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Solutions of the Yang–Baxter equations on quadratic Lie groups: The case of oscillator groups

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ABSTRACT

A Lie group is called quadratic if it carries a bi-invariant semi-Riemannian metric. Oscillator Lie groups constitute a subclass of the class of quadratic Lie groups. In this paper, we determine the Lie bialgebra structures and the solutions of the classical Yang–Baxter equation on a generic class of oscillator Lie algebras. Moreover, we show that any solution of the generalized classical Yang–Baxter equation (resp. classical Yang–Baxter equation) on a quadratic Lie group determines a left invariant locally symmetric (resp. flat) semi-Riemannian metric on the corresponding dual Lie groups.

1. Introduction and main results

The notion of Poisson–Lie group was first introduced by Drinfeld [1,2] and studied by Semenov-Tian-Shansky [3] (see also [4]). It is known that every connected Poisson–Lie group arises from a Lie bialgebra and an important class of Lie bialgebras, the coboundary Lie bialgebras [5], is obtained by solving the generalized Yang–Baxter equation. Recall that a *Lie bialgebra* is a Lie algebra $\mathfrak{g}$ with a linear map $\xi : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g}$ such that:

1. ξ is a 1-cocycle with respect to the adjoint action, i.e.,

$$\xi([u, v]) = \mathrm{ad}_u \xi(v) - \mathrm{ad}_v \xi(u); \tag{1}$$

2. the bracket $[\,,\,]^*$ on the dual $\mathfrak{g}^*$ given by

$$[\alpha, \beta]^*(u) = \xi(u)(\alpha, \beta), \quad u \in \mathfrak{g},\ \alpha, \beta \in \mathfrak{g}^* \tag{2}$$

satisfies the Jacobi identity.

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A Lie bialgebra $(\mathfrak{g}, \xi)$ is called *coboundary Lie bialgebra* if there exists $r \in \wedge^2 \mathfrak{g}$ such that, for any $u \in \mathfrak{g}$, $\xi(u) = \mathrm{ad}_u r$. In this case, the condition (1) is automatically satisfied and (2) holds if and only if r satisfies the *generalized classical Yang–Baxter equation* (GCYBE):

$$\mathrm{ad}_u[r, r] = 0, \quad \forall u \in \mathfrak{g}, \tag{3}$$

where $[r, r] \in \wedge^3 \mathfrak{g}$ is the Schouten bracket (see [6] for instance). A solution of the *classical Yang–Baxter equation* (CYBE) is a bivector $r \in \wedge^2 \mathfrak{g}$ satisfying

$$[r, r] = 0. \tag{4}$$

The quantization of Poisson–Lie groups produces quantum groups; this needs the determination of Lie bialgebra structures on a Lie algebra which is a non trivial problem (see [7–9]). The oscillator groups are the only non commutative simply connected solvable Lie group endowed with an irreducible bi-invariant Lorentzian metric [10]. Their corresponding Lie algebras are called oscillator Lie algebras. The 4-dimensional oscillator Lie group has its origin in the study of the harmonic oscillator which is one of the most simple non-relativistic systems where the Schrödinger equation can be solved completely. Properties of the oscillator Lie groups have been studied in many papers (see for instance [11–16] and [17]).

The aim of the present paper is double. On one hand, we determine the structures of Lie bialgebra and the solutions of the GCYBE and the CYBE on a generic class of oscillator Lie algebras (Theorems 1.1 and 1.2). On the other hand, in the line of Bordemann's work in [18], we prove that any solution of the GCYBE on a quadratic (or orthogonal) Lie group G, defines a locally symmetric (flat when the solution satisfies the CYBE) left invariant semi-Riemannian metric on the dual Lie group associated with the solution. This result, although simple to prove, is new to our knowledge. It is a generalization of Theorem 3.6 in [19] and it implies interesting geometric properties on the tangent bundle of oscillator groups and on the oscillator manifolds, i.e., quotients of oscillator groups by lattices (see [20,21]).

Let us now recall some facts and state our main results.

For $n \in \mathbb{N}^*$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \le \cdots \le \lambda_n$, the λ-oscillator group, denoted by G_λ, is the manifold $\mathbb{R}^{2n+2} = \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ endowed with the product

$$(t, s, z).(t', s', z') = \left(t + t', s + s' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im}\, \bar{z}_j \exp(\mathrm{i}t\lambda_j) z'_j, \dots, z_j + \exp(\mathrm{i}t\lambda_j) z'_j, \dots \right).$$

The Lie algebra of G_λ, denoted by $\mathfrak{g}_\lambda$, admits a basis $\mathbb{B} = \left\{ e_{-1}, e_0, e_i, \check{e}_i, \right\}_{i=1,\dots,n}$ where the brackets are given by

$$[e_{-1}, e_j] = \lambda_j \check{e}_j, \qquad [e_{-1}, \check{e}_j] = -\lambda_j e_j, \qquad [e_j, \check{e}_j] = e_0. \tag{5}$$

The unspecified brackets are either zero or given by antisymmetry.

We call G_λ or $\mathfrak{g}_\lambda$ *generic* if, $0 < \lambda_1 < \cdots < \lambda_n$ and, for any $1 \le i < j < k \le n$, $\lambda_k \ne \lambda_i + \lambda_j$.

We shall denote by S the vector subspace of $\mathfrak{g}_\lambda$ spanned by $\{e_i, \check{e}_i\}_{i=1,\dots,n}$ and by ω the 2-form on $\mathfrak{g}_\lambda$ given by

$$i_{e_{-1}}\omega = i_{e_0}\omega = 0, \qquad \omega(e_i, e_j) = \omega(\check{e}_i, \check{e}_j) = 0 \quad \text{and} \quad \omega(e_i, \check{e}_j) = \delta_{ij}.$$

The restriction of ω to S is a symplectic 2-form and, for any $u, v \in S$,

$$[u, v] = \omega(u, v) e_0. \tag{6}$$

Moreover, S is invariant by the derivation $\mathrm{ad}_{e_{-1}}$ and we have

$$\omega(\mathrm{ad}_{e_{-1}} u, v) + \omega(u, \mathrm{ad}_{e_{-1}} v) = 0 \quad u, v \in S. \tag{7}$$

Notations. For any $r_1, r_2 \in \wedge^2 \mathfrak{g}_\lambda$, let ω_{r_1,r_2} be the element of $\wedge^2 \mathfrak{g}_\lambda$ defined by

$$\omega_{r_1,r_2}(\alpha, \beta) = \frac{1}{2} \left(\omega(r_{1\#}(\alpha), r_{2\#}(\beta)) + \omega(r_{2\#}(\alpha), r_{1\#}(\beta)) \right),$$

where $r_{i\#} : \mathfrak{g}_\lambda^* \longrightarrow \mathfrak{g}_\lambda$ is the endomorphism given by $\beta(r_{i\#}(\alpha)) = r_i(\alpha, \beta)$. We denote (improperly) by $\wedge^2 S$ the space of $r \in \wedge^2 \mathfrak{g}_\lambda$ satisfying $r_\#(e_{-1}^*) = r_\#(e_0^*) = 0$ and by S^* the subspace of $\mathfrak{g}_\lambda^*$ of α such that $\alpha(e_{-1}) = \alpha(e_0) = 0$ and we define $e_{-1}^*, e_0^* \in \mathfrak{g}_\lambda^*$ by $e^*_{-1|S} = e^*_{0|S} = 0$, $e_{-1}^*(e_0) = e_0^*(e_{-1}) = 0$ and $e_{-1}^*(e_{-1}) = e_0^*(e_0) = 1$. We have clearly $\mathfrak{g}_\lambda^* = \mathbb{R}e_{-1}^* \oplus \mathbb{R}e_0^* \oplus S^*$. Finally, for any endomorphism J of $\mathfrak{g}_\lambda$, we denote by $J^\dagger$ the endomorphism of $\wedge^2 \mathfrak{g}_\lambda$, given by

$$J^\dagger r(\alpha, \beta) = r(J^*\alpha, \beta) + r(\alpha, J^*\beta),$$

where $J^* : \mathfrak{g}_\lambda^* \longrightarrow \mathfrak{g}_\lambda^*$ is the dual of J.

We have the following theorem.

Theorem 1.1. *Let $\mathfrak{g}_\lambda$ be a generic oscillator Lie algebra. Then $\xi : \mathfrak{g}_\lambda \longrightarrow \wedge^2 \mathfrak{g}_\lambda$ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$ if and only if there exist $r \in \wedge^2 S$, $u_0 \in S$ and a derivation $J : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda$ commuting with $\mathrm{ad}_{e_{-1}}$ and satisfying $J(e_{-1}) = J(e_0) = 0$ such that, for any $u \in \mathfrak{g}_\lambda$,*

$$\xi(u) = \mathrm{ad}_u^\dagger r + 2e_0 \wedge ((J + \mathrm{ad}_{u_0})(u)),$$

and

$$\omega_{r,\mathrm{ad}_{e_{-1}}^\dagger r} - (J^\dagger \circ \mathrm{ad}_{e_{-1}}^\dagger) r = 0. \tag{8}$$

Moreover, in this case, the Lie bracket on $\mathfrak{g}_\lambda^*$ defined by (2) is given by

$$\begin{cases} [e_0^*, \alpha]^* = 2J^*\alpha - 2(\mathrm{ad}_{e_{-1}}^*\alpha)(u_0)e_{-1}^* + i_{r_\#(\alpha)}\omega, \\ [\alpha, \beta]^* = \mathrm{ad}_{e_{-1}}^\dagger r(\alpha, \beta)e_{-1}^*, \end{cases} \tag{9}$$

where $\alpha, \beta \in S^*$ and e_{-1}^* is a central element.

