) = Mx + b,\ M \in \mathrm{GL}(n), b \in \mathbb{R}^n.$$

The atlas is called an affine atlas and its charts are called affine chart.

An affine manifold is a manifold endowed with an affine atlas or equivalently a flat and torsionless connection.

Definition.

Let M be an affine manifold. A pseudo-Hessian metric on M is a pseudo-Riemannian metric g on M such that for any $m \in M$ there exists an affine chart $(U, (x_i)_{i=1}^n)$ around m and function ϕ on U such that $g_{ij} = g(\partial_{x_i}, \partial_{x_j}) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. The triple (M, ∇, g) is called a pseudo-Hessian manifold.[a]

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Remark.

The metric g is pseudo-Hessian iff

$$\nabla_X(g)(Y, Z) = \nabla_Y(g)(X, Z). \quad \textit{(Codazzi equation)}$$

Let $D \subset \mathbb{R}^n$ a regular open convex cone and define

$$D^* = \{y \in \mathbb{R}^n, \langle y, x \rangle > 0, \quad \forall x \in \bar{D} \setminus \{0\}\}$$

and $\psi : D \longrightarrow \mathbb{R}$ by

$$\psi(x) = \int_{D^*} e^{-\langle y, x \rangle} dy.$$

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Theorem.

$(D, g = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right))$ *is a Hessian manifold, where* $\phi = \ln \psi$. *Moreover,* g *is invariant by the group of automorphisms affine preserving* D.

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For more details see:
Saïd Benayadi & M. Boucetta, On para-Kähler Lie algebroids and generalized pseudo-Hessian structures, arxiv: 1610.09682v1[math.DG]30 oct 2016 (submitted).

Outline

1. Para-Kähler manifolds

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Para-Kähler manifolds

Let (M, g) be a pseudo-Riemannian manifold, ∇ its Levi-Civita connection and $K : TM \longrightarrow TM$ satisfying

$$K^2 = \mathrm{Id}_{TM} \quad \text{and} \quad g(KX, Y) = -g(X, KY).$$

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Proposition.

The following are equivalent:

1. $\nabla_X(K)Y := \nabla_X KY - K\nabla_X Y = 0,$
2. Ω *is closed and*

$$N_K(X, Y) := [KX, KY] - K[KX, Y] - K[X, KY] + K^2[X, Y] = 0.$$

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2. *A symplectic manifold* (M, Ω) *with* $K : TM \longrightarrow TM$ Ω*-skew-symmetric,* $K^2 = \mathrm{Id}_{TM}$ *and* $N_K = 0$.

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2. T^+M and T^-M are totally isotropic with respect to g,
3. For any $X \in \Gamma(TM)$, $Y^+ \in \Gamma(T^+M)$, $Z^- \in \Gamma(T^-M)$

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$$\nabla_X Y^{+} \in \Gamma(T^{+}M) \quad \text{and} \quad \nabla_X Z^{-} \in \Gamma(T^{-}M),$$
4. In particular, $T^{+}M$ and $T^{-}M$ define two foliations $\mathcal{F}^{+}$ and $\mathcal{F}^{-}$ which are transverse, totally isotropic and parallel.

So we get the following geometric definition:

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A para-Kähler manifold is a symplectic manifold (M, Ω) with two foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ which are transverse and Lagrangian.

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$$R^\nabla(X^+, Y^+)Z^- + R^\nabla(Y^+, Z^-)X^+ + R^\nabla(Z^-, X^+)Y^+ = 0.$$

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Thus $R^\nabla(X^+, Y^+) = 0.$

Theorem.

Let M be a para-Kähler manifold. Then the leafs of $\mathcal{F}^\epsilon$ are affine manifolds.

Recall that an affine manifold is a manifold N with a connection ∇ which is flat and torsionless, i.e.,

$$R^\nabla(X,Y) = \nabla_{[X,Y]} - \nabla_X\nabla_Y + \nabla_Y\nabla_X = 0 \quad \text{and} \quad [X,Y] = \nabla_X Y - \nabla_Y X$$

Para-Kähler Lie algebroids

A Lie algebroid over a smooth manifold M is a vector bundle $\pi_A : A \longrightarrow M$ together with a $\mathbb{R}$-Lie algebra structure $[\ ,\ ]_A$ on $\Gamma(A)$ and a vector bundle homomorphism $\rho : A \longrightarrow TM$ called *anchor* such that, for any $a, b \in \Gamma(A)$ and for any $f \in C^\infty(M)$, we have the Leibniz identity

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \tag{1}$$

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An immediate consequence of this definition is that the induced map $\rho : \Gamma(A) \longrightarrow \Gamma(TM)$ is a Lie algebra homomorphism and for any $x \in M$, there is an induced Lie bracket on $\mathfrak{g}_x = \mathrm{K}er(\rho_x) \subset A_x$ which makes it into a Lie algebra.

