Greedy Algorithms

Lecture 5

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Quiz

- You are given n activity schedule $[s_i, f_i]$ for $1 \leq i \leq n$ for one room, where s_i and f_i denote the start and the finishing time of activity i. You are to select the maximum number of activities that can be schedule such that no two activities have an overlapping period. Design an greedy algorithm.

Answer

```
Schedule(s, f):
  Sort (s, f) by f
  S = {1}
  curr = f[1]
  for i = 2 to n:
    if s[i] ≥ curr:
      S = S ∪ i
      curr = f[i]

```

Proof of optimality for homework

Agenda

- Data Structure
  - Heap
  - Graphs
- Greedy Algorithms
  - MST - Minimum Spanning Tree problems
  - Dijkstra's Shortest Path

Heap

- **max-heap**: A complete binary three, where all nodes has key value that are greater than or equal to each of its children.
- **min-heap**: A complete binary three, where all nodes has key value that are less than or equal to each of its children.

Heap Operations

- **Insert**: add a new object to a heap - O(log n)
- **Extract Min**: remove a node in with a minimum key value - O(log n)
- **Heapify**: n batched inserts - O(n)
- **Delete**: remove a node - O(log n)

Application

- If your application requires minimum calculation repeatedly
  - **Selection Sort**: find min O(n^2)
  - **Heap Sort**: same as selection sort but O(n log n)
    - Heapify input array - O(n)
    - Extract-min n times - O(n log n)
- Event manager - "priority queue"
- Median maintenance - extract median

Implementation

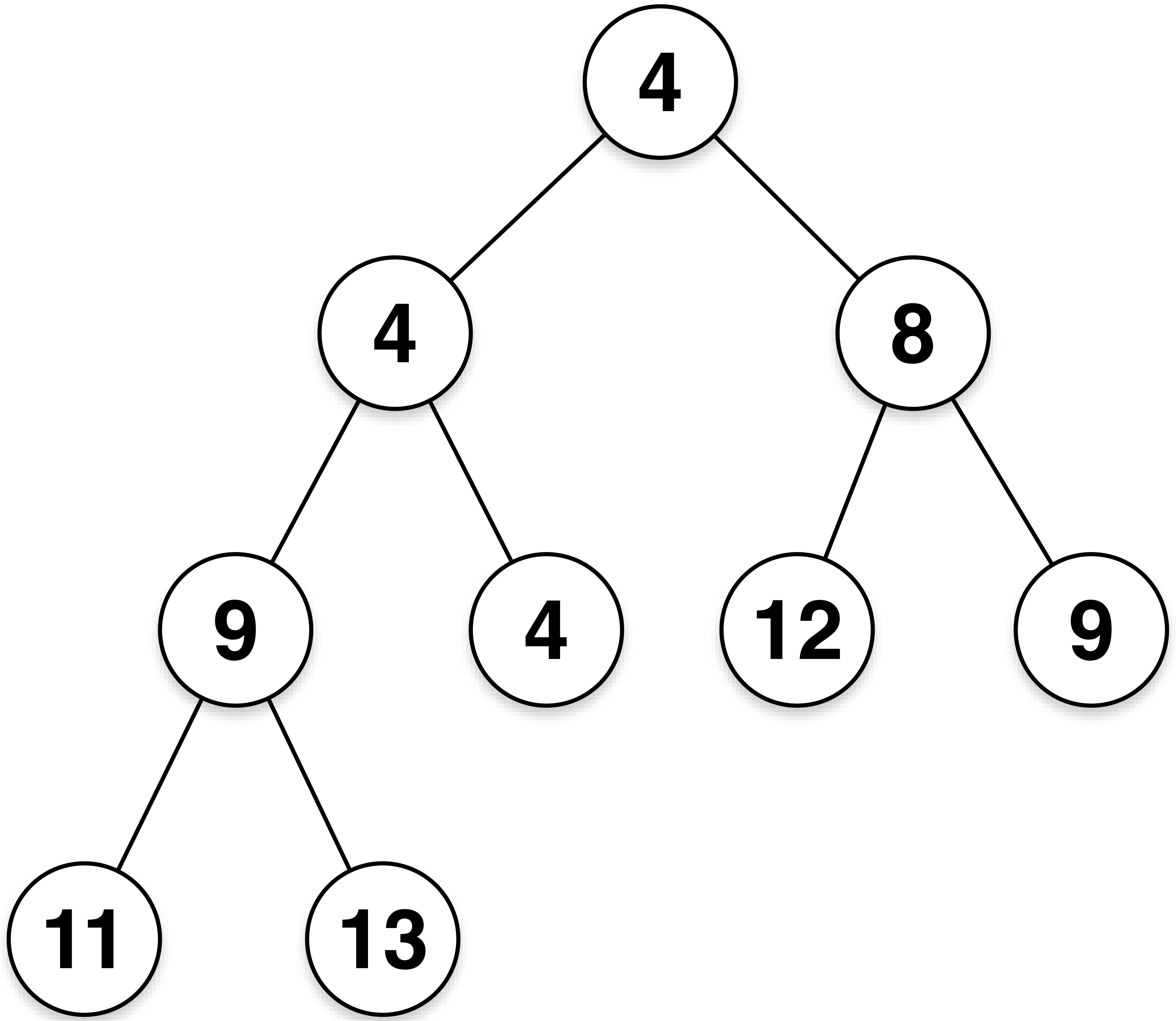
- Usually trees are implemented using pointers and references. However because Heaps are complete binary tree, we can use an array.

- root node: 1^{st} element

- $\text{child}(i) = (2i, 2i+1)$

- $\text{parent}(i) = \begin{cases} i/2 & \text{if } n \text{ is even} \\ \lfloor i/2 \rfloor & \text{if } n \text{ is odd} \end{cases}$

Implementation



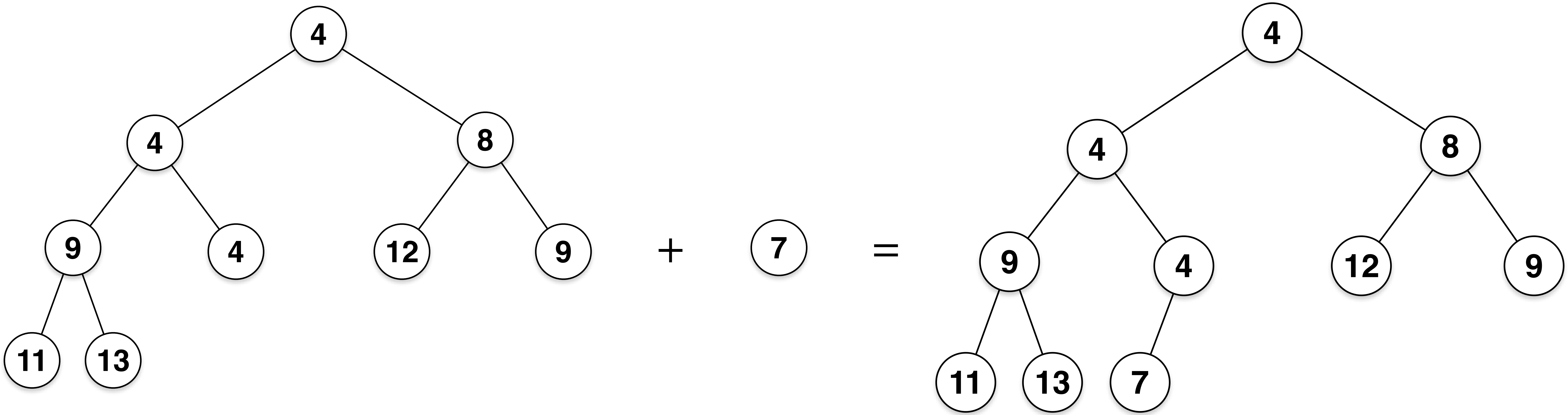
1	2	3	4	5	6	7	8	9
4	4	8	9	4	12	9	11	13

Insert

```
Insert(k):
  Add k to the next place (append)
  if (parent(k).key < k.key):
    done
  else:
    heapify-up (bubble up)
```

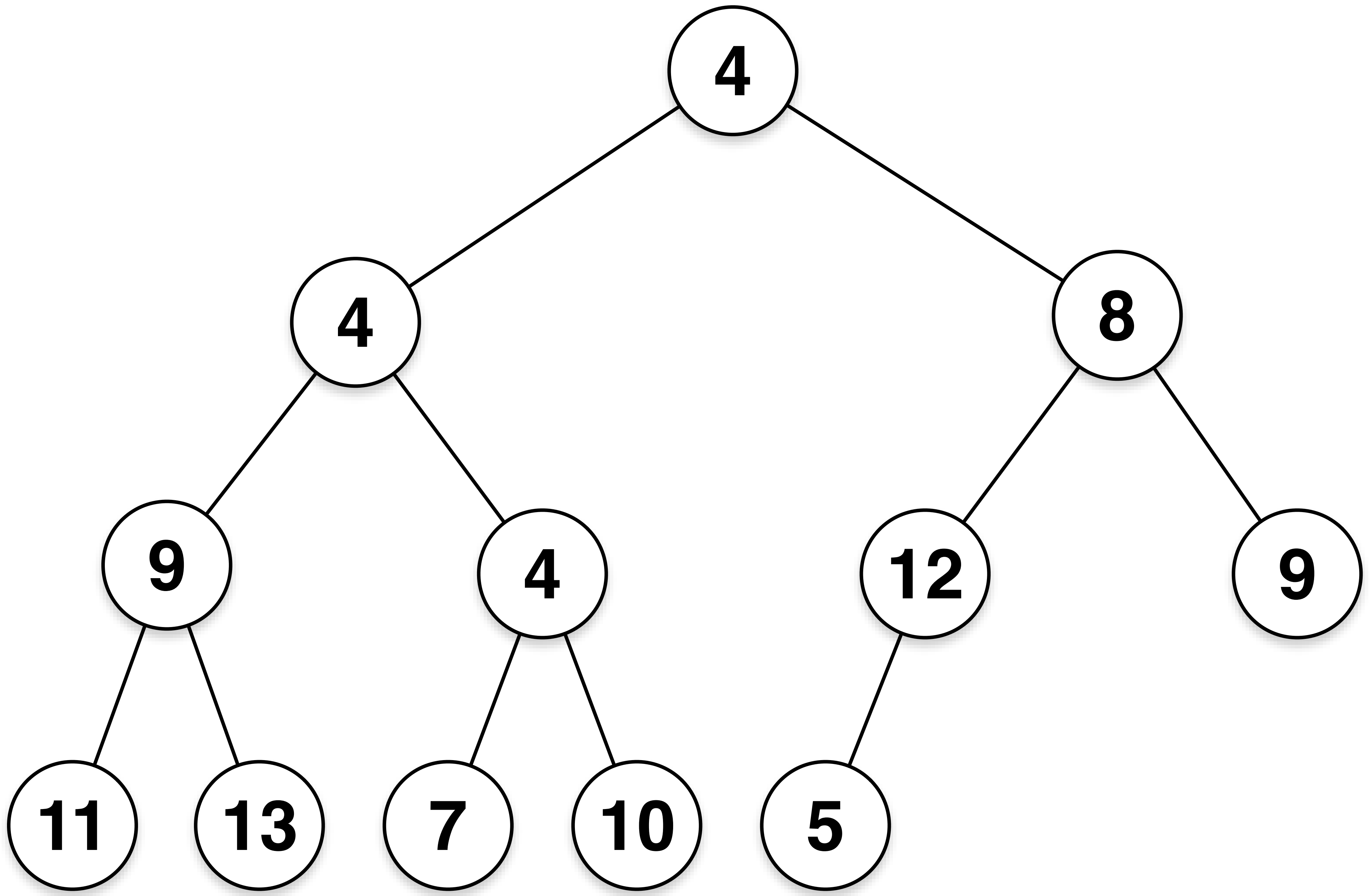
- **Invariant**: all nodes has key value that are less than or equal to each of its children

Example

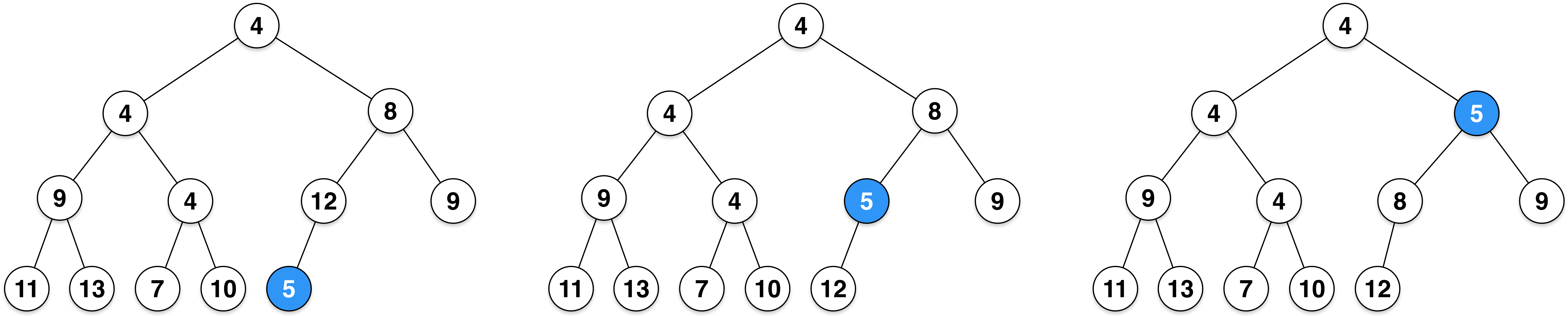


Heapify Up

```
HeapifyUp(A, k):
  while(parent(k).key > k.key):
    swap(parent(k), k)
```



