IOWA STATE UNIVERSITY, COMPUTER SCIENCE

COMS 512 Homework 5

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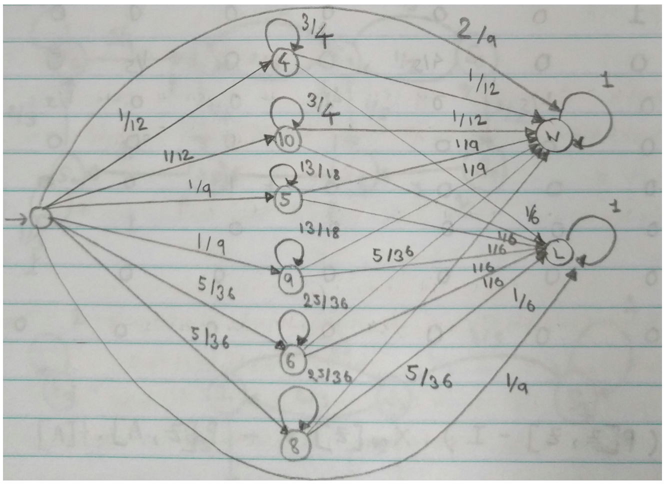
Qustion 1. 1. **DTMC Model of Game of Craps :**



Total number of outcomes of throwing 2 dice = 36
Number of outcomes resulting in 7 and 11 = 8
For 7 we have = {(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)} and for 11 we have = {(5,6),(6,5)}

Hence, probability of winning by throwing 7 or 11 = 2/9
Similarly, number of outcomes of throwing 2,3,12 = 4
Therefore, probability of winning by throwing 2,3,12 = 1/9

When first throw is 4,

chance of repeating = (number of throws not resulting in 4 or 7)/36
= 27/36=3/4

Similarly, chance of loosing = 1/6
chance of winning = 1/12

We have calculated this for 10,5,9,6,8 as well.

2. **Expected number of throws per game :**

   7,11,2,3,12 decides outcome in first throw. We can obtain 7,11,2,3,12 in 12 out of 36 ways.

   The other 24 throws will go to point.

   Number of outcomes for 4 = {(1,3),(3,1),(2,2)} = 3
   Number of outcomes for 5 = {(1,4),(4,1),(2,3),(3,2)} = 4
   Number of outcomes for 6 = {(1,5),(5,1),(2,4),(4,2),(3,3)} = 5
   Number of outcomes for 8 = {(2,6),(6,2),(3,5),(5,3),(4,4)} = 5
   Number of outcomes for 9 = {(3,6),(6,3),(4,5),(5,4)} = 4
   Number of outcomes for 10 = {(4,6),(6,4),(5,5)} = 3

   When the point is 4 or 10,

   Number of wins = 3
   Number of loss = 6

   Therefore 9 out of 36 outcome decides.
   Hence Expected Throws = 36/9 = 4

   When the point is 5 or 9,

   10 out of 36 outcome decides. Expected throws = 3.6

   When point is 6 or 8,

   11 out of 36 outcomes decides. Expected throws = 3.27

Therefore adding these together,
Expected number of throws after point is established,

$$4 * (6/24) + 3.6 * (8/24) + 3.27 * (10/24) = 3.57$$

But 24/36 = 23 throws go up to a point.Therefore expected number of point throws = 3.57 * (2/3) = 2.38

Now there is a first expected throw. The sum of these two gives total expected number of throws = 1 + 2.38 = **3.38**

3. **Probability of winning :**

   Probability of win with an outcome of 7 or 11 = 2/9

   Establish point 4 or 10 an win = (1/12) $*$ (1/3) = (1/36)
   Establish point 5 or 9 an win = (1/9) $*$ (2/5) = (2/45)
   Establish point 6 or 8 an win = (5/36) $*$ (5/11) = (25/396)

   Therefore probability of winning = (2/9) + 2(1/36 + 2/45 + 25/396)
   = 0.222 + 2(0.028 + 0.044 + 0.063)
   = **0.492**

Question 2.

1. Let received = p.

   We have $AFp \equiv \neg EG(\neg p)$

   Now, $EG(\neg p) = EG(\{s_0, s_1\})$
   Now we have, one transient class and one recurrent class. For recurrent class s_2 does not satisfy $\{s_0, s_1\}$.

   For the transient states that satisfies $\{s_0, s_1\}$ is s_0 and s_1. But none of them can reach a recurrent class that satisfy $\{s_0, s_1\}$.

