



Biharmonic and harmonic homomorphisms between Riemannian three dimensional unimodular Lie groups



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ABSTRACT

We classify biharmonic and harmonic homomorphisms $f : (G, g_1) \rightarrow (G, g_2)$ where G is a connected and simply connected three-dimensional unimodular Lie group and g_1 and g_2 are left invariant Riemannian metrics.

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1. Introduction

The theory of biharmonic maps is old and rich and has gained a growing interest in the last decade (see [1,7,9] and others). The theory of harmonic maps into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied in relation to integrable systems by many mathematicians (see for examples [4,10,11]). In particular, harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric are called principal chiral models and intensively studied as toy models of gauge theory in mathematical physics [12]. In the papers [6,8], harmonic inner automorphisms of a compact semi-simple Lie group endowed with a left invariant Riemannian metric where studied. In [2], there is a detailed study of biharmonic and harmonic homomorphisms between Riemannian Lie groups.¹

In this paper, we aim to classify, up to conjugation by automorphisms of Lie groups, harmonic and biharmonic homomorphisms $f : (G, g_1) \rightarrow (G, g_2)$ where G is a non abelian connected and simply-connected three dimensional unimodular Lie group, f is an homomorphism of Lie groups and g_1 and g_2 are two left invariant Riemannian metrics. There are five non abelian connected and simply-connected three-dimensional unimodular Lie groups: the nilpotent Lie group Nil, the special unitary group $SU(2)$, the universal covering group $\tilde{PSL}(2, \mathbb{R})$ of the special linear group, the solvable Lie group Sol and the universal covering group $\tilde{E}_0(2)$ of the connected component of the Euclidean group. Our main results are as follows:

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¹ A biharmonic homomorphism between Riemannian Lie groups is a homomorphism of Lie groups $\phi : G \rightarrow H$ which is also biharmonic where G and H are endowed with left invariant Riemannian metrics.

1. For Nil and Sol we show that a homomorphism is biharmonic if and only if it is harmonic and we classify completely all the harmonic homomorphisms (see [Theorems 3.1, 5.1 and 5.2](#)).
2. For $\tilde{E}_0(2)$ we classify completely all the harmonic homomorphisms (see [Theorem 4.1](#)). For this group there are biharmonic homomorphisms which are not harmonic and we give a complete classification of these homomorphisms (see [Theorem 4.2](#)). To our knowledge, these are the first examples of biharmonic not harmonic homomorphisms between Riemannian Lie groups.
3. For $SU(2)$ and $PSL(2, \mathbb{R})$ we give a complete classification of harmonic homomorphisms (see [Theorems 6.1 and 7.1](#)). We show that these groups have biharmonic homomorphisms which are not harmonic and we give the first examples of these homomorphisms. For $SU(2)$ we recover the results obtained in [\[6,8\]](#) and we complete them.

This work is based on [\[2\]](#), on the results of [\[5\]](#) which gave a complete classification of left invariant Riemannian metrics on three dimensional Lie groups and on the description given in [\[3\]](#) of the automorphisms of $SU(2)$ and $PSL(2, \mathbb{R})$. We proceed by a direct computation and [Proposition 2.2](#) is a useful trick which simplified many computations. Some of our computations were performed using the software Maple.

This paper is divided into seven sections. In Section 2, we give the tools needed in our study and we devote a section to each one of the five groups.

2. Preliminaries

Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds with $m = \dim M$ and $n = \dim N$. We denote by ∇^M and ∇^N the Levi-Civita connections associated respectively to g and h and by $T^\phi N$ the vector bundle over M pull-back of TN by ϕ . It is an Euclidean vector bundle and the tangent map of ϕ is a bundle homomorphism $d\phi : TM \rightarrow T^\phi N$. Moreover, $T^\phi N$ carries a connexion ∇^ϕ pull-back of ∇^N by ϕ and there is a connection on the vector bundle $\text{End}(TM, T^\phi N)$ given by

$$(\nabla_X A)(Y) = \nabla_X^\phi A(Y) - A(\nabla_X^M Y), \quad X, Y \in \Gamma(TM), A \in \Gamma(\text{End}(TM, T^\phi N)).$$

The map ϕ is called harmonic if it is a critical point of the energy $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$. The corresponding Euler–Lagrange equation for the energy is given by the vanishing of the tension field

$$\tau(\phi) = \text{tr}_g \nabla d\phi = \sum_{i=1}^m (\nabla_{E_i} d\phi)(E_i), \quad (1)$$

where $(E_i)_{i=1}^m$ is a local frame of orthonormal vector fields. Note that $\tau(\phi) \in \Gamma(T^\phi N)$. The map ϕ is called biharmonic if it is a critical point of the bienergy of ϕ defined by $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g$. The corresponding Euler–Lagrange equation for the bienergy is given by the vanishing of the bitension field

$$\tau_2(\phi) = -\text{tr}_g (\nabla^\phi)^2 \tau(\phi) - \text{tr}_g R^N(\tau(\phi), d\phi(\cdot)) d\phi(\cdot) = -\sum_{i=1}^m ((\nabla^\phi)_{E_i, E_i}^2 \tau(\phi) + R^N(\tau(\phi), d\phi(E_i)) d\phi(E_i)), \quad (2)$$

where $(E_i)_{i=1}^m$ is a local frame of orthonormal vector fields, $(\nabla^\phi)_{X,Y}^2 = \nabla_X^\phi \nabla_Y^\phi - \nabla_{\nabla_X^M Y}^\phi$ and R^N is the curvature of ∇^N given by

$$R^N(X, Y) = \nabla_X^N \nabla_Y^N - \nabla_Y^N \nabla_X^N - \nabla_{[X, Y]}^N.$$

Let (G, g) be a Riemannian Lie group, i.e., a Lie group endowed with a left invariant Riemannian metric. If $\mathfrak{g} = T_e G$ is its Lie algebra and $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = g(e)$ then there exists a unique bilinear map $A : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called the Levi-Civita product associated to $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ given by the formula:

$$2\langle A_u v, w \rangle_{\mathfrak{g}} = \langle [u, v]^{\mathfrak{g}}, w \rangle_{\mathfrak{g}} + \langle [w, u]^{\mathfrak{g}}, v \rangle_{\mathfrak{g}} + \langle [w, v]^{\mathfrak{g}}, u \rangle_{\mathfrak{g}}. \quad (3)$$

A is entirely determined by the following properties:

1. for any $u, v \in \mathfrak{g}$, $A_u v - A_v u = [u, v]^{\mathfrak{g}}$,
2. for any $u, v, w \in \mathfrak{g}$, $\langle A_u v, w \rangle_{\mathfrak{g}} + \langle v, A_u w \rangle_{\mathfrak{g}} = 0$.

If we denote by u^ℓ the left invariant vector field on G associated to $u \in \mathfrak{g}$ then the Levi-Civita connection associated to (G, g) satisfies $\nabla_{u^\ell} v^\ell = (A_u v)^\ell$. The couple $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ defines a vector say $U^{\mathfrak{g}} \in \mathfrak{g}$ by

$$\langle U^{\mathfrak{g}}, v \rangle_{\mathfrak{g}} = \text{tr}(\text{ad}_v), \quad \text{for any } v \in \mathfrak{g}. \quad (4)$$

One can deduce easily from [\(3\)](#) that, for any orthonormal basis $(e_i)_{i=1}^n$ of \mathfrak{g} ,

$$U^{\mathfrak{g}} = \sum_{i=1}^n A_{e_i} e_i. \quad (5)$$

Note that \mathfrak{g} is unimodular if and only if $U^{\mathfrak{g}} = 0$.

Let $\phi : (G, \mathfrak{g}) \rightarrow (H, \mathfrak{h})$ be a Lie group homomorphism between two Riemannian Lie groups. The differential $\xi : \mathfrak{g} \rightarrow \mathfrak{h}$ of ϕ at e is a Lie algebra homomorphism. There is a left action of G on $\Gamma(T^{\phi}H)$ given by

$$(aX)(b) = T_{\phi(ab)}L_{\phi(a^{-1})}X(ab), \quad a, b \in G, X \in \Gamma(T^{\phi}H).$$

A section X of $T^{\phi}H$ is called left invariant if, for any $a \in G$, $aX = X$. For any left invariant section X of $T^{\phi}H$, we have for any $a \in G$, $X(a) = (X(e))^{\phi}(a)$. Thus the space of left invariant sections is isomorphic to the Lie algebra \mathfrak{h} . Since ϕ is a homomorphism of Lie groups and g and h are left invariant, one can see easily that $\tau(\phi)$ and $\tau_2(\phi)$ are left invariant and hence ϕ is harmonic (resp. biharmonic) iff $\tau(\phi)(e) = 0$ (resp. $\tau_2(\phi)(e) = 0$). Now, one can see easily that

$$\begin{cases} \tau(\xi) := \tau(\phi)(e) = U^{\xi} - \xi(U^{\xi}), \\ \tau_2(\xi) := \tau_2(\phi)(e) = -\sum_{i=1}^n (B_{\xi(e_i)}B_{\xi(e_i)}\tau(\xi) + K^H(\tau(\xi), \xi(e_i))\xi(e_i)) + B_{\xi(U^{\xi})}\tau(\xi), \end{cases} \quad (6)$$

where B is the Levi-Civita product associated to $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$,

$$U^{\xi} = \sum_{i=1}^n B_{\xi(e_i)}\xi(e_i), \quad (7)$$

$(e_i)_{i=1}^n$ is an orthonormal basis of \mathfrak{g} and K^H is the curvature of B given by $K^H(u, v) = [B_u, B_v] - B_{[u, v]}$. So we get the following proposition.

Proposition 2.1. Let $\phi : G \rightarrow H$ be an homomorphism between two Riemannian Lie groups. Then ϕ is harmonic (resp. biharmonic) iff $\tau(\xi) = 0$ (resp. $\tau_2(\xi) = 0$), where $\xi : \mathfrak{g} \rightarrow \mathfrak{h}$ is the differential of ϕ at e .