Heapify Up



Insert Running Time

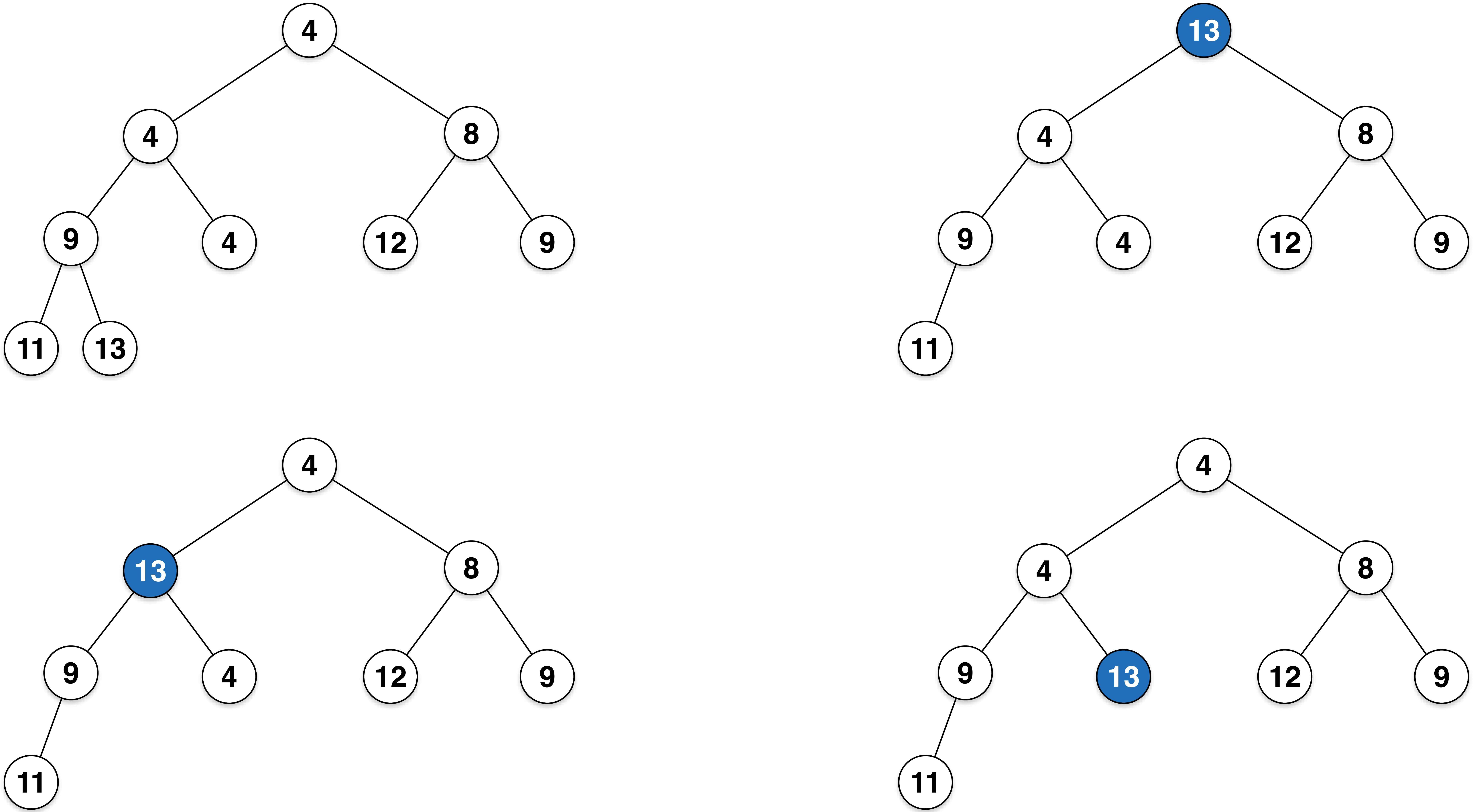
Extract Min

```
ExtractMin(A):
  swap root with the last element
  remove root
  MinHeapify(A, 1)
  return root
```

MinHeapify

```
MinHeapify(A, i):
  left = 2i
  right = 2i + 1
  min = i
  if (A[min] < A[left] && A[min] < A[right])
    return
  if (A[left] < A[right]):
    min = left
  else:
    min = right
  swap(A[i], A[min])
  MinHeapify(A, min)
```

MinHeapify



Heapify

```
Heapify(A):
  Assume A is already a heap
  start = floor(n/2) // first root with children
  for i = start to 1:
    MinHeapify(A, i)
```

Heapify Running Time

- $O(\log n)$??
- If all the subtree at height h has been heapified, then heapifying the sub tree at h+1 level will only require bubbling down the root nodes.
  - $O(h)$ operations (swap) per node
  - Height is measured from bottom up starting at 0
- Notice how most of the heapifying happens at the bottom.

Heapify Running Time

- $O(h)$ operations (swap) per node
- $\text{NodeCount}(h) = \lceil 2^{(\log_2 n - h) - 1} \rceil = \left\lceil \frac{2^{\log n}}{2^{h+1}} \right\rceil = \left\lceil \frac{n}{2^{h+1}} \right\rceil$
- Cost of heapifying the entire tree:

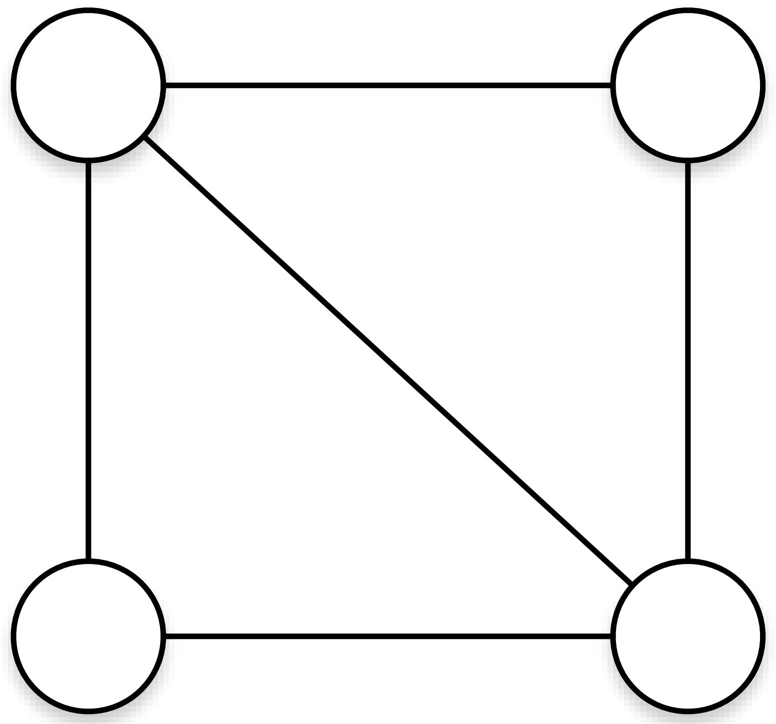
$$\sum_{h=1}^{\lceil \log n \rceil} \frac{n}{2^{h+1}} O(h) = O\left(n \sum_{h=1}^{\lceil \log n \rceil} \frac{h}{2^{h+1}} \right)$$

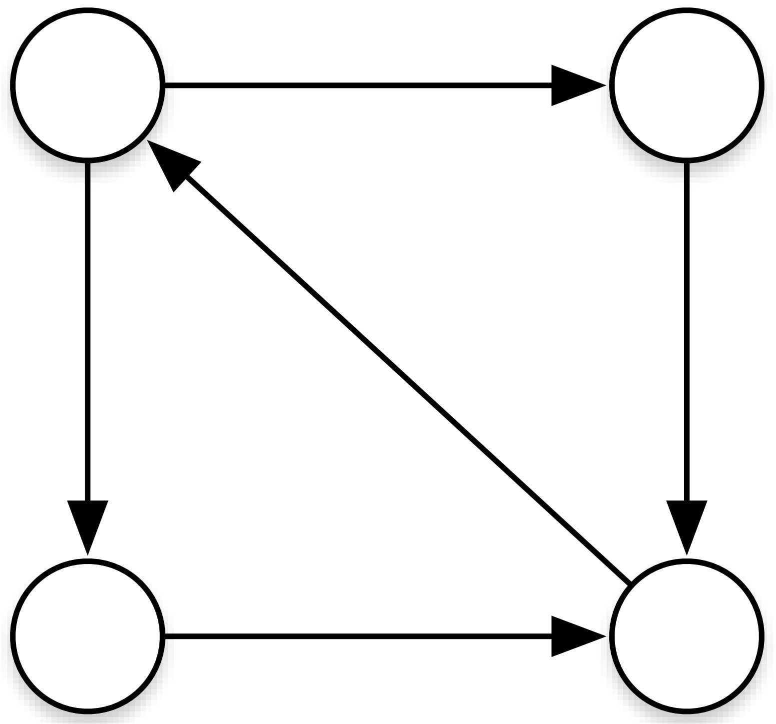
$$\leq O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(2n) = O(n)$$

- http://www.symbolab.com/solver/series-calculator/%5Csum_%7Bn%3D0%7D%5E%7B%5Cinfty%7D%20%5Cfrac%7Bn%7D%7B2%5E%7Bn%7D%7D

Graphs

- **Vertices** (nodes) = V
- **Edges** (pairs of vertices) = E
  - Edges = directed or undirected (pair is ordered or not)
  - Parallel edges = edges that connect the same vertices
- **Degree of a vertex** = # of incident edges
- **Application**:
  - Map Application
  - Web
  - Social Networks
  - Any many more





Graph Size

- Given a graph where $|V| = n$ where no parallel edges allowed
  - Minimum number of edges $= n - 1$
  - Maximum number of edges $= \binom{n}{2} = \dfrac{n^2 - n}{2}$

Sparse vs Dense Graphs

- Let $|V| = n, |E| = m$
- In most applications $m = \Omega(n)$ and $O(n^2)$

Sparse $\Longleftrightarrow$ **Dense**

$m = O(n)$ $\qquad$ $m = O(n^2)$

Graph Representation

- Adjacency Matrix for undirected graph - $\theta(\mathrm{n}^2)$

$$\text{Given } G = (V, E), \text{ use } n \times n \text{ matrix}, A, where$$

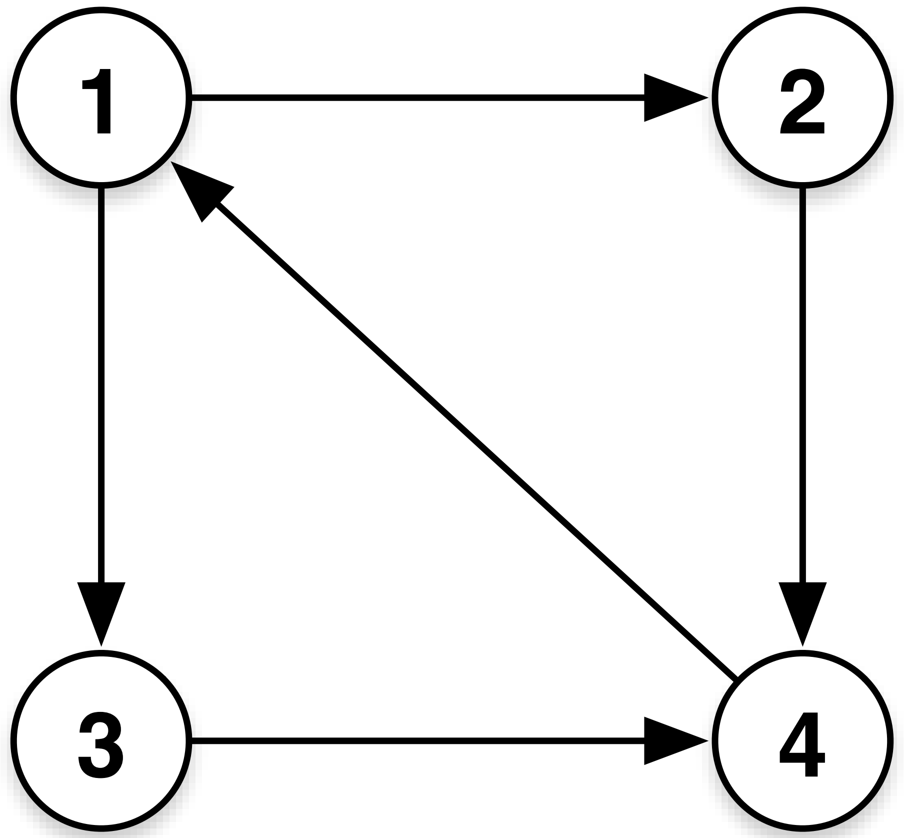
$$A_{ij} = \begin{cases} 1 & \text{if } E(v_i, v_j) \text{ exists} \\ 0 & \text{if } E(v_i, v_j) \text{ doesn't exist} \end{cases}$$

- Graph with parallel edges: $A_{ij} = \#$ of edges between v_i and v_j

- Weighted Edges: $A_{ij} = \text{weight } E(v_i, v_j)$

- Directed Graph: $$A_{ij} = \begin{cases} +1 & E(v_i, v_j) \\ -1 & E(v_j, v_i) \end{cases}$$

