Single-Peaked Jump Schelling Games

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Abstract. Schelling games model the wide-spread phenomenon of residential segregation in metropolitan areas from a game-theoretic point of view. In these games agents of different types each strategically select a node on a given graph that models the residential area to maximize their individual utility. The latter solely depends on the types of the agents on neighboring nodes and it has been a standard assumption to consider utility functions that are monotone in the number of same-type neighbors, i.e., more same-type neighbors yield higher utility. This simplifying assumption has recently been challenged since sociological poll results suggest that real-world agents actually favor diverse neighborhoods.

We contribute to the recent endeavor of investigating residential segregation models with realistic agent behavior by studying Jump Schelling Games with agents having a single-peaked utility function. In such games, there are empty nodes in the graph and agents can strategically jump to such nodes to improve their utility. We investigate the existence of equilibria and show that they exist under specific conditions. Contrasting this, we prove that even on simple topologies like paths or rings such stable states are not guaranteed to exist. Regarding the game dynamics, we show that improving response cycles exist independently of the position of the peak in the utility function. Moreover, we show high almost tight bounds on the Price of Anarchy and the Price of Stability with respect to the recently proposed degree of integration, which counts the number of agents with a diverse neighborhood. Last but not least, we show that computing a beneficial state with high integration is NP-complete and, as a novel conceptual contribution, we also show that it is NP-hard to decide if an equilibrium state can be found via improving response dynamics starting from a given initial state.

1 Introduction

Residential segregation [32], i.e., the emergence of regions in metropolitan areas that are homogeneous in terms of ethnicity or socio-economic status of its inhabitants, has been widely studied. Segregation has many negative consequences for the inhabitants of a city, for example, it negatively impacts their health [1]. The causes of segregation are complex and range from discriminatory laws to individual action. Schelling's classical agent-based model for residential segregation [28,29] specifies a spatial setting where individual agents with a bias towards

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favoring similar agents care only about the composition of their individual local neighborhoods. This model gives a coherent explanation for the widespread phenomenon of residential segregation, since it shows that local choices by the agents yield globally segregated states [15,30]. In Schelling's model two types of agents, placed on a path and a grid, respectively, act according to the following threshold behavior: agents are *content* with their current position if at least a τ-fraction of neighbors, with $\tau \in (0,1)$, is of their own type. Otherwise, they are discontent and want to move, either via swapping with another random discontent agent or via jumping to an empty position. Starting from a random distribution, Schelling showed empirically that the random process drifts towards segregation. This is to be expected if all agents are intolerant, i.e., for $\tau > \frac{1}{2}$. But Schelling's astonishing insight is that this also happens if all agents are tolerant, i.e., for $\tau \leq \frac{1}{2}$.

Many empirical studies have been conducted to investigate the influence of various parameters on the obtained segregation patterns [5,26,27]. In particular, the model has been studied by sociologists [4,11,16] with the help of sophisticated agent-based simulation frameworks such as SimSeg [18]. On the theoretical side, the underlying stochastic process leading to segregation was studied [3,9,21]. Furthermore, Schelling's model recently gained traction within Algorithmic Game Theory, Artificial Intelligence, and Multi-agent Systems [2,6,7,12–14,17,22,23].

Most of these papers are in line with the assumptions made by Schelling and incorporate monotone utility functions, i.e., the agents' utility is monotone in the fraction of same-type neighbors, cf. Fig. 1 (left). Although for $\tau < 1$ it is true that no agent prefers segregation locally, agents are equally content in segregated neighborhoods as they are in neighborhoods that just barely meet their tolerance thresholds. However, recent sociological surveys [31] show that people actually prefer to live in diverse rather than segregated neighborhoods[1]. Based on this, different models in which agents prefer integration have been proposed [26,33,34]. Recently, Bilò et al. [6] introduced the Single-Peaked Swap Schelling Game, where agents have single-peaked utility functions, cf. Fig. 1, and pairs of agents can swap their locations if this is beneficial for both of them.

Based on the model by Bilò et al. [6], we investigate the Jump Schelling Game (JSG), where agents can improve their utility by jumping to empty locations, assuming realistic agents having a single-peaked utility function.

Model. We consider a strategic game played on an undirected, connected graph $G = (V, E)$. For a given node $v \in V$, let $\delta(v)$ be its *degree* and let Δ_G be the *maximum degree* over all nodes $v \in V$. A graph is δ-*regular*, if $\forall v \in V : \delta(v) = \Delta_G$. We denote with $\alpha(G)$ the *independence number* of G, i.e., the cardinality of the maximum independent set in G.

A *Single-Peaked Jump Schelling Game* (G, r, b, p), called the *game*, is defined by a graph G, a pair of integers $r \geq 1$ and $1 \leq b \leq r$, and a single-peaked utility function p. There are two types of agents associated with the colors red and

[1] Respondents (on average 78% white) were asked what they think of "Living in a neighborhood where half of your neighbors were blacks?". In 2018 82% responded "strongly favor", "favor" or "neither favor nor oppose".

