Week 13 Lecture: Applied Machine Learning

L. Jason Anastasopoulos `ljanastas@princeton.edu`

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Dimensionality Reduction Methods

- All of the models discussed used the original predictors in some form.
- Dimensionality reduction methods transform the predictors into variable clusters and then use these transformed variables to fit a model.

Dimensionality Reduction Methods

Consider a linear combination $Z_1, \cdots, Z_M$ of the features $X_1, \cdots, X_1$ such that $M < p$ where:

$$Z_m = \sum_{j=1}^{p} \phi_{jm} X_j$$

For some constants $\phi_1, \cdots, \phi_M; m \in [1, M]$. We can then fit the linear regression model:

$$y_i = \Theta_0 + \sum_{m=1}^{M} \Theta_m z_{im}$$

Dimensionality Reduction Methods

The model

$$y_i = \Theta_0 + \sum_{m=1}^{M} \Theta_m z_{im}$$

now has $M + 1 < p + 1$ predictors and, if chosen well, can result in a better fit through estimating fewer parameters than the original regression model.

Dimensionality Reduction Methods

To be clear take a simple linear regression model with three features:

$$Y = \theta_0 + \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3 + \epsilon$$

Define $z_1 = \phi_1 X_1 + \phi_3 X_3$ and $z_2 = \phi_2 X_2$. We can now estimate the reduced model:

$$\begin{aligned} Y &= \Theta_0 + \Theta_1 z_1 + \Theta_2 z_2 + \epsilon \\ &= \Theta_0 + \Theta_1(\phi_1 X_1 + \phi_3 X_3) + \Theta_2(\phi_2 X_2) + \epsilon \end{aligned}$$

Dimensionality Reduction Methods

- Again the key here is that we are estimating a model with fewer predictors, thus reducing the *dimensionality* of the model.
- This is especially useful in problems where p is large relative to n. Variance will be significantly reduced in this case and this is not uncommon in machine learning problems (ie text analysis)

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- There are several methods for accomplishing this but we will focus on principal components analysis.

Principal Components Analysis (PCA)

$$f : \mathcal{X} \rightarrow \mathcal{F}$$

$$\mathcal{X} \in \mathbb{R}^{nxp}, \mathcal{F} \in \mathbb{R}^{nxm}; p << m$$

- PCA is often discussed in the context of *unsupervised learning* and we'll discuss it in that context later on in the semester.
- It's a popular means of transforming a high dimensional feature space $\mathcal{X}$ into a very low-dimensional space $\mathcal{F}$

Principal Components Analysis (PCA)

- **First principal component** is the dimension along which the data vary the most and would be the most useful for a regression approach.

```
# Predicting political party with votes
library(mlbench)
data(HouseVotes84)
head(HouseVotes84)
```

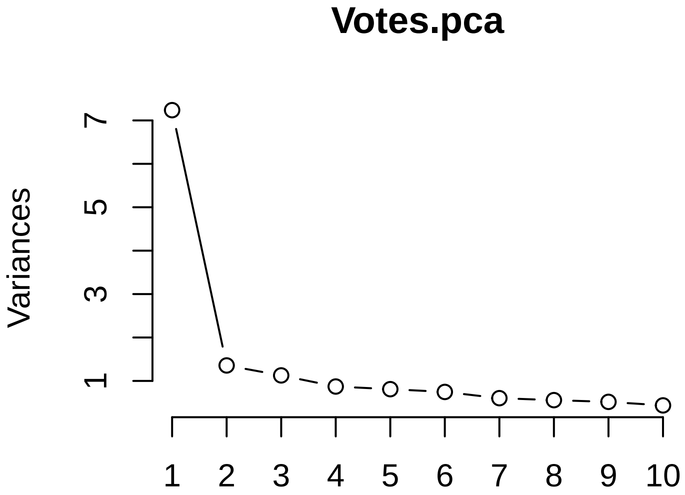
```
##         Class   V1 V2 V3   V4   V5 V6 V7 V8 V9 V10  V11  V12 V13 V14 
## 1 republican    n  y  n    y    y  y  n  n  n   y <NA>    y   y   y
## 2 republican    n  y  n    y    y  y  n  n  n   n    n    y   y   y
## 3   democrat <NA>  y  y <NA>    y  y  n  n  n   n    y    n   y   y
## 4   democrat    n  y  y    n <NA>  y  n  n  n   n    y    n   y   n
## 5   democrat    y  y  y    n    y  y  n  n  n   n    y <NA>   y   y
## 6   democrat    n  y  y    n    y  y  n  n  n   n    n    n   y   y
##    V16
## 1    y
## 2 <NA>
## 3    n
## 4    y
## 5    y
```

