Crisis, contagion and containment policies in financial networks : A dynamic approach

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1 Introduction

We wish to create a model of financial networks that can be used for any structure of network. Each node's state is given by a simplified balance sheet from which an equity can be computed. This is the quantity that determines whether a node is in default. We want to only consider solvency defaults and not liquidity default which cannot happen in our setting. We will then simulate the propagation of different types of shocks on this network in a dynamic fashion going one step further in comparison to the existing literature that uses either a fixed small number of periods ([**10**]) or a "killing cascade" as dynamics ([**1.1**], [**11**])—after an initial shock, banks may default, we check if those defaults bring about new defaults, and so on until no more banks are defaulting. Having a real time dynamics will open the way for the study of dynamic resource allocations strategy on the network to try to contain potential systemic events in the fashion of [**5.2**].

2 Economic description of the model

We focus our analysis on banks and the only type of credit we consider is interbank loans. Banks do not take stakes in one another. The outside economy is simply modelled as a set of risky assets which yields random returns, they could represent for instance loans to companies, to individuals, investment in public projects... as well as financial products.

2.1 Banks

2.1.1 Balance sheet representation

Let n be the number of banks in the network. At each time t, bank j is represented as a simplified balance sheet :

Assets ($\mathcal{A}_t^i$)	Liabilities ($\mathcal{L}_t^i$)
Reserve (R_t^i)	Equity (E_t^i)
Inter-bank loans (L_t^i)	Inter-bank debts (D_t^i)
Portfolio (P_t^i)	

- On the assets side :
  - Reserve R_t^i is the amount of reserves, a fraction or the totality of them may be kept in a central bank.
  - L_t^i corresponds to the cumulative face-value of loans going from bank i to other banks. Also, let us denote by L_t^{ij} the face-value of the loan going from bank i to j. Thus $L_t^i = \sum_{j=1}^n L_t^{ij}$
  - Portfolio P_t^j is the amount of money invested in the economy — more on that in section 2.2.
- On the liabilities side :
  - Equity E_t^i is equal to the assets minus the liabilities. Thus it is when equity goes under 0 that a bank is said to go bankrupt.
  - D_t^i corresponds to the cumulative face-value of the loans coming from other banks to bank i. Thus $D_t^i = \sum_{k=1}^n L_t^{ki}$.

Let us denote by :

- $L_t \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ the matrix of inter-bank loans which entries are the L_t^{ij}. Two important remarks :
  - In order to simplify the graph as much as possible, loans and debts will be netted, which means that $L_t^{ij} > 0 \Rightarrow L_t^{ji} = 0$. Given a matrix L_t, we can obtain its netted version L'_t using component-wise maximum by doing the following operation :

$$L'_t = \max(\mathbf{0}_{n,n}, L_t - L_t^T)$$

    Where $\mathbf{0}_{n,n} \in \mathcal{M}_{n,n}(\mathbb{R})$ is the matrix full of zeros.
  - The matrix L_t contains all the information on the graph at time t, thus no need to have a debt matrix since it is simply equal to L_t^T, the transposed version of L_t.
- $E_t \in \mathcal{M}_{n,1}(\mathbb{R})$ the vector of equities which i-th element is E_t^i
- $R_t \in \mathcal{M}_{n,1}(\mathbb{R})$ the vector of reserves which i-th element is R_t^i

*Remark : the matrix L_t captures the structure of the graph. We will consider it given at first. In the simulation stages, we may test different random graph initialization as done in [**11**]*

2.1.2 Interest rates

We introduce the inter-bank interest rate r_t^i which determines the cost of borrowing for bank i at time t. Its existence is justified by the extra risk taken when lending money. We make several assumptions for now which we may relax later :

- Interest rates are deterministic
- r_t^j does not depend on time, thus we will omit the time-index $r_t^i = r^i$
- $\forall i \in [\![1, n]\!], \quad r^i = r$, thus every bank can borrow money for the same interest rate r to other banks.

2.2 Portfolios and investment opportunities

2.2.1 Risky assets

There are m risky assets in the economy. We take a wide definition of risky assets which entails productive investments such as loans to companies or individuals, investments in public projects... Thus risky assets are not necessarily stocks or financial products although they can be. The only fundamental conditions an asset need to match the definition are :

- to be outside of the network of banks
- to be risky to some extent

Each risky assets has a time-dependant valuation. For a given $l \in [\![1, m]\!]$, let us denote by X_t^l this price. Our model is in discrete time.