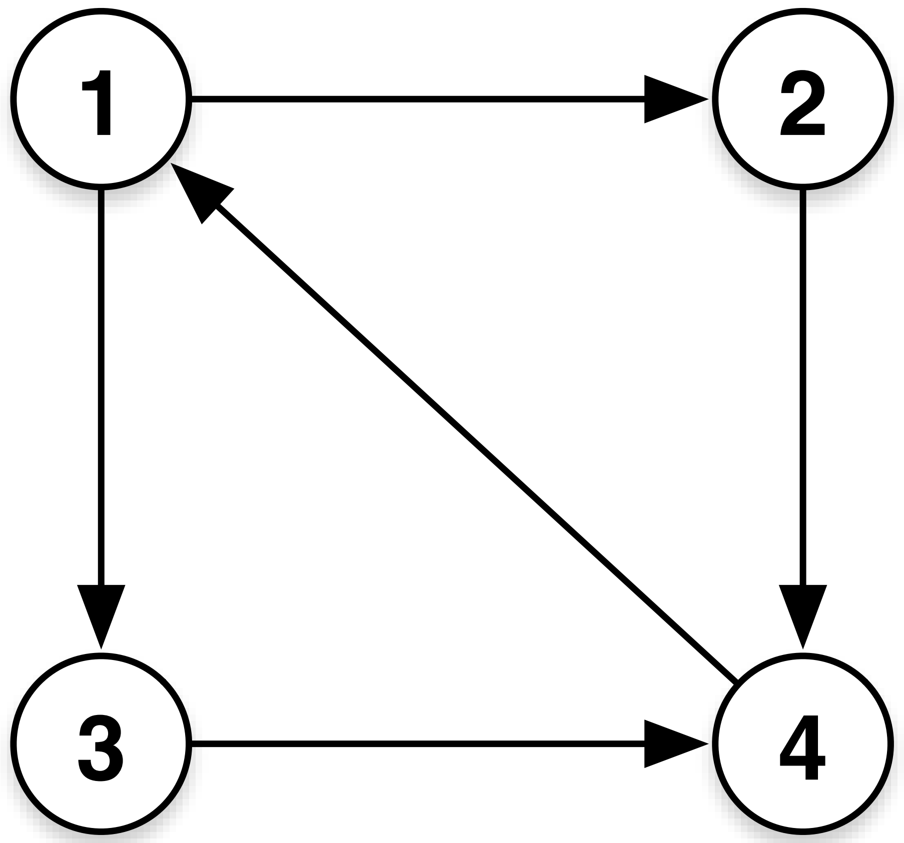
Graph Representation



i\j	1	2	3	4
1	0	+1	+1	-1
2	-1	0	0	+1
3	-1	0	0	+1
4	+1	-1	-1	0

More Graph Representation

- Adjacency List
- For each of the vertices, have a list of vertices its connected to.
- Better for sparse array.

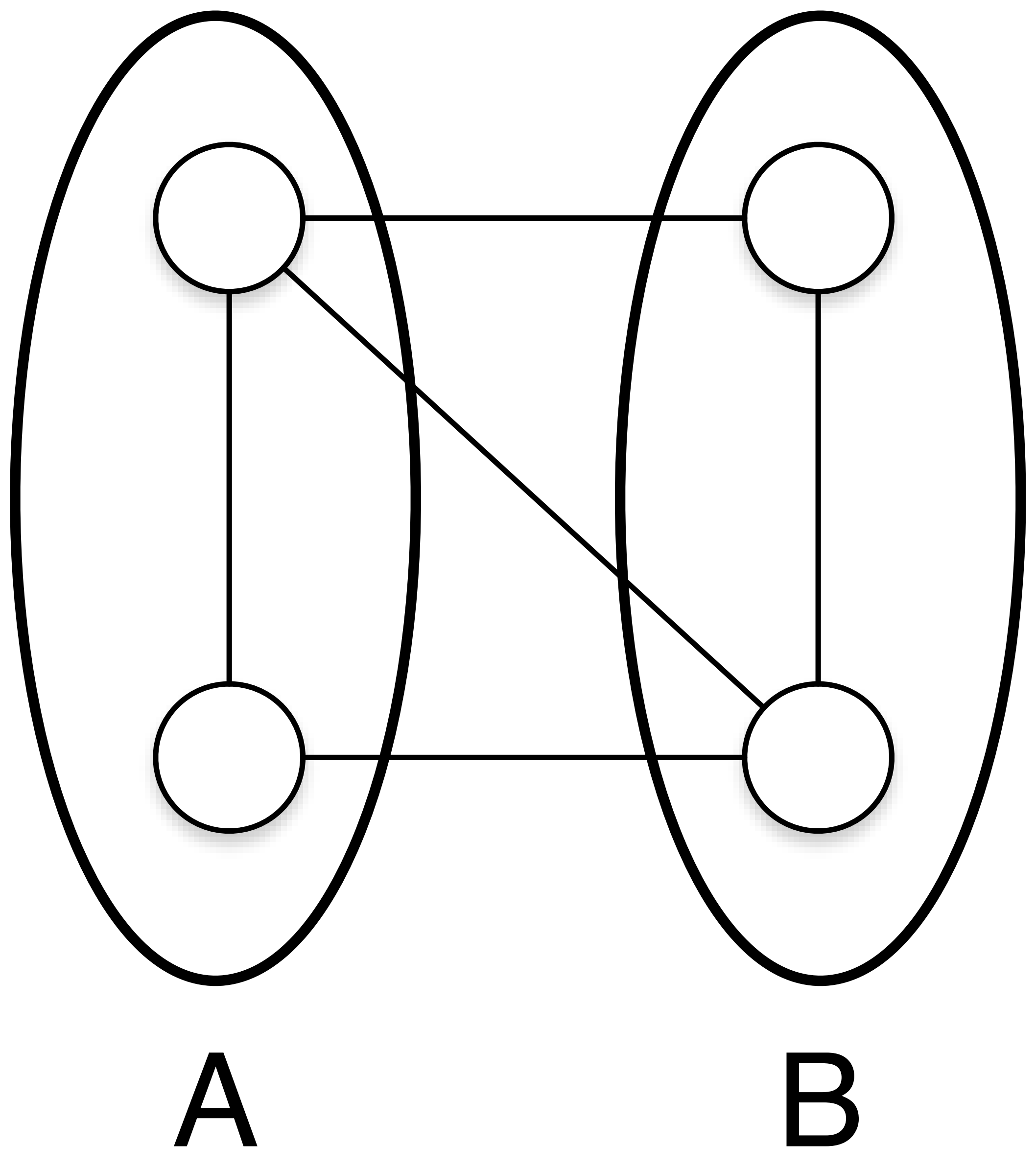


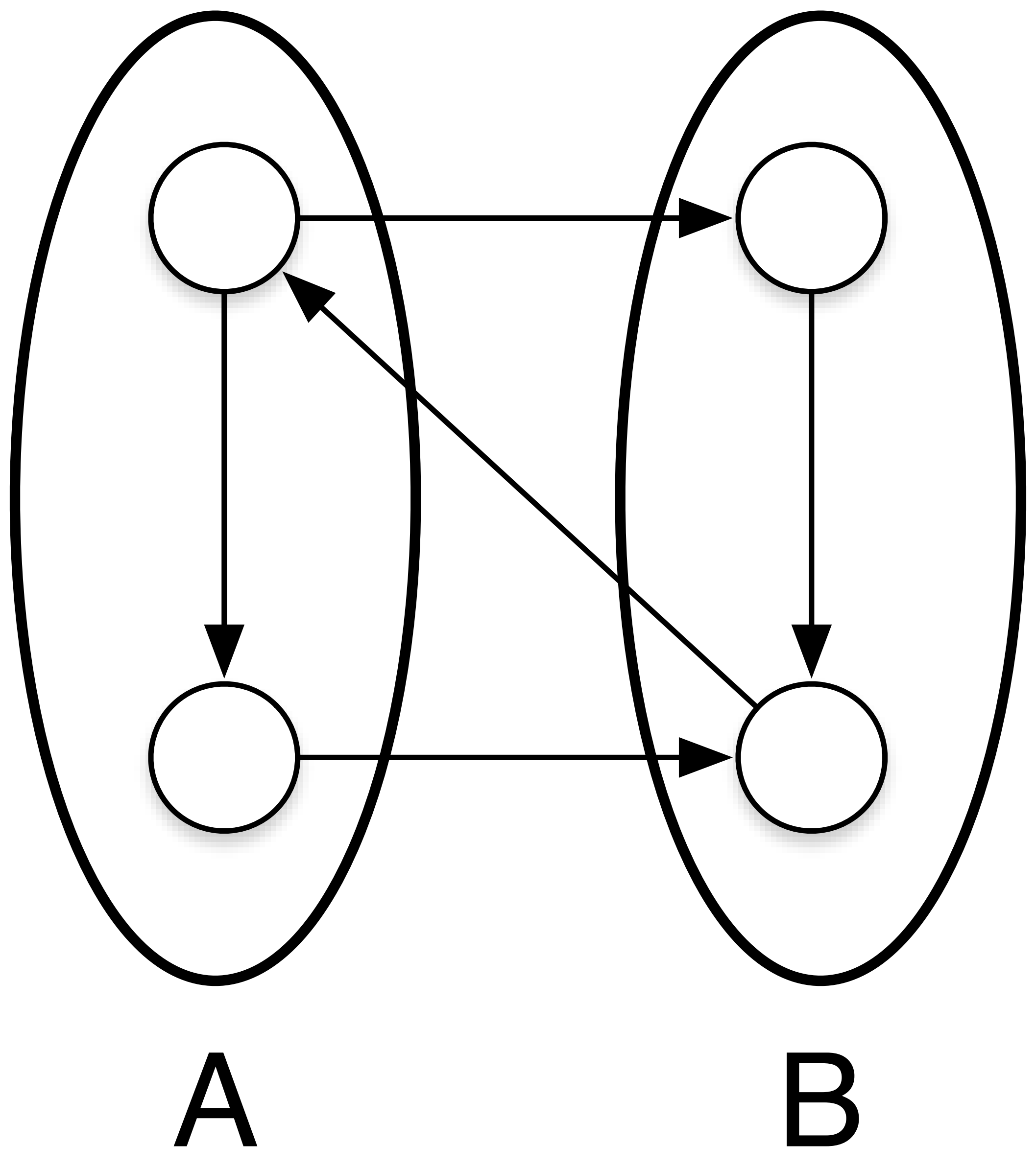
1	2	3
2	4	
3	4	
4	1	

Cut of graphs

- Given a graph $G(V,E)$, a cut of G is a partition of V into two non-empty set (A,B).
- Crossing edges = Set of edges where endpoints are in each of (A,B)
- For directed graphs, count the edges where tail in A and head in B

3 crossing edges





2 crossing edges

Cut of a Graph

- Given a $G(V,E)$, where $|V| = n$, how many possible cuts does G have?
  - $|\text{Set of all possible cuts}| = 2^n$ - 2

MinCut Problem

- Given $G(V,E)$,
  - find a cut with fewest number of crossing edges. (a mincut)
- Brute force algorithm?

MinCut Application

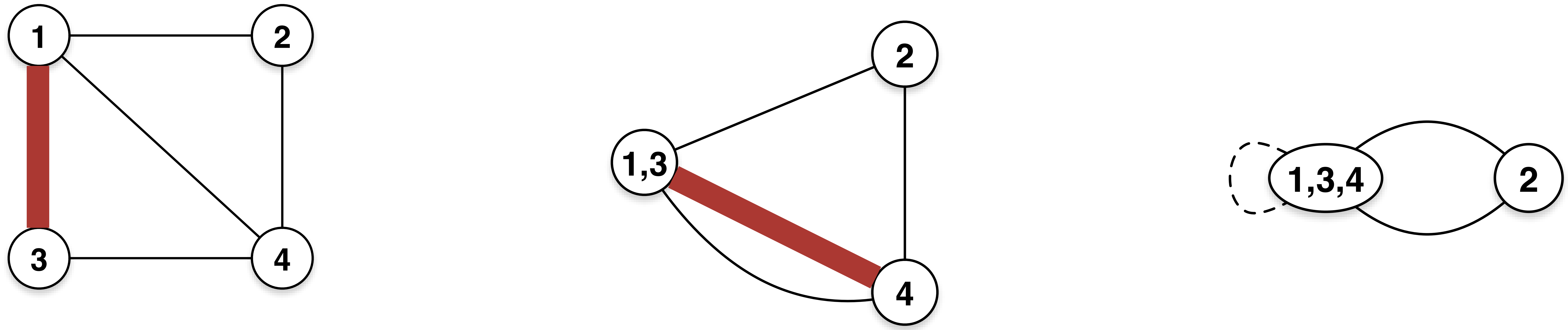
- Finding the weak spot in network (bottleneck)
- Detecting community in a social network
- Image segmentation
  - Graph of pixels
  - Neighboring pixels have weighted edges if two pixels come from a "same" object
- And many many more