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3. *Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket.*
4. *Let (M, π) be a Poisson manifold. The bivector field π defines a bundle homomorphism $\pi_\# : T^*M \longrightarrow TM$ and a bracket on $\Omega^1(M)$ by*

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

such that $(T^*M, M, \pi_\#, [\ ,\ ]_\pi)$ is a Lie algebroid.

Let $\pi_A : A \longrightarrow M$ be a Lie algebroid with anchor map ρ. An A-connection on a vector bundle $\pi_E : E \longrightarrow M$ is an operator $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying:

1. $\nabla_{a+b} s = \nabla_a s + \nabla_b s$ for any $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$;
2. $\nabla_a (s_1 + s_2) = \nabla_a s_1 + \nabla_a s_2$ for any $a \in \Gamma(A)$ and $s_1, s_2 \in \Gamma(E)$;
3. $\nabla_{fa} s = f \nabla_a s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$;
4. $\nabla_a (fs) = f \nabla_a s + \rho(a)(f) s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

If $E = A$, we call ∇ a linear A-connection.

Let ∇ a linear A-connection. Its dual is the connection ∇^* on A^* given by

$$\prec \nabla^*_a \alpha, b \succ = \rho(a). \prec \nabla^*_a \alpha, b \succ - \prec \alpha, \nabla_a b \succ, \; a, b \in \Gamma(A), \alpha \in \Gamma(A^*).$$

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Definition.

A Lie algebroid $(A, M, \rho, [\ ,\ ]_A)$ is called affine if there exists a linear A-connection ∇ such that for any $a, b \in \Gamma(A)$,

$$R^\nabla(a,b) := \nabla_{[a,b]_A} - \nabla_a \nabla_b + \nabla_b \nabla_a = 0 \quad \text{and} \quad [a,b]_A = \nabla_a b - \nabla_b a.$$

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$$\begin{aligned} 2\langle \mathcal{D}_a b, c\rangle_A &= \rho(a).\langle b, c\rangle_A + \rho(b).\langle a, c\rangle_A - \rho(c).\langle a, b\rangle_A \\ &\quad + \langle [c, a]_A, b\rangle_A + \langle [c, b]_A, a\rangle_A + \langle [a, b]_A, c\rangle_A, \end{aligned}$$

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$\mathcal{D}$ is metric, i.e., $\rho(a).\langle b, c\rangle_A = \langle \mathcal{D}_a b, c\rangle_A + \langle b, \mathcal{D}_a c\rangle_A$ and $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]_A$.

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The connection $\mathcal{D}$ is well-known as the *Levi-Civita A-connection* associated to the pseudo-Riemannian metric $\langle\ ,\ \rangle_A$.

Definition.

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2. $\mathcal{D}\Gamma(A^\epsilon) \subset \Gamma(A^\epsilon)$ and A^ϵ is isotropic,
3. $(A^\epsilon, M, \rho^\epsilon, \mathcal{D}^\epsilon)$ is an affine Lie algebroid. (Bianchi's identity)

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The map $A^- \longrightarrow B^*$, $a \mapsto \langle a, .\rangle_A$ is an isomorphism.

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The map $A^- \longrightarrow B^*$, $a \mapsto \langle a, .\rangle_A$ is an isomorphism.
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Proposition.

$(A, M, \rho, \langle\ ,\ \rangle_A, K)$ is isomorphic to $(\Phi(B) = B \oplus B^*, M, \rho_1 \oplus \rho_2, \langle\ ,\ \rangle_0, K_0)$ where:

$$\begin{aligned}
\langle a+\alpha, b+\beta\rangle_0 &= \prec \alpha, b \succ + \prec \beta, a \succ,\\
K_0(a+\alpha) &= a-\alpha,\\
\mathcal{D}_{X+\alpha}(Y+\beta) &= S_X Y + T^*_\alpha Y + S^*_X\beta + T_\alpha\beta,\\
[X+\alpha, Y+\beta]_\phi &= [X,Y]_B + [\alpha,\beta]_{B^*}\\
&\quad + T^*_\alpha Y - T^*_\beta X + S^*_X\beta - S^*_Y\alpha.
\end{aligned}$$

Conversely, let (B, M, ρ_1, S) and (B^*, M, ρ_2, T) two affine Lie algebroids.