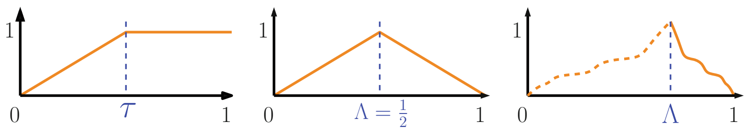

Fig. 1. Left: Schelling's original monotone threshold utility function. Middle+Right: single-peaked utility functions. The dashed line marks the utility of an agent if the fraction of same type neighbors meets the threshold and the peak, respectively

blue. We have r red agents and b blue agents. If $r = b$, we say that the game is *balanced*. Let $c(i)$ the color of agent i.

An agent's *strategy* is her position $v \in V$ on the graph. Each node can only be occupied by at most one agent. The $n = r + b$ strategic agents occupy a strict subset of the nodes in V, i.e., there are $e = |V| - n \geq 1$ *empty nodes*. A *strategy profile* $\sigma \in V^n$ is a vector of n distinct nodes in which the i-th entry $\sigma(i)$ corresponds to the strategy of the i-th agent. We say that an agent i is adjacent to a node v (or an agent j) if G has an edge between $\sigma(i)$ and v (resp. $\sigma(j)$). For convenience, we use σ^{-1} as a mapping from a node $v \in V$ to the agent occupying v or $\ominus$ if v is empty. The set of empty nodes is $\emptyset(\sigma) = \{v \in V \mid \sigma^{-1}(v) = \ominus\}$.

For an agent i, we define $C_i(\sigma) = \{v \in V \setminus \emptyset(V) \mid c(\sigma^{-1}(v)) = c(i)\}$ as the set of nodes occupied by agents of the same color in σ. The *closed neighborhood* of an agent i in a strategy profile σ is $N[i, \sigma] = \{\sigma(i)\} \cup \{v \in V \setminus \emptyset(\sigma) \mid \{v, \sigma(i)\} \in E\}$. The agents care about the fraction $f_i(\sigma)$ of agents of their own color, including themselves, in their closed neighborhood where $f_i(\sigma) = \frac{|N[i,\sigma] \cap C_i(\sigma)|}{|N[i,\sigma]|}$. If $f_i(\sigma) = 1$, we say that agent i is *segregated*. Furthermore, observe that we have $f_i(\sigma) > 0$ for any agent i, since $\sigma(i) \in N[i, \sigma]$. Also, we emphasize that our definition of $f_i(\sigma)$ deviates from similar definitions in related work. In particular, the papers [2,14,17] exclude the respective agent i from her neighborhood, while Kanellopoulos et al. [22] count agent i only in the denominator of $f_i(\sigma)$. The different existing definitions of the homogeneity of a neighborhood all have their individual strengths and weaknesses. We decided to follow the definition of Bilò et al. [6]. The key idea of their definition is that agents contribute to the diversity of their neighborhood. Thus, agents actively strive for integration. We think that this best captures the single-peaked setting.

The *utility* of an agent i is $U_i(\sigma) = p(f_i(\sigma))$, with p being an arbitrary single-peaked function with peak $\Lambda \in (0, 1)$ and the following properties: (1) $p(0) = 0$ and $p(x)$ is strictly monotonically increasing on $[0, \Lambda]$, (2) for all $x \in [\Lambda, 1]$ it holds that $p(x) = p(\frac{\Lambda(1-x)}{1-\Lambda})$, i.e., it is symmetric to the other side of Λ but possibly squeezed. W.l.o.g., we further assume that $p(\Lambda) = 1$. See Fig. 1 (middle and right). We explicitly exclude $\Lambda = 1$ and from our definition follows $p(1) = 0$. Allowing $\Lambda = 1$ would allow monotone utilities, similar to the models in [2,14], where agents actively strive for segregation or passively accept it. In contrast, we assume that agents actively strive for diversity. Thus, a homogeneous neighborhood should not be acceptable. This justifies $p(1) = 0$. Hence, both $\Lambda < 1$ and $p(1) = 0$ model integration-oriented agents and go hand in hand.

The strategic agents attempt to choose their strategy to maximize their utility. The only way in which an agent can change her strategy is to *jump*, i.e., to choose an empty node $v \in \emptyset(\sigma)$ as her new location. We denote the resulting strategy profile after a jump of agent i to a node v as σ_{iv}. A jump is *improving*, if $U_i(\sigma) < U_i(\sigma_{iv})$. A strategy profile σ is a (pure) *Nash Equilibrium* (NE) if and only if there are no improving jumps, i.e., for all agents i and nodes $v \in \emptyset(\sigma)$, we have $U_i(\sigma) \geq U_i(\sigma_{iv})$.

A measure to quantify the amount of segregation in a strategy profile σ is the *degree of integration* (DoI), which counts the number of non-segregated agents, hence $\mathrm{DoI}(\sigma) = |\{i \mid f_i(\sigma) < 1\}|$. For a game (G, r, b, p), let σ^* be a strategy profile that maximizes the DoI and let $\mathrm{NE}(G, r, b, p)$ be its set of Nash Equilibria. We evaluate the impact of the agents' selfishness on the overall social welfare by studying the *Price of Anarchy* (PoA), defined as $\mathrm{PoA}(G, r, b, p) = \frac{\mathrm{DoI}(\sigma^*)}{\min_{\sigma \in NE(G,r,b,p)} \mathrm{DoI}(\sigma)}$ and the *Price of Stability* (PoS), defined as $\mathrm{PoS}(G, r, b, p) = \frac{\mathrm{DoI}(\sigma^*)}{\max_{\sigma \in NE(G,r,b,p)} \mathrm{DoI}(\sigma)}$. If the best (resp. worst) NE has a DoI of 0, the PoS (resp. PoA) is unbounded.