Thus the study of biharmonic and harmonic homomorphisms between connected and simply-connected Lie groups reduces to the study of their differential so, through this paper, we consider homomorphisms $\xi : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{g}, \langle \cdot, \cdot \rangle_2)$ where \mathfrak{g} is a Lie algebra and $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two Euclidean products. We call ξ harmonic (resp. biharmonic) if $\tau(\xi) = 0$ (resp. $\tau_2(\xi) = 0$).

The classification of biharmonic and harmonic homomorphisms will be done up to a conjugation. Two homomorphisms between Euclidean Lie algebras

$$\xi_1 : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{g}, \langle \cdot, \cdot \rangle_2) \quad \text{and} \quad \xi_2 : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1^2) \rightarrow (\mathfrak{g}, \langle \cdot, \cdot \rangle_2^2)$$

are conjugate if there exists two isometric automorphisms $\phi_1 : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1^1) \rightarrow (\mathfrak{g}, \langle \cdot, \cdot \rangle_2^1)$ and $\phi_2 : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1^2) \rightarrow (\mathfrak{g}, \langle \cdot, \cdot \rangle_2^2)$ such that $\xi_2 = \phi_2 \circ \xi_1 \circ \phi_1^{-1}$.

We give now a criteria which will be useful in order to show that an homomorphism is harmonic if and only if it is biharmonic.

Let $\xi : (\mathfrak{g}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{h}, \langle \cdot, \cdot \rangle_2)$ be an homomorphism. We suppose that \mathfrak{g} is unimodular. The following formulas was established in [2, Proposition 2.4]:

$$\begin{aligned} \langle \tau(\xi), u \rangle_2 &= \text{tr}(\xi^* \circ \text{ad}_u \circ \xi), \\ \langle \tau_2(\xi), u \rangle_2 &= \text{tr}(\xi^* \circ (\text{ad}_u + \text{ad}_u^*) \circ \text{ad}_{\tau(\xi)} \circ \xi) - \langle [u, \tau(\xi)], \tau(\xi) \rangle_2, \end{aligned}$$

where $\xi^* : \mathfrak{h} \rightarrow \mathfrak{g}$ and $\text{ad}_u^* : \mathfrak{h} \rightarrow \mathfrak{h}$ are given by

$$\langle \xi^* u, v \rangle_1 = \langle u, \xi v \rangle_2 \quad \text{and} \quad \langle \text{ad}_u^* x, y \rangle_2 = \langle \text{ad}_u y, x \rangle_2, \quad x, y, u \in \mathfrak{h}, v \in \mathfrak{g}.$$

By combining these two formulas, we get

$$\langle \tau_2(\xi), u \rangle_2 = \text{tr}(\xi^* \circ (\text{ad}_u + \text{ad}_u^*) \circ \text{ad}_{\tau(\xi)} \circ \xi) - \text{tr}(\xi^* \circ \text{ad}_{[u, \tau(\xi)]} \circ \xi).$$

So if ξ is biharmonic then $\tau(\xi)$ is solution of the linear system

$$\text{tr}(\xi^* \circ (\text{ad}_u + \text{ad}_u^*) \circ \text{ad}_X \circ \xi) - \text{tr}(\xi^* \circ \text{ad}_{[u, X]} \circ \xi) = 0, \quad u \in \mathfrak{h}. \quad (8)$$

If $\mathbb{B} = (f_1, \dots, f_m)$ is a basis of \mathfrak{h} this system is equivalent to

$$M_{\xi}(\mathbb{B})X = 0,$$

where $M_{\xi}(\mathbb{B}) = (m_{ij})_{1 \leq i \leq j \leq m}$ and

$$m_{ij} = \text{tr}(\xi^* \circ (\text{ad}_{f_i} + \text{ad}_{f_i}^*) \circ \text{ad}_{f_j} \circ \xi) - \text{tr}(\xi^* \circ \text{ad}_{[f_i, f_j]} \circ \xi).$$

We call $M_{\xi}(\mathbb{B})$ the test matrix of ξ in the basis (f_1, \dots, f_n) .

Proposition 2.2. If $\det(M_{\xi}(\mathbb{B})) \neq 0$ then ξ is biharmonic if and only if it is harmonic.

We end this section by describing the main objects of this study, namely, the 3-dimensional unimodular Lie algebras. They are five unimodular simply connected three dimensional unimodular non abelian Lie groups:

1. The nilpotent Lie group Nil known as Heisenberg group whose Lie algebra will be denoted by \mathfrak{n} . We have

$$\text{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra \mathfrak{n} has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are $[X_1, X_2] = X_3$.

2. $SU(2) = \left\{ \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix}, a^2 + b^2 + c^2 + d^2 = 1 \right\}$ and $\mathfrak{su}(2) = \left\{ \begin{pmatrix} iz & y+ix \\ -y+xi & -zi \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\mathfrak{su}(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$

$$X_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, X_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_1, X_2] = X_3, [X_2, X_3] = X_1 \quad \text{and} \quad [X_3, X_1] = X_2.$$

3. The universal covering group $\widetilde{\text{PSL}}(2, \mathbb{R})$ of $\text{SL}(2, \mathbb{R})$ whose Lie algebra is $\text{sl}(2, \mathbb{R})$. The Lie algebra $\text{sl}(2, \mathbb{R})$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_1, X_2] = -X_3, [X_2, X_3] = X_1 \quad \text{and} \quad [X_3, X_1] = X_2.$$

4. The solvable Lie group $\text{Sol} = \left\{ \begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$ whose Lie algebra is $\mathfrak{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$. The Lie algebra \mathfrak{sol} has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_3, X_1] = X_1 \quad \text{and} \quad [X_3, X_2] = -X_2.$$

5. The universal covering group $\widetilde{E}_0(2)$ of the Lie group

$$E_0(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}, \theta, x, y \in \mathbb{R} \right\}.$$

Its Lie algebra is

$$\mathfrak{e}_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \theta, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{e}_0(2)$ has a basis $\mathbb{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where the non-vanishing Lie brackets are

$$[X_3, X_1] = X_2 \quad \text{and} \quad [X_3, X_2] = -X_1.$$

Table 1

Unimodular three dimensional Lie algebras, their homomorphisms and Euclidean metric.

Lie algebra	Non-vanishing Lie brackets	Homomorphisms	Equivalence classes of metrics
\mathfrak{n}	$[X_1, X_2] = X_3$	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}$	$\text{Diag}(\lambda, \lambda, 1), \lambda > 0$
$e_0(2)$	$[X_3, X_1] = X_2, [X_3, X_2] = -X_1$	$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}, \gamma^2 \neq 1$	$\text{Diag}(1, \mu, \nu)$,
		$\begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$	$0 < \mu \leq 1, \nu > 0$
\mathfrak{sof}	$[X_3, X_1] = X_1, [X_3, X_2] = -X_2$	$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \nu > 0, \mu > 1$
		$\begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}, \gamma^2 \neq 1$	$\text{Diag}(1, 1, \nu)$
$\text{sl}(2, \mathbb{R})$	$[X_1, X_2] = -X_3, [X_3, X_1] = X_2, [X_2, X_3] = X_1$	$\text{Rot}_{xy}, \text{Boost}_{xz}, \text{Boost}_{yz}$	$\text{Diag}(\lambda, \mu, \nu)$, $0 < \lambda \leq \mu$ and $\nu > 0$
$\mathfrak{su}(2)$	$[X_1, X_2] = X_3, [X_3, X_1] = X_2, [X_2, X_3] = X_1$	$\text{Rot}_{xy}, \text{Rot}_{xz}, \text{Rot}_{yz}$	$\text{Diag}(\lambda, \mu, \nu)$, $0 < \nu \leq \mu \leq \lambda$

In Table 1, we collect the informations on these Lie algebras we will use in the next sections. For each Lie algebra among the five Lie algebras above, we give the set of its homomorphisms and the equivalence classes of Riemannian metrics carried out by this Lie algebra. These equivalence classes were determined in [5, Theorems 3.3–3.7]. For \mathfrak{n} , \mathfrak{sof} and $e_0(2)$ the homomorphisms can be determined easily. For $\mathfrak{su}(2)$ and $\text{sl}(2, \mathbb{R})$ an homomorphism is necessarily an inner automorphism and these were determined in [3].

$$\text{Rot}_{xy} = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{Rot}_{xz} = \begin{pmatrix} \cos(a) & 0 & \sin(a) \\ 0 & 1 & 0 \\ -\sin(a) & 0 & \cos(a) \end{pmatrix}, \text{Rot}_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & \sin(a) \\ 0 & -\sin(a) & \cos(a) \end{pmatrix}$$

$$\text{Boost}_{xz} = \begin{pmatrix} \cosh(a) & 0 & \sinh(a) \\ 0 & 1 & 0 \\ \sinh(a) & 0 & \cosh(a) \end{pmatrix}, \text{Boost}_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(a) & \sinh(a) \\ 0 & \sinh(a) & \cosh(a) \end{pmatrix}.$$

In the following sections, the computations of $\tau(\xi)$ and $\tau_2(\xi)$ are performed by the software Maple and all the direct computations as well.

3. Harmonic and biharmonic homomorphisms on the 3-dimensional Heisenberg Lie group

The following result gives a complete classification of harmonic and biharmonic homomorphisms of \mathfrak{n} .

Theorem 3.1. A homomorphism of \mathfrak{n} is biharmonic if and only if it is harmonic. Moreover, it is harmonic if and only if it is conjugate to $\xi : (\mathfrak{n}, \langle \cdot, \cdot \rangle_1) \longrightarrow (\mathfrak{n}, \langle \cdot, \cdot \rangle_2)$ where

$$\xi = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ 0 & 0 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a\beta_3 & -a\alpha_3 & 0 \\ b\beta_3 & -b\alpha_3 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}, (\alpha_3, \beta_3) \neq (0, 0)$$

and $\text{Mat}(\langle \cdot, \cdot \rangle_i, \mathbb{B}_0) = \text{Diag}(\lambda_i, \lambda_i, 1)$ and $\lambda_i > 0, i = 1, 2$.