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$(\Phi(B) = B \oplus B^*, M, \rho = \rho_1 \oplus \rho_2, \langle\ ,\ \rangle_0, K_0)$ is a para-Kähler Lie algebroid iff the bracket

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\end{aligned}$$

satisfies Jacobi identity.

Conversely, let (B, M, ρ_1, S) and (B^*, M, ρ_2, T) two affine Lie algebroids.
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satisfies Jacobi identity.

Remark.

If $[\ ,\ ]_\phi$ satisfies Jacobi identity then $\rho : \Gamma(\Phi(B)) \longrightarrow \Gamma(TM)$ is an homomorphism of Lie algebras. In particular

$$\rho([X,\alpha]_\phi) = -\rho_0(T^*_\alpha X) + \rho_1(S^*_X\alpha) = [\rho_0(X), \rho_1(\alpha)].$$

If ρ_0 is invertible then put $\mathrm{r} = \rho_0^{-1} \circ \rho_1 : B^* \longrightarrow B$ and

$$T^*_\alpha X = \mathrm{r}(S^*_X\alpha) - [X, \mathrm{r}(\alpha)]_B,$$

Exact para-Kähler Lie algebroid

Let (B, M, ρ, S) be an affine Lie algebroid, $\mathrm{r} \in \Gamma(A \otimes A)$ symmetric. Let $\mathrm{r}_{\#} : A^* \longrightarrow A$ given by $\beta(\mathrm{r}_{\#}(\alpha)) = \mathrm{r}(\alpha, \beta)$. Put $\rho_{\mathrm{r}} = \rho \circ \mathrm{r}_{\#}$ and, for any $\alpha, \beta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,

$$\begin{aligned}
\prec T_\alpha \beta, X \succ: &= \rho(X).\mathrm{r}(\alpha, \beta) - \mathrm{r}(S_X^* \alpha, \beta) \\
&\quad - \mathrm{r}(\alpha, S_X^* \beta) + \prec S^*_{\mathrm{r}_{\#}(\alpha)} \beta, X \succ \\
&= S_X \mathrm{r}(\alpha, \beta) + \prec S^*_{\mathrm{r}_{\#}(\alpha)} \beta, X \succ, \\
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Problem

Under which conditions $(B^*, M, \rho_{\mathrm{r}}, T)$ is an affine Lie algebroid and $(B \oplus B^*, M, \rho \oplus \rho_{\mathrm{r}}, [\ ,\ ]_\phi, K_0, \langle\ ,\ \rangle_0)$ is a para-Kähler Lie algebroid?

Theorem.

$(B^*, M, \rho_{\mathrm{r}}, T)$ is an affine Lie algebroid and $(B \oplus B^*, M, \rho \oplus \rho_{\mathrm{r}}, [\ ,\ ]_\phi, K_0, \langle\ ,\ \rangle_0)$ is a para-Kähler Lie algebroid iff

$$\begin{aligned} \rho \circ \Delta(r) &= 0, \quad \textit{and} \\ [X, \Delta(r)(\alpha,\beta)]_S &- \Delta(r)(S^*_X\alpha,\beta) - \Delta(r)(\alpha, S^*_X\beta) = 0, \end{aligned}$$

where

$$\Delta(r)(\alpha,\beta) = \mathrm{r}_\#([\alpha,\beta]_T) - [\mathrm{r}_\#(\alpha), \mathrm{r}_\#(\beta)]_S.$$

Theorem.

$(B^*, M, \rho_{\mathrm{r}}, T)$ *is an affine Lie algebroid and* $(B \oplus B^*, M, \rho \oplus \rho_{\mathrm{r}}, [\ ,\ ]_\phi, K_0, \langle\ ,\ \rangle_0)$ *is a para-Kähler Lie algebroid iff*

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where

$$\Delta(r)(\alpha, \beta) = \mathrm{r}_\#([\alpha, \beta]_T) - [\mathrm{r}_\#(\alpha), \mathrm{r}_\#(\beta)]_S.$$

Remark.

If $\Delta(r) = 0$ *then the conditions of the theorem are satisfied. If* ρ *is injective then the conditions of the theorem reduce to* $\Delta(r) = 0$*.*

Contravariant pseudo-Hessian structures

Let (M, ∇) be an affine manifold. Then (TM, M, Id_{TM}, ∇) is an affine Lie algebroid.