For now, those investments do not yield dividend. As a consequence gains (resp. losses) only come from increases (resp. drops) in the valuations. We define the returns :

$$\omega_{t-1}^l = \frac{X_t^l - X_{t-1}^l}{X_{t-1}^l}$$

We model them using independent Gaussian random variables.

$$\omega_{t-1}^l \sim \mathcal{N}(\mu_l, \sigma_l^2)$$

We denote by ω_t the vector of returns. Since its components are Gaussian and independent, this is a Gaussian vector with :

- Mean vector $\mu \in \mathcal{M}_{m,1}(\mathbb{R})$
- Covariance matrix $\Sigma = Diag[(\sigma_l^2)_{1 \leq l \leq m}] \in \mathcal{M}_{m,m}(\mathbb{R}^+)$.

$$\omega_{t-1} \sim \mathcal{N}(\mu, \Sigma)$$

Using returns, we have the following relation for prices evolution :

$$X_t^l = X_{t-1}^l(1 + \omega_t^l)$$

We make the assumptions that returns are independents across time :

$$\forall t \neq t',\ w_{t-1} \perp\!\!\!\perp w_{t-1'}$$

2.2.2 Bank's portfolios and valuation

Banks invest in those risky assets. Let us denote by $Q_t^i \in \mathcal{M}_{1,m}(\mathbb{R}^+)$ the vector which entry Q_t^{il} is number of unit of product l that bank i has in its portfolio at time t. The matrix $Q_t \in \mathcal{M}_{n,m}(\mathbb{R}^+)$ is the matrix which rows are the $(Q_t^i)_{1 \leq j \leq n}$.

As a consequence, the value of i's portfolio is given by : $P_t^i = Q_t^i X_t$. The latter can be written in matrix form : $P_t = Q_t X_t$, with $P_t \in \mathcal{M}_{n,1}(\mathbb{R}^+)$ being the portfolio vector.

3 General dynamics of the system

3.1 Sequence of events

In this subsection, we establish the chronology of events without giving the equations. Since several operations take place within a given time-lapse t the ordering of events need to be precised. The detailed equations will be detailed later—at each stage a reference points to the appropriate section/subsection.

- **Stage 1 : updates** (detailed in 3.3). At the beginning of this stage, the state of a bank is given by the vector $(E_{t-1}^i, D_{t-1}^i, R_{t-1}^i, P_{t-1}^i, L_{t-1}^i)$. The following operations are then carried out
  - If banks have defaulted in the previous periods, the default is now effective and the creditors of the defaulted banks take the corresponding losses.
  - The reserves are updated: banks pay interest rates on inter-bank loans and if banks have defaulted in the previous periods, proceedings from liquidation are added to reserves.
  - The valuation of the risky assets are updated and the value of portfolios are changed accordingly.

  At the end of this stage, the state of a bank is given by a vector of five variables : $(\widehat{E}_t^i, \widehat{D}_t^i, \widehat{R}_t^i, \widehat{P}_t^i, \widehat{L}_t^i)$.

- **Stage 2 : checking for default**. Banks for which $\widehat{E}_t^i \leq \bar{E}^i$ declare default and are liquidated (see section 4 for the detailed processes of liquidation). Although their defaulting is not public information yet, it will become so at the beginning of the next period. The vector $\bar{E}$ which components are the $\bar{E}^i$ is a minimal threshold value for the equity of each bank. It enables us to implicitly include deposits from investors or individuals from outside the system in the balance sheets of the banks.

- **Stage 3 : balance sheet management** (detailed in 3.4) Banks readjust between portfolio and reserves according to (detailed in 3.2):

  - The reserves rule
  - The financial choice for the portfolio

  At the end of the stage, the state of a bank is given by the 5 state variables $(E_t^i, D_t^i, R_t^i, P_t^i, L_t^i)$

3.2 Viability conditions

We need to introduce additional hypothesis, initial conditions and constraints in order to maintain our model's coherence.

- We assume that banks cannot have a negative expected variation of equity. Even though they are unidentified and not included in our model, we can assume that there are shareholders who own the banks. They indeed want their shares to gain value which justify our hypothesis on the variation of equity. This is the subject of section 3.2.1.

- If some banks have borrowed more than they have lent so as to invest in their portfolio section, their reserves will deplete mechanically after a given number of periods. As a consequence, we need to define transfer rules between portfolio and reserves. We thus introduce the reserve rule and the financial choice in 3.2.2.

- We make the assumption that banks can sell freely assets from their portfolio as a regular operation of management without paying any fees or having any price impact. This seems to be a reasonable hypothesis since only small amounts of risky assets are bought and sold in a usual financial market context in order to obtain the chosen portfolio. Also, we do not impose integer-valued quantities.

3.2.1 Bank should have a positive expected net-worth delta

Expected returns vs interest rates Since investment opportunities' returns are uncertain, investment opportunities must offer a risk premium—although loans to other banks are risky too they are obviously less so . This implies a first condition on the means μ of the vector of returns :

$$\forall l \in [\![1, m]\!],\ \mu_l \geq r$$

Balance sheets coherence Thus it can be interesting for a bank to borrow money from other banks in order to invest. Although it may do so only in such a way that it gains money on average—where $\mathcal{F}_t$ is the information available up to time t :

$$\mathbb{E}[E_t^i|\mathcal{F}_{t-1}] \geq E_{t-1}^i$$

Let us remark first that since $E_{t-1}^i \geq \bar{E}^i$, this implies that a bank that has not defaulted in $t-1$ cannot be expected to default in t.