Predicting political party from votes, 1984

```
##
## Call:
## lm(formula = Party ~ ., data = data.frame(Votes))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.82054 -0.04439  0.01879  0.08784  0.70224
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.7671523  0.1423941   5.388 1.20e-07 ***
## V1          -0.0264965  0.0213872  -1.239 0.216080
## V2          -0.0282866  0.0200216  -1.413 0.158459
## V3          -0.1988778  0.0296838  -6.700 6.76e-11 ***
## V4           0.6314606  0.0322082  19.606  < 2e-16 ***
## V5           0.0768241  0.0379082   2.027 0.043339 *
## V6          -0.0481934  0.0272663  -1.768 0.077872 .
## V7           0.0642615  0.0288318   2.229 0.026355 *
## V8           0.0559900  0.0352681   1.588 0.113144
## V9          -0.0855327  0.0314344  -2.721 0.006780 **
## V10          0.0478126  0.0187176   2.554 0.010990 *
## V11         -0.1240756  0.0199903  -6.207 1.30e-09 ***
```

Predicting political party from votes, 1984

- Can the votes be explained with a single dimension?



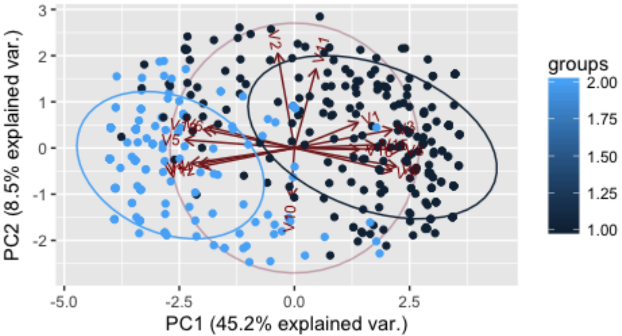
Predicting political party from votes, 1984

- Can the votes be explained with a single dimension?

```r
summary(Votes.pca)
```

```
## Importance of components:
##                           PC1     PC2     PC3     PC4     PC5     PC
## Standard deviation     2.6901 1.16470 1.06151 0.93320 0.90006 0.8638
## Proportion of Variance 0.4523 0.08478 0.07043 0.05443 0.05063 0.0466
## Cumulative Proportion  0.4523 0.53706 0.60749 0.66192 0.71255 0.7591
##                            PC7     PC8     PC9    PC10    PC11     PC
## Standard deviation     0.77631 0.74664 0.71935 0.66043 0.64191 0.576
## Proportion of Variance 0.03767 0.03484 0.03234 0.02726 0.02575 0.020
## Cumulative Proportion  0.79686 0.83170 0.86404 0.89130 0.91705 0.937
##                           PC13    PC14    PC15    PC16
## Standard deviation     0.56507 0.52220 0.48542 0.40914
## Proportion of Variance 0.01996 0.01704 0.01473 0.01046
## Cumulative Proportion  0.95777 0.97481 0.98954 1.00000
```

Predicting political party from votes, 1984



Predicting political party from votes, 1984

- Took 16 dimensions, reduced to 1 or 2 that still explain about 50% of the variance.
- Can use these dimensions in regression for comparison.
- Let's just use dimensions one and two

Predicting political party from votes, 1984

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Predicting political party from votes, 1984

$$Party = \Theta_0 + \Theta\pi_1 + \Theta_2\pi_2$$

```
pi1<-Votes.pca$x[,1]
pi2<-Votes.pca$x[,2]
summary(lm(Party~pi1 + pi2))
```

```
##
## Call:
## lm(formula = Party ~ pi1 + pi2)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -0.9021 -0.1181  0.0211  0.1560  0.9177
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.386207   0.012918 107.312   <2e-16 ***
## pi1         -0.144037   0.004807 -29.961   <2e-16 ***
## pi2         -0.105903   0.011104  -9.538   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Problems with PCA

- Very sensitive to scaling
- Is a good idea to standardize the predictors.