Proof. The first part of the theorem is a consequence of [2, Theorem 6.5]. On the other hand, according to Table 1, and homomorphism $\xi : (\mathfrak{n}, \langle \cdot, \cdot \rangle_1) \longrightarrow (\mathfrak{n}, \langle \cdot, \cdot \rangle_2)$ has, up to a conjugation, the form

$$\xi = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} \quad \text{and} \quad \langle \cdot, \cdot \rangle_i = \text{Diag}(\lambda_i, \lambda_i, 1), \lambda_i > 0, i = 1, 2.$$

Then

$$\tau(\xi) = \frac{\alpha_3\beta_1 + \beta_3\beta_2}{\lambda_2\lambda_1}X_1 - \frac{\alpha_3\alpha_1 + \beta_3\alpha_2}{\lambda_2\lambda_1}X_2$$

and the second part of the theorem follows. \square

4. Harmonic and biharmonic homomorphisms on $\widetilde{E}_0(2)$

The situation on $e_0(2)$ is different and there exists biharmonic homomorphisms which are not harmonic. The following two theorems give a complete classification of harmonic and biharmonic homomorphisms on $e_0(2)$.

Theorem 4.1. A homomorphism of $e_0(2)$ is harmonic if and only if it is conjugate to $\xi : (e_0(2), \langle \cdot, \cdot \rangle_1) \longrightarrow (e_0(2), \langle \cdot, \cdot \rangle_2)$

where $\text{Mat}(\langle \cdot, \cdot \rangle_i, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & v_i \end{pmatrix}, 0 < \mu_i \leq 1, v_i > 0$ and either

1. $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ with $\gamma^2 \neq 1$ and $(a \neq 0, b \neq 0, \gamma = 0, \mu_2 = 1), ((a, b) \neq (0, 0), ab = 0, \gamma = 0)$ or $(a = b = 0)$
2. $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$ and $(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1)$ or $(a = b = 0, \mu_2 = 1)$,
3. $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$ and $(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1)$ or $(a = b = 0, \mu_2 = 1)$.

Proof. According to Table 1, and homomorphism $\xi : (e_0(2), \langle \cdot, \cdot \rangle_1) \longrightarrow (e_0(2), \langle \cdot, \cdot \rangle_2)$ has, up to a conjugation, the form $\text{Mat}(\langle \cdot, \cdot \rangle_i, \mathbb{B}_0) = \text{Diag}(1, \mu_i, v_i), i = 1, 2, 0 < \mu_i \leq 1, v_i > 0$ and

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \gamma^2 \neq 1, \xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}.$$

• $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ with $\gamma^2 \neq 1$. We have

$$\tau(\xi) = -\frac{\gamma \mu_2 b}{v_1} X_1 + \frac{\gamma a}{\mu_2 v_1} X_2 + \frac{ba(\mu_2 - 1)}{v_2 v_1} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a \neq 0, b \neq 0, \gamma = 0, \mu_2 = 1), ((a, b) \neq (0, 0), ab = 0, \gamma = 0) \text{ or } (a = b = 0).$$

• $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$. We have

$$\tau(\xi) = -\frac{\mu_2 b}{v_1} X_1 + \frac{a}{\mu_2 v_1} X_2 + \frac{(\mu_2 - 1)(\alpha \beta v_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 v_1 v_2} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1) \text{ or } (a = b = 0, \mu_2 = 1).$$

• $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$. We have

$$\tau(\xi) = \frac{\mu_2 b}{v_1} X_1 - \frac{a}{\mu_2 v_1} X_2 + \frac{(\mu_2 - 1)(\alpha \beta v_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 v_1 v_2} X_3$$

and $\tau(\xi) = 0$ if and only if

$$(a = b = 0, \alpha = 0), (a = b = 0, \beta = 0), (a = b = 0, \mu_1 = 1) \text{ or } (a = b = 0, \mu_2 = 1). \quad \square$$

Theorem 4.2. A homomorphism of $e_0(2)$ is biharmonic not harmonic if and only if it is conjugate to $\xi : (e_0(2), \langle \cdot, \cdot \rangle_1) \longrightarrow$

$(e_0(2), \langle \cdot, \cdot \rangle_2)$ where $\text{Mat}(\langle \cdot, \cdot \rangle_i, \mathbb{B}_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & v_i \end{pmatrix}, 0 < \mu_i \leq 1, v_i > 0$ and either:

1. $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$ and $(a^2 = b^2, ab \neq 0)$,

$$2. (\mu_1 \neq 0, \mu_2 \neq 1), \xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} \text{ and } (a = b = 0, \alpha^2 = \beta^2, \alpha\beta \neq 0) \text{ or}$$

$$\left(a = \epsilon b \mu_2 \sqrt{\mu_1}, b \neq 0, \alpha^2 = \beta^2 = \frac{\sqrt{\mu_1}(\mu_2^2 v_2 + a^2(\mu_2 - 1)^2)}{\mu_2 v_1 (\mu_2 - 1)^2 (1 - \mu_1)}, \beta = \epsilon \alpha, \epsilon = \pm 1. \right)$$

$$3. (\mu_1 \neq 0, \mu_2 \neq 1), \xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix} \text{ and } (a = b = 0, \alpha^2 = \beta^2, \alpha\beta \neq 0) \text{ or}$$

$$\left(a = \epsilon b \mu_2 \sqrt{\mu_1}, b \neq 0, \alpha^2 = \beta^2 = \frac{\sqrt{\mu_1}(\mu_2^2 v_2 + a^2(\mu_2 - 1)^2)}{\mu_2 v_1 (\mu_2 - 1)^2 (1 - \mu_1)}, \beta = \epsilon \alpha, \epsilon = \pm 1. \right).$$

Proof. As in the proof of [Theorem 4.1](#), according to [Table 1](#), and homomorphism $\xi : (e_0(2), \langle \cdot, \cdot \rangle_1) \longrightarrow (e_0(2), \langle \cdot, \cdot \rangle_2)$ has, up to a conjugation, the form $\text{Mat}(\langle \cdot, \cdot \rangle_i, \mathbb{B}_0) = \text{Diag}(1, \mu_i, v_i)$, $i = 1, 2, 0 < \mu_i \leq 1, v_i > 0$ and

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \gamma^2 \neq 1, \xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}.$$

• $\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}$ with $\gamma^2 \neq 1$. We have

$$\begin{cases} \tau_2(\xi) = -\frac{b\gamma((\gamma^2 v_2 + a^2)\mu_2^2 - 2a^2\mu_2 + a^2)}{v_1^2 v_2} X_1 + \frac{\gamma a(b^2\mu_2(\mu_2 - 1)^2 + \gamma^2 v_2)}{v_1^2 \mu_2^2 v_2} X_2 \\ + \frac{((\gamma^2 v_2 + a^2 - b^2)\mu_2^2 + (\gamma^2 v_2 - a^2 + b^2)\mu_2 + \gamma^2 v_2)b(\mu_2 - 1)a}{v_1^2 v_2^2 \mu_2} X_3. \end{cases}$$

If $\gamma = 0$ then

$$\tau_2(\xi) = \frac{(a - b)(a + b)(\mu_2 - 1)^2 ba}{v_1^2 v_2^2} X_3$$

and ξ is biharmonic not harmonic if and only if $a^2 = b^2$ and $ab \neq 0$.

If $\gamma \neq 0$ and $b \neq 0$ and $\tau_2(\xi) = 0$ then

$$(\gamma^2 v_2 + a^2)\mu_2^2 - 2a^2\mu_2 + a^2 = 0.$$

The discriminant of this equation on μ_2 is $\Delta = -4a^2\gamma^2 v_2 \leq 0$ and this equation has no solution. It is also clear that if $\gamma \neq 0$ and $a \neq 0$ then $\tau_2(\xi) \neq 0$. In conclusion ξ is biharmonic not harmonic if and only if

$$(\gamma = 0, a^2 = b^2, ab \neq 0).$$

• $\xi = \begin{pmatrix} \alpha & -\beta & a \\ \beta & \alpha & b \\ 0 & 0 & 1 \end{pmatrix}$. We have

$$\begin{cases} \tau(\xi) = -\frac{\mu_2^2 b}{v_1} X_1 + \frac{a}{\mu_2 v_1} X_2 + \frac{(\mu_2 - 1)(\alpha\beta v_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 v_1 v_2} X_3, \\ \tau_2(\xi) = A_1 X_1 + A_2 X_2 + A_3 X_3, \\ A_1 = -\frac{(b\mu_1(\mu_2 - 1)^2 a^2 + \beta\alpha v_1(\mu_2 - 1)^2(\mu_1 - 1)a + b\mu_1\mu_2^2 v_2)}{\mu_1 v_1^2 v_2}, \\ A_2 = \frac{((\alpha v_1(\mu_1 - 1)\beta + ab\mu_1)b\mu_2^3 - 2(\alpha v_1(\mu_1 - 1)\beta + ab\mu_1)b\mu_2^2 + (\alpha v_1(\mu_1 - 1)\beta + ab\mu_1)b\mu_2 + a\mu_1 v_2)}{v_1^2 \mu_2^2 v_2 \mu_1}, \\ \mu_1^2 v_1^2 v_2^2 \mu_2 A_3 = \mu_2 b v_1^2 (\mu_1 - 1)^2 (\mu_2 - 1)^2 \alpha^3 + \mu_2 v_1 ab\mu_1 (\mu_2 - 1)^2 (\mu_1 - 1) \alpha^2 \\ + \mu_2 \beta v_1 (\mu_1 - 1) (-\beta^2 \mu_1 v_1 + a^2 \mu_1 - b^2 \mu_1 + \beta^2 v_1) (\mu_2 - 1)^2 \alpha + ab\mu_1 (-\beta^2 \mu_1 \mu_2^2 v_1 + \mu_1 \mu_2^2 v_2 + a^2 \mu_1 \mu_2^2 \\ - b^2 \mu_1 \mu_2^2 + \beta^2 \mu_1 \mu_2 v_1 + \beta^2 \mu_2^2 v_1 + \mu_1 \mu_2 v_2 - a^2 \mu_1 \mu_2 + b^2 \mu_1 \mu_2 - \beta^2 \mu_2 v_1 + \mu_1 v_2) (\mu_2 - 1) \end{cases}$$