Applying expectation conditionally on $\mathcal{F}_t$ to (6) we get:

$$\mathbb{E}[E_t^i|\mathcal{F}_{t-1}] = E_{t-1}^i + rL_{t-1}^i - rD_{t-1}^i + \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l \mathbb{E}[\omega_t^l|\mathcal{F}_{t-1}].$$

Which is equivalent to:

$$\mathbb{E}[E_t^i|\mathcal{F}_{t-1}] - E_{t-1}^i = rL_{t-1}^i - rD_{t-1}^i + \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l \mu_l.$$

As a consequence,

$$\mathbb{E}[E_t^i|\mathcal{F}_{t-1}] \geq E_{t-1}^i \Leftrightarrow L_{t-1}^i - rD_{t-1}^i + \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l \mu_l \geq 0.$$

Rearanging the terms, the condition condition is:

$$rL_{t-1}^i + \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l \mu_l \geq rD_{t-1}^i.$$

Actually this condition may prove too difficult to enforce at each future period. We will thus consider only its initial version in $t = 0$:

$$rL_0^i + \sum_{l=1}^{m} Q_0^{il} X_0^l \mu_l \geq rD_0^i. \tag{1}$$

3.2.2 Regulation rules and management conditions

Reserve rule We want to ensure that banks can pay their interest rates the next day. We thus introduce the following regulatory rule imposed by some prudential authority.

$$R_t^i + r(L_t^i - D_t^i) \geq 0 \tag{2}$$

Proposition A bank that has not defaulted in t necessarily has enough liquidity in t to comply with the reserve rule.

To prove this, let us distinguish between two case.

- $\widehat{D}_t^i \leq \widehat{L}_t^i$. In that case, a bank only receives interest rates and thus automatically complies with the reserve rule. In other words, since $R_t^i \geq 0$

$$\widehat{L}_t^i - \widehat{D}_t^i \geq 0 \Rightarrow r(\widehat{L}_t^i - \widehat{D}_t^i) \geq 0 \Rightarrow \widehat{R}_t^i + r(\widehat{L}_t^i - \widehat{D}_t^i) \geq 0.$$

- $\widehat{D}_t^i > \widehat{L}_t^i$. The bank has not defaulted in t, thus:

$$\widehat{E}_t^i > \bar{E}^i \Rightarrow \widehat{E}_t^i > 0 \Leftrightarrow \widehat{R}_t^i + \widehat{P}_t^i + \widehat{L}_t^i - \widehat{D}_t^i > 0.$$

  We have:

$$\widehat{D}_t^i - \widehat{L}_t^i > \widehat{R}_t^i + \widehat{P}_t^i.$$

  Since $r < 1$:

$$\widehat{D}_t^i - \widehat{L}_t^i > r(\widehat{D}_t^i - L_t^i).$$

  As a consequence:

$$\widehat{R}_t^i + \widehat{P}_t^i > r(\widehat{D}_t^i - \widehat{L}_t^i).$$

We can thus conclude that a bank that has not defaulted in t always has enough liquidity to be able to reallocate its liquid assets in t so as to comply with the reserve rule. It can do so by selling partially its portfolio in order to increase its reserves.

Thus, this rule enables us to avoid liquidity defaults, and being solvent is a sufficient condition to be able to comply with it. Since we want to account only for solvency defaults, our model is coherent. Indeed, default can be defined as either having not enough cash to honor one's financial obligation (liquidity default) or not having enough equity (solvency default)—or both. We have shown here that since the only illiquid assets are inter-bank loans, if equity is positive in t then by construction liquidity default cannot happen in t.

Portfolio management condition Since the value of the assets in the portfolio will vary, so will the size of the portfolio in relation to the rest of the balance-sheet quantities. As a basic rule of management, we may state that each bank wants to maximize its investment in the portfolio under the constraint that it amounts to a given fraction of the total of its assets $\mathcal{A}_t$ and that it respects the reserve rule. Let us define then by α^i the target investment percentage of bank j which we can interpret as a behavioral parameter. We assume for now that the share of wealth invested in each asset remains constant. As a consequence, we can formulate the problem in term of P_t only :

$$\max_{\substack{P_t^i \leq \alpha^i \mathcal{A}_t^i \\ R_t^i + r(L_t^i - D_t^i) \geq 0 \\ P_t^i + R_t^i = \widehat{P}_t^i + \widehat{R}_t^i}} P_t^i$$

3.3 Stage 1 : updates

3.3.1 Reserves

The evolution of reserves is given by:

$$\widehat{R}_t^i = R_{t-1}^i + r\widehat{L}_t^i - r\widehat{D}_t^i \tag{3}$$

3.3.2 Portfolio

Using the returns, the dynamic of the portfolio is given by :

$$\widehat{P}_t^i = \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l (1 + \omega_t^l)$$

$$\widehat{P}_t^i = P_{t-1}^i + \sum_{l=1}^{m} Q_{t-1}^{il} X_{t-1}^l \omega_t^l$$

Although it is simpler to define $\Delta X_t = X_t - X_{t-1}t$ and write the previous dynamic relation using dot products:

$$\widehat{P}_t^i = P_{t-1}^i + Q_{t-1}^i \Delta X_t \tag{4}$$

3.3.3 Equity

In math form, the accounting definition of equity is :

$$\widehat{E}_t^i = \widehat{R}_t^i + \widehat{P}_t^i + \widehat{L}_t^i - \widehat{D}_t^i \tag{5}$$