From the expression on the brackets above, we can deduce immediately the following result.

Corollary 1.1. *Let $\mathfrak{g}_\lambda$ be a generic oscillator Lie algebra and $\xi(u) = \mathrm{ad}_u^\dagger r + 2e_0 \wedge ((J + \mathrm{ad}_{u_0})(u))$ a Lie bialgebra structure on $\mathfrak{g}_\lambda$. Denote by $2p$, the dimension of the kernel of the restriction of $\mathrm{ad}_{e_{-1}}^\dagger r$ to S^*. Then $(\mathfrak{g}_\lambda^*, [\,,\,]^*)$ is isomorphic to the semi-direct product of $\mathbb{R}e_0^*$ by the ideal $\mathcal{H}_{2(n-p)+1} \oplus \mathbb{R}^{2p}$ where $\mathcal{H}_{2(n-p)+1}$ is the $2(n-p)+1$-dimensional Heisenberg Lie algebra, $\mathbb{R}^{2p}$ the $2p$-dimensional Abelian Lie algebra and the action of e_0^* is given by the first relation in (9). In particular, $(\mathfrak{g}_\lambda^*, [\,,\,]^*)$ is solvable. Moreover, $(\mathfrak{g}_\lambda^*, [\,,\,]^*)$ is unimodular if and only if $\sum_{i=1}^n r(e_i, \check{e}_i) = 0$.*

Theorem 1.2. *Let $\mathfrak{g}_\lambda$ be a generic oscillator Lie algebra. Then, we have the following.*

1. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the GCYBE if and only if there exist $u_0 \in S$, $r_0 \in \wedge^2 S$ and $\alpha \in \mathbb{R}$ such that $r = 2\alpha e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0$ and*

$$\omega_{r_0,\mathrm{ad}_{e_{-1}}^\dagger r_0} + \alpha(\mathrm{ad}_{e_{-1}}^\dagger \circ \mathrm{ad}_{e_{-1}}^\dagger) r_0 = 0. \tag{10}$$

2. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the CYBE if and only if there exist $u_0 \in S$, $r_0 \in \wedge^2 S$ and $\alpha \in \mathbb{R}$ such that $r = \alpha e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0$ and*

$$\omega_{r_0,r_0} + \alpha \mathrm{ad}_{e_{-1}}^\dagger r_0 = 0. \tag{11}$$

There are some comments on Theorems 1.1 and 1.2.

1. On an oscillator algebra not necessarily generic, the 1-cocycles (resp. the r) verifying the conditions of Theorem 1.1 (resp. Theorem 1.2) also define Lie bialgebra structures (resp. solution of the GCYBE) on the algebra.
2. Theorems 1.1 and 1.2 reduce the problem of finding Lie bialgebra structures or solutions of Yang–Baxter equations on a generic oscillator Lie algebra for solving (8), (10) and (11). Or these equations involve only the symplectic space (S, ω) and the restrictions of the derivations J and $\mathrm{ad}_{e_{-1}}$ to S.

The second part of our study involves bi-invariant semi-Riemannian metrics on Lie groups and solutions of the GCYBE.

Let (G, k) be a Lie group endowed with a bi-invariant semi-Riemannian metric. The value of k at identity induces on the Lie algebra $\mathfrak{g}$ of G an adjoint invariant non degenerate bilinear symmetric form $\langle\,,\,\rangle$. Such a Lie algebra is called a *quadratic* (or *orthogonal*) Lie algebra. For instance, reductive Lie algebras and oscillator Lie algebras are quadratic (see [10]).

Let r be a solution of (3) on a quadratic Lie algebra $(\mathfrak{g}, \langle\,,\,\rangle)$. Then r defines on $\mathfrak{g}^*$ a Lie bracket by

$$[\alpha, \beta]_r = \mathrm{ad}_{r_\#(\beta)}^*\alpha - \mathrm{ad}_{r_\#(\alpha)}^*\beta. \tag{12}$$

Consider the bilinear form on $\mathfrak{g}^*$ given by

$$\langle\alpha, \beta\rangle^* = \langle\phi^{-1}(\alpha), \phi^{-1}(\beta)\rangle,$$

where $\phi(u) = \langle u, .\rangle$.

Let us denote by G_r^* a Lie group with Lie algebra $(\mathfrak{g}^*, [\,,\,]_r)$, by k^* the left invariant semi-Riemannian metric whose value at the identity is $\langle\,,\,\rangle^*$ and by ∇^* its Levi-Civita connexion. With this notations, we have the following result.

Theorem 1.3. *Let (G, k) be a Lie group endowed with a bi-invariant pseudo-Riemannian metric and let r be a solution of (3) on G. Then, we have the following.*

1. *(G^*_r, k^*) is a locally symmetric pseudo-Riemannian manifold, i.e.,*
$$\nabla^* R = 0,$$
where R is the curvature of k^*. In particular, R vanishes identically when r is a solution of (4).
2. *If k^* is flat then it can be complete if and only if G^*_r is unimodular. If the two conditions are verified then G^*_r is solvable.*

Remark 1. 1. The second part of Theorem 1.3 can be paraphrased as follows: any symplectic Lie subgroup S of a quadratic Lie group (G, k) determines a unique simply connected Lie group G^* such that the left invariant semi-Riemannian metric k^* is flat.
2. The Killing form k of a semisimple Lie group G determines a flat left invariant semi-Riemannian metric on an every dual Lie group of G. In particular, if G is compact then $\mathfrak{g}^*$ is a semi-direct product of two Abelian Lie algebras according to Milnor (see [22]).
3. If r is a solution of (4), the sequence
$$0 \longrightarrow \ker r_\# \longrightarrow \mathfrak{g}^* \longrightarrow \operatorname{Im} r_\# \longrightarrow 0$$
is an exact sequence of Lie algebras where $\ker r_\#$ is an Abelian ideal of $\mathfrak{g}^*$ and $\operatorname{Im} r_\#$ is a symplectic Lie algebra. In particular, if G is solvable or if $\operatorname{Im} r$ is k-nondegenerate then G^*_r is solvable. The sequence is also an exact sequence of left symmetric algebras (see [19]).
4. Note that knowing the links between the solutions of the CYBE on $(\mathfrak{g}, k)$ and the corresponding solutions on their orthogonal double extensions, in the sense of Medina and Revoy, is a very interesting problem.

The paper is organized as follows. Section 2 is devoted to the study of Lie bialgebra structures on oscillator Lie algebras and culminates by proving Theorems 1.1–1.3. In Section 3, we give the solutions of (8), (10) and (11) when $\dim \mathfrak{g}_\lambda \le 6$. In Section 4, we give some examples. For a study of left invariant affine structures the reader is refered to [23,24] and [25].

2. Lie bialgebra structures on oscillator Lie algebras

This section is devoted to the study of bialgebra structures on oscillator Lie algebras. It culminates with a proof of Theorems 1.1–1.3.

Let $\mathbb{B}^* = \left\{e^*_{-1}, e^*_0, e^*_i, \check{e}^*_i\right\}_{i=1,\ldots,n}$ be the dual basis of $\mathbb{B}$. From (5), we get that the non vanishing $\operatorname{ad}^*_u \alpha$ with $u \in \mathbb{B}$ and $\alpha \in \mathbb{B}^*$ are
$$\begin{aligned}
&\operatorname{ad}^*_{e_{-1}} e^*_i = -\lambda_i \check{e}^*_i, \qquad \operatorname{ad}^*_{e_{-1}} \check{e}^*_i = \lambda_i e^*_i,\\
&\operatorname{ad}^*_{e_i} e^*_0 = \check{e}^*_i, \qquad \operatorname{ad}^*_{e_i} \check{e}^*_i = -\lambda_i e^*_{-1},\\
&\operatorname{ad}^*_{\check{e}_i} e^*_0 = -e^*_i, \qquad \operatorname{ad}^*_{\check{e}_i} e^*_i = \lambda_i e^*_{-1}.
\end{aligned} \tag{13}$$
An important class of 1-cocycles are the coboundaries, i.e., $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ such that, for any $u \in \mathfrak{g}_\lambda$, $\xi(u) = \operatorname{ad}^\dagger_u r$ where $r \in \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$. In this case, the bracket defined by (2) is given by
$$[\alpha, \beta]^* = \operatorname{ad}^*_{r_\#(\beta)} \alpha - \operatorname{ad}^*_{r_\#(\alpha)} \beta, \quad \alpha, \beta \in \mathfrak{g}^*_\lambda.$$
In the proof of Theorem 1.1, we need to compute the expression of this bracket in the basis $\mathbb{B}^*$. A direct computation using (13) gives:
$$\begin{cases}
[e^*_{-1}, \alpha]^* = -\operatorname{ad}^*_{r_\#(e^*_{-1})} \alpha, \quad \alpha \in \mathfrak{g}^*_\lambda,\\
[e^*_0, e^*_i]^* = -\lambda_i r(e^*_0, \check{e}^*_i) e^*_{-1} + \lambda_i r(e^*_0, e^*_{-1}) \check{e}^*_i + \displaystyle\sum_{j=1}^n \left(r(e^*_i, e^*_j)\check{e}^*_j - r(e^*_i, \check{e}^*_j) e^*_j \right),\\
[e^*_0, \check{e}^*_i]^* = \lambda_i r(e^*_0, e^*_i) e^*_{-1} - \lambda_i r(e^*_0, e^*_{-1}) e^*_i + \displaystyle\sum_{j=1}^n \left(r(\check{e}^*_i, e^*_j)\check{e}^*_j - r(\check{e}^*_i, \check{e}^*_j) e^*_j \right),\\
[e^*_i, e^*_j]^* = \left(-\lambda_i r(\check{e}^*_i, e^*_j) - \lambda_j r(e^*_i, \check{e}^*_j)\right) e^*_{-1} + \lambda_i r(e^*_{-1}, e^*_j)\check{e}^*_i + \lambda_j r(e^*_i, e^*_{-1})\check{e}^*_j,\\
[e^*_i, \check{e}^*_j]^* = \left(-\lambda_i r(\check{e}^*_i, \check{e}^*_j) + \lambda_j r(e^*_i, e^*_j)\right) e^*_{-1} + \lambda_i r(e^*_{-1}, \check{e}^*_j)\check{e}^*_i - \lambda_j r(e^*_i, e^*_{-1}) e^*_j,\\
[\check{e}^*_i, \check{e}^*_j]^* = \left(\lambda_i r(e^*_i, \check{e}^*_j) + \lambda_j r(\check{e}^*_i, e^*_j)\right) e^*_{-1} - \lambda_i r(e^*_{-1}, \check{e}^*_j) e^*_i - \lambda_j r(\check{e}^*_i, e^*_{-1}) e^*_j.
\end{cases} \tag{14}$$
Notation. For any linear map $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ and for any $i, j = 1, \ldots, n$, we denote by $\alpha_{-1,0}, \alpha_{-1,i}, \check{\alpha}_{-1,i}, a_{0,i}, \check{a}_{0,i}, b_{i,j}, \check{b}_{i,j}, c_{i,j}$, the elements of $\mathfrak{g}^*_\lambda$ whose values at $u \in \mathfrak{g}_\lambda$ are given by
$$\begin{aligned}
&\alpha_{-1,0}(u) = \xi(u)(e^*_{-1}, e^*_0), \qquad \alpha_{-1,i}(u) = \xi(u)(e^*_{-1}, e^*_i), \qquad \check{\alpha}_{-1,i}(u) = \xi(u)(e^*_{-1}, \check{e}^*_i),\\
&a_{0,i}(u) = \xi(u)(e^*_0, e^*_i), \qquad \check{a}_{0,i}(u) = \xi(u)(e^*_0, \check{e}^*_i),\\
&b_{i,j}(u) = \xi(u)(e^*_i, e^*_j), \qquad \check{b}_{i,j}(u) = \xi(u)(\check{e}^*_i, \check{e}^*_j), \qquad c_{i,j}(u) = \xi(u)(e^*_i, \check{e}^*_j).
\end{aligned}$$