   Therefore no state satisfies $EG(\{s_0, s_1\})$.And all states satisfy $\neg EG(\neg p)$, thus AFp.

   Therefore the model satisfies AFp.

2. We are given the initial probability distribution p_0=$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
   Therefore $p_o[Z]$=$\begin{bmatrix} 1 & 0 \end{bmatrix}$.

We have transition probability matrix, P= $\begin{bmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ 0 & 0 & 1 \end{bmatrix}$

We have P[Z,Z] = $\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}$

From the section 14.4 of the lecture we know that, $n(P[Z,Z]-I) = -p0[Z]$.

Let the vector n=[a,b].

Now, $(P[Z,Z]-I)$= $\begin{bmatrix} -1 & 1 \\ \alpha & -1 \end{bmatrix}$

Therefore, $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} -1 & 1 \\ \alpha & -1 \end{bmatrix}$=- $\begin{bmatrix} 1 & 0 \end{bmatrix}$

By solving the equation we get $a = 1/(1-\alpha)$ and b=a.
Therefore average time = $2/(1-\alpha)$.

3. Initial probability distribution is given as $p_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

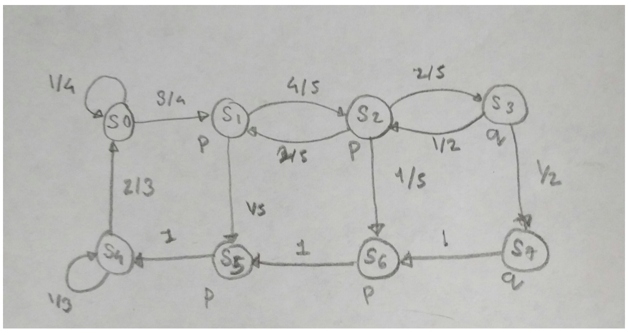
   Now we need to calculate the limiting probability p_∞ which is given by the formula p_∞=[0,n.P[Z,A]+$p_0[A]$]

   From the previous problem we have calculated n to be [$1/(1$-$\alpha)$,$1/(1$-$\alpha)$]

   For this DTMC,
   P[Z,A]= $\begin{bmatrix} 0 \\ (1-\alpha) \end{bmatrix}$

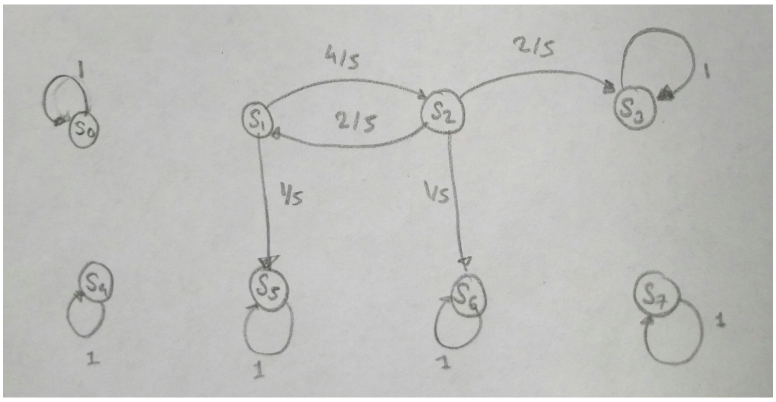
so we obtain n.P[Z,A] = [$1/(1$-$\alpha)^2$]
Therefore the probability that message will be received at time t is given by $1/(1$-$\alpha)^2$.

Question 3. We have, $P_{>1/2} pUq$.



States that satisfy q=$\{s_3, s_7\}$
States that does not satisfy $EpUq = \{s_0, s_4, s_5, s_6\}$

We make this states absorbing states. The modified DTMC is,



Transition Probability matrix is,

Transient States = $\{s_1, s_2\}$
Absorbing States = $\{s_0, s_3, s_4, s_5, s_6, s_7\}$

$$P = \begin{bmatrix} 0 & 4/5 & 0 & 0 & 0 & 1/5 & 0 & 0 \\ 2/5 & 0 & 0 & 2/5 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we want to measure $X_\infty[i] = \lim_{t\to\infty} P^t.f$

The absorbing part is $X_\infty[A] = f(a)$

f(a)= $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

The transient part is $X_\infty[Z] = N.P[Z, A].f[A]$
$\Rightarrow N^{-1}.X_\infty[Z] = P[Z, A].f[A]$
$\Rightarrow (P[Z, Z] - I).X_\infty[Z] = -P[Z, A].f[A]$