Putting all above dynamic equation together yields:

$$\widehat{E}_t^i = R_{t-1}^i + r\widehat{L}_t^i - r\widehat{D}_t^i + \widehat{L}_t^i - \widehat{D}_t^i + P_{t-1}^i + Q_{t-1}^i \Delta X_t$$

If no bank has defaulted in $t-1$, we have that $\widehat{L}_t^i = L_{t-1}^i$ and $\widehat{D}_t^i = D_{t-1}^i$, and thus:

$$E_t^i = R_{t-1}^i + P_{t-1}^i + L_{t-1}^i - D_{t-1}^i + r(L_{t-1}^i - D_{t-1}^i) + Q_{t-1}^i \Delta X_t$$

We can thus deduce the following recursion formula for equity in the case where no bank has defaulted in the previous period:

$$\Leftrightarrow E_t^i = E_{t-1}^i + r(\widehat{L}_{t-1}^i - \widehat{D}_{t-1}^i) + Q_{t-1}^i \Delta X_t \tag{6}$$

3.4 Stage 3 : balance sheet management

3.4.1 Reserve rule and portoflio rule in practice

Let us distinguish two cases

- $\widehat{R}_t^i < r(\widehat{D}_t^i - \widehat{L}_t^i)$ (reserve rule not matched)
- $\widehat{R}_t^i \geq r(\widehat{D}_t^i - \widehat{L}_t^i)$ (reserve rule matched)

Let us also define the portfolio valuation objective P_t^j which corresponds to the valuation of the portfolio the bank wish to reach.

Reserve rule matched No need to rebalance to comply with the reserve rule, although if the objective valuation of the portfolio is to be increased, we must keep in mind that the new portfolio objective must also comply with the reserve rule. We distinguish again different cases :

- $\widehat{P}_t^i > \alpha^i \mathcal{A}_t^i$. In that case, a portion $\widehat{P}_t^i - \alpha^i \mathcal{A}_t^i$ of the portfolio must be sold. No other actions are required, thus:
$$P_t^i = \alpha^i \mathcal{A}_t^i.$$
- $\widehat{P}_t^i \leq \alpha^i \mathcal{A}_t^i$. In that case the bank wants to increase its portfolio to saturate the constraint if possible given the reserve rule. Thus:
$$P_t^i = \min\left(\alpha^i \mathcal{A}_t^i, \quad \widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i)\right).$$

Reserve rule not matched The bank have to sell a portion of its portfolio to comply with the reserve rule. However, it may sell more if its portfolio is still too large according to the financial choice. Let us distinguish two cases :

- $\widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i) \leq \alpha^i \mathcal{A}_t^i$. In that case, the bank sells only the amount necessary to comply with the reserve rule — since $\widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i) < 0$ this is indeed selling . As a consequence:
$$P_t^i = \widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i).$$
- $\widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i) > \alpha^i \mathcal{A}_t^i$. In that case, the bank sells enough portfolio assets to comply with the financial choice. Which is enough to comply also with the reserve rule since we have: $\widehat{P}_t^i - \alpha^i \mathcal{A}_t^i > -\widehat{R}_t^i + r(\widehat{D}_t^i - \widehat{L}_t^i)$. As a consequence:
$$P_t^i = \alpha^i \mathcal{A}_t^i.$$

Synthesis We can actually unify all four cases in the following formula :

$$P_t^i = \min(\widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i), \quad \alpha^i \mathcal{A}_t^i). \tag{7}$$

The new amount of reserves is also deduced easily from P_t^i:

$$R_t^i = \widehat{R}_t^i + \widehat{P}_t^i - P_t^i. \tag{8}$$

3.4.2 Find quantities to match a given portfolio valuation objective

Now that P_t^j is optimized, we show here how to adjust the quantities invested in each asset to match this portfolio valuation objective. Given prices X_t and quantities Q_{t-1}^j, we seek to find the vector of quantities Q_t^j for which $Q_t^j X_{t-1} = P_t^j$ while keeping the relative quantities constant. We show how to do this in the appendices A yielding the following result:

$$\forall l, \quad Q_t^{il} = Q_{t-1}^{il} \left(1 + \frac{P_t^i - \widehat{P}_t^i}{\widehat{P}_t^i} \right). \tag{9}$$

Using (7) into (9), we deduce the new quantities:

$$\forall l, \quad Q_t^{il} = Q_{t-1}^{il} \left(1 + \frac{\min(\widehat{P}_t^i + \widehat{R}_t^i - r(\widehat{D}_t^i - \widehat{L}_t^i), \quad \alpha^i \mathcal{A}_t^i) - \widehat{P}_t^i}{\widehat{P}_t^i} \right). \tag{10}$$

4 Taking into account bankruptcy

4.1 Hypothesis and definitions

Set of defaulting banks A time t, if the capital of a non-empty set of banks to drop below a given threshold, those banks declare bankruptcy at time t. Let $\mathcal{D}_t$ be the set of banks that declare bankruptcy at time t. Thus :

$$j \in \mathcal{D}_t \Leftrightarrow \{\widehat{E}_t^i \leq \bar{E}^i\} \cap \{E_{t-1}^i \geq \bar{E}^i\}$$

We also define the set of banks that have defaulted up to time $t+1$:

$$\mathcal{D}_{0:t} = \bigcup_{s=0}^{t} \mathcal{D}_s$$

Symmetrically, we use the notation $\overline{\mathcal{D}_t}$ the complementary in the set of banks of $\mathcal{D}_t$. We use the same notation for the complementary of $\mathcal{D}_{0:t}$ which we shall then denote by $\overline{\mathcal{D}_{0:t}}$.