Proposition 2.1. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a 1-cocycle with respect to the adjoint action. Then $\xi(e_0) = 0$.*

Proof. Since e_0 is a central element in $\mathfrak{g}_\lambda$, the cocycle condition implies that $\mathrm{ad}_u\xi(e_0) = 0$, for any $u \in \mathfrak{g}_\lambda$. From this relation and (13), one can deduce easily that $\xi(e_0) = 0$. □

Proposition 2.2. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a 1-cocycle with respect to the adjoint action. Then, for any $i, j = 1, \dots, n$ with $i \neq j$,*

$$b_{i,j}(e_i) = b_{i,j}(e_j) = 0,$$
$$c_{j,j}(\check{e}_i) = c_{j,j}(e_i) = c_{i,j}(e_i) = c_{i,j}(\check{e}_j) = c_{i,i}(e_{-1}) = 0,$$
$$\check{b}_{i,j}(\check{e}_i) = \check{b}_{i,j}(\check{e}_j) = 0.$$

Proof. Let $i, j = 1, \dots, n$ with $i \neq j$. The cocycle condition, Proposition 2.1, (5) and (13) imply

$$\begin{aligned} 0 &= \mathrm{ad}_{\check{e}_i}\xi(\check{e}_j)(e_0^*, \check{e}_j^*) - \mathrm{ad}_{\check{e}_j}\xi(\check{e}_i)(e_0^*, \check{e}_j^*) = -\xi(\check{e}_j)(e_i^*, \check{e}_j^*) + \xi(\check{e}_i)(e_j^*, \check{e}_j^*) \\ &= -c_{i,j}(\check{e}_j) + c_{j,j}(\check{e}_i) \\ 0 &= \mathrm{ad}_{\check{e}_i}\xi(e_j)(e_0^*, e_j^*) - \mathrm{ad}_{e_j}\xi(\check{e}_i)(e_0^*, e_j^*) = -\xi(e_j)(e_i^*, e_j^*) - \xi(\check{e}_i)(\check{e}_j^*, e_j^*) \\ &= -b_{i,j}(e_j) + c_{j,j}(\check{e}_i) \\ 0 &= \mathrm{ad}_{e_j}\xi(\check{e}_j)(e_0^*, e_i^*) - \mathrm{ad}_{\check{e}_j}\xi(e_j)(e_0^*, e_i^*) = \xi(\check{e}_j)(\check{e}_j^*, e_i^*) + \xi(e_j)(e_j^*, e_i^*) \\ &= -c_{i,j}(\check{e}_j) - b_{i,j}(e_j). \end{aligned}$$

We deduce that $b_{i,j}(e_i) = c_{j,j}(\check{e}_i) = c_{i,j}(\check{e}_j) = 0$. On the other hand,

$$\begin{aligned} 0 &= \mathrm{ad}_{e_i}\xi(e_j)(e_0^*, e_j^*) - \mathrm{ad}_{e_j}\xi(e_i)(e_0^*, e_j^*) = \xi(e_j)(\check{e}_i^*, e_j^*) - \xi(e_i)(\check{e}_j^*, e_j^*) \\ &= -c_{j,i}(e_j) + c_{j,j}(e_i) \\ 0 &= \mathrm{ad}_{e_i}\xi(\check{e}_j)(e_0^*, \check{e}_j^*) - \mathrm{ad}_{\check{e}_j}\xi(e_i)(e_0^*, \check{e}_j^*) = \xi(\check{e}_j)(\check{e}_i^*, \check{e}_j^*) + \xi(e_i)(e_j^*, \check{e}_j^*) \\ &= \check{b}_{i,j}(\check{e}_j) + c_{j,j}(e_i) \\ 0 &= \mathrm{ad}_{\check{e}_j}\xi(e_j)(e_0^*, \check{e}_i^*) - \mathrm{ad}_{e_j}\xi(\check{e}_j)(e_0^*, \check{e}_i^*) = -\xi(e_j)(e_j^*, \check{e}_i^*) - \xi(\check{e}_j)(\check{e}_j^*, \check{e}_i^*) \\ &= -c_{j,i}(e_j) + \check{b}_{i,j}(\check{e}_j). \end{aligned}$$

Thus $\check{b}_{i,j}(\check{e}_i) = c_{j,j}(e_i) = c_{i,j}(e_i) = 0$. To complete the proof, we need to show that $c_{i,i}(e_{-1}) = 0$. Indeed, by applying (1), respectively, to (e_{-1}, e_i) and $(e_{-1}, \check{e}_i)$, we get

$$\begin{aligned} \lambda_i\xi(\check{e}_i)(e_0^*, e_i^*) &= \mathrm{ad}_{e_{-1}}\xi(e_i)(e_0^*, e_i^*) - \mathrm{ad}_{e_i}\xi(e_{-1})(e_0^*, e_i^*) \\ &= -\lambda_i\xi(e_i)(e_0^*, \check{e}_i^*) - \xi(e_{-1})(\check{e}_i^*, e_i^*), \\ -\lambda_i\xi(e_i)(e_0^*, \check{e}_i^*) &= \mathrm{ad}_{e_{-1}}\xi(\check{e}_i)(e_0^*, \check{e}_i^*) - \mathrm{ad}_{\check{e}_i}\xi(e_{-1})(e_0^*, \check{e}_i^*) \\ &= \lambda_i\xi(\check{e}_i)(e_0^*, e_i^*) + \xi(e_{-1})(e_i^*, \check{e}_i^*). \end{aligned}$$

Hence,

$$c_{i,i}(e_{-1}) = \lambda_i(a_{0,i}(\check{e}_i) + \check{a}_{0,i}(e_i)) = -\lambda_i(\check{a}_{0,i}(e_i) + a_{0,i}(\check{e}_i)),$$

and then $c_{i,i}(e_{-1}) = 0$. □

Proposition 2.3. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a 1-cocycle on $\mathfrak{g}_\lambda$ with respect to the adjoint action and let $k, l = 1, \dots, n$. Then, for any $1 \le j \le n$ such that $j \neq k$ and $j \neq l$, we have:*

1. *if $\lambda_j \neq \lambda_k + \lambda_l$ and $\lambda_j \neq |\lambda_l - \lambda_k|$, then*
$$b_{k,l}(e_j) = b_{k,l}(\check{e}_j) = \check{b}_{k,l}(e_j) = \check{b}_{k,l}(\check{e}_j) = c_{k,l}(e_j) = c_{k,l}(\check{e}_j) = 0;$$
2. *if $\lambda_j = \lambda_k + \lambda_l$, then there exist $a, b \in \mathbb{R}$ such that*
$$(b_{k,l}(e_j), c_{k,l}(e_j), c_{l,k}(e_j), \check{b}_{k,l}(e_j)) = (-a, b, -b, a)$$
$$(b_{k,l}(\check{e}_j), c_{k,l}(\check{e}_j), c_{l,k}(\check{e}_j), \check{b}_{k,l}(\check{e}_j)) = (-b, -a, a, b);$$
3. *If $\lambda_j = |\lambda_l - \lambda_k|$, then there exists $a, b \in \mathbb{R}$ such that*
$$(b_{k,l}(e_j), c_{k,l}(e_j), c_{l,k}(e_j), \check{b}_{k,l}(e_j)) = (a, -b, -b, a)$$
$$(b_{k,l}(\check{e}_j), c_{k,l}(\check{e}_j), c_{l,k}(\check{e}_j), \check{b}_{k,l}(\check{e}_j)) = (\epsilon b, \epsilon a, \epsilon a, \epsilon b),$$
where ϵ is the sign of $\lambda_l - \lambda_k$.