A game has the *finite improvement property* (FIP) if and only if, starting from any strategy profile σ, the game will always reach a NE in a finite number of steps. As proven by Monderer and Shapley [25], this is equivalent to the game being a *generalized ordinal potential game*. In particular, the FIP does not hold if there is a cycle of strategy profiles $\sigma^0, \sigma^1, \ldots, \sigma^k = \sigma^0$, such that for any $k' < k$, there is an agent i and empty node $v \in \emptyset(\sigma^{k'})$ with $\sigma^{k'+1} = \sigma^{k'}_{iv}$ and $U_i(\sigma^{k'}) < U_i(\sigma^{k'+1})$. These cycles are known as *improving response cycles* (IRCs).

Related Work. Game-theoretic models for residential segregation were first studied by Chauhan et al. [14] and later extended by Echzell et al. [17]. There, agents have a monotone utility function as shown in Fig. 1 (left). Additionally, agents may also have location preferences. Agarwal et al. [2] consider a simplified model using the most extreme monotone threshold-based utility function with $\tau = 1$. They prove results on the existence of equilibria, in particular, that equilibria are not guaranteed to exist on trees, and on the complexity of deciding equilibrium existence. Also, they introduce the DoI as social welfare measure and they study the PoA in terms of utilitarian social welfare and in terms of the DoI. The complexity results were extended by Kreisel et al. [24], in particular, they show that deciding the existence of NE in the swap version as well as in the jump version of the simplified model is NP-hard. Bilò et al. [7] strengthen the PoA results for the swap version w.r.t. the utilitarian social welfare function and investigate the model on almost regular graphs, grids and paths. Additionally, they introduce a variant with locality. Chan et al. [13] study a variant of the JSG with $\tau = 1$, where the agents' utility is a function of the neighborhood composition and of the social influence. Kanellopoulos et al. [23] considered a generalized variant, where the agent types are linearly ordered. Another novel variant of the JSG, that includes an agent when counting the neighborhood size, was investigated by Kanellopoulos et al. [22]. This subtle change leads to agents preferring locations with more own-type neighbors. Bullinger et al. [12] measure social welfare via the number of agents with non-zero utility, they prove hardness results for computing the social optimum and discuss other solution concepts.

Most related is the recent work by Bilò et al. [6], which studies the same model as we do, but there only pairs of agents can improve their utility by swapping their locations. They find that equilibria are not guaranteed to exist in general, but they do exist for $\Lambda = \frac{1}{2}$ on bipartite graphs and for $\Lambda \leq \frac{1}{2}$ on almost regular graphs. The latter is shown via an ordinal potential function, i.e., convergence of IRDs is guaranteed. For the PoA they prove an upper bound of $\min\{\Delta(G), \frac{n}{b+1}\}$ and give almost tight lower bounds for bipartite graphs and regular graphs. Also, they lower bound the PoS by $\Omega(\sqrt{n\Lambda})$ and give constant bounds on bipartite and almost regular graphs. Note that due to the existence of empty nodes in our model, our results cannot be directly compared.

Also related are hedonic diversity games [8,10,20] where selfish agents form coalitions and the utility of an agent only depends on the type distribution of her coalition. For such games, single-peaked utility functions yield favorable game-theoretic properties.

Our Contribution. We investigate Jump Schelling Games with agents having a single-peaked utility function. In contrast to monotone utility functions studied in earlier works, this assumption better reflects recent sociological poll results on real-world agent behavior [31]. Moreover, this transition to a different type of utility function is also interesting from a technical point of view since it yields insights into the properties of Schelling-type systems under different preconditions.

Regarding the existence of pure NE, we provide a collection of positive and negative results. On the negative side, we show that NE are not guaranteed to exist on the simplest possible topologies, i.e., on paths and rings with single-peaked utilities with $\Lambda \geq \frac{1}{2}$. This is in contrast to the version with monotone utilities where for the case of rings NE always exist. On the positive side, we give various conditions that enable NE existence, e.g., such states are guaranteed to exist if the underlying graph has a sufficiently large independent set, or if it has sufficiently many degree 1 nodes. The situation is worse for the convergence of game dynamics. We show that even on regular graphs IRCs exist independently of the position of the peak in the utility function. Moreover, this even holds for the special case with $\Lambda = \frac{1}{2}$ and only a single empty node. These negative results for $\Lambda \leq \frac{1}{2}$ also represent a marked contrast to the swap version, where convergence is guaranteed for this case on almost regular graphs.

With regard to the quality of the equilibria, we focus on the DoI as social cost function. This measure has gained popularity since it can be understood as a simple proxy for the obtained segregation strength. For the PoA w.r.t. the DoI, we establish that the technique from Bilò et al. [6] can be carried over to our setting. This yields the same PoA upper bound of $\min\{\Delta(G), n/(b + 1)\}$.