and the test matrix is given by

$$M_{\xi}(\mathbb{B}_0) = \begin{pmatrix} \frac{\mu_2}{v_1} & 0 & -\frac{a(\mu_2-1)}{v_1} \\ 0 & \frac{1}{v_1} & \frac{b(\mu_2-1)}{v_1} \\ -\frac{\mu_2 a}{v_1} & -\frac{b}{v_1} & \frac{(\mu_2-1)((\alpha^2-\beta^2)v_1+a^2-b^2)\mu_1+v_1(-\alpha^2+\beta^2)}{\mu_1 v_1} \end{pmatrix} \quad \text{and}$$

$$\det(M_{\xi}(\mathbb{B})) = \frac{\mu_2(\mu_2-1)(\alpha^2-\beta^2)(\mu_1-1)}{v_1^2 \mu_1}.$$

Note first that if $\mu_2 = 1$ then $\tau_2(\xi) = -\frac{b}{v_1^2}X_1 + \frac{a}{v_1^2}X_2$ and ξ is biharmonic if and only if it is harmonic. If $\mu_1 = 1$ then

$$A_1 = -\frac{(b(\mu_2-1)^2 a^2 + b\mu_2^2 v_2)}{v_1^2 v_2} \quad \text{and} \quad A_2 = \frac{(b^2 a \mu_2^3 - 2b^2 a \mu_2^2 + b^2 a \mu_2 + a v_2)}{v_1^2 \mu_2^2 v_2}$$

and one can see easily $A_1 = A_2 = 0$ if and only if $a = b = 0$ and hence ξ is biharmonic if and only if ξ is biharmonic.

We suppose now that $\mu_1 < 1$ and $\mu_2 < 1$. So $\det(M_{\xi}(\mathbb{B}_0)) = 0$ if and only if $\alpha^2 = \beta^2$. According to Proposition 2.2, if $\alpha^2 \neq \beta^2$ then ξ is biharmonic if and only if it is harmonic. We have also that if $\alpha = \beta = 0$ then $\tau_2(\xi) = 0$ if and only if $a = b = 0$.

Suppose that $\alpha^2 = \beta^2$ and $\alpha \neq 0$. If $a = b = 0$ then $\tau_2(\xi) = 0$ and ξ is biharmonic not harmonic. Suppose $(a, b) \neq 0$. Then the rank of $M_{\xi}(\mathbb{B}_0)$ is equal to 2 and its kernel has dimension one and

$$v = a(\mu_2 - 1)X_1 - b(\mu_2 - 1)\mu_2 X_2 + \mu_2 X_3$$

v is a generator of the kernel of $M_{\xi}(\mathbb{B}_0)$. But if ξ is biharmonic then $\tau(\xi)$ is in the kernel of $M_{\xi}(\mathbb{B}_0)$ and hence it is a multiple of v . Recall that

$$\tau(\xi) = -\frac{\mu_2 b}{v_1}X_1 + \frac{a}{\mu_2 v_1}X_2 + \frac{(\mu_2 - 1)(\epsilon \alpha^2 v_1(\mu_1 - 1) + ab\mu_1)}{\mu_1 v_1 v_2}X_3 \quad \text{and} \quad \epsilon = \pm 1.$$

But $(v, \tau(\xi))$ are linearly dependent if and only if

$$\begin{cases} \frac{(\mu_2 - 1) v_2 \mu_1 (-b^2 \mu_2^3 + a^2)}{\mu_2} = 0, \\ -b(\mu_2 - 1)^2 \mu_2 \epsilon (\mu_1 - 1) v_1 \alpha^2 - a \mu_1 (b^2 \mu_2^3 - 2b^2 \mu_2^2 + b^2 \mu_2 + v_2) = 0, \\ b \mu_1 (\mu_2 - 1)^2 a^2 + \alpha^2 \epsilon v_1 (\mu_2 - 1)^2 (\mu_1 - 1) a + b \mu_1 \mu_2^2 v_2 = 0. \end{cases}$$

Since $(a, b) \neq (0, 0)$, this is equivalent to

$$\begin{cases} a^2 = b^2 \mu_2^3, \\ \alpha^2 = -\frac{\mu_1 a (b^2 \mu_2 (\mu_2 - 1)^2 + v_2)}{b (\mu_2 - 1)^2 \mu_2 \epsilon (\mu_1 - 1) v_1} = -\frac{\mu_1 b (\mu_2^2 v_2 + a^2 (\mu_2 - 1)^2)}{\epsilon v_1 (\mu_2 - 1)^2 (\mu_1 - 1) a} \end{cases}$$

and this is equivalent to

$$\begin{cases} a^2 = b^2 \mu_2^3, \\ \alpha^2 = -\frac{\mu_1 b (\mu_2^2 v_2 + a^2 (\mu_2 - 1)^2)}{\epsilon v_1 (\mu_2 - 1)^2 (\mu_1 - 1) a}. \end{cases}$$

So $a = \epsilon b \mu_2 \sqrt{\mu_2}$ and we get the desired result.

The case of $\xi = \begin{pmatrix} \alpha & \beta & a \\ \beta & -\alpha & b \\ 0 & 0 & -1 \end{pmatrix}$ can be treated identically. \square

5. Harmonic and biharmonic homomorphisms on Sol

Theorem 5.1. A homomorphism of sol is harmonic if and only if it is conjugate to $\xi : (\text{sol}, (\cdot, \cdot)_1) \rightarrow (\text{sol}, (\cdot, \cdot)_2)$ where:

1. $\langle \cdot, \cdot \rangle_i = \text{Diag}(1, 1, v_i)$, $i = 1, 2$ and $v_i > 0$ and either

$$[\xi = \xi_1, (a = b = 0) \text{ or } (\gamma = 0, a^2 = b^2)], [\xi = \xi_2, (a = b = 0, \alpha^2 = \beta^2)] \text{ or } [\xi = \xi_3, (a = b = 0, \alpha^2 = \beta^2)].$$

2. $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_1 > 0$, $\mu_2 > 1$ and either

$$\left[\xi = \xi_1, (a = b = 0) \text{ or } \left(\gamma = 0, \mu_2 = \frac{a^2}{b^2} \right) \right],$$

$$\begin{cases} \xi = \xi_2, (a = b = \alpha = \beta = 0) \text{ or } \left(a = b = 0, \mu_2 = \frac{\alpha^2}{\beta^2} \right) \\ \xi = \xi_3, (a = b = \alpha = \beta = 0) \text{ or } \left(a = b = 0, \mu_2 = \frac{\alpha^2}{\beta^2} \right). \end{cases}$$

$$3. (\cdot, \cdot)_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, (\cdot, \cdot)_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix} \text{ and } \nu_i > 0, \mu_1 > 1,$$

$$\begin{cases} \xi = \xi_1, (a = b = 0) \text{ or } (\gamma = 0, a^2 = b^2) \\ \xi = \xi_2, (a = b = \alpha = \beta = 0) \text{ or } \left(a = b = 0, \mu_1 = \frac{\beta^2}{\alpha^2} \right) \\ \xi = \xi_3, (a = b = \alpha = \beta = 0) \text{ or } \left(a = b = 0, \mu_1 = \frac{\beta^2}{\alpha^2} \right). \end{cases}$$

$$4. (\cdot, \cdot)_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, (\cdot, \cdot)_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix} \text{ and } \nu_i > 0, \mu_i > 1 \text{ and either}$$

$$\begin{cases} \xi = \xi_1, (a = b = 0) \text{ or } \left(\gamma = 0, \mu_2 = \frac{a^2}{b^2} \right) \\ [\xi = \xi_2, (a = b = \alpha = \beta = 0) \text{ or } (a = b = 0, (\alpha, \beta) \neq (0, 0), \alpha^2 \mu_1 = \beta^2 \mu_2)] \text{ or} \\ [\xi = \xi_3, (a = b = \alpha = \beta = 0) \text{ or } (a = b = 0, (\alpha, \beta) \neq (0, 0), \alpha^2 \mu_1 \mu_2 = \beta^2)]. \end{cases}$$

The homomorphisms $\xi_i, i = 1..3$ are given by

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \xi_2 = \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \xi_3 = \begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}.$$

Proof. We use Table 1 to get all the conjugation classes of homomorphisms of \mathfrak{sol} and for each one we compute $\tau(\xi)$.