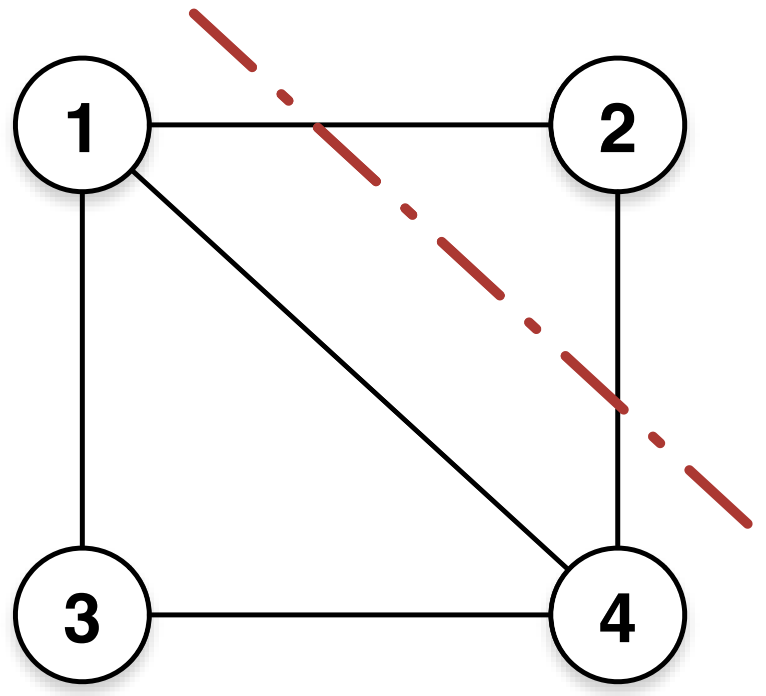
Random Contraction Algorithm

```
RCA(G):
  V = G(V)
  while(|V| > 2) {
     e(v_i,v_j) = random(E)
     remove e
     fuse (v_i,v_j)
  }
  return the cut represented by final 2 vertices

Fuse(v_i,v_j):
  create a brand new edge, v_new
  take all the edges connecting v_i, v_j then connect them to v_new
  remove any self-loops created
```

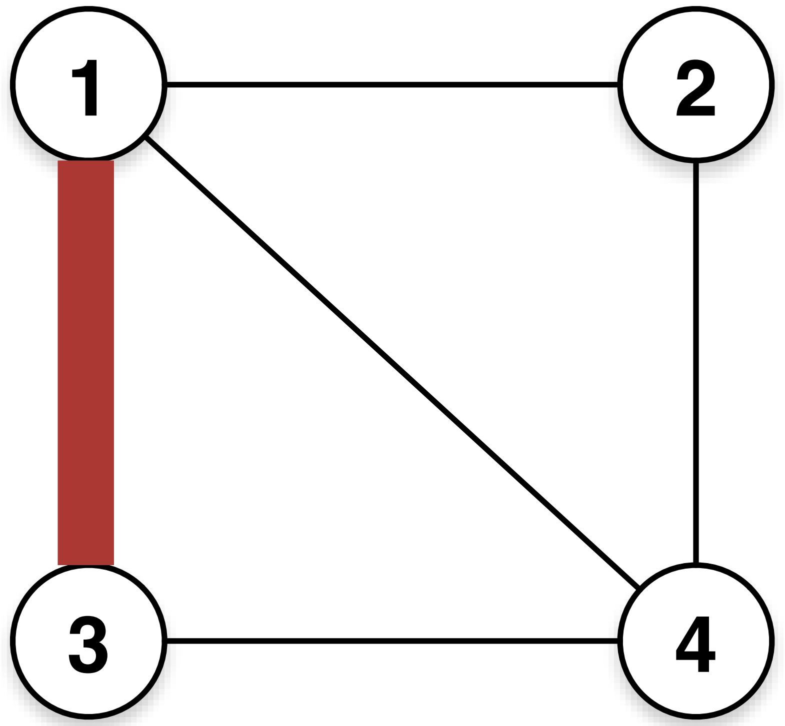
Example

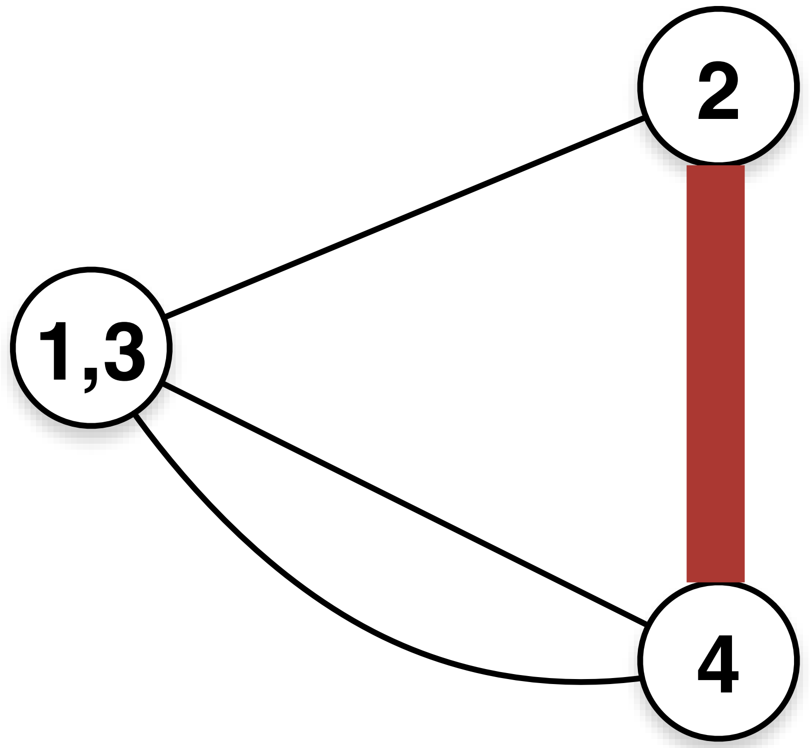


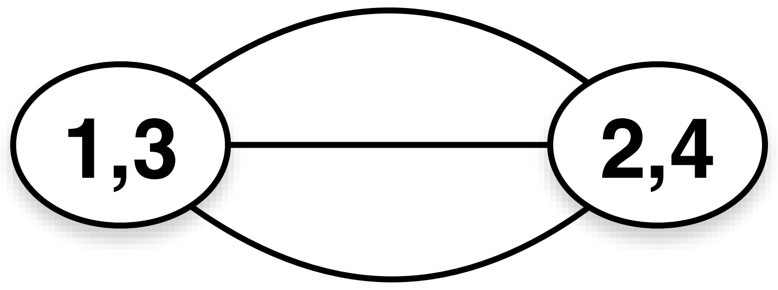


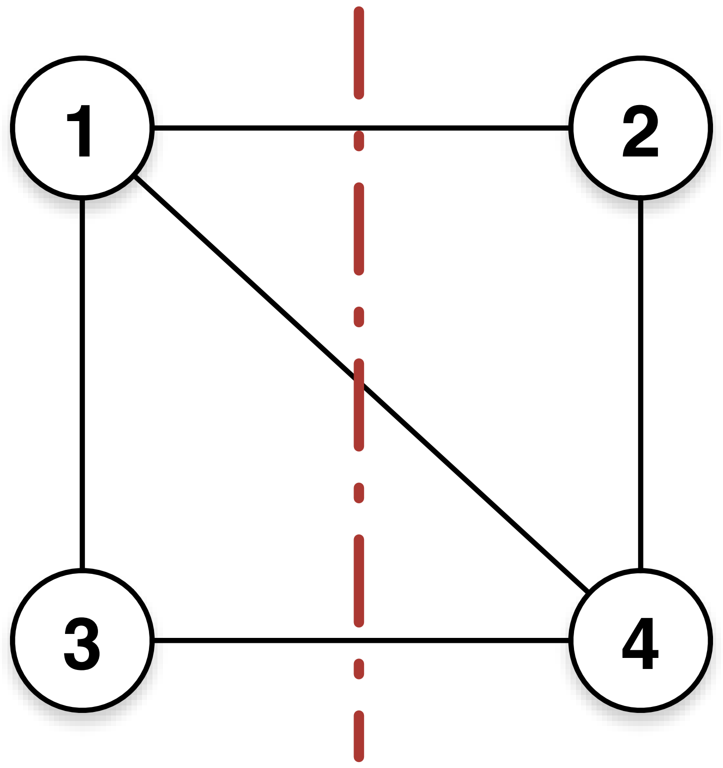
= Cut of ({1,3,4}, {2}) = 2 crossing edges

Example 2









= Cut of ({1,3}, {2,4}) = 3 crossing edges!

Random Contraction Algorithm

- Doesn't work!
- Is it useful?
  - What is the probability that it'll create MinCut?

RCA Analysis

- Fix a graph, $G(V,E)$ with n vertices and m edges
- Fix a minimum cut (A,B)
- Let $F = \{\text{crossing edges}\}$, $|F| = k$
- If the algorithm chooses an edge from F to contract, RCA will fail.
- Converse: If RCA never fails, then none of the edges in F gets chosen.
- P(Output of RCA is (A,B)) = P(RCA never contracts an edge in F)

RCA Analysis Cont.

- Computing P(RCA never contracts an edge in F) $= PF$
- Let S_i = event that an edge in F gets contracted at i^{th} iteration
- $PF = P(\neg S_1 \wedge \neg S_2 \wedge \ldots \wedge \neg S_{n\text{-}2})$
- $P(S_1) = k/m$
- $P(\neg S_1) = 1 - k/m$
- $P(\neg S_2 | \neg S_1) =$?? how does m change??
- It'd be nice if we can instead track the probability in terms of n

RCA Analysis Cont.

- Note, degree of each vertex in G is at least k
  - Proof idea: Note each vertex v defines a cut $(\{v\},\ V$ - $\{v\})$
- This implies that

$$\sum_V \text{degree}(v) = 2m \geq kn$$

$$m \geq \frac{kn}{2}$$

- Finally, $P(S_i) = \dfrac{k}{m} \leq \dfrac{2}{n}$

RCA Analysis Cont.

- $P(\neg S_1 \wedge \neg S_2) = P(\neg S_2 | \neg S_1) \cdot P(\neg S_1)$

$$\geq \left(1 - \frac{k}{\#\text{ of remaining edges}}\right) \cdot \left(1 - \frac{2}{n}\right)$$

- Note, contracted vertex creates a cut with a degree of at least k

$$\#\text{ of remaining edges} \geq \frac{k(n-1)}{2}$$

$$P(\neg S_1 \wedge \neg S_2) \geq \left(1 - \frac{2}{n-1}\right) \cdot \left(1 - \frac{2}{n}\right)$$

RCA Analysis Cont.

$$P(\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge ... \wedge S_{n-2}) =$$

$$P(\neg S_1) \cdot P(\neg S_2 | \neg S_1) \cdot P(\neg S_3 | \neg S_1 \wedge \neg S_2) \cdot ... \cdot P(\neg S_{n-2} | \neg S_1 \wedge ... \wedge \neg S_{n-3})$$

$$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) ... \left(1 - \frac{2}{n-(n-4)}\right)\left(1 - \frac{2}{n-(n-3)}\right)$$

$$= \left(\frac{n-2}{n}\right) \cdot \left(\frac{n-3}{n-1}\right) \cdot \left(\frac{n-4}{n-2}\right) \cdot ... \cdot \left(\frac{2}{4}\right) \cdot \left(\frac{1}{3}\right) \cdot = \frac{2}{n(n-1)} \geq \frac{1}{n^2}$$

- Low success probability but it's not trivial!
- Blind random cut picking has a success rate of $\frac{1}{2^n}$

RCA Solution

- **Solution**: run RCA many times to increase the probability!
- How many times? Assume N trials
- Let T_i = event that the MinCut is found on i^{th} try.
- All T_is are independent
- $P(\text{All } N \text{ trials fail}) = P(\neg T_1 \wedge \neg T_2 \wedge \ldots \wedge \neg T_N)$
  $= P(\neg T_1) P(\neg T_2) \ldots \mathrm{P}(\neg T_N)$
- Since the success probability is bounded above $1/n^2$
- Therefore, the failure probability is bounded below 1 - $(1/n^2)$

RCA Solution Cont.

- $P(\text{All } N \text{ trials fail}) = P(\neg T_1 \wedge \neg T_2 \wedge \ldots \wedge \neg T_N)$
  $= P(\neg T_1)P(\neg T_2) \ldots \mathrm{P}(\neg T_N) \leq (1 - 1/n^2)^N$
- Note: $1 + x \leq e^x$
- $P(\text{All } N \text{ trials fail}) \leq \left(e^{-1/n^2}\right)^N$
- If we do n^2 trials,

$$P(\text{All } n^2 \text{ trials fail}) \leq \left(e^{-1/n^2}\right)^{n^2} = \frac{1}{e}$$

- If we do n^2 ln n trials,

$$P(\text{All } n^2 \ln n \text{ trials fail}) \leq \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}$$

Multiple RCA Running time

- Running Time:
  - $O(\text{number of trials} \times \text{running single trial})$
    $= \Omega(\mathrm{n}^2 \times \mathrm{m})$
  - Slow but much better than doing brute force
- There are LOT of clever tricks to shave off time that gives us $O(n^2)$