Subsequently, we give almost matching PoA lower bounds and we prove that also the lower bounds for the PoS almost match this high upper bound. On the positive side, we show that on graphs with a sufficiently large independent set, the PoS depends on the ratio of the largest and the smallest node degree, which implies a PoS of 1 on regular graphs that also holds for rings with a single empty node.

Last but not least, we consider the complexity aspects of our model. Analogously to previous work on the Jump Schelling Game with monotone utilities and to work on Swap Schelling Games with single-peaked utilities, we focus on the hardness of computing a strategy profile with a high degree of integration. Using a novel technique relying on the Max SAT problem, we show that this problem is NP-complete, improving on an earlier result by Agarwal et al. [2]. Moreover, as a novel conceptual contribution, we investigate the hardness of finding an equilibrium state via improving response dynamics. As one of our main results, we show that this problem is NP-hard. So far, researchers have studied the complexity of deciding the existence of an equilibrium for a given instance of a Schelling Game. We depart from this, since even if it can be decided efficiently that for some instance an equilibrium exists, guiding the agents towards this equilibrium from a given initial state is complicated, since this would involve a potentially very complex centrally coordinated relocation of many agents in a single step. In contrast, reaching an equilibrium via a sequence of improving moves is much easier to coordinate, since in every step the respective move can be recommended and, since this is an improving move, the agents will follow this advice.

Overall we find that making the model more realistic by employing single-peaked utilities entails a significantly different behavior of the model compared to the variant with monotone utilities but also compared to Single-Peaked Swap Schelling Games.

We refer to the full version [19] for all omitted details.

2 Game Dynamics

We show that even on very simple graph classes improving response dynamics are not guaranteed to converge to stable states. Moreover, we provide IRCs for the entire range of Λ. Note, that given an IRC for a game on a graph G IRCs exist for all games on any graph H that contains G as a node-induced subgraph since we can add empty nodes to G to obtain H without interfering with the IRC. We start with an IRC for $\Lambda \geq \frac{1}{2}$.

Theorem 1. *For $\Lambda \geq \frac{1}{2}$, the FIP does not hold even on rings and paths with $e \geq 2$.*

Proof. Consider a game with five nodes, two red agents and one blue agent on a ring or path. We start with a strategy profile in which the blue agent is adjacent to both red agents. An illustration is given in Fig. 2. As $\Lambda \geq \frac{1}{2}$, the blue agent prefers to be in a neighborhood with only one of the red agents. Hence,

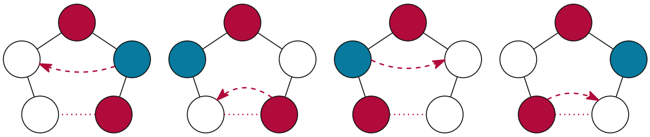

Fig. 2. IRC on a ring (path without the dotted edge) for $\Lambda \geq \frac{1}{2}$.

an improving jump from the blue agent results in one segregated red agent. As a consequence the red agent jumps to the node adjacent to the blue agent. Further, observe that at no point in this cycle does any other agent have an improving jump and none of the jumping agents have an alternative improving jump (except for symmetry). $\square$

We now show that IRCs also exist if $\Lambda \leq \frac{1}{2}$.

Theorem 2. *For $\Lambda \leq \frac{1}{2}$, the FIP does not hold even on regular graphs.*

Even for $\Lambda = \frac{1}{2}$ and $e = 1$ no convergence is guaranteed.

Theorem 3. *For $\Lambda = \frac{1}{2}$, the FIP does not hold even on regular graphs with $e = 1$.*

On the positive side, convergence is guaranteed on rings.

Theorem 4. *On rings, the game with $e = 1$ and $\Lambda = \frac{1}{2}$ is an ordinal potential game. It converges after at most n steps.*

Proof. We claim that for each improving jump of an agent i to a node v, we have $\mathrm{DoI}(\sigma_{iv}) \geq \mathrm{DoI}(\sigma) + 1$. Hence, $\mathrm{DoI}(\sigma)$ is an ordinal potential function and a NE must be reached after at most n improving jumps.

Assume there is an agent i with an improving jump to v, i.e., $U_i(\sigma) < U_i(\sigma_{iv})$. We claim $U_i(\sigma) = 0$. Assume $U_i(\sigma) > 0$, i.e., either $p\left(\frac{1}{2}\right) = 1$ or $p\left(\frac{1}{3}\right) = p\left(\frac{2}{3}\right)$. In the first case, agent i already has the highest possible utility and thus no incentive to jump. In the second case $\left(U_i(\sigma) = p\left(\frac{1}{3}\right)\right)$, we must have $U_i(\sigma_{iv}) = 1$. But since v is the only empty node this is only possible if $\sigma(i)$ and v are adjacent. However, this requires $|N[i, \sigma]| = 2 \neq 3$.