• $(\cdot, \cdot)_i = \text{Diag}(1, 1, \nu_i), i = 1, 2$ and $\nu_i > 0$. We have

$$\begin{cases} \tau(\xi_1) = -\frac{\gamma a}{\nu_1} X_1 + \frac{\gamma b}{\nu_1} X_2 + \frac{a^2 - b^2}{\nu_2 \nu_1} X_3, \\ \tau(\xi_2) = -\frac{a}{\nu_1} X_1 + \frac{b}{\nu_1} X_2 + \frac{(\alpha^2 - \beta^2) \nu_1 + a^2 - b^2}{\nu_2 \nu_1} X_3, \\ \tau(\xi_3) = \frac{a}{\nu_1} X_1 - \frac{b}{\nu_1} X_2 + \frac{(-\alpha^2 + \beta^2) \nu_1 + a^2 - b^2}{\nu_2 \nu_1} X_3. \end{cases}$$

$$\bullet (\cdot, \cdot)_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}, (\cdot, \cdot)_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix} \text{ and } \nu_i > 0, \mu_2 > 1. \text{ We have}$$

$$\begin{cases} \tau(\xi_1) = -\frac{((a+2b)\mu_2+a)\gamma}{(\mu_2-1)\nu_1} X_1 + \frac{\gamma(b\mu_2+2a+b)}{(\mu_2-1)\nu_1} X_2 + \frac{-b^2\mu_2+a^2}{\nu_2\nu_1} X_3, \\ \tau(\xi_2) = -\frac{(a+2b)\mu_2+a}{(\mu_2-1)\nu_1} X_1 + \frac{b\mu_2+2a+b}{(\mu_2-1)\nu_1} X_2 + \frac{(-\beta^2\mu_2+\alpha^2)\nu_1-b^2\mu_2+a^2}{\nu_2\nu_1} X_3, \\ \tau(\xi_3) = \frac{(a+2b)\mu_2+a}{(\mu_2-1)\nu_1} X_1 - \frac{\mu_2b+2a+b}{(\mu_2-1)\nu_1} X_2 + \frac{(-\alpha^2\mu_2+\beta^2)\nu_1-b^2\mu_2+a^2}{\nu_2\nu_1} X_3. \end{cases}$$

• $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_1 > 1$. We have

$$\left\{ \begin{array}{l} \tau(\xi_1) = -\frac{\gamma a}{\nu_1} X_1 + \frac{\gamma b}{\nu_1} X_2 + \frac{a^2 - b^2}{\nu_2 \nu_1} X_3, \\ \tau(\xi_2) = -\frac{a}{\nu_1} X_1 + \frac{b}{\nu_1} X_2 + \frac{(\alpha^2 \nu_1 + a^2 - b^2) \mu_1 - \beta^2 \nu_1 - a^2 + b^2}{\nu_2 (\mu_1 - 1) \nu_1} X_3, \\ \tau(\xi_3) = \frac{a}{\nu_1} X_1 - \frac{b}{\nu_1} X_2 + \frac{(-\alpha^2 \nu_1 + a^2 - b^2) \mu_1 + \beta^2 \nu_1 - a^2 + b^2}{\nu_2 (\mu_1 - 1) \nu_1} X_3. \end{array} \right.$$

• $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & \nu_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & \nu_2 \end{pmatrix}$ and $\nu_i > 0$, $\mu_i > 1$. We have

$$\left\{ \begin{array}{l} \tau(\xi_1) = -\frac{((a+2b)\mu_2+a)\gamma}{(\mu_2-1)\nu_1} X_1 + \frac{\gamma(b\mu_2+2a+b)}{(\mu_2-1)\nu_1} X_2 + \frac{-b^2\mu_2+a^2}{\nu_2\nu_1} X_3, \\ \tau(\xi_2) = -\frac{(a+2b)\mu_2+a}{(\mu_2-1)\nu_1} X_1 + \frac{b\mu_2+2a+b}{(\mu_2-1)\nu_1} X_2 + \frac{(\alpha^2\nu_1-b^2\mu_2+a^2)\mu_1+(-\beta^2\nu_1+b^2)\mu_2-a^2}{\nu_2(\mu_1-1)\nu_1} X_3, \\ \tau(\xi_3) = \frac{(a+2b)\mu_2+a}{(\mu_2-1)\nu_1} X_1 - \frac{b\mu_2+2a+b}{(\mu_2-1)\nu_1} X_2 + \frac{((-a^2\nu_1-b^2)\mu_2+a^2)\mu_1+b^2\mu_2+\beta^2\nu_1-a^2}{\nu_2(\mu_1-1)\nu_1} X_3. \end{array} \right.$$

One can check that $\tau(\xi_i) = 0$ are equivalent to the conditions given in the theorem. \square

Theorem 5.2. A homomorphism of \mathfrak{sol} is biharmonic if and only if it is harmonic.

Proof. As above, we put

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & \gamma \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \alpha & 0 & a \\ 0 & \beta & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \xi_3 = \begin{pmatrix} 0 & \beta & a \\ \alpha & 0 & b \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\xi : (\mathfrak{sol}, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{sol}, \langle \cdot, \cdot \rangle_2)$ a homomorphism. Table 1 gives all the possible conjugation classes of ξ and we will show that for each case ξ is biharmonic if and only if ξ is harmonic.

• $\xi = \xi_1$ and $\langle \cdot, \cdot \rangle_i = \text{Diag}(1, 1, \nu_i)$, $i = 1, 2$ and $\nu_i > 0$. We have

$$\tau_2(\xi) = -2 \frac{(1/2 \gamma^2 \nu_2 + a^2 - b^2) \alpha \gamma}{\nu_1^2 \nu_2} X_1 - 2 \frac{(-1/2 \gamma^2 \nu_2 + a^2 - b^2) \gamma b}{\nu_1^2 \nu_2} X_2 + \frac{\gamma^2 (a^2 - b^2) \nu_2 + 2 a^4 - 2 b^4}{\nu_2^2 \nu_1^2} X_3.$$

One can see easily that $\tau_2(\xi) = 0$ if and only if $(a = b = 0)$ or $(\gamma = 0, a^2 = b^2)$ which, according to Theorem 5.1, is equivalent to ξ is harmonic.

• $\xi = \xi_2$ and $\langle \cdot, \cdot \rangle_i = \text{Diag}(1, 1, \nu_i)$, $i = 1, 2$ and $\nu_i > 0$. We have

$$\left\{ \begin{array}{l} \tau_2(\xi) = -2 \frac{a((\alpha^2 - \beta^2) \nu_1 + a^2 - b^2 + 1/2 \nu_2)}{\nu_1^2 \nu_2} X_1 - 2 \frac{((\alpha^2 - \beta^2) \nu_1 + a^2 - b^2 - 1/2 \nu_2) b}{\nu_1^2 \nu_2} X_2 \\ + \frac{2 \alpha^4 \nu_1^2 - 2 \beta^4 \nu_1^2 + 4 a^2 \alpha^2 \nu_1 - 4 b^2 \beta^2 \nu_1 + 2 a^4 - 2 b^4 + a^2 \nu_2 - b^2 \nu_2}{\nu_1^2 \nu_2^2} X_3. \end{array} \right.$$

We have also

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} \nu_1^{-1} & 0 & -2 \frac{a}{\nu_1} \\ 0 & \nu_1^{-1} & -2 \frac{b}{\nu_1} \\ -\frac{a}{\nu_1} & -\frac{b}{\nu_1} & \frac{2 a^2 \nu_1 + 2 b^2 \nu_1 + 2 a^2 + 2 b^2}{\nu_1} \end{bmatrix} \quad \text{and} \quad \det(M_\xi(\mathbb{B}_0)) = 2 \frac{\alpha^2 + \beta^2}{\nu_1^2}.$$

According to Proposition 2.2, if $(\alpha, \beta) \neq (0, 0)$ then ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

• $\xi = \xi_3$ and $\langle \cdot, \cdot \rangle_i = \text{Diag}(1, 1, \nu_i)$, $i = 1, 2$ and $\nu_i > 0$. We have

$$\left\{ \begin{array}{l} \tau_2(\xi) = \frac{a(-2 \alpha^2 \nu_1 + 2 \beta^2 \nu_1 + 2 a^2 - 2 b^2 + \nu_2)}{\nu_1^2 \nu_2} X_1 + \frac{b(-2 \alpha^2 \nu_1 + 2 \beta^2 \nu_1 + 2 a^2 - 2 b^2 - \nu_2)}{\nu_1^2 \nu_2} X_2 \\ + \frac{-2 \alpha^4 \nu_1^2 + 2 \beta^4 \nu_1^2 + 4 a^2 \beta^2 \nu_1 - 4 \alpha^2 \beta^2 \nu_1 + 2 a^4 - 2 b^4 + a^2 \nu_2 - b^2 \nu_2}{\nu_1^2 \nu_2^2} X_3 \end{array} \right.$$

and

$$M_{\xi}(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & 0 & 2 \frac{a}{v_1} \\ 0 & v_1^{-1} & 2 \frac{b}{v_1} \\ \frac{a}{v_1} & \frac{b}{v_1} & 2 \frac{a^2 v_1 + 2 b^2 v_1 + 2 a^2 + 2 b^2}{v_1} \end{bmatrix} \quad \text{and} \quad \det(M_{\xi}(\mathbb{B}_0)) = 2 \frac{\alpha^2 + \beta^2}{v_1^2}$$

and the situation is similar to the precedent cases.