Proof. By applying (1) to (e_{-1}, e_j) and $(e_{-1}, \check{e}_j)$, we get

$$\mathrm{ad}_{e_{-1}}\xi(e_j) - \mathrm{ad}_{e_j}\xi(e_{-1}) = \lambda_j\xi(\check{e}_j)$$
$$\mathrm{ad}_{e_{-1}}\xi(\check{e}_j) - \mathrm{ad}_{\check{e}_j}\xi(e_{-1}) = -\lambda_j\xi(e_j).$$

By evaluating these relations, respectively, on (e_k^*, e_l^*), $(e_k^*, \check{e}_l^*)$, $(\check{e}_k^*, e_l^*)$, $(\check{e}_k^*, \check{e}_l^*)$, we deduce that

$$BX = -\lambda_j \check{X} \quad \text{and} \quad B\check{X} = \lambda_j X. \tag{15}$$

where

$$B = \begin{pmatrix} 0 & \lambda_l & \lambda_k & 0 \\ -\lambda_l & 0 & 0 & \lambda_k \\ -\lambda_k & 0 & 0 & \lambda_l \\ 0 & -\lambda_k & -\lambda_l & 0 \end{pmatrix},$$

$X = (b_{k,l}(e_j), c_{k,l}(e_j), -c_{l,k}(e_j), \check{b}_{k,l}(e_j))$ and $\check{X} = (b_{k,l}(\check{e}_j), c_{k,l}(\check{e}_j), -c_{l,k}(\check{e}_j), \check{b}_{k,l}(\check{e}_j))$.

Eq. (15) implies that

$$B^2 X = -\lambda_j^2 X \quad \text{and} \quad B^2 \check{X} = -\lambda_j^2 \check{X}.$$

Now the eigenvalues of B^2 are $-(\lambda_l - \lambda_k)^2$ and $-(\lambda_l + \lambda_k)^2$ and the corresponding eigenspaces are span$\{(1, 0, 0, 1), (0, -1, 1, 0)\}$ and span$\{(-1, 0, 0, 1), (0, 1, 1, 0)\}$. Hence, we have the following.

1. If $\lambda_j^2 \neq (\lambda_l + \lambda_k)^2$ and $\lambda_j^2 \neq (\lambda_l - \lambda_k)^2$ then $X = \check{X} = 0$.
2. If $\lambda_j = \lambda_l + \lambda_k$ then
$$X = a(-1, 0, 0, 1) + b(0, 1, 1, 0),$$
$$\check{X} = \check{a}(-1, 0, 0, 1) + \check{b}(0, 1, 1, 0)$$
and (15) is equivalent to $a = -\check{b}$ and $b = \check{a}$ and we get the desired relation.
3. If $\lambda_j = |\lambda_l - \lambda_k|$ then
$$X = a(1, 0, 0, 1) + b(0, -1, 1, 0),$$
$$\check{X} = \check{a}(1, 0, 0, 1) + \check{b}(0, -1, 1, 0)$$
and (15) is equivalent to $(\lambda_l - \lambda_k)a = -\lambda_j \check{b}$ and $(\lambda_k - \lambda_l)b = -\lambda_j \check{a}$ and we get the desired relation. □

Proposition 2.4. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a 1-cocycle on $\mathfrak{g}_\lambda$ with respect to the adjoint action. Then for any $1 \leq i, j \leq n$,*

$$\begin{cases} \alpha_{-1,0}(e_{-1}) = 0, \\ c_{i,i}(e_{-1}) = 0, \\ a_{0,i}(e_i) = \check{a}_{0,i}(\check{e}_i), \\ a_{0,i}(\check{e}_i) = -\check{a}_{0,i}(e_i), \\ b_{i,j}(e_{-1}) = \lambda_j \check{a}_{0,j}(\check{e}_i) - \lambda_i a_{0,j}(e_i), \\ \check{b}_{i,j}(e_{-1}) = \lambda_j a_{0,j}(e_i) - \lambda_i \check{a}_{0,j}(\check{e}_i), \\ c_{j,i}(e_{-1}) = \lambda_i a_{0,j}(\check{e}_i) + \lambda_j \check{a}_{0,j}(e_i), \\ c_{i,j}(e_{-1}) = -\lambda_i \check{a}_{0,j}(e_i) - \lambda_j a_{0,j}(\check{e}_i). \end{cases}$$

Proof. Let $i \neq j$. By applying (1) to (e_{-1}, e_i) and $(e_{-1}, \check{e}_i)$, we get

$$\mathrm{ad}_{e_{-1}} \xi(e_i) - \mathrm{ad}_{e_i} \xi(e_{-1}) = \lambda_i \xi(\check{e}_i),$$
$$\mathrm{ad}_{e_{-1}} \xi(\check{e}_i) - \mathrm{ad}_{\check{e}_i} \xi(e_{-1}) = -\lambda_i \xi(e_i).$$

The evaluation of these two relations, respectively, on (e_0^*, e_i^*), (e_0^*, e_j^*), $(e_0^*, \check{e}_i^*)$, $(e_0^*, \check{e}_j^*)$ gives the desired relations. □

Lemma 2.1. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a bialgebra structure on $\mathfrak{g}_\lambda$. Then e_{-1}^* is a central element of the dual Lie algebra $\mathfrak{g}_\lambda^*$.*

Proof. Remark that e_{-1}^* is a central element if and only if $\alpha_{-1,0} = \alpha_{-1,i} = \check{\alpha}_{-1,i} = 0$ for $i = 1, \ldots, n$.

Note first that, according to Proposition 2.1, $\alpha_{-1,0}(e_0) = \alpha_{-1,i}(e_0) = \check{\alpha}_{-1,i}(e_0) = 0$ for $i = 1, \ldots, n$.

Let $1 \leq i, j, k \leq n$ such that $i \neq j$. By using the 1-cocycle condition, (5) and (13) we get

$$0 = \mathrm{ad}_{e_i} \xi(e_j)(\check{e}_i^*, e_k^*) - \mathrm{ad}_{e_j} \xi(e_i)(\check{e}_i^*, e_k^*) = -\lambda_i \xi(e_j)(e_{-1}^*, e_k^*) = -\lambda_i \alpha_{-1,k}(e_j),$$
$$0 = \mathrm{ad}_{e_i} \xi(\check{e}_j)(\check{e}_i^*, e_i^*) - \mathrm{ad}_{\check{e}_j} \xi(e_i)(\check{e}_i^*, e_i^*) = -\lambda_i \xi(\check{e}_j)(e_{-1}^*, e_i^*) = -\lambda_i \alpha_{-1,i}(\check{e}_j),$$
$$0 = \mathrm{ad}_{\check{e}_i} \xi(\check{e}_j)(\check{e}_k^*, e_i^*) - \mathrm{ad}_{\check{e}_j} \xi(\check{e}_i)(\check{e}_k^*, e_i^*) = \lambda_i \xi(\check{e}_j)(\check{e}_k^*, e_{-1}^*) = -\lambda_i \check{\alpha}_{-1,k}(\check{e}_j),$$
$$0 = \mathrm{ad}_{\check{e}_i} \xi(e_j)(\check{e}_i^*, e_i^*) - \mathrm{ad}_{e_j} \xi(\check{e}_i)(\check{e}_i^*, e_i^*) = \lambda_i \xi(e_j)(\check{e}_i^*, e_{-1}^*) = -\lambda_i \check{\alpha}_{-1,i}(e_j).$$

Thus $\alpha_{-1,k}(e_j) = \alpha_{-1,i}(\check{e}_j) = \check{\alpha}_{-1,k}(\check{e}_j) = \check{\alpha}_{-1,i}(e_j) = 0$.

On the other hand,

$$\begin{aligned}
0 &= \mathrm{ad}_{e_j}\xi(e_i)(\check{e}_j^*, \check{e}_i^*) - \mathrm{ad}_{e_i}\xi(e_j)(\check{e}_j^*, \check{e}_i^*) = -\lambda_j\xi(e_i)(e_{-1}^*, \check{e}_i^*) + \lambda_i\xi(e_j)(\check{e}_j^*, e_{-1}^*)\\
&= -\lambda_j\check{\alpha}_{-1,i}(e_i) - \lambda_i\check{\alpha}_{-1,j}(e_j),\\
0 &= \mathrm{ad}_{e_i}\xi(\check{e}_j)(\check{e}_i^*, e_j^*) - \mathrm{ad}_{\check{e}_j}\xi(e_i)(\check{e}_i^*, e_j^*) = -\lambda_i\xi(\check{e}_j)(e_{-1}^*, e_j^*) + \lambda_j\xi(e_i)(e_{-1}^*, \check{e}_i^*)\\
&= -\lambda_i\alpha_{-1,j}(\check{e}_j) + \lambda_j\check{\alpha}_{-1,i}(e_i),\\
0 &= \mathrm{ad}_{\check{e}_j}\xi(e_j)(\check{e}_j^*, e_j^*) - \mathrm{ad}_{e_j}\xi(\check{e}_j)(\check{e}_j^*, e_j^*) = \lambda_j\xi(e_j)(\check{e}_j^*, e_{-1}^*) + \lambda_j\xi(\check{e}_j)(e_{-1}^*, e_j^*)\\
&= -\lambda_j\check{\alpha}_{-1,j}(e_j) + \lambda_j\alpha_{-1,j}(\check{e}_j).
\end{aligned}$$

We deduce that $\alpha_{-1,i}(\check{e}_i) = \check{\alpha}_{-1,i}(e_i) = 0$ and hence,

$$[e_{-1}^*, e_i^*]^* = \alpha_{-1,i}(e_{-1})e_{-1}^* \quad \text{and} \quad [e_{-1}^*, \check{e}_i^*]^* = \check{\alpha}_{-1,i}(e_{-1})e_{-1}^*. \tag{16}$$

To complete the proof, we will show that $\alpha_{-1,i}(e_{-1}) = \check{\alpha}_{-1,i}(e_{-1}) = 0$ and $\alpha_{-1,0} = 0$.