Therefore, in σ, agent i is not adjacent to any agent of the other color and in σ_{iv} adjacent to at least one agent of the other color. Thus, any agent adjacent to i that has a utility larger than 0 in σ still has a utility larger than 0 in σ_{iv}. Also, no agent adjacent to v can drop to utility 0 because of i jumping to v. Thus, we have $\mathrm{DoI}(\sigma) + 1 \leq \mathrm{DoI}(\sigma_{iv})$. $\square$

3 Existence of Equilibria

A fundamental question is if NE always exist. We start with a negative result that even on rings existence of equilibria is not guaranteed for $\Lambda \geq \frac{1}{2}$. However, in certain cases, we can provide existential results. In particular, equilibria exist if the underlying graph has an independent set that is large enough or if the graph contains sufficiently many leaf nodes. Moreover, for regular graphs, we show that equilibria exist if $e = 1$ and r is large enough. The following non-existence result for rings follows from Theorem 1.

Corollary 1. *Even on rings, the existence of equilibria for the game is not guaranteed for* $\Lambda \geq \frac{1}{2}$.

If the independence number is at least the number of blue agents plus the number of empty nodes, existence of NE is guaranteed. This result is similar to the swap version [6].

Theorem 5. *Every game on a graph with an independent set of size* $\alpha(G) \geq b+e$ *has a NE.*

Thus, if r is large enough NE always exist on bipartite graphs.

Corollary 2. *Every game with* $r \geq \frac{|V|}{2}$ *on a bipartite graph admits an efficiently computable NE.*

Next, we show that for $\Lambda \geq \frac{1}{2}$ games with a low number of empty nodes and a low difference between the number of red and blue agents proportional to the number of empty nodes admit a NE. To this end, we consider a special kind of independent sets.

Definition 1. *A maximum degree independent set (max-deg IS) is an independent set* I*, such that* $\forall u \in I, v \in V \setminus I : \delta(v) \leq \delta(u)$*. The size of the largest max-deg IS of a graph* G *is* $\alpha^{\max\delta}(G)$*.*

Note that for any graph, it holds that $\alpha^{\max\delta}(G) \geq 1$.

Theorem 6. *Let* G *be a graph with* $e \leq \alpha^{\max\delta}(G)$ *and* $e \leq \frac{r-b}{\Delta_G}$*. For* $\Lambda \geq \frac{1}{2}$*, the game has a NE.*

Proof. Let I be a max-deg IS of size e. Since $e \leq \alpha^{\max\delta}(G)$, this exists. We place red agents on all nodes adjacent to nodes in I. For this, we need at most $\Delta_G \cdot e$ red agents. We have r' red agents left and $r' \geq r - \Delta_G \cdot e \geq b$.

We claim that we can place the remaining agents on the remaining nodes, such that every blue agent is adjacent to at least one red agent. For this, consider the layer graph rooted at an imaginary node that results from merging all e nodes in I, cf. Fig. 3. Let the root layer be layer 0. Note that therefore, layer 1 is fully occupied by the $r - r'$ red agents we placed in the first step. Let L_2 be the set of nodes in all even layers (except for layer 0) and L_3 be the set of nodes in all odd layers (except for layer 1).

All nodes in L_2 (resp. L_3) have at least one adjacent node not in L_2 (resp. L_3). Also, we have $|L_2| + |L_3| = r' + b$. Hence, $|L_2|$ or $|L_3|$ is at least $\frac{r'+b}{2}$. Since $r' \geq b$, it follows $b \leq \frac{r'+b}{2}$, so there is $L \in \{L_2, L_3\}$ with $|L| \geq b$. We place all blue nodes in L and all red nodes on the remaining empty spots in L_2, L_3. Then, every blue node has at least one red neighbor.

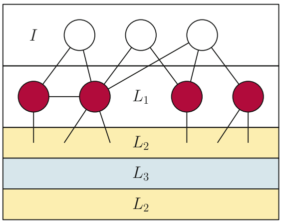

Fig. 3. The layer graph. We first have an independent set I of e nodes, then a layer L_1 of at most $\Delta_G \cdot e$ red agents. The nodes in the following layers are either part of L_2 (for even layers) or L_3 (for odd layers). (Color figure online)

The placement σ is stable. As all empty nodes are adjacent to only red nodes, no red agent wants to jump. Let i be a blue agent and u be an empty node. By construction, $\sigma(i) \notin I$ and $u \in I$. At least one neighbor of $\sigma(i)$ is red, hence i has a non-zero utility. Since $\Lambda \geq \frac{1}{2}$, the worst non-zero utility is $p\left(\frac{1}{\delta(\sigma(i))+1}\right)$. Thus, $U_i(\sigma) \geq p\left(\frac{1}{\delta(\sigma(i))+1}\right)$ and since all neighbors of u are red, $U_i(\sigma_{iu}) = p\left(\frac{1}{\delta(u)+1}\right)$. As I is a max-deg IS, we have $\delta(\sigma(i)) \leq \delta(u)$. Furthermore, it follows from $\Lambda \geq \frac{1}{2}$ that $p\left(\frac{1}{\delta(u)+1}\right) \leq p\left(\frac{1}{\delta(\sigma(i))+1}\right) = U_i(\sigma)$. Hence, i has no improving jump. □

Note that for regular graphs any independent set is a max-deg IS, i.e., $\alpha^{\max \delta}(G) = \alpha(G) \geq \frac{|V|}{\delta+1}$.