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$$\begin{aligned} \prec \mathcal{D}_\alpha\beta, X \succ &= \nabla_X h(\alpha,\beta) + \prec \nabla^*_{h_\#(\alpha)}\beta, X \succ, \\ [\alpha,\beta]_{\mathcal{D}} &= \nabla^*_{h_\#(\alpha)}\beta - \nabla^*_{h_\#(\beta)}\alpha. \end{aligned}$$

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Proposition.

$$\begin{aligned} \prec \gamma, \Delta(h)(\alpha, \beta) \succ &= \nabla_{h_\#(\beta)} h(\alpha, \gamma) - \nabla_{h_\#(\alpha)} h(\beta, \gamma), \\ \prec \gamma, h_\# (\mathcal{D}_\alpha \beta) \succ &= \prec \gamma, \nabla_{h_\#(\alpha)} h_\#(\beta) \succ + \prec \beta, \Delta(h)(\alpha, \gamma) \succ . \end{aligned}$$

Theorem.

$(T^*M, M, h_\#, \mathcal{D})$ *is an affine Lie algebroid and* $(TM \oplus T^*M, M, \mathrm{Id}_{TM} \oplus h_\#, [\ ,\ ]_\phi, K_0, \langle\ ,\ \rangle_0)$ *is a para-Kähler Lie algebroid iff*

$$\nabla_{h_\#(\beta)} h(\alpha, \gamma) = \nabla_{h_\#(\alpha)} h(\beta, \gamma). \tag{2}$$

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Proposition.

If (M, ∇, g) is a pseudo-Hessian manifold then $h = g^{-1}$ satisfies (2).

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Definition.

A contravariant pseudo-Hessian structure on an affine manifold M is a symmetric bivector field h satisfying (2).

Theorem.

Let (M, ∇, h) be a contravariant pseudo-Hessian manifold. Then $\mathrm{Im} h_{\#}$ is integrable and defines a singular foliation on M such that for every leaf L we have:

(i) *For every vector fields X, Y tangent to L, $\nabla_X Y$ is tangent to L,*

(ii) *L has a natural pseudo-Hessian structure.*

The following theorem can be compared to Darboux-Weinstein near a regular point in Poisson geometry (see[**?**]).

Theorem.

Let (M, ∇, h) be a contravariant pseudo-Hessian structure and $x \in M$ such the rank of $h_{\#}$ is constant in a neighborhood of x. Then there exists a chart $(x_1, \ldots, x_r, y_1, \ldots, y_{n-r})$ and a function $f(x, y)$ such that

$$h = \sum_{i,j=1}^{r} h_{ij} \partial_{x_i} \otimes \partial_{x_j}, \quad \nabla_{\partial_{x_i}} \partial_{x_j} = 0, \; i, j = 1, \ldots, r,$$

and the matrix (h_{ij}) is invertible and its inverse is the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Canonical contravariant pseudo-Hessian structure on the dual of a commutative associative algebra

Let $(\mathcal{A},.)$ be a finite dimensional commutative associative algebra. We define a symmetric bivector h on $\mathcal{A}^*$ by putting

$$h(\alpha,\beta)(\mu) = \prec \mu, \alpha(\mu).\beta(\mu) \succ, \quad \alpha,\beta \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*,\mathcal{A}), \mu \in \mathcal{A}^*.$$

We denote by ∇^0 the canonical affine connection of $\mathcal{A}^*$ given by $\nabla^0_X Y(\mu) = d_\mu Y(X(\mu))$ where $X, Y : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ are regarded as vector fields on $\mathcal{A}^*$. For any $u \in \mathcal{A}$, we denote by u^* the linear function on $\mathcal{A}^*$ given by $u^*(\mu) = \prec \mu, u \succ$, by X_u the vector field on $\mathcal{A}^*$ given by $X_u = h_\#(du^*)$ and by $L_u : \mathcal{A} \longrightarrow \mathcal{A}$ the left multiplication by u.

Let $(\mathcal{A}, .)$ be a finite dimensional commutative associative algebra. We define a symmetric bivector h on $\mathcal{A}^*$ by putting

$$h(\alpha,\beta)(\mu) = \prec \mu, \alpha(\mu).\beta(\mu) \succ, \quad \alpha,\beta \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*, \mathcal{A}), \mu \in \mathcal{A}^*.$$

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Theorem.

$(\mathcal{A}^*, \nabla^0, h)$ is a contravariant pseudo-Hessian manifold and the singular foliation associated to $\mathrm{Im}h_\#$ is given by the orbits of the linear action Φ of the abelian Lie group $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(u, \mu) = \exp(L_u^*)(\mu)$.