Since ξ is a bialgebra structure, the dual bracket satisfies the Jacobi identity. Let us apply this identity to $e_{-1}^*, e_0^*, \check{e}_i^*$ for $i = 1, \dots, n$. We have

$$\begin{aligned}
[[e_{-1}^*, e_0^*]^*, e_i^*]^*(e_i) &= \alpha_{-1,0}(e_{-1})[e_{-1}^*, e_i^*]^*(e_i) + \alpha_{-1,0}(\check{e}_i)[\check{e}_i^*, e_i^*]^*(e_i)\\
&\quad + \sum_j \alpha_{-1,0}(e_j)[e_j^*, e_i^*]^*(e_i) + \sum_{j\neq i}\alpha_{-1,0}(\check{e}_j)[\check{e}_j^*, e_i^*]^*(e_i)\\
&= -\alpha_{-1,0}(\check{e}_i)c_{i,i}(e_i) + \sum_j \alpha_{-1,0}(e_j)b_{j,i}(e_i) - \sum_{j\neq i}\alpha_{-1,0}(\check{e}_j)c_{i,j}(e_i)\\
&= -\alpha_{-1,0}(\check{e}_i)c_{i,i}(e_i) \quad (\text{see Proposition 2.2})
\end{aligned}$$

$$[[e_i^*, e_{-1}^*]^*, e_0^*]^* \stackrel{(16)}{=} -\alpha_{-1,i}(e_{-1})[e_{-1}^*, e_0^*]^*,$$
$$[[e_0^*, e_i^*]^*, e_{-1}^*]^*(e_i) = 0.$$

The last equality is a consequence of the fact that $\xi(e_0) = 0$ and (16). Hence, the Jacobi identity implies

$$\alpha_{-1,0}(\check{e}_i)c_{i,i}(e_i) + \alpha_{-1,i}(e_{-1})\alpha_{-1,0}(e_i) = 0.$$

Now

$$\begin{aligned}
c_{i,i}(e_i) &= \xi(e_i)(e_i^*, \check{e}_i^*) \stackrel{(13)}{=} -\xi(e_i)(\mathrm{ad}_{\check{e}_i}^* e_0^*, \check{e}_i^*)\\
&= -\mathrm{ad}_{\check{e}_i}\xi(e_i)(e_0^*, \check{e}_i^*) \stackrel{(1)}{=} -\mathrm{ad}_{e_i}\xi(\check{e}_i)(e_0^*, \check{e}_i^*)\\
&\stackrel{(13)}{=} -\lambda_i\alpha_{-1,0}(\check{e}_i),\\
\alpha_{-1,i}(e_{-1}) &= \xi(e_{-1})(e_{-1}^*, e_i^*) \stackrel{(13)}{=} -\xi(e_{-1})(e_{-1}^*, \mathrm{ad}_{\check{e}_i}^* e_0^*)\\
&= -\mathrm{ad}_{\check{e}_i}\xi(e_{-1})(e_{-1}^*, e_0^*) \stackrel{(1)}{=} -\left(\mathrm{ad}_{e_{-1}}\xi(\check{e}_i)(e_{-1}^*, e_0^*) + \lambda_i\xi(e_i)(e_{-1}^*, e_0^*)\right)\\
&= -\lambda_i\alpha_{-1,0}(e_i).
\end{aligned}$$

Hence, $(\alpha_{-1,0}(\check{e}_i))^2 + (\alpha_{-1,0}(e_i))^2 = 0$, and then $\alpha_{-1,0}(\check{e}_i) = \alpha_{-1,0}(e_i) = 0$. Note that, we have also shown that

$$c_{i,i}(e_i) = \alpha_{-1,i}(e_{-1}) = 0, \tag{17}$$

and in the same way we can show that

$$c_{i,i}(\check{e}_i) = \check{\alpha}_{-1,i}(e_{-1}) = 0. \tag{18}$$

To complete the proof, note that $\alpha_{-1,0}(e_{-1}) = 0$ according to Proposition 2.4. □

Proposition 2.5. *Let $\xi : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ be a bialgebra structure on $\mathfrak{g}_\lambda$. Then, for $i, j = 1, \dots, n$ with $i \neq j$,*

$$\begin{aligned}
&c_{i,i} = 0,\\
&b_{i,j}(e_i) = b_{i,j}(\check{e}_i) = b_{i,j}(e_j) = b_{i,j}(\check{e}_j) = 0,\\
&\check{b}_{i,j}(e_i) = \check{b}_{i,j}(\check{e}_i) = \check{b}_{i,j}(e_j) = \check{b}_{i,j}(\check{e}_j) = 0,\\
&c_{i,j}(e_i) = c_{i,j}(\check{e}_i) = c_{i,j}(e_j) = c_{i,j}(\check{e}_j) = 0.
\end{aligned}$$

Proof. The vanishing of $c_{i,i}$ is a consequence of Propositions 2.1 and 2.2 and (17)–(18).

Note that in Proposition 2.2, we have shown

$$b_{i,j}(e_i) = b_{i,j}(e_j) = c_{i,j}(e_i) = c_{i,j}(\check{e}_j) = \check{b}_{i,j}(\check{e}_i) = \check{b}_{i,j}(\check{e}_j) = 0.$$

On the other hand, by using (1) and (13), we get

$$\begin{aligned} b_{i,j}(\check{e}_i) &= \xi(\check{e}_i)(e_i^*, e_j^*) = -\xi(\check{e}_i)(e_i^*, \mathrm{ad}^*_{\check{e}_j} e_0^*) \\ &= -\mathrm{ad}_{\check{e}_j}\xi(\check{e}_i)(e_i^*, e_0^*) = -\mathrm{ad}_{\check{e}_i}\xi(\check{e}_j)(e_i^*, e_0^*) \\ &= -\lambda_i \xi(\check{e}_j)(e_{-1}^*, e_0^*) = 0. \end{aligned}$$

The same calculation gives the other relations. $\square$

We end this section by proving Theorems 1.1–1.3.

Proof of Theorem 1.1. Suppose that $\xi(u) = \mathrm{ad}_u^\dagger r + 2e_0 \wedge (J + \mathrm{ad}_{u_0})(u)$. Since e_0 is central and $J + \mathrm{ad}_{u_0}$ is a derivation, ξ is a 1-cocycle with respect to the adjoint action. Remark that for any $u \in \mathfrak{g}_\lambda$ and any $\alpha \in \mathfrak{g}_\lambda$, $r(\mathrm{ad}_u^* e_0^*, \alpha) = i_{r_\#(\alpha)}\omega(u)$. From this relation and by a straightforward computation one can show that the bracket on $\mathfrak{g}_\lambda$ associated to ξ is given by (9). Moreover, ξ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$ if and only if this bracket satisfies the Jacobi identity which is equivalent to

$$[e_0^*, [\alpha, \beta]^*]^* + [\alpha, [\beta, e_0^*]^*]^* + [\beta, [e_0^*, \alpha]^*]^* = 0,$$

for all $\alpha, \beta \in S^*$. Now $[e_0^*, [\alpha, \beta]^*]^* = 0$ and

$$\begin{aligned} [\alpha, [\beta, e_0^*]^*]^* &= -2[\alpha, J^*\beta]^* - [\alpha, i_{r_\#(\beta)}\omega]^* \quad (J^*\beta, i_{r_\#(\beta)}\omega \in S^*) \\ &= -2(\mathrm{ad}^\dagger_{e_{-1}} r)(\alpha, J^*\beta)e_{-1}^* - (\mathrm{ad}^\dagger_{e_{-1}} r)(\alpha, i_{r_\#(\beta)}\omega)e_{-1}^* \\ &= -2(\mathrm{ad}^\dagger_{e_{-1}} r)(\alpha, J^*\beta)e_{-1}^* - \omega(r_\#(\beta), (\mathrm{ad}^\dagger_{e_{-1}} r)_\#(\alpha))e_{-1}^*. \end{aligned}$$

In conclusion, the Jacobi identity holds if and only if,

$$\omega((\mathrm{ad}^\dagger_{e_{-1}} r)_\#(\alpha), r_\#(\beta)) + \omega(r_\#(\alpha), (\mathrm{ad}^\dagger_{e_{-1}} r)_\#(\beta)) - 2(J^\dagger \circ \mathrm{ad}^\dagger_{e_{-1}})r(\alpha, \beta) = 0$$

which is equivalent to (8).