Graph Search

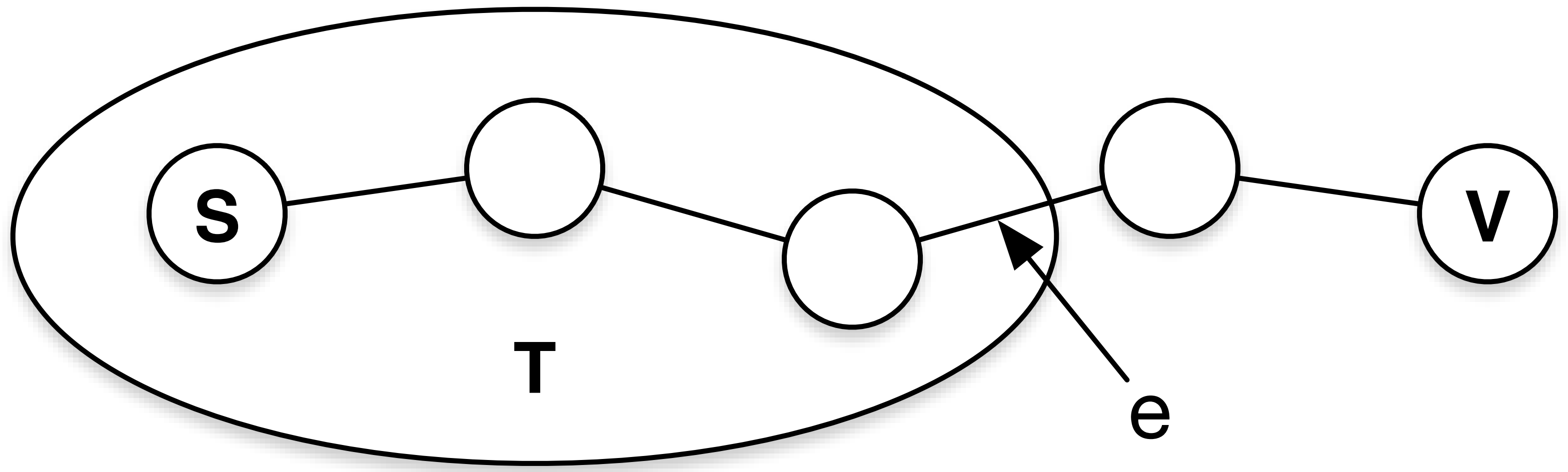
- **Goal**: Find all reachable vertices given a starting vertex
- Some definition:
  - **Connected Graph**: a graph there is a path between every pair of vertices
  - **Disconnected Graph**: a graph that is not connected

Algorithm Blueprint

```
Search(G, vs):
  X = V - vs  // unexplored
  T = {vs}    // explored
  while true:
    pick an edge (u,v) such that u ∈ T and v ∈ X
    T = T ∪ v
    X = X - v
    if no such edge exist halt
```

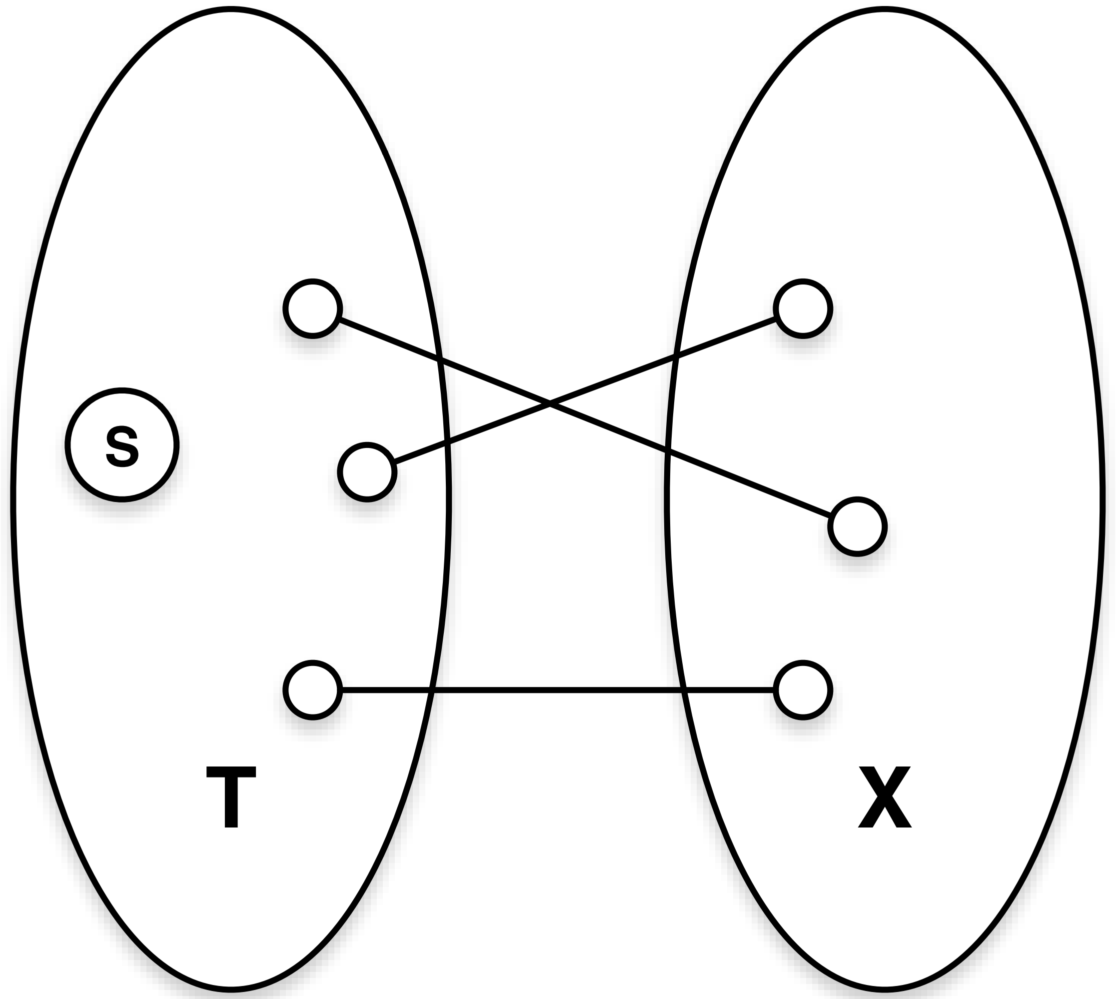
Proof of the algorithm

- **Claim**: Given the output of Search algorithm, T:
  - for all $v \in T \Leftrightarrow G$ has a path from s to v
- **Proof:**
  - $\Longrightarrow$ use induction (summary: in order for v to be included in T, we traversed to it)
  - $\Longleftarrow$ By contradiction. Suppose G has a path from s to v but $v \notin T$
  - But the algorithm then would have terminate without traversing into e
  - Contradiction. QED



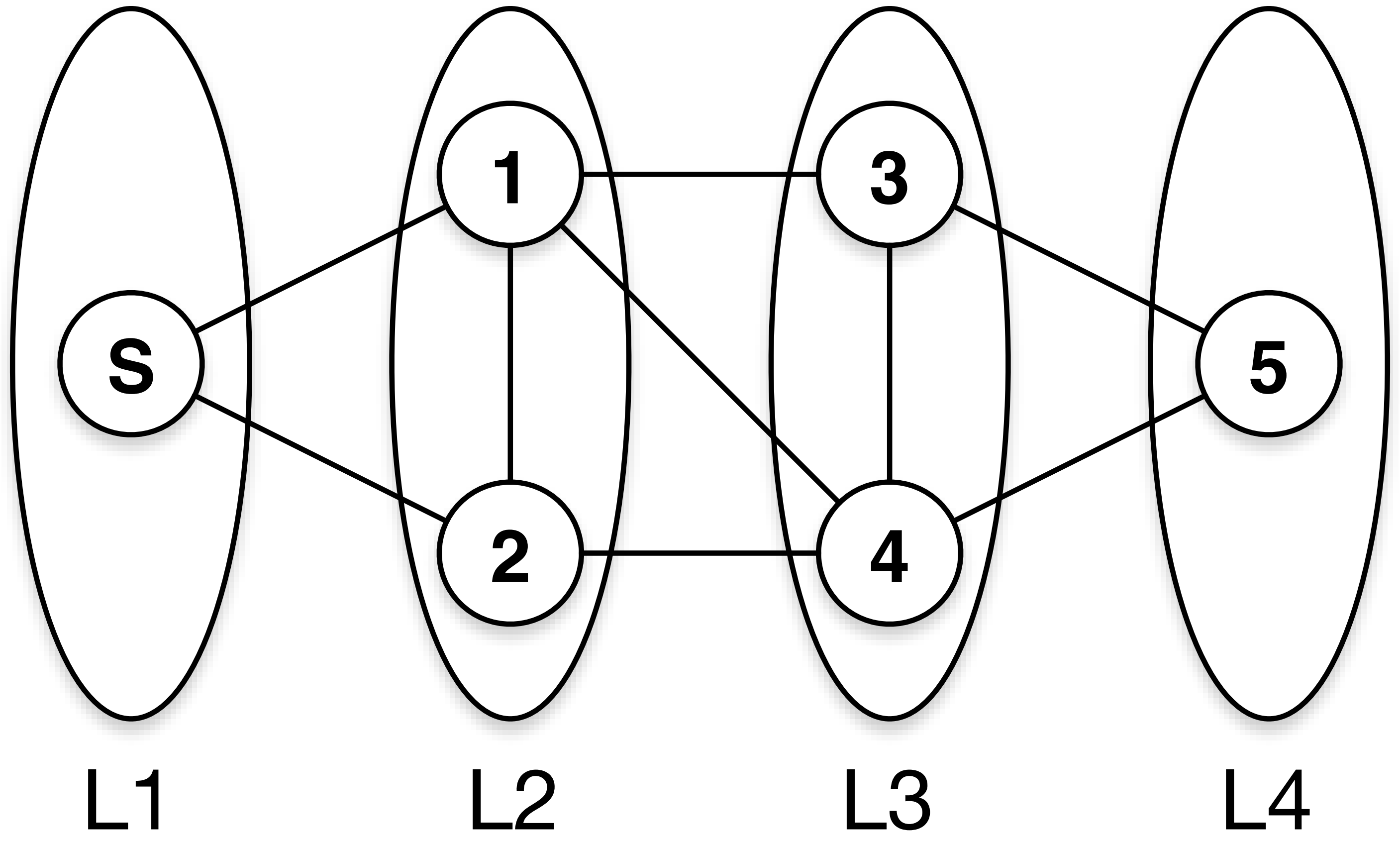
BFT vs DFT

- How do we pick the edge?
- **Breadth First Search** (BFT)
  - Search by "layer"
  - Can compute shortest path
  - Compute connected components of undirected graph
  - $O(m+n)$ using a queue
- **Depth First Search** (DFT)
  - Go deep as possible then backtrack
  - Gives you a topological ordering
  - Compute connected components of directed graph
  - $O(m+n)$ using recursion



BFT

```
BFS(G,vₛ):
  T = {vₛ}  // explored
  Q = {vₛ}
  X = V - vₛ  // unexplored
  while Q ≠ ∅:
    v = dequeue(Q)
    for each edge (v, u):
      if u ∈ X:
        T = T ∪ u
        enqueue(Q, u)
  return T
```



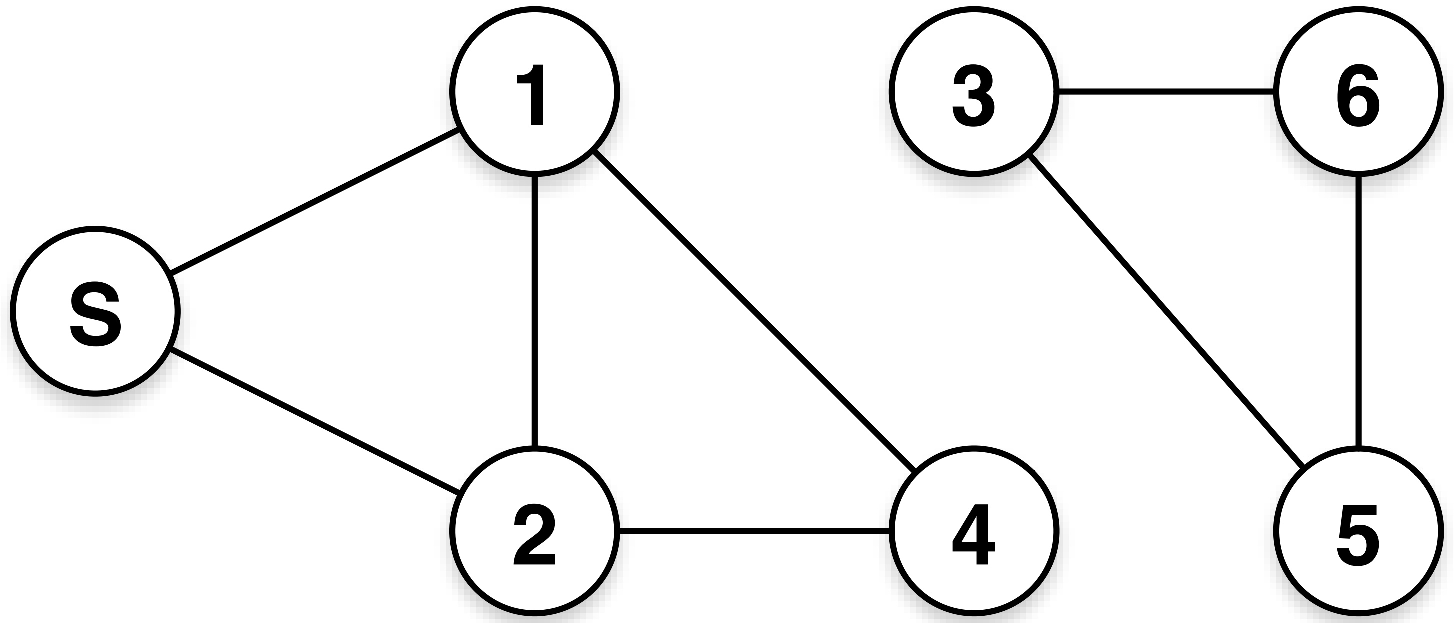
BFS Application - Shortest Path

```
ShortestPaths(G,v_s):
  T = {v_s}
  Q = {v_s}
  X = V - v_s
  D[v_s] = 0
  D[all other v] = infinity
  while Q ≠ ∅:
    v = dequeue(Q)
    for each edge (v, u):
      if u ∈ X:
        T = T ∪ u
        enqueue(Q, u)
        D[u] = D[v] + 1
  return T, D
```