Corollary 3. *Any game on a δ-regular graph G with $e \leq \alpha(G)$, $r \geq b + \delta \cdot e$ and $\Lambda \geq \frac{1}{2}$ has NE.*

Next, we show that graphs with a large number of leaves admit NE. In particular, this applies to trees with many leaves, e.g., stars.

Theorem 7. *Every game with $\Lambda \geq \frac{1}{2}$ on a graph with at least b nodes of degree one admits NE.*

While even for regular graphs with $e = 1$ the FIP is violated, we can guarantee the existence of NE with further conditions.

Theorem 8. *For any game on a δ-regular graph with $\Lambda \geq \frac{1}{2}$, $r \geq \delta$ and $e = 1$, equilibria exist and can be computed efficiently.*

4 Price of Anarchy and Stability

We study the PoA and PoS of the game with respect to the DoI. We already showed that the existence of equilibria is not guaranteed for many instances, yet, we still give bounds that apply whenever equilibria do exist.

4.1 Price of Anarchy

We start with a necessary condition that holds for any NE.

Lemma 1. *No NE contains segregated agents of different colors.*

As shown in [6], Lemma 1 can be used to get a bound on the PoA for the swap version. The proofs do not rely on swaps and thus carry over.

Lemma 2. *For any game and σ, we have DoI(σ) $\leq \min((\Delta_G + 1)b, n)$.*

With this, we get the same upper bound as in [6].

Theorem 9 ([6])**.** *For any game, the PoA is at most* $\min\left(\Delta_G, \frac{n}{b+1}\right)$.

It still remains to be shown that this upper bound is tight. We show that this is, asymptotically with respect to Δ_G, the case for general graphs.

Theorem 10. *For any Λ, there exists a game with PoA $\frac{n}{b+1} = \Delta_G - 1$.*

Proof. For some $\delta \geq 4$, consider the game (G, r, b, p) with $b = \delta - 1, r = b^2$ depicted in Fig. 4. The graph G has a node v adjacent to a set B of b nodes. Further, v is adjacent to another node, which lies on a path of altogether b nodes, which at the same time represent the root of a tree. Hence, each node on this path is adjacent to one node in B', each of which is adjacent to b nodes in total. Observe that $\Delta_G = \delta$ and $e = b + 1$.

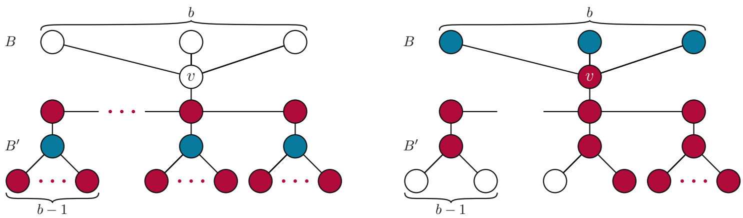

Fig. 4. A game with $b = \Delta_G - 1$ and $r = b^2$. The middle row is a path of b many nodes occupied by red agents. The dots in the lower row are representative for the rest of the $b-1$ many leaf nodes. Left: Optimum σ^* with $\mathrm{DoI}(\sigma^*) = \Delta_G(\Delta_G - 1) = n$. Right: NE σ with $\mathrm{DoI}(\sigma) = b + 1 = \Delta_G$. (Color figure online)

There is an optimal strategy profile σ^* in which all nodes in B' are occupied by blue agents and all nodes outside of $B \cup B' \cup \{v\}$ are occupied by red agents. We have that $\mathrm{DoI}(\sigma^*) = \Delta_G(\Delta_G - 1) = n$. Also, there is a NE σ in which the blue agents occupy B and $b + 1$ of the leaf nodes adjacent to nodes in B' are empty. Since each blue agent is adjacent to exactly one red agent, we have for any red agent i and empty node u that $U_i(\sigma) = U_i(\sigma_{iu}) = p\left(\frac{1}{2}\right)$. Thus, we have that σ is a NE. Only the blue agents and one red agent are not segregated, hence it holds that $\mathrm{DoI}(\sigma) = b + 1 = \Delta_G$. Thus, we have that $\mathrm{PoA}(G, r, b, p) \geq \frac{n}{b+1} = \frac{\Delta_G(\Delta_G - 1)}{\Delta_G}$. $\square$

We use a similar construction as in [6] to also obtain a lower bound for a regular graph. Yet, in our case the bound holds for all values of Λ.

Theorem 11. *For every $\delta \geq 2$ and Λ, a game (G, r, b, p) on a δ-regular graph with $PoA(G, r, b, p) \geq \frac{\delta(\delta+1)}{2\delta+1} = \frac{\delta+1}{2} - \frac{\delta+1}{4\delta+2}$ exists.*

If $r = b$, Theorem 9 yields that PoA $\leq \frac{2b}{b+1} < 2$. This is tight.

Theorem 12. *For any Λ and b, there is a balanced game (G, b, b, p) with $PoA(G, b, b, p) = \frac{2b}{b+1}$.*

4.2 Price of Stability

We give bounds on the PoS under different conditions. First, we observe from Theorem 9, that for any game the PoS is at most $\min\left(\Delta_G, \frac{n}{b+1}\right)$. We now present a lower bound which is asymptotically tight for $b = 1$.

