Regression II

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$$Y_i = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_p X_{pi} + \epsilon_i$$

Here

- β_0 is the mean of Y when all the Xs are equal to zero
- β_i is the change in the mean of Y when we increase X_i by one while holding all the other Xs fixed

In matrix formulation,

$$\mathbf{Y} = X\beta + \epsilon$$

where

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 1 & x_{11} & x_{21} & \ldots & x_{p1} \\ 1 & x_{12} & x_{22} & \ldots & x_{p2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} & \ldots & x_{pn} \end{pmatrix}$$

Example: antique grandfather clocks

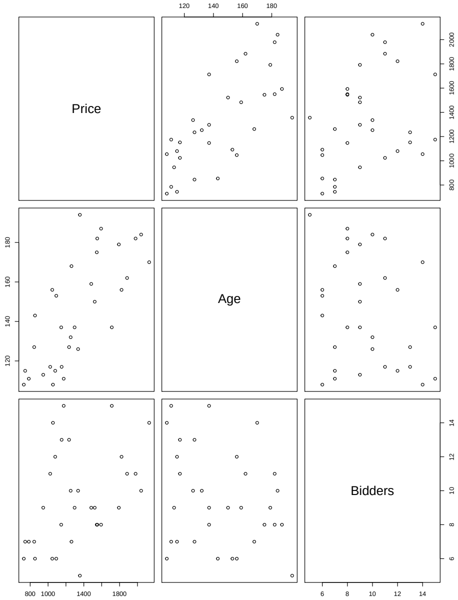
- The data give the selling price at auction of 32 antique grandfather clocks. Also recorded is the age of the clock and the number of people who made a bid.
- The variables are
  1. Age : Age of the clock (years)
  2. Bidders: Number of individuals participating in the bidding
  3. Price: Selling price (pounds sterling)

	Age	Bidders	Price
1	127	13	1235
2	115	12	1080
3	127	7	845
4	150	9	1522
5	156	6	1047
6	182	11	1979
7	156	12	1822
8	132	10	1253
9	137	9	1297

Example: antique grandfather clocks

10	113	9	946
11	137	15	1713
12	117	11	1024
13	137	8	1147
14	153	6	1092
15	117	13	1152
16	126	10	1336
17	170	14	2131
18	182	8	1550
19	162	11	1884
20	184	10	2041
21	143	6	854
22	159	9	1483
23	108	14	1055
24	175	8	1545
25	108	6	729
26	179	9	1792
27	111	15	1175
28	187	8	1593
29	111	7	785
30	115	7	744
31	194	5	1356
32	168	7	1262

Example (Clocks continued)



To estimate β we minimize

$$\sum_{i=1}^{n} \epsilon_i^2 = \epsilon^T \epsilon = (\mathbf{Y} - X\beta)^T (\mathbf{Y} - X\beta)$$

The solution is

$$\mathbf{b} = (X^T X)^{-1} X^T \mathbf{Y}$$

Under the assumption we made before

$$\mathbf{b} \sim N(\beta, \sigma^2 (X^T X)^{-1})$$

This implies in particular that

- $E(\mathbf{b}) = \beta$, that is, $\mathbf{b}$ is an unbiased estimator of β.
- $Var(b) = \sigma^2 (X^T X)^{-1}$ and is estimated by

$$s_b = MSE(X^T X)^{-1}$$

- An estimate of the variance of β_i is

$$s_{b_i} = MSE(X^T X)_{ii}^{-1}$$

where $(X^T X)_{ii}^{-1}$ is the ith diagonal element of $(X^T X)^{-1}$.

```
> attach(data)
> fit<-lm(Price~ Age+Bidders)
> fit

Call:
lm(formula = Price ~ Age + Bidders)

Coefficients:
(Intercept)          Age      Bidders
   -1336.72        12.74        85.82
```

The regression equation is

$$\widehat{Price} = -1336.72 + 12.74\text{Age} + 85.82\text{Bidders}$$

- A $100(1-\alpha)\%$ confidence interval for β_i is

$$b_i \pm t_{n-1}(\alpha/2)s_{b_i}$$

  The interpretation of this confidence interval is: We are $100(1-\alpha)\%$ confident that when we increase X_i by one unit while holding all the other Xs fixed, the average, Y changes by an amount in this interval.

```
> confint(fit)
                  2.5 %     97.5 %
(Intercept) -1691.27514 -982.16896
Age            10.89062   14.58177
Bidders        68.00986  103.62040
```

We are 95% that when we increase age by one year while holding the number of bidders fixed, on average the price goes by an amount between 10.89 and 12.58 pounds sterling.