Conversely, suppose that $\mathfrak{g}_\lambda$ is generic and ξ is a Lie bialgebra structure on $\mathfrak{g}_\lambda$. By gathering all the relations shown in Propositions 2.1–2.5 and Lemma 2.1 we get that the bracket on $\mathfrak{g}_\lambda^*$ associated to ξ is given by (the vanishing brackets are omitted)

$$\begin{aligned} [e_0^*, e_i^*]^* &= a_{0,i}(e_{-1})e_{-1}^* + \sum_{j=1}^n \left(a_{0,i}(\check{e}_j)\check{e}_j^* + a_{0,i}(e_j)e_j^*\right), \\ [e_0^*, \check{e}_i^*]^* &= \check{a}_{0,i}(e_{-1})e_{-1}^* + \sum_{j=1}^n \left(\check{a}_{0,i}(\check{e}_j)\check{e}_j^* + \check{a}_{0,i}(e_j)e_j^*\right), \\ [e_i^*, e_j^*]^* &= \left(\lambda_j \check{a}_{0,j}(\check{e}_i) - \lambda_i a_{0,j}(e_i)\right) e_{-1}^*, \\ [e_i^*, \check{e}_j^*]^* &= \left(-\lambda_i \check{a}_{0,j}(e_i) - \lambda_j a_{0,j}(\check{e}_i)\right) e_{-1}^* \\ &= \left(\lambda_j a_{0,i}(\check{e}_j) + \lambda_i \check{a}_{0,i}(e_j)\right) e_{-1}^* \quad i \neq j, \\ [\check{e}_i^*, \check{e}_j^*]^* &= \left(\lambda_j a_{0,j}(e_i) - \lambda_i \check{a}_{0,j}(\check{e}_i)\right) e_{-1}^*. \end{aligned}$$

The skew-symmetry of $[e_i^*, e_j^*]^*$ and $[\check{e}_i^*, \check{e}_j^*]^*$ and the two expressions of $[e_i^*, \check{e}_j^*]^*$ give

$$\begin{cases} \lambda_j(\check{a}_{0,j}(\check{e}_i) - a_{0,i}(e_j)) - \lambda_i(a_{0,j}(e_i) - \check{a}_{0,i}(\check{e}_j)) = 0, \\ -\lambda_i(\check{a}_{0,j}(\check{e}_i) - a_{0,i}(e_j)) + \lambda_j(a_{0,j}(e_i) - \check{a}_{0,i}(\check{e}_j)) = 0, \\ \lambda_j(a_{0,i}(\check{e}_j) + a_{0,j}(\check{e}_i)) + \lambda_i(\check{a}_{0,i}(e_j) + \check{a}_{0,j}(e_i)) = 0, \\ \lambda_i(a_{0,i}(\check{e}_j) + a_{0,j}(\check{e}_i)) + \lambda_j(\check{a}_{0,i}(e_j) + \check{a}_{0,j}(e_i)) = 0. \end{cases}$$

Since λ_i are mutually distinct, these relations are equivalent to

$$\check{a}_{0,j}(\check{e}_i) - a_{0,i}(e_j) = a_{0,j}(e_i) - \check{a}_{0,i}(\check{e}_j) = a_{0,i}(\check{e}_j) + a_{0,j}(\check{e}_i) = \check{a}_{0,i}(e_j) + \check{a}_{0,j}(e_i) = 0. \tag{19}$$

Put

$$\begin{cases} r_0(e_0^*, e_i^*) = \dfrac{1}{\lambda_i}\check{a}_{0,i}(e_{-1}), \quad r_0(e_0^*, \check{e}_i^*) = -\dfrac{1}{\lambda_i}a_{0,i}(e_{-1}), \\ r_0(e_i^*, e_j^*) = a_{0,i}(\check{e}_j), \qquad r_0(\check{e}_i^*, \check{e}_j^*) = -\check{a}_{0,i}(e_j), \\ r_0(e_i^*, \check{e}_j^*) = -a_{0,i}(e_j), \qquad i_{e_{-1}^*} r_0 = 0, \quad a_i = \check{a}_{0,i}(e_i), \end{cases}$$

and define J by $Je_0 = J_{e_{-1}} = 0$, $Je_i = a_i\check{e}_i$ and $J\check{e}_i = -a_i e_i$. From (19), $r_0 \in \wedge^2 \mathfrak{g}_\lambda$. The endomorphism J is a derivation which commutes with $\mathrm{ad}_{e_{-1}}$ and comparing the above brackets with those given in (14) one can see that $\xi = \mathrm{ad}^\dagger r_0 + e_0 \wedge J$. To complete the proof note that since $i_{e_{-1}^*} r_0 = 0$, there exist $r \in \wedge^2 S$ and $u_0 \in S$ such that $r_0 = e_0 \wedge u_0 + r$. $\square$

Proof of Theorem 1.2. Let $r \in \wedge^2 \mathfrak{g}_\lambda$ be a solution of the GCYBE. Then ξ given by $\xi(u) = \mathrm{ad}_u^\dagger r$ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$. By Lemma 2.1, for any $i = 1, \ldots, n$, $\mathrm{ad}_{e_{-1}}^\dagger r(e_{-1}^*, e_i) = \mathrm{ad}_{e_{-1}}^\dagger r(e_{-1}^*, \check{e}_i) = 0$. These relations are equivalent to $r(e_{-1}^*, e_i^*) = r(e_{-1}^*, \check{e}_i^*) = 0$ and hence,

$$r = r(e_0^*, e_{-1}^*)e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0,$$

where $r_0 \in \wedge^2 S$ and $u_0 \in S$. Thus

$$\xi(u) = \mathrm{ad}_u^\dagger r = \mathrm{ad}_u^\dagger r_0 + 2e_0 \wedge \left(-\alpha \mathrm{ad}_{e_{-1}} - \frac{1}{2}\mathrm{ad}_{u_0}\right)(u),$$

where $r(e_0^*, e_{-1}^*) = 2\alpha$. Now according to Theorem 1.1, ξ defines a Lie bialgebra structure if and only if r_0 satisfies (8) with $J = -\alpha \mathrm{ad}_{e_{-1}}$. This is equivalent to the fact that r_0 satisfies (10).

Let $r \in \wedge^2 \mathfrak{g}_\lambda$ be a solution of the CYBE. In particular, r is a solution of the GCYBE and from 1,

$$r = \alpha e_0 \wedge e_{-1} + e_0 \wedge u_0 + r_0,$$

where $r_0 \in \wedge^2 S$ and $u_0 \in S$. Now

$$[r, r] = [r_0, r_0] + 2\alpha e_0 \wedge \mathrm{ad}_{e_{-1}}^\dagger r_0 + 2e_0 \wedge \mathrm{ad}_{u_0}^\dagger r_0.$$

Now, by using (6) and the formula (see [6])

$$[r_0, r_0](\alpha, \beta, \gamma) = 2\alpha\left([r_{0\#}(\beta), r_{0\#}(\gamma)]\right) + 2\beta\left([r_{0\#}(\gamma), r_{0\#}(\alpha)]\right) + 2\gamma\left([r_{0\#}(\alpha), r_{0\#}(\beta)]\right) \tag{20}$$

one can check easily that

$$[r_0, r_0] = 2e_0 \wedge \omega_{r_0, r_0}.$$

Thus $[r, r] = 0$ if and only if

$$e_0 \wedge \left(\omega_{r_0, r_0} + \alpha \mathrm{ad}_{e_{-1}}^\dagger r_0 + \mathrm{ad}_{u_0}^\dagger r_0\right) = 0.$$

Since $r_0 \in \wedge^2 S$, $e_0 \wedge \mathrm{ad}_{u_0}^\dagger r_0 = 0$, ω_{r_0, r_0}, $\mathrm{ad}_{e_{-1}}^\dagger r_0 \in \wedge^2 S$ and hence, the equation above is equivalent to (11). □

Proof of Theorem 1.3. We identify $(\mathfrak{g}^*, [\,,\,]_r)$ to the Lie algebra of left invariant vector fields on G_r^* and $\langle\,,\,\rangle^*$ to the restriction of k^* to $\mathfrak{g}^*$. From the definition $[\,,\,]_r$ given by (12) and since ad_u^* is skew-symmetric for any $u \in \mathfrak{g}$, one can easily see that, for any $\alpha, \beta \in \mathfrak{g}^*$,

$$\nabla_\alpha^* \beta = -\mathrm{ad}_{r_\#(\alpha)}^* \beta.$$

Thus the curvature of ∇^* is given by

$$R(\alpha, \beta)\gamma = \nabla_{[\alpha,\beta]_r}^* \gamma - [\nabla_\alpha^*, \nabla_\beta^*]\gamma = \mathrm{ad}_{[r_\#(\alpha), r_\#(\beta)] - r_\#([\alpha,\beta]_r)}^* \gamma.$$

Now, from formula (20) one can deduce that

$$\gamma([r_\#(\alpha), r_\#(\beta)] - r_\#([\alpha, \beta]_r)) = \frac{1}{2}[r, r](\alpha, \beta, \gamma).$$

From this relation, we deduce that if r is a solution of (4) then R vanishes identically. We return to the general case and we denote by $u_r(\alpha, \beta)$ the element of $\mathfrak{g}$ defined by

$$\gamma(u_r(\alpha, \beta)) = \frac{1}{2}[r, r](\alpha, \beta, \gamma). \tag{21}$$

Thus

$$R(\alpha, \beta)\gamma = \mathrm{ad}_{u_r(\alpha,\beta)}^* \gamma.$$

Let us compute $\nabla^* R$. We have, for any $\alpha, \beta, \gamma, \rho \in \mathfrak{g}^*$,

$$\begin{aligned}(\nabla_\rho^* R)(\alpha, \beta, \gamma) &= \nabla_\rho^*(R(\alpha, \beta, \gamma)) - R(\nabla_\rho^*\alpha, \beta, \gamma) - R(\alpha, \nabla_\rho^*\beta, \gamma) - R(\alpha, \beta, \nabla_\rho^*\gamma)\\ &= -\mathrm{ad}_{r_\#(\rho)}^* \circ \mathrm{ad}_{u_r(\alpha,\beta)}^*\gamma + \mathrm{ad}_{u_r(\mathrm{ad}_{r_\#(\rho)}^*\alpha, \beta)}^*\gamma + \mathrm{ad}_{u_r(\alpha, \mathrm{ad}_{r_\#(\rho)}^*\beta)}^*\gamma + \mathrm{ad}_{u_r(\alpha,\beta)}^* \circ \mathrm{ad}_{r_\#(\rho)}^*\gamma\\ &= \mathrm{ad}_{[r_\#(\rho), u_r(\alpha,\beta)]}^*\gamma + \mathrm{ad}_{u_r(\mathrm{ad}_{r_\#(\rho)}^*\alpha, \beta)}^*\gamma + \mathrm{ad}_{u_r(\alpha, \mathrm{ad}_{r_\#(\rho)}^*\beta)}^*\gamma\\ &= \mathrm{ad}_{w(\alpha,\beta)}^*\gamma,\end{aligned}$$

where

$$w(\alpha, \beta) = [r_\#(\rho), u_r(\alpha, \beta)] + u_r(\mathrm{ad}_{r_\#(\rho)}^*\alpha, \beta) + u_r(\alpha, \mathrm{ad}_{r_\#(\rho)}^*\beta).$$