- $\xi = \xi_1$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_1 > 0, \mu_2 > 1$.

$$\left\{ \begin{array}{l} \tau(\xi) = -\frac{\gamma ((a+2b)\mu_2+a)}{(v_2-1)v_1}X_1 + \frac{\gamma (b\mu_2+2a+b)}{(v_2-1)v_1}X_2 + \frac{-b^2\mu_2+a^2}{v_2v_1}X_3, \\ \tau_2(\xi) = -2 \left(\frac{-b^2(a-b)\mu_2^3+(a^3-a^2b)+(1/2\gamma^2v_2+b^2)a+2b\gamma^2v_2-b^3}{v_1^2v_2(\mu_2-1)^2} \right) \mu_2^2 + (3a\gamma^2v_2+2b\gamma^2v_2-a^3+a^2b)\mu_2 + 1/2a\gamma^2v_2 X_1 \\ + 2 \left(\frac{\gamma (b^3\mu_2^3-b(-1/2\gamma^2v_2+a^2+ab+b^2)\mu_2^2+(ab^2+(3\gamma^2v_2+a^2)b+2a\gamma^2v_2+a^3)\mu_2+2a\gamma^2v_2+1/2b\gamma^2v_2-a^3)}{v_1^2v_2(\mu_2-1)^2} \right) X_2 \\ + 2 \left(\frac{(-b^2\mu_2+a^2)(b^2\mu_2^2+(1/2\gamma^2v_2+a^2-b^2)\mu_2+3/2\gamma^2v_2-a^2)}{v_1^2v_2^2(\mu_2-1)} \right) X_3 \end{array} \right.$$

Suppose that ξ is biharmonic not harmonic. Then, by virtue of [Theorem 5.1](#), $(a, b) \neq 0$ and $(\gamma \neq 0 \text{ or } \mu_2 \neq \frac{a^2}{b^2})$. If $\mu_2 = \frac{a^2}{b^2}$ then a direct computation shows that

$$\tau_2(\xi) = -\frac{(a+b)^2\gamma^3a}{(a-b)^2v_1^2}X_1 + \frac{b(a+b)^2\gamma^3}{(a-b)^2v_1^2}X_2$$

and since $(a, b) \neq (0, 0)$, $\gamma \neq 0$ and $\mu_2 > 1$ this is impossible so we must have $\mu_2 \neq \frac{a^2}{b^2}$. In this case, since the last coordinate of $\tau_2(\xi)$ vanishes, we get

$$\left(\frac{1}{2}\gamma^2v_2+a^2 \right) \mu_2 + \frac{3}{2}\gamma^2v_2-a^2+(\mu_2^2-\mu_2)b^2=\frac{1}{2}\gamma^2v_2\mu_2+a^2(\mu_2-1)+\frac{3}{2}\gamma^2v_2+(\mu_2^2-\mu_2)b^2=0$$

But since $\mu_2 > 1$, this is equivalent to $\gamma = a = b = 0$ which is a contradiction. Finally, ξ is biharmonic if and only if it is harmonic.

- $\xi = \xi_2$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_1 > 0, \mu_2 > 1$. We have

$$M_{\xi}(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & -v_1^{-1} & -2 \frac{a}{v_1} \\ -v_1^{-1} & \frac{\mu_2}{v_1} & -2 \frac{b\mu_2}{v_1} \\ \frac{-a+b}{v_1} & \frac{-b\mu_2+a}{v_1} & \frac{2v_1\beta^2\mu_2+2a^2v_1+2b^2\mu_2+2a^2}{v_1} \end{bmatrix} \quad \text{and} \quad \det(M_{\xi}(\mathbb{B}_0)) = 2 \frac{(\mu_2-1)(\beta^2\mu_2+a^2)}{v_1^2}$$

If $(\alpha, \beta) \neq (0, 0)$ then, according to [Proposition 2.2](#), ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_1 > 0, \mu_2 > 1$. We have

$$M_{\xi}(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & -v_1^{-1} & 2 \frac{a}{v_1} \\ -v_1^{-1} & \frac{\mu_2}{v_1} & 2 \frac{b\mu_2}{v_1} \\ \frac{a-b}{v_1} & \frac{b\mu_2-a}{v_1} & \frac{2\alpha^2\mu_2v_1+2b^2\mu_2+2\beta^2v_1+2a^2}{v_1} \end{bmatrix} \quad \text{and} \quad \det(M_{\xi}(\mathbb{B}_0)) = 2 \frac{(\mu_2-1)(\alpha^2\mu_2+\beta^2)}{v_1^2}$$

If $(\alpha, \beta) \neq (0, 0)$ then, according to [Proposition 2.2](#), ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_1$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_1 > 0, \mu_1 > 1$. We have

$$\tau_2(\xi) = -2 \frac{\gamma (1/2\gamma^2v_2+a^2-b^2)a}{v_1^2v_2}X_1 - 2 \frac{b(-1/2\gamma^2v_2+a^2-b^2)\gamma}{v_1^2v_2}X_2 + \frac{\gamma^2(a^2-b^2)v_2+2a^4-2b^4}{v_2^2v_1^2}X_3$$

One can see easily that $\tau_2(\xi) = 0$ if and only if $(a = b = 0)$ or $(\gamma = 0, a^2 = b^2)$ which, according to [Theorem 5.1](#), is equivalent to ξ is harmonic.

- $\xi = \xi_2$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_i > 0$, $\mu_1 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & 0 & -2 \frac{a}{v_1} \\ 0 & v_1^{-1} & -2 \frac{b}{v_1} \\ -\frac{a}{v_1} & -\frac{b}{v_1} & \frac{(2a^2v_1+2a^2+2b^2)\mu_1+2\beta^2v_1-2a^2-2b^2}{(v_1-1)v_1} \end{bmatrix} \text{ and } \det(M_\xi(\mathbb{B}_0)) = 2 \frac{\alpha^2\mu_1+\beta^2}{v_1^2(\mu_1-1)}.$$

If $(\alpha, \beta) \neq (0, 0)$ then, according to [Proposition 2.2](#), ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_i > 0$, $\mu_1 > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & 0 & 2 \frac{a}{v_1} \\ 0 & v_1^{-1} & 2 \frac{b}{v_1} \\ \frac{a}{v_1} & \frac{b}{v_1} & \frac{(2\alpha^2v_1+2a^2+2b^2)\mu_1+2\beta^2v_1-2a^2-2b^2}{(v_1-1)v_1} \end{bmatrix} \text{ and } \det(M_\xi(\mathbb{B}_0)) = 2 \frac{\alpha^2\mu_1+\beta^2}{v_1^2(\mu_1-1)}.$$

The situation is similar to the precedent case.

- $\xi = \xi_1$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_i > 0$, $\mu_i > 1$. We have

$$\left\{ \begin{array}{l} \tau_2(\xi) = -2 \frac{\gamma(-b^2(a-b)\mu_2^3 + (a^3 - a^2b + (1/2)\gamma^2v_2 + b^2)a + 2b\gamma^2v_2 - b^3)\mu_2^2 + (3a\gamma^2v_2 + 2b\gamma^2v_2 - a^3 + a^2b)\mu_2 + 1/2a\gamma^2v_2}{v_1^2v_2(\mu_2 - 1)^2} X_1 \\ + 2 \frac{\gamma(b^3\mu_2^3 - b(-1/2)\gamma^2v_2 + a^2 + ab + b^2)\mu_2^2 + (ab^2 + (3)\gamma^2v_2 + a^2)b + 2a\gamma^2v_2 + a^3)\mu_2 + 2a\gamma^2v_2 + 1/2b\gamma^2v_2 - a^3}{v_1^2v_2(\mu_2 - 1)^2} X_2 + \\ 2 \frac{(-b^2\mu_2 + a^2)(b^2\mu_2^2 + (1/2)\gamma^2v_2 + a^2 - b^2)\mu_2 + 3/2\gamma^2v_2 - a^2}{v_1^2v_2^2(\mu_2 - 1)} X_3 \end{array} \right.$$

One can see that $\tau_2(\xi)$ is the same as in the case $\xi = \xi_1$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and we can

use the same arguments to conclude.

- $\xi = \xi_2$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_i > 0$, $\mu_i > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & -v_1^{-1} & -2 \frac{a}{v_1} \\ -v_1^{-1} & \frac{\mu_2}{v_1} & -2 \frac{\mu_2b}{v_1} \\ -\frac{a+b}{v_1} & -\frac{\mu_2b+a}{v_1} & \frac{(2a^2v_1+2b^2\mu_2+2a^2)\mu_1+(2\beta^2v_1-2b^2)\mu_2-2a^2}{v_1(\mu_1-1)} \end{bmatrix} \text{ and } \det(M_\xi(\mathbb{B}_0)) = 2 \frac{(\mu_2-1)(\alpha^2\mu_1+\beta^2\mu_2)}{v_1^2(\mu_1-1)}.$$

If $(\alpha, \beta) \neq (0, 0)$ then, according to [Proposition 2.2](#), ξ is biharmonic if and only if it is harmonic. If $\alpha = \beta = 0$ then $\xi = \xi_1$ with $\gamma = 1$ and we can use the arguments used in the precedent case to conclude.

- $\xi = \xi_3$ and $\langle \cdot, \cdot \rangle_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}$, $\langle \cdot, \cdot \rangle_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}$ and $v_i > 0$, $\mu_i > 1$. We have

$$M_\xi(\mathbb{B}_0) = \begin{bmatrix} v_1^{-1} & -v_1^{-1} & 2 \frac{a}{v_1} \\ -v_1^{-1} & \frac{\mu_2}{v_1} & 2 \frac{\mu_2b}{v_1} \\ \frac{a-b}{v_1} & \frac{\mu_2b-a}{v_1} & \frac{((2\alpha^2v_1+2b^2)\mu_2+2a^2)\mu_1+(2\beta^2v_1-2b^2)\mu_2+2\beta^2v_1-2a^2}{v_1(\mu_1-1)} \end{bmatrix} \text{ and } \det(M_\xi(\mathbb{B}_0)) = 2 \frac{(\mu_2-1)(\alpha^2\mu_1\mu_2+\beta^2)}{v_1^2(\mu_1-1)}.$$

The situation is similar to the precedent case. \square

6. Harmonic and biharmonic homomorphisms of $\mathfrak{su}(2)$

The following proposition is a consequence of [2, Proposition 2.5].

Proposition 6.1. Let $\xi : (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_1) \longrightarrow (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_2)$ be an automorphism. If $\langle \cdot, \cdot \rangle_1$ or $\langle \cdot, \cdot \rangle_2$ is bi-invariant then ξ is harmonic.