Proceedings from liquidation and claim coefficient Let us firstly introduce two quantities that we will use across this section.

- We define the proceedings from liquidation π_t^j . This is the cash liquidation value of j's balance sheet after it declares default and is liquidated at time t
- We define the claim coefficient of creditor i on bank j by :

$$\Psi_t^{ij} = \frac{\widehat{L}_t^{ij}}{\sum_{s=1}^{n} \widehat{L}_t^{sj}}$$

In this section, we will present two possible procedures to determine π_t^j : internal settlement in section 4.2 and intervention of a third party in section 4.3. Although we firstly present some considerations of the price impact of fire sales.

Impact of fire sale on risky assets' valuation When a portfolio is sold in a fire sale context, an important volume is sold and the selling is done in the urgency. It is as a consequence interesting — realistic — to add a fire sale impact to the valuation of the risky assets sold. There are two dimensions to this impact :

- When a bank is liquidated, it implies that the liquidation value of a portfolio is less than its face value. We model this using the fire sale constant ξ that is introduced in the next section.
- The liquidation of an important volume of a given asset is also bound to have a long term price impact. Such long term impact would be an interesting add-in to our model since for now the price impact is local — affects only the portfolio liquidation of one bank at a time — and memory-less — does not have long term impact on the prices.

4.2 Internal settlement

In the internal settlement case, the loans of a defaulting banks j are redistributed to its creditor proportionally to their claim on j.

1. In $t-1$ at stage 2 :
   - if $\widehat{E}_{t-1}^j \leq \bar{E}^j$, j declares default. Which triggers the following liquidation steps.
   - π_{t-1}^j is computed. Portfolio is sold according to its valuation with a discount coefficient $0 < \xi < 1$ applied due to fire-sale. Reserves are included.
   $$\pi_{t-1}^j = \xi \widehat{P}_{t-1}^j + \widehat{R}_{t-1}^j$$
2. In t at stage 1, the default becomes public information which brings about the following modifications :
   - j's loans are added to the loans of the creditors of j proportionally to Ψ_{t-1}^{ij} :
   $$\forall i \notin \mathcal{D}_{0:t-1}, \quad \forall k \notin \mathcal{D}_{0:t-1}, \quad \widehat{L}_t^{ik} = L_{t-1}^{ik} + \Psi_{t-1}^{ij} L_{t-1}^{jk}$$
   - The loans matrix is modified to account for j's default:
   $$\forall i, \quad \widehat{L}_t^{ij} = 0$$
   $$\forall k, \quad \widehat{L}_t^{jk} = 0$$

The aggregated loans and debts are then computed according to their definition:

$$\forall i, \quad \widehat{L}_t^i = \sum_{s=1}^{n} \widehat{L}_t^{is}$$
$$\forall i, \quad \widehat{D}_t^i = \sum_{s=1}^{n} \widehat{L}_t^{si}$$

- j's balance sheet quantities are set to zero :

$$(E_t^j, D_t^j, R_t^j, P_t^j, L_t^j, Q_t^j) = (0, 0, 0, 0, 0, 0_{\mathbb{R}^m})$$

- Proceedings of liquidation are distributed to the creditors of j proportionally to their claim on j. This implies a modified version of the equation (3):

$$\widehat{R}_t^i = R_{t-1}^i + r(\widehat{L}_t^i - \widehat{D}_t^i) + \Psi_{t-1}^{ij} \pi_{t-1}^j$$

Let us highlight that since the loans of a defaulting bank are redistributed internally to its non defaulting creditors, the amount of interests paid on loans by each non defaulting bank is not affected by the defaults of other banks. On the other hand, the amount of interests received can decrease since the loans to a defaulting banks are lost.

4.3 Introduction of a third party

Introduction of the liquidator Another option is to introduce a special actor in the system which we call the liquidator. It only intervenes when a bank is liquidated and cannot go bankrupt (it has infinite reserves). Although for reason that will become clear, the liquidator must be integrated to the graph. We shall as a consequence give it a special index: 0.