Theorem 13. *For any $\Lambda \geq \frac{1}{2}$, there is a game on a tree with $PoS = \frac{\Delta_G}{2} = \frac{n-2}{2} = \frac{n-2}{b+1}$.*

Next, we study balanced games. Here, the PoS is upper bounded by a PoA of at most 2. We show that this bound is tight for $\Lambda \geq \frac{1}{2}$.

Theorem 14. *For $\Lambda \geq \frac{1}{2}$, a game (G, b, b, p) with $PoS(G, b, b, p) \geq 2 - \varepsilon$ for any $\varepsilon > 0$ exists.*

Proof. Consider the balanced game (G, b, b, p) as shown in Fig. 5. The graph G has two sets A and B of b nodes each and the i*-th* node in A is connected to the i-th node in B. Furthermore, the first node $v \in A$ is connected to all nodes in both, A and B. Additionally, the node v is adjacent to $2b$ leaf nodes Z.

In the optimal strategy profile σ^*, all red nodes are located on A and all blue nodes on B. Thus, it holds that $\mathrm{DoI}(\sigma^*) = 2b$. We claim that there is no equilibrium σ in which v is empty or any agent of the opposite color of $\sigma^{-1}(v)$ is adjacent to any agent of $c(\sigma^{-1}(v))$ other than $\sigma^{-1}(v)$.

Suppose that v is empty. There are $2b - 1$ nodes in $B \cup A \setminus \{v\}$. Thus by counting, there must be an agent i on a node in Z. As v is empty, we have

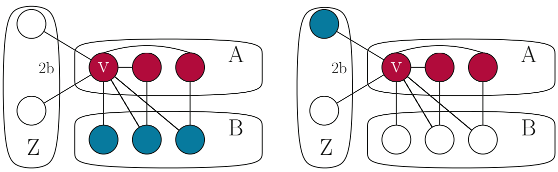

Fig. 5. The PoS of a balanced game with b agents per type. A and B contain b nodes each. Left: Optimum. Right: Best NE, where b nodes in Z are occupied by blue agents. (Color figure online)

that $U_i(\sigma) = 0$. Yet, it holds that $U_i(\sigma_{iv}) = p(\frac{b}{2b}) > 0$, so agent i has an improving jump. W.l.o.g., let there be a red agent on v. Suppose that a blue agent i is adjacent to an additional red agent that is not $\sigma(v)$. Then, we have that $U_i(\sigma) = p(\frac{1}{3})$. As there are $2b$ nodes in Z and neither $\sigma^{-1}(v)$ nor i are on a node in Z, there is an empty node $u \in Z$ and since $U_i(\sigma_{iu}) = p(\frac{1}{2}) > U_i(\sigma)$ agent i has an improving jump. Thus, v is not empty and all red agents except for $\sigma^{-1}(v)$ are segregated. For any equilibrium, it holds that at most the agent on v and the agents of a different color may be non-segregated, i.e., $\mathrm{DoI}(\sigma) \leq b + 1$. Figure 5 shows such a NE: All nodes in A are occupied by red agents, all nodes in B are empty and b nodes in Z are blue. Clearly, no red agent can improve and any blue agent jumping to a node in B will have a utility of either $p(\frac{1}{2})$ or $p(\frac{1}{3})$ which is no improvement. Hence, $\mathrm{PoS}(G, b, b, p) \geq \frac{2b}{b+1}$. Thus, for any $\varepsilon > 0$, we can achieve a $PoS \geq 2 - \varepsilon$ by choosing b large enough. □

Earlier, in Theorem 5, we proved the existence of equilibria for graphs that have an independent set of size at least $b+e$. Now, we show that on such graphs, we can also bound the PoS.

Theorem 15. *For any game (G, r, b, p) with $b + e \leq \alpha(G)$, we have $PoS(G, r, b, p) \leq \frac{\Delta_G+1}{\delta_G+1}$.*

Theorem 15 applies to δ-regular graphs. Note that for any δ-regular graph, we have $\alpha(G) \geq \frac{|V|}{\delta+1}$.

Corollary 4. *For any game on a δ-regular graph with $b + e \leq \alpha(G)$, we have $PoS(G, r, b, \Lambda) = 1$.*

Furthermore, in Theorem 4, we prove that any game on a ring with $\Lambda = \frac{1}{2}$ and $e = 1$ converges to a NE by proving that $\mathrm{DoI}(\sigma)$ is an ordinal potential function. It follows that every strategy profile that maximizes the degree of integration must be a NE.

Corollary 5. *For any game $(G, r, b, \frac{1}{2})$ on a ring with $e = 1$, we have $PoS(G, r, b, \frac{1}{2}) = 1$.*

4.3 Quality of Equilibria w.r.t. the Utilitarian Welfare

While our main focus in this work is on the quality of equilibria with respect to the degree of integration as social welfare, we close this section by pointing out, that our results on the PoA and PoS with respect to the degree of integration also imply bounds on the PoA and the PoS with respect to the standard utilitarian welfare (PoA^U and PoS^U for short), assuming that p is linear. Remember, that the utilitarian social welfare simply is the sum over the utilities of all the agents.

In particular, for a fixed peak Λ and a fixed maximum degree δ, a constant bound on PoA yields a constant bound on PoA^U, as the following theorem demonstrates.