Now, for any $\mu \in \mathfrak{g}^*$, we have

$$\begin{aligned}\mu\left(w(\alpha,\beta)\right) &= \mathrm{ad}^*_{r_\#(\rho)}\mu(u_r(\alpha,\beta)) + \mu(u_r(\mathrm{ad}^*_{r_\#(\rho)}\alpha,\beta)) + \mu(u_r(\alpha,\mathrm{ad}^*_{r_\#(\rho)}\beta))\\ &\overset{(21)}{=} \frac{1}{2}[r,r](\alpha,\beta,\mathrm{ad}^*_{r_\#(\rho)}\mu) + \frac{1}{2}[r,r](\mathrm{ad}^*_{r_\#(\rho)}\alpha,\beta,\mu) + \frac{1}{2}[r,r](\alpha,\mathrm{ad}^*_{r_\#(\rho)}\beta,\mu)\\ &= \frac{1}{2}(\mathrm{ad}_{r_\#(\rho)}[r,r])(\alpha,\beta,\mu) \overset{(3)}{=} 0.\end{aligned}$$

This achieves to show that $\nabla^* R = 0$ and the first part of the theorem follows.

According to a result of Aubert and Medina (see [26]), a flat left invariant semi-Riemannian metric on a Lie group is complete if and only if the group is unimodular and in this case the group is solvable. $\square$

3. Solutions of (8), (10) and (11) when dim $G_\lambda \leq 6$

In this section, we determine the bialgebra structures and the solutions of the GYBE and the CYBE on $\mathfrak{g}_\lambda$ generic with $\dim \mathfrak{g}_\lambda \leq 6$ according to Theorems 1.1–1.2.

Note first that if $J : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}_\lambda$ is a derivation commuting with $\mathrm{ad}_{e_{-1}}$ and satisfying $J(e_{-1}) = J(e_0) = 0$ then there exists $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that for any $i = 1, \ldots, n$ such that

$$J(e_i) = a_i \check{e}_i \quad \text{and} \quad J(\check{e}_i) = -a_i e_i.$$

We shall denote such an endomorphism by J_a.

When $\dim \mathfrak{g}_\lambda = 4$, the situation is simple. We have $\mathbb{B} = \{e_{-1}, e_0, e_1, \check{e}_1\}$, $S = \mathrm{span}\{e_1, \check{e}_1\}$ and $\wedge^2 S = \mathrm{span}\{e_1 \wedge \check{e}_1\}$. Moreover, $\mathrm{ad}^\dagger_{e_{-1}}(e_1 \wedge \check{e}_1) = J^\dagger_a(e_1 \wedge \check{e}_1) = 0$ and $\omega_{e_1\wedge\check{e}_1, e_1\wedge\check{e}_1} = e_1 \wedge \check{e}_1$. By applying Theorems 1.1 and 1.2, we get the following proposition.

Proposition 3.1. *Let $\lambda \in \mathbb{R}$ and let $\mathfrak{g}_\lambda$ be the associated 4-dimensional oscillator Lie algebra. Then, we have the following.*

1. *$\xi : \mathfrak{g}_\lambda \longrightarrow \wedge^2 \mathfrak{g}_\lambda$ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$ if and only if there exist $a, \alpha \in \mathbb{R}$ and $u_0 \in S$ such that*
$$\xi(u) = \alpha \mathrm{ad}^\dagger_u(e_1 \wedge \check{e}_1) + e_0 \wedge (J_a + \mathrm{ad}_{u_0})(u).$$
2. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the GCYBE if and only if $r = e_0 \wedge u + \alpha e_1 \wedge \check{e}_1$, where $\alpha \in \mathbb{R}$ and $u \in \mathfrak{g}_\lambda$.*
3. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the CYBE if and only if $r = e_0 \wedge u$, where $\alpha \in \mathbb{R}$ and $u \in \mathfrak{g}_\lambda$.*

We return now to the general case. Consider a generic $2n + 1$-dimensional oscillator Lie algebra $\mathfrak{g}_\lambda$ and fix $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. For any $1 \leq i, j \leq n$, put

$$r_{ij} = e_i \wedge e_j, \qquad \check{r}_{ij} = \check{e}_i \wedge \check{e}_j, \qquad s_{ij} = e_i \wedge \check{e}_j, \qquad \check{s}_{ij} = \check{e}_i \wedge e_j, \qquad t_i = e_i \wedge \check{e}_i.$$

A direct computation gives

$$\begin{aligned}&\omega_{r_{ij},r_{ij}} = \omega_{\check{r}_{ij},\check{r}_{ij}} = \omega_{s_{ij},s_{ij}} = \omega_{\check{s}_{ij},\check{s}_{ij}} = 0,\\ &\omega_{r_{ij},s_{ij}} = \omega_{r_{ij},\check{s}_{ij}} = \omega_{\check{r}_{ij},s_{ij}} = \omega_{\check{r}_{ij},\check{s}_{ij}} = 0,\\ &\omega_{r_{ij},\check{r}_{ij}} = -\omega_{s_{ij},\check{s}_{ij}} = \frac{1}{2}(t_i + t_j),\\ &\omega_{t_i,r_{ij}} = \frac{1}{2}r_{ij}, \qquad \omega_{t_i,\check{r}_{ij}} = \frac{1}{2}\check{r}_{ij}, \qquad \omega_{t_i,s_{ij}} = \frac{1}{2}s_{ij}, \qquad \omega_{t_i,\check{s}_{ij}} = \frac{1}{2}\check{s}_{ij}\\ &\omega_{t_j,r_{ij}} = \frac{1}{2}r_{ij}, \qquad \omega_{t_j,\check{r}_{ij}} = \frac{1}{2}\check{r}_{ij}, \qquad \omega_{t_j,s_{ij}} = \frac{1}{2}s_{ij}, \qquad \omega_{t_j,\check{s}_{ij}} = \frac{1}{2}\check{s}_{ij},\\ &\omega_{t_i,t_i} = t_i, \qquad \omega_{t_i,t_j} = 0.\end{aligned}$$

Fix $1 \leq i < j \leq n$ and let us find the solutions of (8), (10) and (11) of the form $r = p_{ij} + c_i t_i + c_j t_j$, where $p_{ij} \in \mathrm{span}\{r_{ij}, \check{r}_{ij}, s_{ij}, \check{s}_{ij}\}$. In order to simplify the computations, let us introduce the following new basis:

$$E_{ij} = s_{ij} + \check{s}_{ij}, \qquad \check{E}_{ij} = -r_{ij} + \check{r}_{ij}, \qquad F_{ij} = -s_{ij} + \check{s}_{ij}, \qquad \check{F}_{ij} = r_{ij} + \check{r}_{ij}.$$

We have

$$J^\dagger_a(E_{ij}) = (a_i + a_j)\check{E}_{ij}, \qquad J^\dagger_a(\check{E}_{ij}) = -(a_i + a_j)E_{ij}, \qquad J^\dagger_a(F_{ij}) = (a_j - a_i)\check{F}_{ij}, \qquad J^\dagger_a(\check{F}_{ij}) = (a_i - a_j)F_{ij}.$$

Since $\mathrm{ad}^\dagger_{e_{-1}} = J^\dagger_\lambda$, similar relations hold for $\mathrm{ad}^\dagger_{e_{-1}}$. On the other hand, one can easily see that $(E_{ij}, \check{E}_{ij}, F_{ij}, \check{F}_{ij})$ is ω-orthogonal and

$$\omega_{E_{ij},E_{ij}} = \omega_{\check{E}_{ij},\check{E}_{ij}} = -\omega_{F_{ij},F_{ij}} = -\omega_{\check{F}_{ij},\check{F}_{ij}} = -(t_i + t_j).$$

Let $(a, \check{a}, b, \check{b})$ be the coordinates of p_{ij} in $(E_{ij}, \check{E}_{ij}, F_{ij}, \check{F}_{ij})$. From the relations above we get

$$\begin{aligned}
\omega_{r,r} + \alpha \mathrm{ad}^\dagger_{e_{-1}} r &= c_i^2 t_1 + c_j^2 t_j + (c_i + c_j) p_{ij} + (b^2 + \check{b}^2 - a^2 - \check{a}^2)(t_i + t_j) \\
&\quad + \alpha(\lambda_i + \lambda_j)\left(-\check{a} E_{ij} + a \check{E}_{ij}\right) + \alpha(\lambda_j - \lambda_i)\left(-\check{b} F_{ij} + b \check{F}_{ij}\right), \\
\omega_{r, \mathrm{ad}^\dagger_{e_{-1}} r} - (J_a^\dagger \circ \mathrm{ad}^\dagger_{e_{-1}}) r &= \frac{1}{2}(c_i + c_j)(\lambda_i + \lambda_j)\left(-\check{a} E_{ij} + a \check{E}_{ij}\right) + \frac{1}{2}(c_i + c_j)(\lambda_j - \lambda_i)\left(-\check{b} F_{ij} + b \check{F}_{ij}\right) \\
&\quad + (\lambda_i + \lambda_j)(a_i + a_j)\left(a E_{ij} + \check{a} \check{E}_{ij}\right) + (\lambda_j - \lambda_i)(a_j - a_i)\left(b F_{ij} + \check{b} \check{F}_{ij}\right), \\
\omega_{r, \mathrm{ad}^\dagger_{e_{-1}} r} + \alpha(\mathrm{ad}^\dagger_{e_{-1}} \circ \mathrm{ad}^\dagger_{e_{-1}}) r &= \frac{1}{2}(c_i + c_j)(\lambda_i + \lambda_j)\left(-\check{a} E_{ij} + a \check{E}_{ij}\right) + \frac{1}{2}(c_i + c_j)(\lambda_j - \lambda_i)\left(-\check{b} F_{ij} + b \check{F}_{ij}\right) \\
&\quad - \alpha(\lambda_i + \lambda_j)^2\left(a E_{ij} + \check{a} \check{E}_{ij}\right) - \alpha(\lambda_j - \lambda_i)^2\left(b F_{ij} + \check{b} \check{F}_{ij}\right).
\end{aligned}$$