The balance sheet of the liquidator has a special format :

Assets ($\mathcal{A}_t^0$)	Liabilities ($\mathcal{L}_t^0$)
Reserves ($R_t^0 =$ «∞»)	Equity (E_t^0)
Inter-bank loans (L_t^0)	(Debts ($D_t^0 = 0$))
(Portfolio ($P_t^0 = 0$))	

Liquidation with the liquidator When a bank j goes bankrupt at time $t-1$, the liquidator buys the totality of the bankrupt bank's loans to other banks with a discount $0 < \zeta < 1$. Although, we have to be careful since the bankrupt bank in question may have lent to other defaulting bank. The liquidator does not buy those value-less loans. For the rest the procedure is similar to the previous one :

1. In $t-1$ at stage 2 :

   - if $\widehat{E}^j_{t-1} \leq \bar{E}^j$, j declares default. Which triggers the following liquidation steps.
   - π^j_{t-1} is computed. Portfolio is sold according to its valuation with a discount coefficient $0 < \xi < 1$ applied due to fire-sale. Reserves are included. Finally cash from the loans sold to the liquidator are incorporated

$$\pi^j_{t-1} = \xi \widehat{P}^j_{t-1} + \widehat{R}^j_{t-1} + \zeta \sum_{k \in \overline{\mathcal{D}_{t-1}}} \widehat{L}^{jk}_{t-1}$$

2. In t at stage 1, the default becomes public information which brings about the following modifications :

   - The liquidator's balance sheets is updated to incorporate the loans buy-outs (bank j's debtor now owns money to the liquidator) :

$$\begin{aligned}
\widehat{R}^0_t &= \widehat{R}^0_{t-1} - \zeta \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\widehat{L}^0_t &= L^0_{t-1} + \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\widehat{E}^0_t &= E^0_{t-1} + (1-\zeta) \sum_{k \in \overline{\mathcal{D}_{t-1}}} L^{jk}_{t-1} \\
\forall k \in \overline{\mathcal{D}_{t-1}}, \quad \widehat{L}^{0k}_t &= L^{0k}_{t-1} + L^{jk}_{t-1}
\end{aligned}$$

   - The loans matrix is modified :

$$\forall i, \quad \widehat{L}^{ij}_t = 0$$
$$\forall k, \quad \widehat{L}^{jk}_t = 0$$

   The aggregated loans and debts are then computed according to their definition:

$$\forall i, \quad \widehat{L}^i_t = \sum_{s=1}^{n} \widehat{L}^{is}_t$$
$$\forall i, \quad \widehat{D}^i_t = \sum_{s=1}^{n} \widehat{L}^{si}_t$$

   - j's balance sheet quantities are set to zero :

$$(E^j_t, D^j_t, R^j_t, P^j_t, L^j_t, Q^j_t) = (0, 0, 0, 0, 0, 0_{\mathbb{R}^m})$$

   - Proceedings of liquidation are distributed to the creditors of j proportionally to their claim on j. This implies a modified version of the equation (3):

$$\widehat{R}^i_t = R^i_{t-1} + r(\widehat{L}^i_t - \widehat{D}^i_t) + \Psi^{ij}_{t-1} \pi^j_{t-1}$$

5 Synthesis and aggregation of the model

In this section we summarize the fundamental equations of the model.

5.1 Preliminaries

- Definition of the returns:

$$\forall l \in [\![1, m]\!], \quad \omega_t^l = \frac{X_t^l - X_{t-1}^l}{X_{t-1}^l}.$$

- No expected losers initial condition

$$\forall i \in [\![0, n]\!], \quad rL_0^i + \sum_{l=1}^{m} Q_0^{il} X_0^l \mu_l \geq rD_0^i.$$

5.2 Stage 1 of period t

- Update balance sheet of the liquidator

$$\begin{aligned}
\widehat{R}_t^0 &= R_{t-1}^0 - \zeta \sum_{j\in\mathcal{D}_{t-1}} \sum_{k\in\overline{\mathcal{D}_{t-1}}} L_{t-1}^{jk} \\
\widehat{L}_t^0 &= L_{t-1}^0 + \sum_{j\in\mathcal{D}_{t-1}} \sum_{k\in\overline{\mathcal{D}_{t-1}}} L_{t-1}^{jk} \\
\widehat{E}_t^0 &= E_{t-1}^0 + (1-\zeta) \sum_{j\in\mathcal{D}_{t-1}} \sum_{k\in\overline{\mathcal{D}_{t-1}}} L_{t-1}^{jk} \\
\forall k \in \overline{\mathcal{D}_{t-1}}, \quad \forall j \in \mathcal{D}_{t-1}, \quad \widehat{L}_t^{0k} &= L_{t-1}^{0k} + L_{t-1}^{jk}.
\end{aligned}$$

- The loans matrix is modified :

$$\forall i \in [\![0, n]\!], \quad \forall j \in \mathcal{D}_{t-1}, \quad \forall i, \quad \widehat{L}_t^{ij} = 0$$
$$\forall k \in [\![0, n]\!], \quad \forall j \in \mathcal{D}_{t-1}, \quad \forall k, \quad \widehat{L}_t^{jk} = 0.$$

The aggregated loans and debts are then computed according to their definition:

$$\forall i \in [\![0, n]\!], \quad \widehat{L}_t^i = \sum_{s=1}^{n} \widehat{L}_t^{is}$$
$$\forall i \in [\![0, n]\!], \quad \widehat{D}_t^i = \sum_{s=1}^{n} \widehat{L}_t^{si}.$$

- Zero out the defaulting bank's balance sheet

$$\forall j \in \mathcal{D}_{t-1}, \quad (E_t^j, D_t^j, R_t^j, P_t^j, L_t^j, Q_t^j) = (0, 0, 0, 0, 0, 0_{\mathbb{R}^m}).$$