- Compute Distance(v) = fewest # of edges on a path from s to v
- Add a Distance data structure to keep track of the hops
- Distance$(v) \Leftrightarrow i^{\text{th}}$ layer
- You can modify the data structure of Distance to keep track of the path

BFS Application - Connectivity

- **Problem**: Compute all connected components of a graph

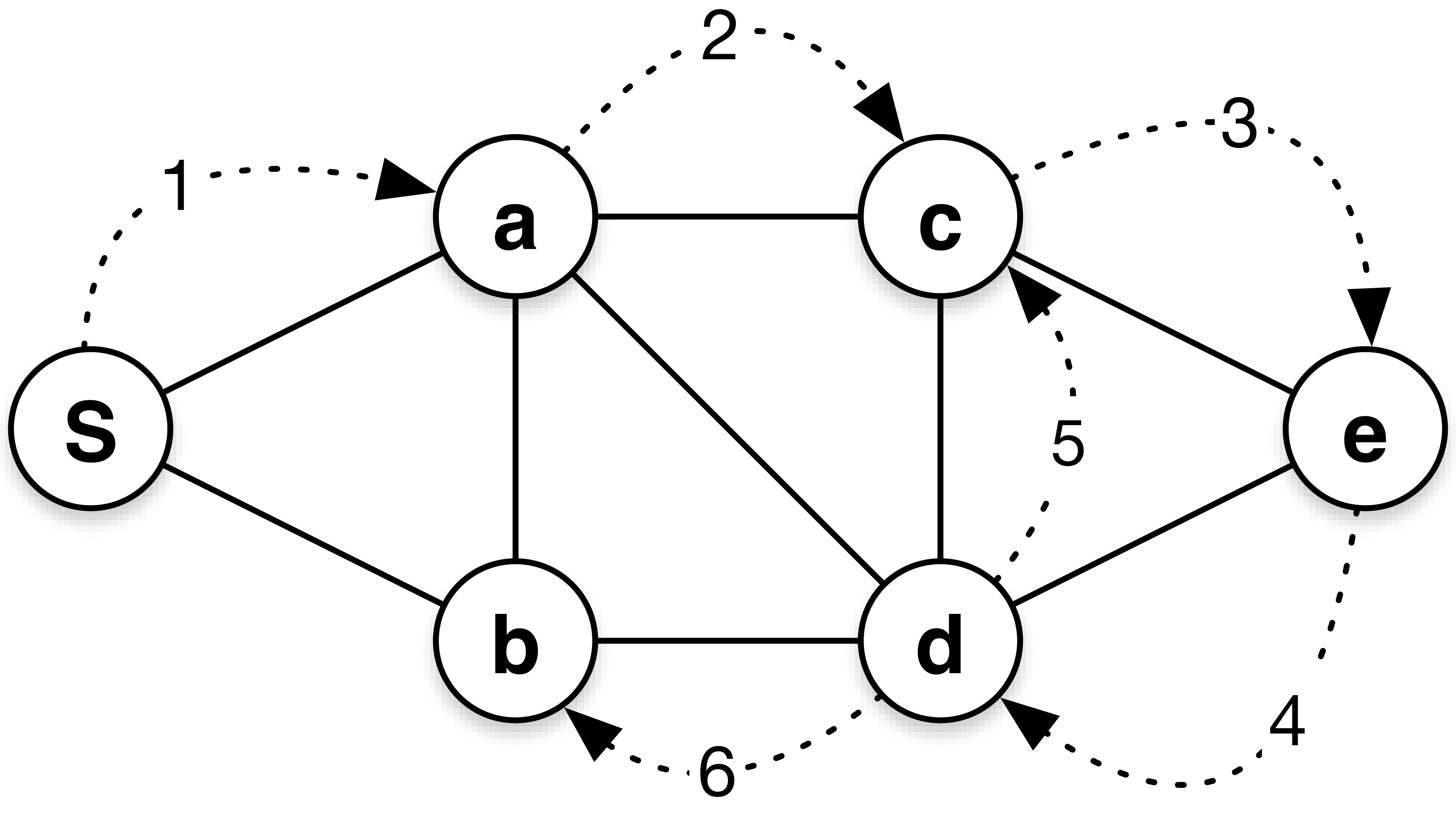



```
FindConnected(G):
  X = V  // unexplored
  j = 1
  for i = 1 to n:
    if i ∈ V:
      T_j = T_j + BFS(G, i)
      X = X - T_j
  return all T_j
```

DFS

```
DFS(G, X, vs):
  X = X + vs
  for each edge (vs, u):
    if u ∉ X:
      DFS(G, X, u)

DFS(G, [], vs)
```

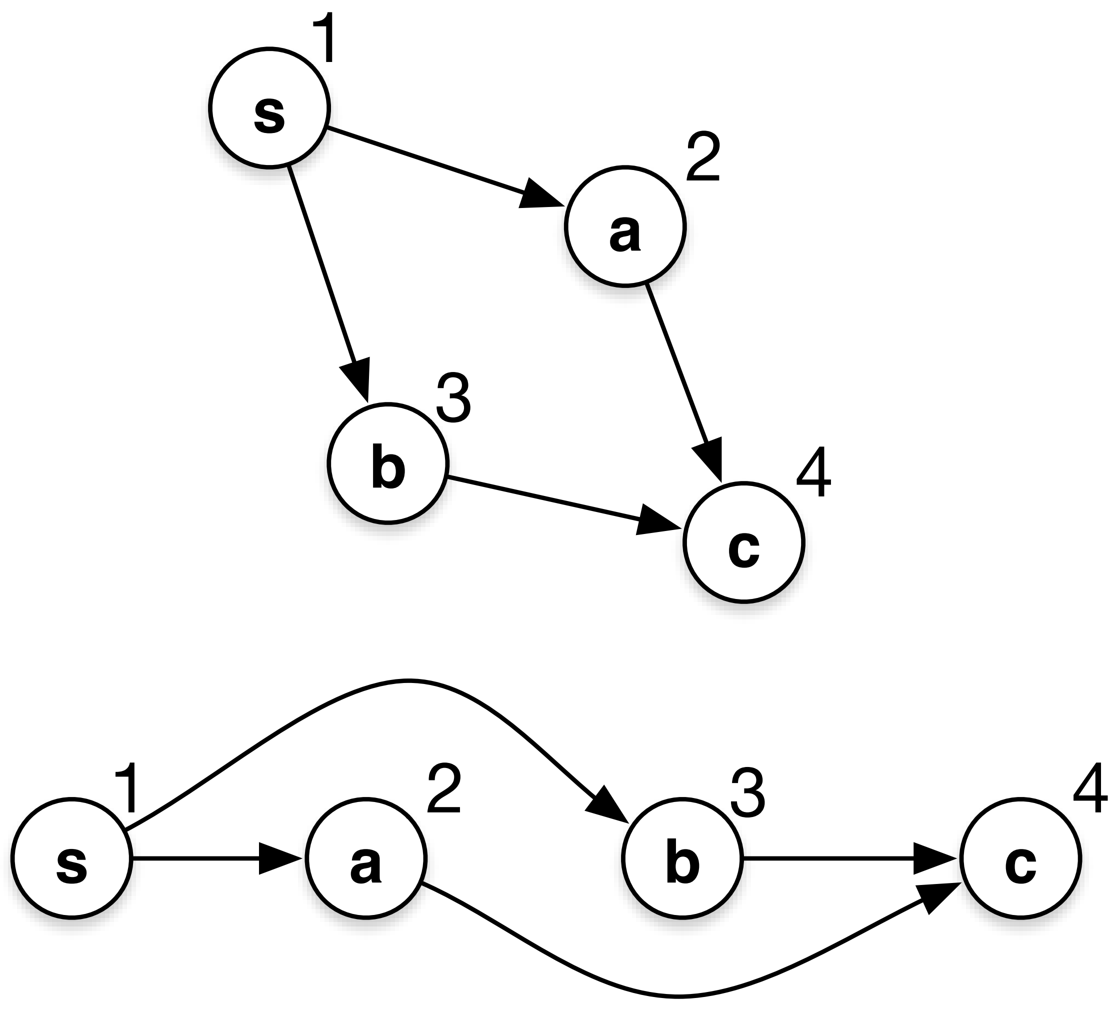


DFS Application - Topological Ordering

- **Problem**: Given a directed graph of $G(V,E)$,

$$\forall v \in G, \text{find } f(v) \text{ where } \forall (u,v) \in G \implies f(u) < f(v)$$

- **Application**: Find all the prerequisite for a given goal/destination



- **Note**: If g has directed cycle, then no topological ordering can be found
- **Theorem**: no directed cycle $\Rightarrow$ topological ordering can be found

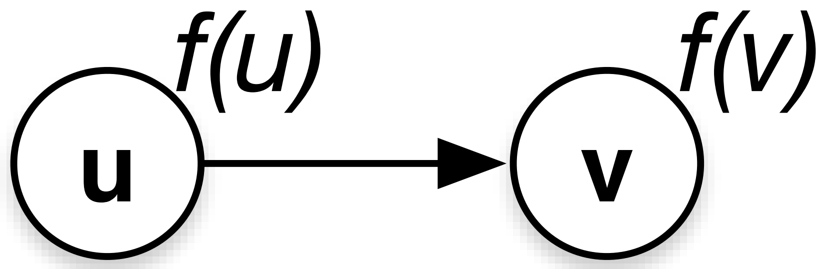
DFS Application - Topological Ordering

```
TopoOrder(G):
  X = {}
  global count = |V|
  f = {}
  for each v ∈ G:
    if v ∉ X:
      DFS'(G, X, v)
```

```
DFS'(G, X, v_s):
  X = X + v_s
  for each edge (v_s, u):
    if u ∉ X:
      DFS(G, X, u)
  f(v_s) = count
  count = count - 1
```

Topological Ordering Proof of Correctness

- **Claim**: The algorithm produces f values such that if (u,v) is an edge, then $f(u) < f(v)$

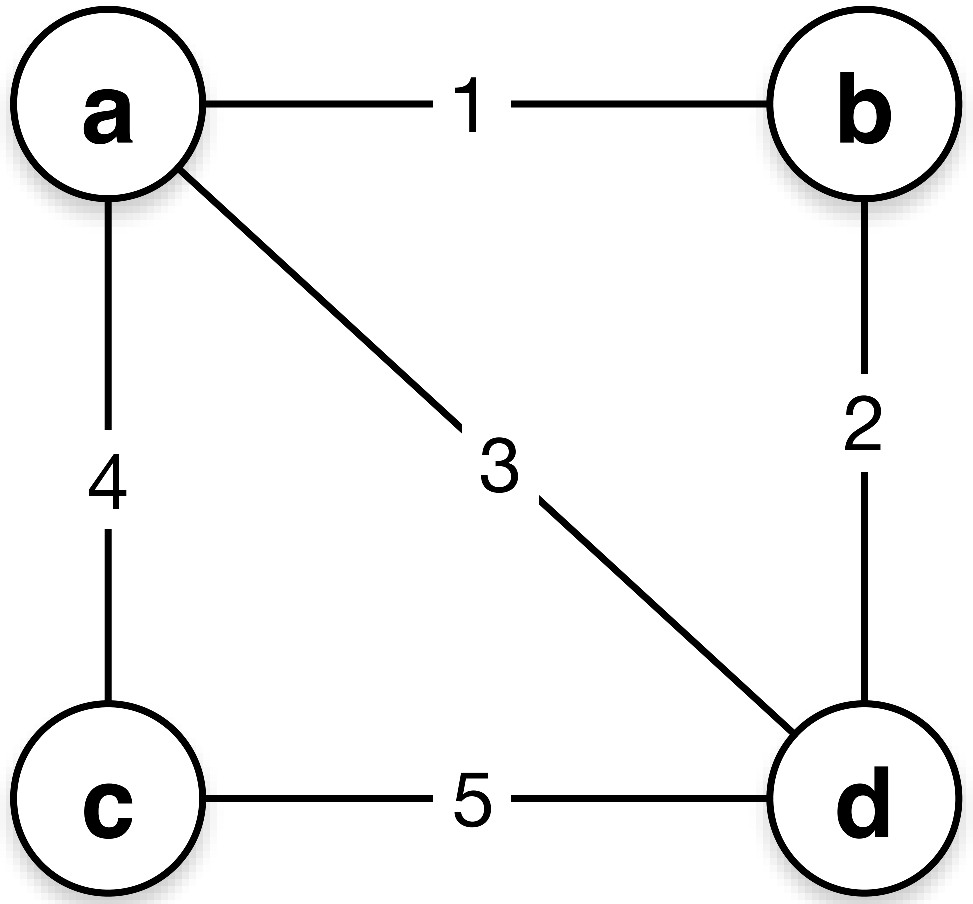


- **Case 1**: if u is visited by DFS before v, then DFS(G, X, v) call finishes before DFS(G, X, u) due to the recursive nature. Thus $f(u) < f(v)$

- **Case 2**: if v is visited by DFS before u, since there is no cycle the call stack that caused by DFS(G, X, v) will complete before calling DFS(G, X, u). Thus $f(u) < f(v)$