Let compute now all the mathematical objects above in the case where $M = \{\exp(L_a^*)(\mu), a \in \mathcal{A}\}$ is an orbit of the pseudo-Hessian foliation associated to the contravariant pseudo-Hessian manifold $(\mathcal{A}^*, \nabla^0, h)$ appearing in Theorem 25. Note that $T_\nu M = \{X_a(\nu), a \in \mathcal{A}\}$. As above, we denote by ∇ the affine connection on M, g the pseudo-Riemannian metric, D the Levi-Civita connection. The following proposition is a consequence of an easy and straightforward computation.

Proposition.

For any $a, b, c \in \mathcal{A}$ and any $\nu \in \mathcal{A}^*$,

$$\begin{aligned} g(X_a(\nu), X_b(\nu)) &= \prec \nu, a.b \succ, \\ \nabla_{X_a} X_b &= X_{a.b}, D_{X_a} X_b = \frac{1}{2} X_{a.b}. \end{aligned}$$

Example

$\mathcal{A} = \mathbb{R}^n$. The product is given by $e_i e_i = e_i$ for $i = 1, \dots, n$. We have

$$\Phi\left(\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} x_i e_i^*\right) = \sum_{i=1}^{n} e^{a_i} x_i e_i^*.$$

Moreover, for any $i = 1, \dots, n$, $X_{e_i} = x_i \partial_{x_i}$. The orbit of a point $x \in \mathcal{A}^*$ is $M_x = \{\sum_{i=1}^{n} e^{a_i} x_i e_i^*, a_i \in \mathbb{R}\}$. It is a convex cone and one can see easily that if $\phi : \mathcal{A}^* \longrightarrow \mathbb{R}$ is the function given by

$$\phi(u) = \sum_{i=1}^{n} u_i \ln |u_i|,$$

then the restriction of $\nabla d\phi$ to M_x together with the restriction of ∇ to M_x define the pseudo-Hessian structure on M_x.

Example

$\mathcal{A} = \mathbb{C}$. We have

$$X_{e_1} = \alpha\partial_\alpha + \beta\partial_\beta \quad \text{and} \quad X_{e_2} = \beta\partial_\alpha - \alpha\partial_\beta.$$

We deduce that we have two orbits the origin and $\mathcal{A}^* \setminus \{0\}$. Let describe the pseudo-Hessian structure of $M := \mathcal{A}^* \setminus \{0\}$. The pseudo-Hessian metric g satisfies

$$g(X_{e_1}, X_{e_1}) = \alpha,\ g(X_{e_1}, X_{e_2}) = \beta,\ g(X_{e_2}, X_{e_2}) = -\alpha$$

and hence

$$g = \frac{1}{\alpha^2 + \beta^2}(\alpha d\alpha^2 + 2\beta d\alpha d\beta - \alpha d\beta^2).$$

Thus (M, ∇, g) is a Lorentzian Hessian manifold. Moreover, the metric g is flat.

Example

Now we look for a function f on M such that $g = \nabla df$, i.e.,

$$\frac{\partial^2 f}{\partial \alpha^2} = \frac{\alpha}{\alpha^2 + \beta^2}, \ \frac{\partial^2 f}{\partial \beta^2} = \frac{-\alpha}{\alpha^2 + \beta^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial \alpha \partial \beta} = \frac{\beta}{\alpha^2 + \beta^2}$$

The function f given by

$$f(\alpha, \beta) = \frac{1}{2} \alpha \ln(\alpha^2 + \beta^2) + \beta \arctan\left(\frac{\alpha}{\beta}\right)$$

satisfies these equations on the open set $\{\beta \neq 0\}$. Note that this function is harmonic.

Example

We take $\mathcal{A} = \mathbb{R}^4$ with the commutative associative product given by

$$e_1 e_1 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_1 e_4 = e_4,\ e_2 e_2 = e_3,\ e_2 e_3 = e_4.$$

We have

$$X_{e_1} = x\partial_x + y\partial_y + z\partial_z + t\partial_t,\ X_{e_2} = y\partial_x + z\partial_y + t\partial_z,\ X_{e_3} = z\partial_x + t\partial_y \quad \text{and} \quad X_{e_4} = t\partial_x.$$

Thus $\{t > 0\}$ and $\{t < 0\}$ are orbits and hence carry a pseudo-Hessian structure. The metric is given by

$$g = \frac{1}{t}\left(2dxdt + 2dydz - \frac{2z}{t}dydt - \frac{z}{t}dz^2 + \frac{2(z^2 - yt)}{t^2}dzdt + \frac{2zyt - xt^2 - z^3}{t^3}dt^2\right).$$

The signature of this metric is $(+,+,-,-)$. One can check easily that g is the restriction of $\nabla d\phi$ to M, where

$$\phi(x, y, z, t) = -\frac{z^3}{6t^2} + \frac{yz}{t} + x\ln|t|.$$