- Update reserves

$$\forall i \in [\![0,n]\!], \quad \widehat{R}^i_t = R^i_{t-1} + r(\widehat{L}^i_t - \widehat{D}^i_t) + \sum_{j \in \mathcal{D}_{t-1}} \Psi^{ij}_{t-1}\pi^j_{t-1}.$$

- Update portfolios

$$\forall i \in [\![0,n]\!], \quad \widehat{P}^i_t = P^i_{t-1} + Q^i_{t-1}\Delta X_t.$$

- Update equities

$$\forall i \in [\![0,n]\!], \quad \widehat{E}^i_t = R^i_{t-1} + r(\widehat{L}^i_t - \widehat{D}^i_t) + \sum_{j \in \mathcal{D}_{t-1}} \Psi^{ij}_{t-1}\pi^j_{t-1} + \widehat{L}^i_t + \widehat{P}^i_t - \widehat{D}^i_t.$$

  Developing, we can show that there are three possible sources of losses of equity. Let us consider i given:

$$\widehat{E}^i_t = R^i_{t-1} + r(\widehat{L}^i_t - \widehat{D}^i_t) + \sum_{j \in \mathcal{D}_{t-1}} \Psi^{ij}_{t-1}\pi^j_{t-1} + L^i_{t-1} + \widehat{L}^i_t - L^i_{t-1} + \widehat{P}^i_t - \widehat{D}^i_t$$

  Using the fact that $\widehat{D}^i_t = D^i_{t-1}$, we can get the following formula:

$$\widehat{E}^i_t = E^i_{t-1} + \underbrace{r(\widehat{L}^i_t - D^i_{t-1})}_{\downarrow\ net\ loans\ revenues} + \underbrace{\sum_{j \in \mathcal{D}_{t-1}} \Psi^{ij}_{t-1}\pi^j_{t-1} + \widehat{L}^i_t - L^i_{t-1}}_{defaults} + \underbrace{Q^i_{t-1}\Delta X_t}_{portfolio}.$$

5.3 Stage 2 of period t

- Computation of the proceedings of liquidation

$$\forall j \in \mathcal{D}_{t-1}, \quad \pi^j_t = \xi\widehat{P}^j_t + \widehat{R}^j_t + \zeta \sum_{k \in \overline{\mathcal{D}_{t-1}}} \widehat{L}^{jk}_t.$$

- Compute the claim coefficients

$$\forall i \in [\![0,n]\!], \quad \forall j \in \mathcal{D}_{t-1}, \quad \Psi^{ij}_t = \frac{\widehat{L}^{ij}_t}{\sum_{s=1}^n \widehat{L}^{sj}_t}.$$

5.4 Stage 3 of period t

- Portfolio management : set the definitive portfolio valuation for $t+1$

$$\forall i \in \overline{\mathcal{D}_{t-1}}, \quad P^i_t = \min(\widehat{P}^i_t + \widehat{R}^i_t - r(\widehat{D}^i_t - \widehat{L}^i_t), \quad \alpha^i \mathcal{A}^i_t)$$

- Update the quantities consequently

$$\forall i \in \overline{\mathcal{D}_{t-1}}, \ \forall l \in [\![0,m]\!], \ Q^{il}_t = Q^{il}_{t-1}\left(1 + \frac{\min(\widehat{P}^i_t + \widehat{R}^i_t - r(\widehat{D}^i_t - \widehat{L}^i_t), \quad \alpha^i \mathcal{A}^i_t) - \widehat{P}^i_t}{\widehat{P}^i_t}\right)$$

- Set the definitive reserves for $t+1$ accordingly

$$\forall i \in \overline{\mathcal{D}_{t-1}}, \quad R_t^i = \widehat{R}_t^i + \widehat{P}_t^i - P_t^i$$

Appendices

A Find quantities to match a given portfolio valuation

We show here how to adjust the quantities invested in each asset to match a portfolio valuation objective. This will be useful in the other rebalancing stages.

Given a portfolio value objective P_t^i, a portfolio current valuation $\widehat{P}_t^i$, prices X_{t-1} and quantities Q_{t-1}^i, we seek to find the vector of quantities Q_t^i for which $Q_t^i X_{t-1} = P_t^i$ while keeping the relative quantities constant i.e :

$$\forall l \in [\![1, m]\!], \quad \frac{Q_t^{il}}{\sum_{c=1}^m Q_t^{ic}} = \frac{Q_{t-1}^{il}}{\sum_{c=1}^m Q_{t-1}^{ic}}$$

Let us define :

$$Q_{t-1}^{\bar{i}} = \sum_{c=1}^{m} Q_{t-1}^{ic}$$

$$\delta^{il} = \frac{Q_{t-1}^{il}}{Q_{t-1}^{\bar{i}}}$$

It is easy to verify that given the constraints, we only have one degree of freedom to modify the quantities and thus finding Q_t^i boils down to finding η^i such that :

$$\sum_{l=1}^{m} (Q_{t-1}^{il} + \delta^{il}\eta^i) X_t^l = P_t^i$$

The resulting solution quantities being :

$$\forall l, \quad Q_t^{il} = (Q_{t-1}^{il} + \delta^{il}\eta^i)$$

Developing and solving in η^i yields :

$$\eta^i = Q_{t-1}^{\bar{i}} \frac{P_t^i - \widehat{P}_t^i}{\widehat{P}_t^i}$$

Thus :

$$\forall l, \quad Q_t^{il} = Q_{t-1}^{il} \left(1 + \frac{P_t^i - \widehat{P}_t^i}{\widehat{P}_t^i}\right)$$

B Graph initialization

The aim of this appendix is to find a way to initialize the loans and debts given a graph structure so as to match a given net loans/debts objective.