Now, $P[Z, Z] - I = \begin{bmatrix} 0 & 4/5 \\ 2/5 & 0 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, $-P[Z,A].f[A]$

$$=\begin{bmatrix}0 & 0 & 0 & 1/5 & 0 & 0\\ 0 & 2/5 & 0 & 0 & 1/5 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix} =- \begin{bmatrix}0\\2/5\end{bmatrix}$$

Now, $(P[Z,Z]-I)^{-1}$=(-25/17) $\begin{bmatrix}1 & 4/5\\ 2/5 & 1\end{bmatrix}$

Therefore $X_\infty[Z]$= $\begin{bmatrix}8/17\\10/17\end{bmatrix}$

X_∞= $\begin{bmatrix}0 & 8/17 & 10/17 & 1 & 0 & 0 & 0 & 1\end{bmatrix}$

Therefore the following set of states satisfy the property,
h=$\begin{bmatrix}0 & 0 & 1 & 1 & 0 & 0 & 0 & 1\end{bmatrix}$

Therefore states s_2, s_3, s_7 satisfy the property.

Question 4.

1. From the definition of absorbing DTMC, every state is either transient or absorbing. Or in other words it is a DTMC in which every state can reach an absorbing state.
   An Absorbing state is such state that once entered, can not be left.

   If X is an absorbing DTMC then m is an absorbing state. Which means $\alpha = 1$.Because we know that absorbing state loops into itself with probability 1.

   If $\alpha = 1$ then in DTMC X' the edge from m to m+1 will have probability 1.
   And the state m+1 will loop into itself with probability $\alpha = 1$.
   The outward edge from m+1 to m will have probability $(1-\alpha) = 0$.
   Which means once in state m , the system will always enter state m+1 which is an absorbing state and it can not left (m+1) as the probability at the outward edge is 0.

   This is in line with the definition of absorbing chain. Therefore X' is an absorbing chain given X is an absorbing chain.

2. p_0 is the initial probability distribution of absorbing DTMC X.
   p_0' is the initial probability distribution of absorbing DTMC X' where p_0'=[p_0,0]

   n is the vector of expected number of visits to each state in X.
   n' is the vector of expected number of visits to each state in X'.

   In X m is an absorbing state. Let (m-1) be a transient state just before m. Let the number of visits of state (m-1) i.e n[m-1] = a. On reaching m from (m-1) it will be absorbed and can not return.

In X', number of visits of (m-1) until it reaches state m is also n[m-1]=a. Now if it returns from m to m-1 then n[m-1] is not a anymore.
But in X',(m+1) is an absorbing state, hence $\alpha = 1$.Now there is an edge from m to m+1 with probability α. If $\alpha = 1$, then m can not have any more outward edges. That means there will be no more outward edges from m. Therefore once it reaches m it can not return back to m-1 and number of visits n[m-1] remains unchanged.
Therefore n[m-1]=n'[m-1] and in general, n[i]=n'[i] for all $i \leq m$.

If i is an absorbing state in X and X', then n[i]=n'[i] is trivially true.
Therefore, n[i]=n'[i] for all $i \leq m$.

3. n_m is the vector of expected number of visits to each state in x_m.

   Now,$n[m+1] = \alpha/(1-\alpha).n[m]$ from part 2.
   Here our starting state m is 2.i.e expected number of visits to state 2 is n[2].
   Therefore n[3]=$\alpha/(1-\alpha)n[2]$

   Similarly we have n[4]=$\alpha/(1-\alpha)^2 n[2]$ and
   n[m]=$\alpha/(1-\alpha)^{m-2} n[2]$

   Therefore vector n_m is given by,
   $[n[2], \alpha/(1-\alpha)n[2], \alpha/(1-\alpha)^{m-2} n[2], ..., \alpha/(1-\alpha)^{m-2} n[2]]$

4. t_m is the mean time to absorbtion for x_m.

   We have, $t_m = \Sigma_{m=2}^{m} n[m]$
   = n[2]+n[3]+.........+n[m]
   = $n[2][1+\alpha/(1-\alpha)+.....+\alpha/(1-\alpha)^{m-2}]$
   When n[m]=Expected number of visits to a state m.
   From part 2 we know that $n[m+1] = \alpha/(1-\alpha)n[m]$.Using this formula we get this expression of t_m.