Any homomorphism of $\mathfrak{su}(2)$ is an automorphism and it is a product $\xi_3(a) \circ \xi_2(b) \circ \xi_1(c)$ where

$$\xi_1(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & \sin(a) \\ 0 & -\sin(a) & \cos(a) \end{pmatrix}, \xi_2(a) = \begin{pmatrix} \cos(a) & 0 & \sin(a) \\ 0 & 1 & 0 \\ -\sin(a) & 0 & \cos(a) \end{pmatrix}, \xi_3(a) = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\xi_i : (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_1) \longrightarrow (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_2)$ with $\langle \cdot, \cdot \rangle_j = \text{Diag}(\lambda_j, \mu_j, v_j)$ then

$$\left\{ \begin{array}{l} \tau(\xi_1(a)) = -\frac{\sin(a) \cos(a) (\mu_2 - v_2) (\mu_1 - v_1)}{\lambda_2 \mu_1 v_1} X_1, \\ \tau_2(\xi_1(a)) = -2 \frac{(\mu_2 - v_2)^2 (-v_1 + \mu_1)^2 \cos(a) ((\cos(a))^2 - 1/2) \sin(a)}{\mu_1^2 v_1^2 \lambda_2^2} X_1, \\ \tau(\xi_2(a)) = \frac{\sin(a) \cos(a) (\lambda_2 - v_2) (\lambda_1 - v_1)}{\mu_2 \lambda_1 v_1} X_2, \\ \tau_2(\xi_2(a)) = \frac{(2(\cos(a))^2 - 1) \cos(a) (\lambda_2 - v_2)^2 (-v_1 + \lambda_1)^2 \sin(a)}{\lambda_1^2 v_1^2 \mu_2^2} X_2, \\ \tau(\xi_3(a)) = -\frac{\cos(a) \sin(a) (\lambda_2 - \mu_2) (\lambda_1 - \mu_1)}{\lambda_1 \mu_1 v_2} X_3, \\ \tau_2(\xi_3(a)) = -2 \frac{\cos(a) \sin(a) ((\cos(a))^2 - 1/2) (\mu_2 - \lambda_2)^2 (\mu_1 - \lambda_1)^2}{\lambda_1^2 \mu_1^2 v_2^2} X_3. \end{array} \right.$$

So we get:

Proposition 6.2.

1. If $\mu_2 = v_2$ or $\mu_1 = v_1$ then $\xi_1(a)$ is harmonic.
2. If $\mu_2 \neq v_2$ and $\mu_1 \neq v_1$ then $\xi_1(a)$ is harmonic if and only if $\sin(2a) = 0$ and $\xi_1(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.
3. If $\lambda_2 = v_2$ or $\lambda_1 = v_1$ then $\xi_2(a)$ is harmonic.
4. If $\lambda_2 \neq v_2$ and $\lambda_1 \neq v_1$ then $\xi_2(a)$ is harmonic if and only if $\sin(2a) = 0$ and $\xi_2(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.
5. If $\lambda_2 = \mu_2$ or $\lambda_1 = \mu_1$ then $\xi_3(a)$ is harmonic.
6. If $\lambda_2 \neq \mu_2$ and $\lambda_1 \neq \mu_1$ then $\xi_3(a)$ is harmonic if and only if $\sin(2a) = 0$ and $\xi_3(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.

Theorem 6.1. We consider the automorphism

$$\xi = \xi_3(a) \circ \xi_2(b) \circ \xi_1(c) : (\mathfrak{su}(2), \text{diag}(\lambda_1, \mu_1, v_1)) \longrightarrow (\mathfrak{su}(2), \text{diag}(\lambda_2, \mu_2, v_2)), \quad 0 \leq v_i < \mu_i \leq \lambda_i, i = 1, 2.$$

1. If $(0 < v_1 < \mu_1 < \lambda_1, 0 < v_2 < \mu_2 < \lambda_2)$ or $(0 < v_1 < \mu_1 < \lambda_1, 0 < v_2 < \mu_2 < \lambda_2)$ then ξ is harmonic if and only if one of the following condition holds:

- (i) $\cos(b) = 0, \sin(b) = 1$ and $\sin(2(a - c)) = 0$,
- (ii) $\cos(b) = 0, \sin(b) = -1$ and $\sin(2(a + c)) = 0$,
- (iii) $\sin(b) = 0$ and $\sin(a) = 0$.

2. If $(0 < v_1 < \mu_1 < \lambda_1, 0 < v_2 = \mu_2 < \lambda_2)$ then ξ is harmonic if and only if one of the following condition holds:

- (i) $\cos(b) = 0, \sin(b) = 1$ and $\sin(2(a - c)) = 0$,
- (ii) $\cos(b) = 0, \sin(b) = -1$ and $\sin(2(a + c)) = 0$,
- (iii) $\sin(b) = 0$ and $\sin(a) = 0$.

3. If $(0 < v_1 < \mu_1 = \lambda_1, 0 < v_2 < \mu_2 < \lambda_2)$ then ξ is harmonic if and only if one of the following condition holds:

- (i) $\cos(b) = 0, \sin(b) = 1$ and $\sin(2(a - c)) = 0$,
- (ii) $\cos(b) = 0, \sin(b) = -1$ and $\sin(2(a + c)) = 0$,
- (iv) $\sin(b) = 0, \sin(2c) = 0$ and $\cos(a) = 0$.

- (iii) $\sin(b) = 0$ and $\sin(a) = \sin(2c) = 0$.
(iv) $\sin(b) = 0$ and $\cos(a) = \sin(2c) = 0$.
4. If $(0 < v_1 < \mu_1 < \lambda_1, 0 < v_2 < \mu_2 = \lambda_2)$ then ξ is harmonic if and only if $\cos(b) = 0$ or $(\sin(b) = \sin(2c) = 0)$.
5. If $(0 < v_1 = \mu_1 < \lambda_1, 0 < v_2 < \mu_2 < \lambda_2)$ then ξ is harmonic if and only if $\cos(b) = 0$ or $(\sin(b) = \sin(2a) = 0)$.
6. If $(0 < v_1 = \mu_1 < \lambda_1, 0 < v_2 = \mu_2 < \lambda_2)$ then ξ is harmonic if and only if $(\cos(b) = 0), (\cos(a) = 0)$ or $(\sin(b) = \sin(a) = 0)$.
7. If $(0 < v_1 = \mu_1 < \lambda_1, 0 < v_2 < \mu_2 = \lambda_2)$ then ξ is harmonic if and only if $\sin(2b) = 0$.
8. If $(0 < v_1 < \mu_1 = \lambda_1, 0 < v_2 < \mu_2 = \lambda_2)$ then ξ is harmonic if and only if $\cos(b)\cos(c) = 0$ or $(\sin(b) = \sin(c) = 0)$.
9. If $(0 < v_1 < \mu_1 = \lambda_1, 0 < v_2 = \mu_2 < \lambda_2)$ then ξ is harmonic if and only if one of the following situations holds

- (i) $\cos(b) = 0, \sin(b) = 1$ and $\sin(2(a - c)) = 0$,
(ii) $\cos(b) = 0, \sin(b) = -1$ and $\sin(2(a + c)) = 0$,
(iii) $\cos(c) = 0$ and $\sin(2a) = 0$,
(iv) $\sin(b) = \sin(c) = 0$,
(v) $\cos(a) = (-1)^k \frac{\sin(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}}$ and $\sin(a) = (-1)^{k+1} \frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}}$.

Proof. We have

$$\begin{aligned} \tau(\xi) &= A_1 X_1 + A_2 X_2 + A_3 X_3, \\ \lambda_2 \lambda_1 \mu_1 v_1 A_1 &= \cos(b) (\sin(a) \sin(b) \lambda_1 (\mu_1 - v_1) (\cos(c))^2 - \sin(c) \lambda_1 \cos(a) (\mu_1 - v_1) \cos(c) + \sin(a) \sin(b) \\ &\quad \times v_1 (\lambda_1 - \mu_1)) (\mu_2 - v_2) \\ &= \cos(b) (\mu_2 - v_2) R, \\ \mu_2 \lambda_1 \mu_1 v_1 A_2 &= \cos(b) (\sin(b) (\lambda_1 (\mu_1 - v_1) (\cos(c))^2 + v_1 (\lambda_1 - \mu_1)) \cos(a) + \cos(c) \sin(a) \sin(c) \lambda_1 (\mu_1 - v_1)) \\ &\quad \times (\lambda_2 - v_2) \\ &= (\lambda_2 - v_2) \cos(b) S, \\ z &:= -v_2 \lambda_1 \mu_1 v_1 A_3 = (\lambda_2 - \mu_2) (2 \cos(c) \sin(b) \sin(c) \lambda_1 (\mu_1 - v_1) (\cos(a))^2 \\ &\quad + (\lambda_1 ((\cos(b))^2 - 2) (\mu_1 - v_1) (\cos(c))^2 + v_1 (\lambda_1 - \mu_1) (\cos(b))^2 + \lambda_1 (\mu_1 - v_1)) \sin(a) \cos(a) - \cos(c) \\ &\quad \times \sin(b) \sin(c) \lambda_1 (\mu_1 - v_1)). \end{aligned}$$

On the other hand, the following relations are straightforward to establish:

$$\begin{cases} R \cos(a) - S \sin(a) = -\lambda_1 (\mu_1 - v_1) \sin(c) \cos(c), \\ R \sin(a) + S \cos(a) = \sin(b) (\lambda_1 (\mu_1 - v_1) (\cos(c))^2 + v_1 (\lambda_1 - \mu_1)) \end{cases} \quad (9)$$

and if $\cos(b) = 0$ then

$$z = \begin{cases} \frac{1}{2} \sin(2(c - a)) (\lambda_2 - \mu_2) \lambda_1 (\mu_1 - v_1) & \text{if } \sin(b) = 1, \\ \frac{1}{2} \sin(2(c + a)) (\lambda_2 - \mu_2) \lambda_1 (\mu_1 - v_1) & \text{if } \sin(b) = -1. \end{cases} \quad (10)$$

Suppose that $(0 < v_1 < \mu_1 < \lambda_1, 0 < v_2 < \mu_2 < \lambda_2)$. Then ξ is harmonic if and only if

$$R \cos(b) = S \cos(b) = z = 0.$$

We distinguish two cases:

- $\cos(b) = 0$. Then ξ is harmonic if and only if $z = 0$ and, by virtue of (10), we get the desired result.
- $\cos(b) \neq 0$ then from (9) $\sin(b) = 0$ and $\sin(c) \cos(c) = 0$ and one can check easily that ξ is harmonic if and only if $\cos(a) \sin(a) = 0$.