We consider the equity vector E_0 given (which means that we fix each bank's initial equity). For simplicity's sake let us consider that the vector of initial reserves R_0 is given by a fraction $0 < \tau < 1$ of the initial equity. Thus

$$R_0 = \tau E_0.$$

For each bank i, this leaves us $(1-\tau)E_0^i$ to distribute between $L_0^i - D_0^i$ and P_0^i. We will determine first $L_0^i - D_0^i$ for each bank which will give us the ability to make sure that the condition of positive expected equity is matched :

$$rL_0^i + \sum_{l=1}^{m} Q_0^{il} X_0^l \mu_l \geq rD_0^i.$$

Given a matrix of adjacency of a given graph structure A (non weighted and non directed graph, thus A is symmetric and only contains 0s and 1s). We wish to determine a matrix of link weights W. The entry W^{ij} corresponds to the net loans/debts situation from i to j. If $W^{ij} = 100$ for instance it means that i lent 100 to j. If $W^{ij} = -100$ on the other hand it means that i owes 100 to j. It is obvious that the matrix L can be recovered from W.

Let us also consider an objective net loans/debts Y which ith entry correspond to the net loans/debt situation we want to impose on bank i. Meaning that $y^i = L_0^i - D_0^i$.

We thus wish to find $W \in \mathbb{R}^{n \times n}$ such that :

$$\forall i, \quad y_i = \sum_{s=1}^{n} A^{is} W^{is}$$

Since W is anti-symmetric and its diagonal terms must all be zeros, only determining its superior triangle coefficient without the diagonal is sufficient. The previous condition can be expressed equivalently by an under-determined linear system with $\frac{(n-1)n}{2}$ unknowns and n equations.

For simplicity's sake we show this transformation only in the case $n = 4$. Let us consider a matrix A :

$$A = \begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ A^{21} & A^{22} & A^{23} & A^{24} \\ A^{31} & A^{32} & A^{33} & A^{34} \\ A^{41} & A^{42} & A^{43} & A^{44} \end{bmatrix}$$

Let us define then the transformation $\mathcal{T}$:

$$\mathcal{T}(A) = \begin{bmatrix} A^{12} & A^{13} & A^{14} & 0 & 0 & 0 \\ A^{12} & 0 & 0 & A^{23} & A^{24} & 0 \\ 0 & A^{13} & 0 & A^{23} & 0 & A^{34} \\ 0 & 0 & A^{14} & 0 & A^{24} & A^{34} \end{bmatrix}$$

Using this transformation we can characterize the conditions on W using the following linear system:

$$\mathcal{T}(A) \times \begin{bmatrix} W^{12} \\ W^{13} \\ W^{14} \\ W^{23} \\ W^{24} \\ W^{34} \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix}$$

Let us define by w the vector $w = \begin{bmatrix} W^{12} \\ W^{13} \\ W^{14} \\ W^{23} \\ W^{24} \\ W^{34} \end{bmatrix}$

And let us denote by B the transformed matrix : $B = \mathcal{T}(A)$

Our system is thus :

$$Bw = y$$

We need to pick a vector in the vector space of solutions of this linear system. We can for instance use the pseudo inverse to find the minimum l_2 norm solution.

Nevertheless, if we only minimize the l_2 norm, we are not assured that all the link that should be present (non null coefficient in the matrix A) will effectively have non null weights in the solution. One idea would be to add an inequality constraint on the absolute values of the coefficients of W to the pseudo-inverse problem yielding the following modified problem (we define $m = \frac{n(n-1)}{2}$):

$$\begin{array}{lcc} & \min_{w \in \mathbb{R}^m} & w^T w \\ subject\ \ to\ \ : & Bw = y & \\ & \forall i \in [\![1, m]\!], & |w_i| \geq \kappa \end{array}$$

Where $\kappa \geq 0$ is a chosen threshold.

We can use the following equivalent formulation in order to get rid of the non differentiable equality constraints functions :

$$
\begin{array}{rll}
 & \min_{(z^+,z^-)\in\mathbb{R}^{2m}} & (z^+ - z^-)^T(z^+ - z^-) \\
subject\ \ to\ \ : & B(z^+ - z^-) = y & \\
 & \forall i \in [\![1, m]\!], & z_i^+ + z_i^- \geq \kappa \\
 & \forall i \in [\![1, m]\!], & z_i^+ z_i^- = 0 \\
 & z^+ \geq 0 & \\
 & z^- \geq 0 &
\end{array}
$$