Minimum Spanning Tree

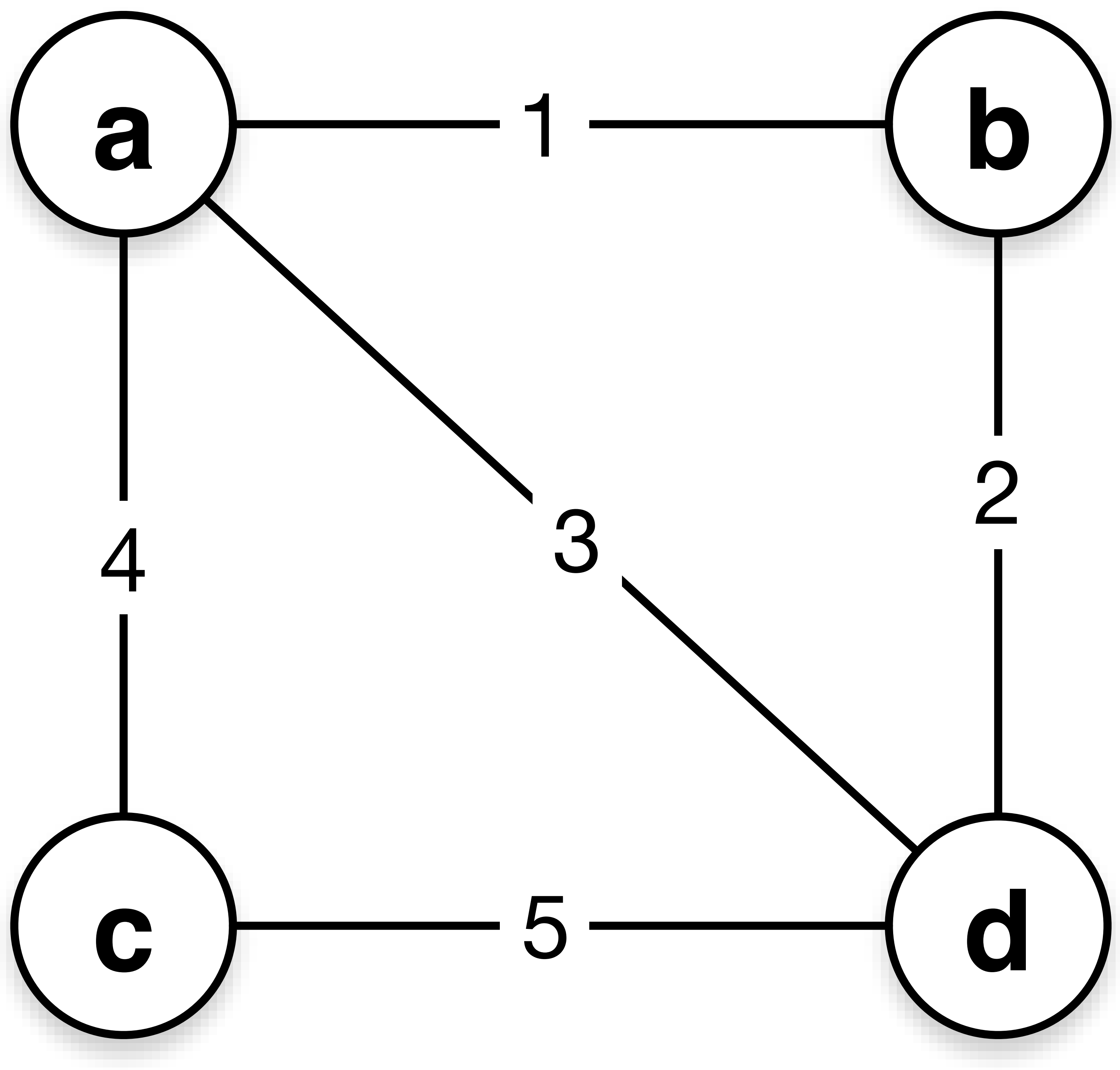
- **Problem**: Connect all vertices together deeply as possible
- **Input**: undirected graph $G=(V,E)$ and c_e for $e \in E$
- **Output**:
  - Minimum Cost Tree $T \subseteq E$ that spans V
  - No cycle
  - Connected
- **Assumptions**:
  - G is connected to begin with
  - C is unique



Prim's Algorithm

```
Prim(G):
  X = {s}  // s is chosen arbitrarily
  T = {}
  while X ≠ V:
    let e = (u,v)
      where e is the cheapest crossing edge of cut (X, V-X)
    T = T + e
    X = X + v
  return T
```

Prim's Algorithm



Proof of Correctness

- **Claim**: Prim's algorithm correctly computes an MST
- **Part I**: Prim's algorithm produces a spanning tree T*
  - Spanning = all vertices are included
  - Tree = no cycles
- **Part II**: T* is a MST
  - Minimal cost

Definitions to recall

- **Connected Graph**: a graph there is a path between every pair of vertices

Part I

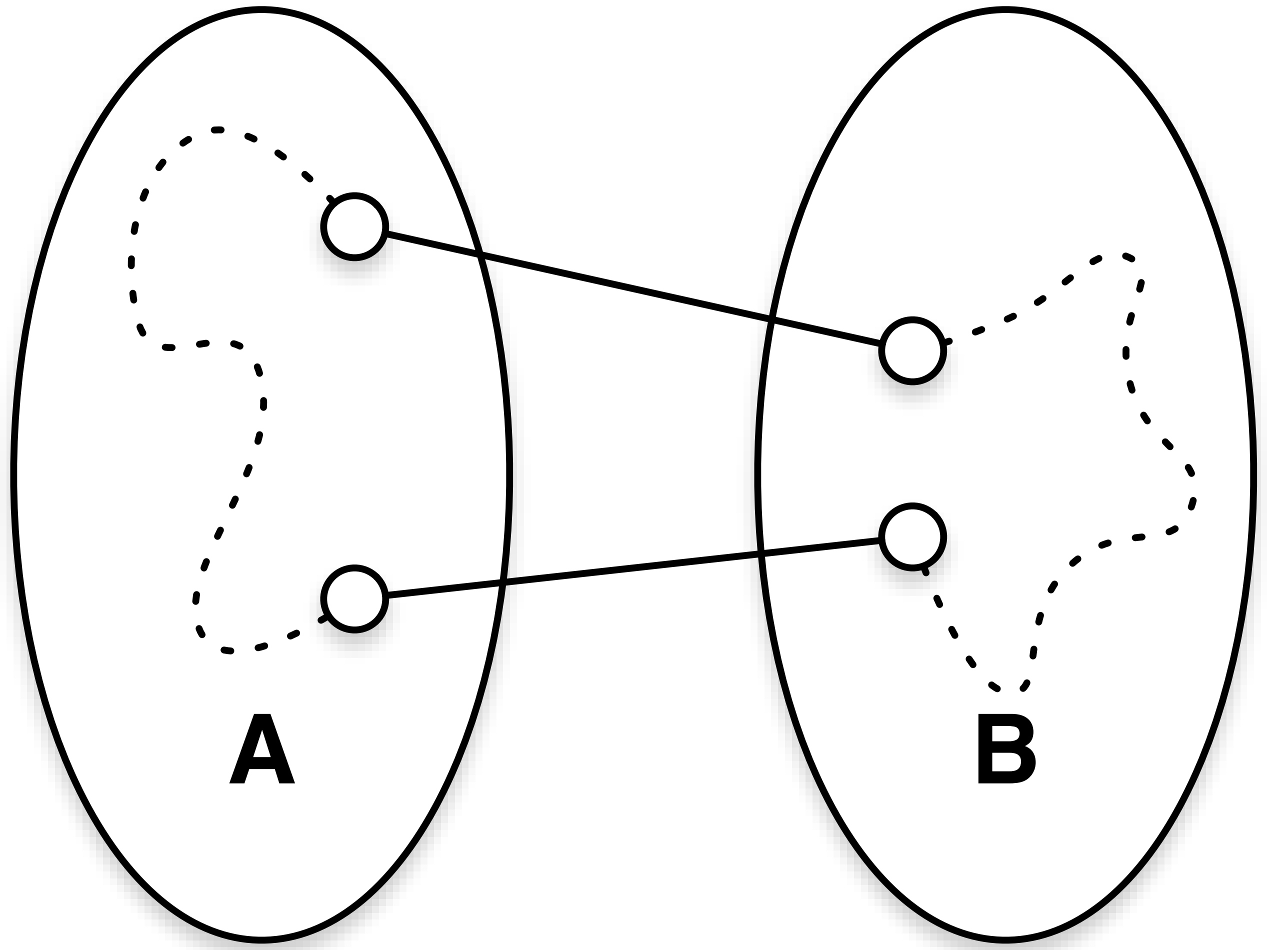
- **Empty cut lemma**:
  - Graph, G, is not connected $\Leftrightarrow$ $\exists$ cut (A,B) of G with no crossing edges
- **Proof of $\Leftarrow$**:
  - Assume $\exists$ cut (A,B) with no crossing edges
  - Pick any $u \in$ A and $v \in$ B
  - Since there are no crossing edges given cut (A,B), there is no edge (u,v)
  - $\therefore$ by the definition of connected graph, G is not connected

Part I cont.

- **Empty cut lemma**:
  - Graph, G, is not connected $\Leftrightarrow$ $\exists$ cut (A,B) of G with no crossing edges
- **Proof of $\Longrightarrow$**:
  - Assume Graph, G, is not connected
  - Pick any $u \in$ G
  - Create a cut of (A, B) such that
    - A = { all vertices reachable from u }
    - B = { all other vertices }
  - Then cut (A, B) has no crossing edges

Part 1 cont.

- **Double Crossing Lemma**:
  - Suppose cycle, $C \subseteq E$, has an edge crossing a cut (A,B) then there must exist another edge that crosses that cut.
  - Proof by contradiction
- **Lemma 3**:
  - If e is the only edge crossing a cut (A,B) then e is not part of any cycle.
  - Proof using Double Crossing Lemma



Part 1 cont.

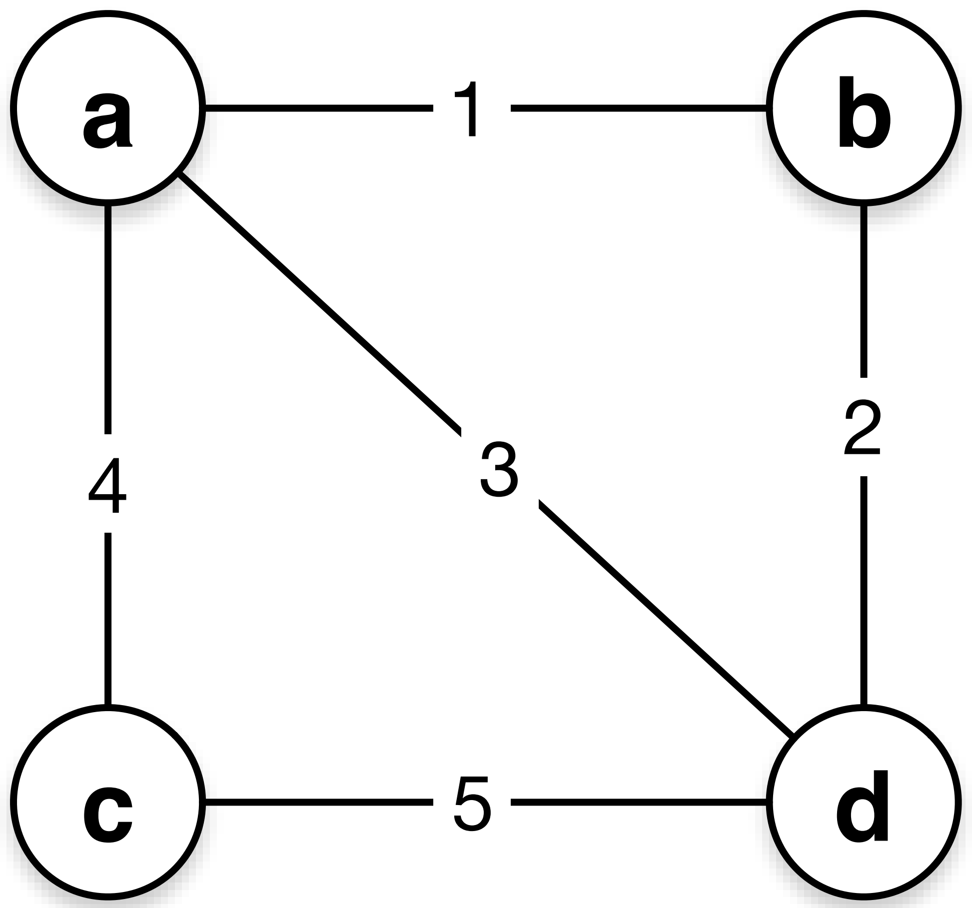
- **Part I**: Prim's algorithm produces a **spanning** tree
  - The algorithm chooses only edges stemming from X. Therefore the algorithm maintains the invariant of T spans X (meaning T includes all the vertices in X).
  - Algorithm must halt (X eventually = V),
    otherwise the cut of (X, V - X) will have no crossing edges. If the cut has no crossing edges, by Empty Cut Lemma, G must be disconnected. This is a contradiction,
    thus the algorithm halts with X = V.
  - $\therefore$ Prim's algorithm produces a T that spans V

Part I cont.