Theorem 16. *Let p be a linear function. For any game $\Gamma = (G, r, b, \Lambda)$, the following holds:*

- $PoA(\Gamma) \leq a \Rightarrow PoA^U(\Gamma) \leq a \cdot \max(\Lambda, (1-\Lambda)) \cdot (\Delta_G + 1)$.
- $PoS(\Gamma) \leq s \Rightarrow PoS^U(\Gamma) \leq s \cdot \max(\Lambda, (1-\Lambda)) \cdot (\Delta_G + 1)$.

For the PoA, this bound is asymptotically tight, i.e., $PoA^U(G, b, b, \frac{1}{2}) = PoA(G, b, b, \frac{1}{2}) \cdot \frac{1}{2} \cdot \Delta_G$ *holds.*

5 Computational Complexity

We discuss the computational complexity of finding equilibria via improving response dynamics and the complexity of computing strategy profiles with a high DoI. Especially the former question is particularly interesting, since finding equilibria via improving moves can be easily coordinated within a society of selfish agents. In contrast, centrally switching from some initial state directly to an equilibrium state requires much more coordination and also that the agents trust the central coordinator.

Settling the complexity of the equilibrium decision problem seems to be very challenging and we leave this as an open problem. However, our hardness proof for finding equilibria via improving response dynamics can be seen as a first step towards proving that deciding the existence of equilibria is NP-hard as well. Moreover, we note in passing that if we would allow for stubborn agents, as in [2], then we can prove that deciding if an equilibrium exists is indeed NP-hard. We suspect that this assumption may be removed, similarly to the approach of [24].

5.1 Finding Equilibria via Improving Response Dynamics

We investigate the problem of finding equilibria. We consider the problem of deciding whether an equilibrium for a given game can be reached through *improving response dynamics (IRDs)* from a given initial strategy profile σ_0. We show that this problem is NP-hard for any value of Λ.

Theorem 17. *For any fixed* $\Lambda = \frac{x}{y} \in (0, 1)$*, it is NP-hard to decide if a given game played on a graph* G *with* r *red and* b *blue agents can reach a NE through IRDs starting from a given initial placement* σ_0*.*

5.2 Existence of Strategy Profiles with High DoI

We aim for finding a strategy profile with a DoI larger than some threshold d. This problem is indifferent to the utilities of the agents and thus the same for any Jump Schelling Game. For $d = n$, the hardness of this problem has been studied before by [2]. However, their focus lies on swap games and therefore assumes $|V| = n$. As noted by the authors this result can be generalized to $|V| > n$ by adding isolated empty nodes. We improve on their result by showing that the hardness holds in a more realistic setting without isolated nodes.

Theorem 18. *Given a JSG with* r *red and* b *blue agents on a connected graph* $G = (V, E)$ *with* $|V| > n$*, it is NP-complete to decide if there is a strategy profile* σ^* *with* $DoI(\sigma^*) \geq d$*.*

6 Discussion

Our paper sheds light on Jump Schelling Games with non-monotone agent utilities. With this, we strengthen the recent trend of investigating more realistic residential segregation models.

6.1 Comparison with Single-Peaked Swap Schelling Games

Similarly to other variants of Schelling games, we also observe that our jump version behaves very differently compared to the swap version studied by Bilò et al. [6] and novel techniques are required. The main difference in jump games is that structural properties of the underlying graph cannot be exploited. The reason is that empty nodes are not counted when computing an agent's utility and hence it is impossible to distinguish between an empty node or a missing node. We do carry over some ideas from Single-Peaked Swap Schelling Games, e.g., the PoA upper bound proof, or the idea of considering independent sets, but the main part of our paper, e.g., all lower bound proofs and the proofs of our hardness results, follow entirely new approaches.

We obtained predominantly negative results with regard to convergence towards equilibria, in particular the finite improvement property does not hold for any $\Lambda \in (0, 1)$, not even on regular graphs or trees. This is in stark contrast to the swap version, which converges to equilibria even on almost regular graphs for $\Lambda \leq \frac{1}{2}$. Furthermore, on regular graphs with $\Lambda = \frac{1}{2}$, instances of our jump version exist that do not admit equilibria. Also, although we get similar PoA bounds, compared to the swap version, we find that the PoS of the jump version tends to be worse, in particular, while the swap version has a PoS of at most 2 on bipartite graphs for $\Lambda = \frac{1}{2}$, there exists a tree that enforces a PoS that is linear in n for our jump version for this setting.

6.2 The Variant with Self-exclusive Neighborhoods

To enable a better comparison with the models by Chauhan et al. [14] and Agarwal et al. [2], that do not count the agent herself in the computation of the fraction of same-type neighbors, we also considered a variant of our model with self-exclusive neighborhoods, i.e., where the agent herself is not contained in her neighborhood. This self-exclusive variant behaves in some aspects very similarly to our model: the FIP does not hold and there is no equilibrium existence guarantee on regular graphs. Regarding the PoA it gets even worse, since equilibria exist where every agent has utility 0, implying an unbounded PoA. This also holds for the PoA with respect to the utilitarian social welfare. Moreover, also the PoA and the PoS with respect to the utilitarian welfare is unbounded.

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