1. (a) For $\alpha \neq 0$, $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (11) if and only if $r = 0$.
   (b) For $\alpha = 0$, $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (11) if and only if $r = aE_{ij} + \check{a}\check{E}_{ij} + bF_{ij} + \check{b}\check{F}_{ij} + c(t_i - t_j)$ and $c^2 = a^2 + \check{a}^2 - b^2 - \check{b}^2$.
2. (a) For $\alpha = 0$, $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (10) if and only if $r = c_i t_i + c_j t_j$ or $r = aE_{ij} + \check{a}\check{E}_{ij} + bF_{ij} + \check{b}\check{F}_{ij} + c(t_i - t_j)$.
   (b) For $\alpha \neq 0$, $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (10) if and only if $r = c_i t_i + c_j t_j$.
3. (a) If $a_1 + a_2 \neq 0$ and $a_2 - a_1 \neq 0$ then $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (8) if and only if $r = c_i t_i + c_j t_j$.
   (b) If $a_1 + a_2 = 0$ and $a_2 - a_1 \neq 0$ then $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (8) if and only if $r = c_i t_i + c_j t_j$ or $r = aE_{ij} + \check{a}\check{E}_{ij} + c(t_i - t_j)$.
   (c) If $a_1 + a_2 \neq 0$ and $a_2 - a_1 = 0$ then $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (8) if and only if $r = c_i t_i + c_j t_j$ or $r = bF_{ij} + \check{b}\check{F}_{ij} + c(t_i - t_j)$.
   (d) If $a_1 = a_2 = 0$ then $r = p_{ij} + c_i t_i + c_j t_j$ is a solution of (8) if and only if $r = c_i t_i + c_j t_j$ or $r = aE_{ij} + \check{a}\check{E}_{ij} + bF_{ij} + \check{b}\check{F}_{ij} + c(t_i - t_j)$.

   When $\dim \mathfrak{g}_\lambda = 6$, we have constructed all the solutions. Let us summarize this case.

Proposition 3.2. *Let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 < \lambda_2$ and let $\mathfrak{g}_\lambda$ be the associated 6-dimensional oscillator Lie algebra. Then, we have the following.*

1. *$\xi : \mathfrak{g}_\lambda \longrightarrow \wedge^2 \mathfrak{g}_\lambda$ defines a Lie bialgebra structure on $\mathfrak{g}_\lambda$ if and only if ξ has one of the following forms:*
   (a) *$\xi(u) = \mathrm{ad}_u(c_1 t_1 + c_2 t_2) + e_0 \wedge (J_a + \mathrm{ad}_{u_0})(u)$, where $c_1, c_2 \in \mathbb{R}$, $a \in \mathbb{R}^2$ and $u_0 \in S$;*
   (b) *$\xi(u) = \mathrm{ad}_u(dE_{12} + \check{d}\check{E}_{12} + c(t_1 - t_2)) + e_0 \wedge (J_{(a,-a)} + \mathrm{ad}_{u_0})(u)$, where $a, d, \check{d}, c \in \mathbb{R}$ and $u_0 \in S$;*
   (c) *$\xi(u) = \mathrm{ad}_u(dF_{12} + \check{d}\check{F}_{12} + c(t_1 - t_2)) + e_0 \wedge (J_{(a,a)} + \mathrm{ad}_{u_0})(u)$, where $a, d, \check{d}, c \in \mathbb{R}$ and $u_0 \in S$;*
   (d) *$\xi(u) = \mathrm{ad}_u(dE_{12} + \check{d}\check{E}_{12} + bF_{12} + \check{b}\check{F}_{12} + c(t_1 - t_2)) + e_0 \wedge (\mathrm{ad}_{u_0})(u)$, where $b, \check{b}, d, \check{d}, c \in \mathbb{R}$ and $u_0 \in S$;*
2. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the GCYBE if and only if $r = e_0 \wedge u + c_1 t_1 + c_2 t_2$, where $u \in \mathfrak{g}_\lambda$ and $c_1, c_2 \in \mathbb{R}$, or $r = e_0 \wedge u + aE_{12} + \check{a}\check{E}_{12} + bF_{12} + \check{b}\check{F}_{12} + c(t_1 - t_2)$, where $u \in S$ and $a, \check{a}, b, \check{b}, c \in \mathbb{R}$.*
3. *A bivector $r \in \wedge^2 \mathfrak{g}_\lambda$ is a solution of the CYBE if and only if $r = e_0 \wedge u$, where $\alpha \in \mathbb{R}$ and $u \in \mathfrak{g}_\lambda$, or $r = e_0 \wedge u + aE_{12} + \check{a}\check{E}_{12} + bF_{12} + \check{b}\check{F}_{12} + c(t_1 - t_2)$ where $u \in S$ and $c^2 = a^2 + \check{a}^2 - b^2 - \check{b}^2$.*

By using Corollary 1.1 and the proposition above, one can prove easily the following result.

Proposition 3.3. *Let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 < \lambda_2$ and let $\mathfrak{g}_\lambda$ be the associated 6-dimensional oscillator Lie algebra. Then for any solution r of the CYBE $(\mathfrak{g}_\lambda^*, [\,,\,]_r)$ is unimodular.*

4. Examples

1. From Proposition 3.2, it is clear that

   $$r = e_0 \wedge e_1 + e_1 \wedge \check{e}_2 + \check{e}_1 \wedge e_2 + e_1 \wedge \check{e}_1 - e_2 \wedge \check{e}_2$$

   is a solution of (4) on $\mathfrak{g}_\lambda$ $(\lambda = (\lambda_1, \lambda_2))$. By using (14), we can check that the bracket $[\,,\,]_r$ on $\mathfrak{g}_\lambda^*$ associated to r is given by

   $$\begin{aligned}
   &[e_0^*, e_1^*]_r = -e_1^* - e_2^*, \qquad [e_0^*, e_2^*]_r = e_1^* + e_2^*, \qquad [e_0^*, \check{e}_1^*]_r = \lambda_1 e_{-1}^* - \check{e}_1^* + \check{e}_2^*, \\
   &[e_0^*, \check{e}_2^*]_r = -\check{e}_1 + \check{e}_2, \qquad [e_2^*, \check{e}_2^*]_r = [e_1^*, \check{e}_1^*]_r = [e_1^*, \check{e}_2^*]_r = [e_2^*, \check{e}_1^*]_r = 0, \\
   &[e_1^*, e_2^*]_r = -[\check{e}_1^*, \check{e}_2^*]_r = -(\lambda_1 + \lambda_2) e_{-1}^*.
   \end{aligned}$$

   Consequently, the simply connected Lie group G_λ^* is unimodular and a semi-direct product of the (normal) five dimensional Heisenberg group $\mathbb{R}$. Moreover the flat and complete left invariant Lorentzian metric given by Theorem 1.3 is determined by

   $$\mathbf{k}_\lambda^*(e_0^*, e_{-1}^*) = 1, \qquad \mathbf{k}_\lambda^*(e_i^*, e_i^*) = \mathbf{k}_\lambda^*(\check{e}_i^*, \check{e}_i^*) = \lambda_i, \quad i = 1, 2.$$

2. Let $G = \mathrm{GL}(n+1, \mathbb{R})$ be the linear group endowed with the quadratic structure k given by the orthogonal sum of $\mathbb{R}$ and $(\mathrm{SL}(n+1, \mathbb{R}), k')$ where k' is the Killing form. The classical affine Lie group $H := \mathrm{Aff}(n, \mathbb{R})$ is a closed symplectic subgroup of G and hence, it defines a solution r of the CYBE on G. The simply connected Lie group dual of G, relative to r, is endowed with a flat left invariant semi-Riemannian metric having the same signature as k.
3. Let S be a simple Lie group and $\mathfrak{s}$ its Lie algebra. Let us endow the cotangent bundle $G = T^*S$ with the Lie group structure, semidirect product of the Abelian group $\mathfrak{s}^*$ by S via the coadjoint action. Denote by $t^*(\mathfrak{s})$ its Lie algebra. The group G admits, in general, many quadratic structures. Let k' be the canonical hyperbolic structure on G. Every graph of a linear map $r : \mathfrak{s}^* \longrightarrow \mathfrak{s}$ which is a Lagrangian subalgebra, relative to k', of $t^*(\mathfrak{s})$ is a solution of the CYBE on $\mathfrak{s}$ (see [27]). From Theorem 1.3, any Lie subgroup of G whose Lie algebra is a Lagrangian subspace of $t^*(\mathfrak{s})$ is endowed with a flat left invariant semi-Riemannian metric.

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