- **Part I**: Prim's algorithm produces a spanning **tree** (no cycle)
  - Whenever an edge, e, gets added to T, e is the first edge to cross the cut (X, V - X). By Lemma 3, e does not create a cycle
  - $\therefore$ Prim's algorithm produces a tree

Part II

- **Part II**: T* is a MST
  - Minimal cost
- **The Cut Property**
  - Given $e \in$ G, suppose $\exists$ cut (A,B) | e is the cheapest crossing edge, then $e \in$ MST(G)



Part II

- **Claim**: Cut Property $\Longrightarrow$ Prim's Algorithm produces MST(G)
- Every edge e $\in$ T* is chosen as the cheapest crossing edge of cut (X, V - X).
- By the cut property, T* $\subseteq$ MST(G).
- From Part I, since T* is a spanning tree of G, T* = MST(G). QED

Proof of cut property

- **The Cut Property**
  - Given $e \in$ G, suppose $\exists$ cut (A,B) $|$ e is the cheapest crossing edge, then $e \in$ MST(G)
- By contradiction.
  - Suppose e is the cheapest crossing edge of a cut (A,B) of G, yet $e \notin$ MST(G)

Proof of cut property

- Suppose e is the cheapest crossing edge of a cut (A,B) of G, yet $e \notin$ MST(G), T*
- Since T* doesn't include e,
  then it must include another edge, $f \mid c_f > c_e$, $f \in$ T* and cuts (A,B) and is part of T*. Otherwise, T* is not connected.
- (We want to use a swap method here, but if e and f are part of a cycle, we can't just swap.)
- Since $f \in$ T*, and T* is a spanning tree, T* $+$ e would create a cycle.

Proof of cut property

- By Double Cut Lemma, there must exist another edge, $e' \mid c_{e'} > c_e$ and creates the cycle with e
- (Now we can swap e with e')
- Note, T = T* + e - e' is a spanning tree.
- But cost of T < T* which is a constradiction. QED.

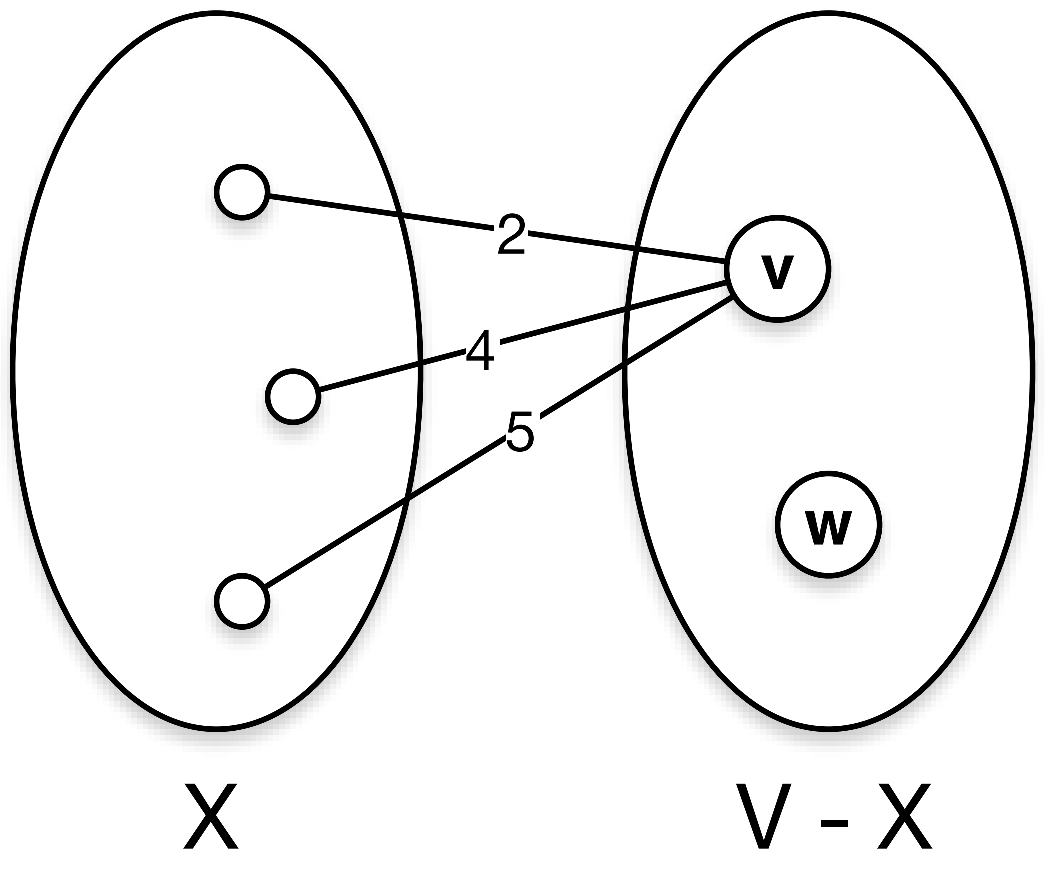
Prim's Algorithm Running Time

```
Prim(G):
  X = {s}  // s is chosen arbitrarily
  T = {}
  while X ≠ V:
    let e = (u,v)
      where e is the cheapest crossing edge of cut (X, V-X)
    T = T + e
    X = X + v
  return T
```

Running time of Prim's

- O(n $\times$ m) - Literal implementation
- We can use MinHeap where most of its operations are in $O(\log\ n)$
- Use Heap to store edges $\Longrightarrow O(m \log\ n)$
- But faster to store vertices in Heap with following invariant
  - Invariant #1: Elements in heap $= v \in$ V - X
  - Invariant #2: for $v \in$ V - X:

$$\text{key}_v = \text{cheapest edge } (u,v)$$

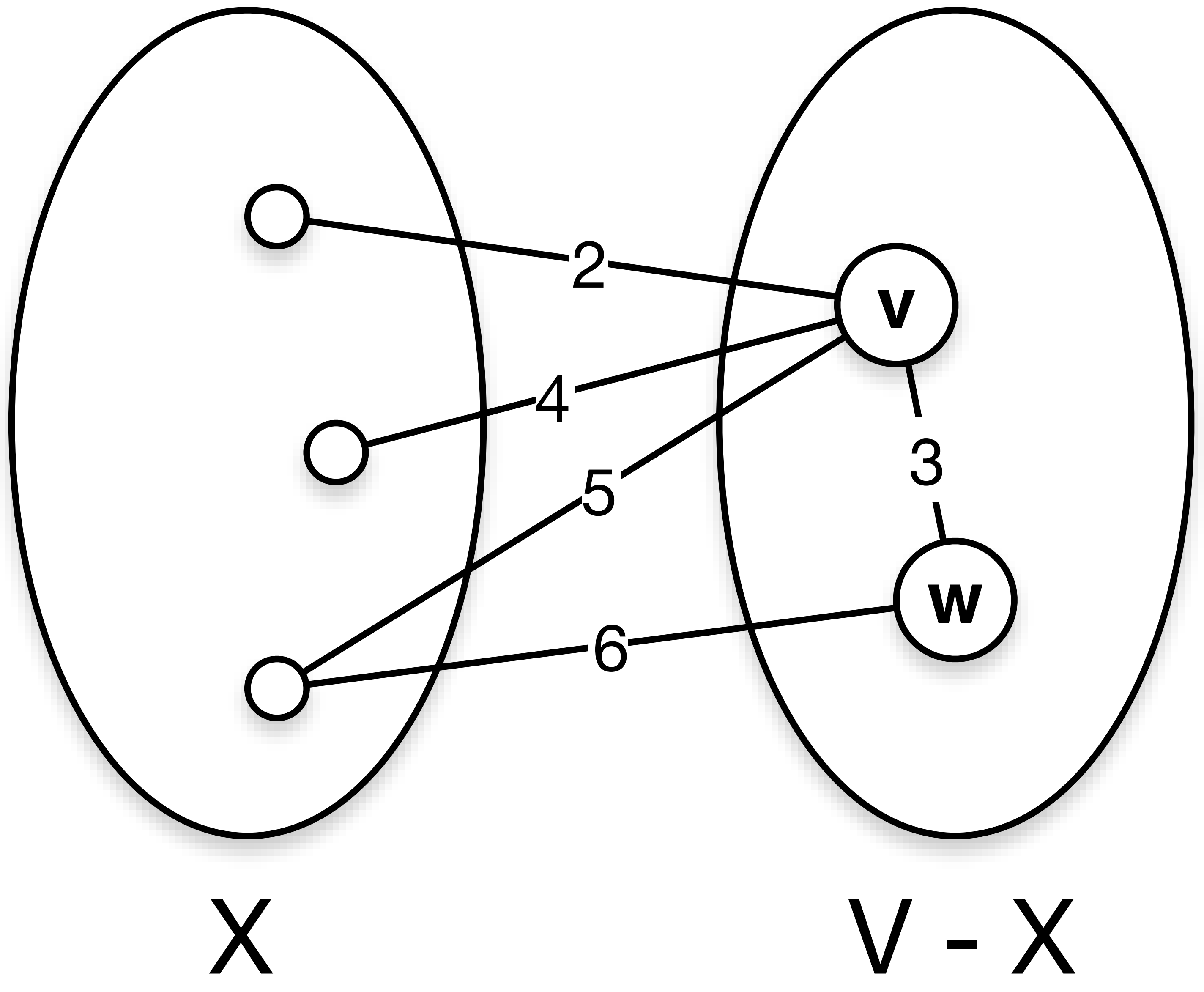


Prim's with MinHeap of Vertices

- **Preprocessing** (Initialization) of heap:
  - Initial cut = ({s}, V - {s})
  - Find all the edges that cross that cut and create heap:
    - O(m + n log n) or O(m + n) if you use heapify
    - Since m ≥ n - 1, O(m + n) = O(n)

Prim's with MinHeap of Vertices

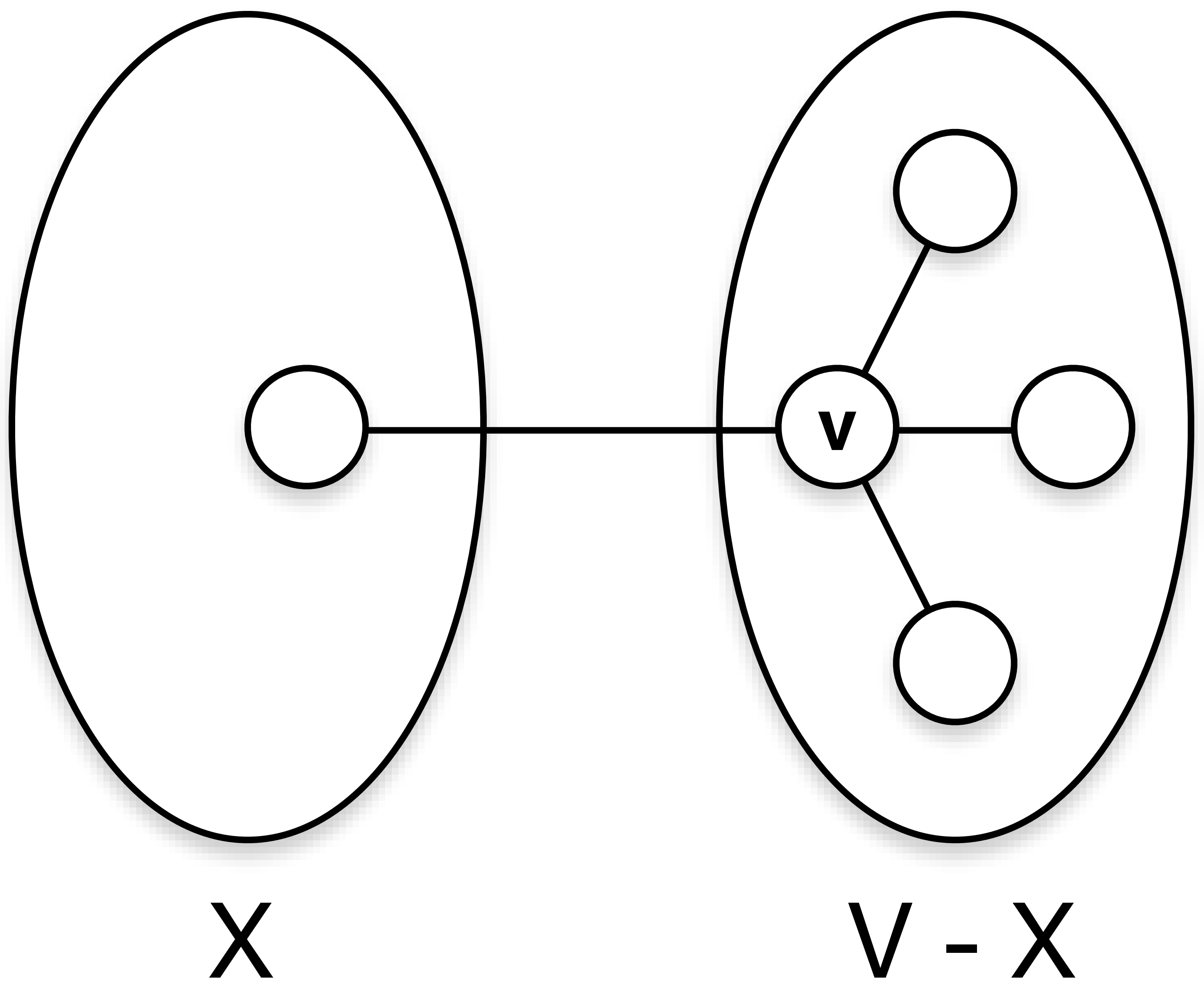
- **During execution**:
  - To pick the cheapest edge, single ExtractMin call to heap will give you the right edge.
  - $O(\log\ n)$
- **Keeping the invariant**
  after ExtractMin has been called



Prim's with MinHeap of Vertices

- **Keeping the invariant** after ExtractMin has been called
- Extra metadata needed to delete

```
When v is added to X:
  for each edge (v,w) ∈ E:
    if w ∈ V - X:
      delete w from heap
      key[w] = min{key[w], c_{v,w}}
      insert w into heap
```



Final Running Time of Prim

- Preprocessing - O(n)
- One ExtractMin called per each vertex - O(n log n)
- Each edge triggers at most one delete/insert - O(m log n)
- Entire running time = O(m log n)