Except the last case, all the other cases can be deduced in the same way. Let us complete the proof by treating the last case. We suppose that $(0 < v_1 < \mu_1 = \lambda_1, 0 < v_2 = \mu_2 < \lambda_2)$. Then

$$\begin{cases} \tau(\xi) = \frac{\cos(b) \cos(c) (\lambda_1 - v_1) (\lambda_2 - \mu_2) R_1}{\mu_2 \lambda_1 v_1} X_2 - \frac{2 (\lambda_1 - v_1) (\lambda_2 - \mu_2) S_1}{\mu_2 \lambda_1 v_1} X_3, \\ R_1 = \sin(a) \sin(c) + \cos(a) \sin(b) \cos(c), \\ S_1 = \sin(b) (\cos(a))^2 \sin(c) \cos(c) + \frac{1}{2} \sin(a) (1 + ((\cos(b))^2 - 2) (\cos(c))^2) \cos(a) - \frac{1}{2} \sin(b) \sin(c) \cos(c). \end{cases}$$

If $\cos(b) = 0$ then

$$S_1 = \begin{cases} \frac{1}{4} \sin(2(c-a)) & \text{if } \sin(b) = 1, \\ -\frac{1}{4} \sin(2(c+a)) & \text{if } \sin(b) = -1 \end{cases}$$

and we get (i) and (ii).

If $\cos(c) = 0$ then $S_1 = \frac{1}{4} \sin(2a)$ and we get (iii).

If $\sin(b) = \sin(c) = 0$ the $S_1 = R_1 = 0$ and hence ξ is harmonic.

Suppose now that $\cos(b) \neq 0$, $\cos(c) \neq 0$ and $(\sin(b), \sin(c)) \neq (0, 0)$. Then ξ is harmonic if and only if $R_1 = S_1 = 0$.

We have

$$R_1 = \sin(a) \frac{\sin(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}} + \cos(a) \frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}} = \sin(a + \alpha)$$

where

$$\cos(\alpha) = \frac{\sin(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}} \quad \text{and} \quad \sin(\alpha) = \frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}}.$$

So $R_1 = 0$ if and only if $a + \alpha = k\pi$ where $k \in \mathbb{Z}$. Thus

$$\begin{aligned} \cos(a) &= (-1)^k \cos(\alpha) = (-1)^k \frac{\sin(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}} \quad \text{and} \\ \sin(a) &= -(-1)^{k+1} \frac{\sin(b)\cos(c)}{\sqrt{\sin^2(c) + \sin^2(b)\cos^2(c)}}. \end{aligned}$$

If we replace $\cos(a)$ and $\sin(a)$ in S_1 , we get $S_1 = 0$ which completes the proof. \square

The situation for biharmonic homomorphisms is more complicated. We have the following non trivial biharmonic homomorphism which is not harmonic.

Example 1. The homomorphism $\xi = \xi_3(a) \circ \xi_2(b) \circ \xi_1(c) : (\mathfrak{su}(2), \text{diag}(\lambda_1, \mu_1, \nu_1)) \rightarrow (\mathfrak{su}(2), \text{diag}(\lambda_2, \mu_2, \nu_2))$ is biharmonic not harmonic if

$$\mu_1 = \nu_1, \mu_2 = \nu_2 \quad \text{and} \quad \cos(a) = \cos(b) = \left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

7. Harmonic and biharmonic homomorphisms of $\mathfrak{sl}(2, \mathbb{R})$

Any homomorphism of $\mathfrak{sl}(2, \mathbb{R})$ is an automorphism and it is a product $\xi_3(a) \circ \xi_2(b) \circ \xi_1(c)$ where

$$\xi_1(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(a) & \sinh(a) \\ 0 & \sinh(a) & \cosh(a) \end{pmatrix}, \xi_2(a) = \begin{pmatrix} \cosh(a) & 0 & \sinh(a) \\ 0 & 1 & 0 \\ \sinh(a) & 0 & \cosh(a) \end{pmatrix}, \xi_3(a) = \begin{pmatrix} \cos(a) & \sin(a) & 0 \\ -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\xi_i : (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_1) \rightarrow (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_2)$ with $\langle \cdot, \cdot \rangle_j = \text{Diag}(\lambda_j, \mu_j, \nu_j)$ then

$$\begin{cases} \tau(\xi_1(a)) = -\frac{\cosh(a) \sinh(a) (\mu_2 + \nu_2) (\nu_1 + \mu_1)}{\lambda_2 \mu_1 \nu_1} X_1, \\ \tau_2(\xi_1(a)) = -2 \frac{(\mu_2 + \nu_2)^2 ((\cosh(a))^2 - 1/2) (\nu_1 + \mu_1)^2 \cosh(a) \sinh(a)}{\mu_1^2 \nu_1^2 \lambda_2^2} X_1 \\ \tau(\xi_2(a)) = \frac{\cosh(a) \sinh(a) (\lambda_2 + \nu_2) (\nu_1 + \lambda_1)}{\mu_2 \lambda_1 \nu_1} X_2, \\ \tau_2(\xi_2(a)) = \frac{(2 (\cosh(a))^2 - 1) \cosh(a) (\lambda_2 + \nu_2)^2 (\nu_1 + \lambda_1)^2 \sinh(a)}{\lambda_1^2 \nu_1^2 \mu_2^2} X_2, \\ \tau(\xi_3(a)) = -\frac{\sin(a) \cos(a) (\lambda_2 - \mu_2) (-\mu_1 + \lambda_1)}{\nu_2 \lambda_1 \mu_1} X_3, \\ \tau_2(\xi_3(a)) = -2 \frac{\sin(a) \cos(a) (-\lambda_2 + \mu_2)^2 (\mu_1 - \lambda_1)^2 ((\cos(a))^2 - 1/2)}{\lambda_1^2 \mu_1^2 \nu_2^2} X_3. \end{cases}$$

So we get:

Proposition 7.1.

1. $\xi_1(a)$ is biharmonic if and only if it is harmonic if and only if $a = 0$, i.e., $\xi_1 = \text{Id}$.
2. $\xi_2(a)$ is biharmonic if and only if it is harmonic if and only if $a = 0$, i.e., $\xi_2 = \text{Id}$.
3. If $\lambda_2 = \mu_2$ or $\lambda_1 = \mu_1$ then $\xi_3(a)$ is harmonic.
4. If $\lambda_2 \neq \mu_2$ and $\lambda_1 \neq \mu_1$ then $\xi_3(a)$ is harmonic if and only if $(\sin(2a) = 0)$ and $\xi_3(a)$ is biharmonic not harmonic if and only if $\cos(a)^2 = \frac{1}{2}$.

Theorem 7.1. The automorphism

$$\xi = \xi_3(a) \circ \xi_2(b) \circ \xi_1(c) : (\text{sl}(2, \mathbb{R}), \text{diag}(\lambda_1, \mu_1, \nu_1)) \longrightarrow (\text{sl}(2, \mathbb{R}), \text{diag}(\lambda_2, \mu_2, \nu_2), 0) < \lambda_i \leq \mu_i, \nu_i > 0$$

is harmonic if and only if $\xi_2(b) = \xi_1(c) = \text{Id}_{\text{sl}(2, \mathbb{R})}$ and $\xi_3(a)$ is harmonic.

Proof. We have

$$\left\{ \begin{array}{l} \tau(\xi) = \frac{(\mu_2 + \nu_2)R}{\lambda_2 \lambda_1 \mu_1 \nu_1} X_1 + \frac{(\lambda_2 + \nu_2)S}{\mu_2 \lambda_1 \mu_1 \nu_1} X_2 + \frac{(\lambda_2 - \nu_2)Q}{\nu_2 \lambda_1 \mu_1 \nu_1} X_3, \\ R = \cosh(b) (\sinh(b) \lambda_1 \sin(a) (\mu_1 + \nu_1) (\cosh(c))^2 - \sinh(c) \lambda_1 \cos(a) (\mu_1 + \nu_1) \cosh(c) - \sinh(b) \nu_1 \sin(a) \\ \times (\lambda_1 - \mu_1)), \\ S = \cosh(b) (\sinh(b) \lambda_1 \cos(a) (\mu_1 + \nu_1) (\cosh(c))^2 + \sinh(c) \lambda_1 \sin(a) (\mu_1 + \nu_1) \cosh(c) - \sinh(b) \nu_1 \cos(a) \\ \times (\lambda_1 - \mu_1)), \\ Q = -2 \cosh(c) \sinh(b) \sinh(c) \lambda_1 (\mu_1 + \nu_1) (\cos(a))^2 + \cosh(c) \sinh(b) \sinh(c) \lambda_1 (\mu_1 + \nu_1) \\ + \sin(a) (\lambda_1 ((\cosh(b))^2 - 2) (\mu_1 + \nu_1) (\cosh(c))^2 - \nu_1 (\lambda_1 - \mu_1) (\cosh(b))^2 + \lambda_1 (\mu_1 + \nu_1)) \cos(a). \end{array} \right.$$

On the other hand, one can show easily

$$\left\{ \begin{array}{l} \cos(a)R - \sin(a)S = -\cosh(b) \sinh(c) \cosh(c) \lambda_1 (\mu_1 + \nu_1), \\ \sin(a)R + \cos(a)S = (\lambda_1 (\mu_1 + \nu_1) (\cosh(c))^2 + \nu_1 (\mu_1 - \lambda_1)) \cosh(b) \sinh(b). \end{array} \right.$$

So ξ is harmonic if and only if

$$\sinh(b) = \sinh(c) = Q = 0$$

and we get the desired result. \square

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