To test $H_0 : \beta_i = \beta_{i0}$ against $H_a : \beta_i \neq \beta_{i0}$, the test statistic is

$$t = \frac{b_i - \beta_{i0}}{s_{b_i}}$$

and we reject H_0 if

$$|t| > t_{n-p-1}(\alpha/2) \quad \text{or if} \quad p - value < \alpha$$

```
 > summary(fit)
lm(formula = Price ~ Age + Bidders)

Coefficients:
                Estimate       Std. Error    t value     Pr(>|t|)
(Intercept)    -1336.7221      173.3561     -7.711      1.67e-08 ***
Age               12.7362        0.9024     14.114      1.60e-14 ***
Bidders           85.8151        8.7058      9.857      9.14e-11 ***

Residual standard error: 133.1 on 29 degrees of freedom
Multiple R-squared:  0.8927,Adjusted R-squared:  0.8853
F-statistic: 120.7 on 2 and 29 DF,  p-value: 8.769e-15
```

p-values very small we reject $H_0 : \beta_i = 0$ against $H_a : \beta_i \neq 0$

- The ANOVA table is given by

Source	df	SS	MS	F
Model	p	SSR	MSR=SSR/p	MSR/MSE
Error	n-p-1	SSE	MSE=SSE/(n-p-1)	
Total	n-1	SST		

- The coefficient of determination is

$$R^2 = \frac{SSR}{SST}$$

- Adjusted

$$R^2_{adj} = 1 - \frac{n-1}{n-p-1}\frac{SSE}{SST}$$

  can be used for model selection
- MSE is an estimate of σ^2

- To test $H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0$ against H_a : at least one of these βs is not zero, we reject H_0 is $F > F(1-\alpha, p, n-p-1)$ or if $p - value < \alpha$.
- The test statistic is given by

$$F = \frac{(SSE_R - SSE_F)/(df_R - df_f)}{SSE_F/df_F}$$

  and we reject H_0 if

$$F > F(1-\alpha, df_R - df_F, df_F)$$

  or if $p - value < \alpha$.
- In the example, to test $H_0 : \beta_1 = \beta_2 = 0$ against H_a : at least one of them is not equal to zero, we

```
F-statistic: 120.7 on 2 and 29 DF,  p-value: 8.769e-15
```

  Since the p-value is very small we reject H_0

Partial F test

- Suppose we want to test $H_0 : \beta_1 = \beta_2 = \ldots = \beta_k = 0, k < p$ against H_a : Not H_0.
- In this case we have two models:
  - a reduced model(the model in which $\beta_1 = \beta_2 = \ldots = \beta_k = 0$) and
  - a full model in which we have all the βs

Lack of fit test

- we have up to this point assumed that $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \ldots, n, \epsilon_i$s are iid $N(0, \sigma^2)$ and made inference about β_0 and β_1
- The goal of the lack of fit test is to test a specific type of regression function fits the data.
- The lack fit test assumes that the observations for a given x are
  1. independent of each other
  2. normally distributed
  3. the distribution of y given x have the same variance σ^2.
- We want to test

$$\begin{aligned} H_0 &: \mu_x = \beta_0 + \beta_1 x \\ H_1 &: \mu_x \neq \beta_0 + \beta_1 x \end{aligned}$$

- To carry out a lack of fit test requires repeat observations at one or more x levels data

x	y	mean under H_0	mean under H_a
x_1	$y_{11}, y_{12}, \ldots, y_{1n_1}$	$\beta_0 + \beta_1 x_1$	μ_{x_1}
x_2	$y_{21}, y_{22}, \ldots, y_{2n_2}$	$\beta_0 + \beta_1 x_2$	μ_{x_2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_c	$y_{c1}, y_{c2}, \ldots, y_{cn_c}$	$\beta_0 + \beta_1 x_c$	μ_{x_2}

- Under H_a, the model is $y_{ij} = \mu_i + \epsilon_{ij}, i = 1, 2, \ldots, c, j = 1, 2, \ldots, n_i$
- The estimate of of μ_i is $\bar{y}_i = \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i}$
- $SSE_R = \sum_{i=1}^{c} \sum_{j=1}^{n_i} (y_{ij} - \hat{y}_{ij})^2$ where $\hat{y}_{ij} = b_0 + b_1 x_i$.
- $SSE_F = \sum_{i=1}^{c} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$
- The partial F-test is

$$F = \frac{SSE_R - SSE_F}{df_R - df_F} \div \frac{SSE_F}{df_F}$$

- Reject H_0 is $F > F(1 - \alpha, df_R - df_F, df_F)$ or if $p - value < \alpha$
- $df_F = \sum_{i=1}^{c} n_i - c$ and $df_R = \sum_{i=1}^{c} n_i - 2$
- The difference $SSE_R - SSE_F$ is called the lack of fit sum of squares and is denoted by SSLF
- The test statistic is sometimes expressed as

$$F = \frac{MSLF}{MSE}$$

```
   x      y
0.01  127.6
0.48  124.0
0.71  110.8
0.95  103.9
1.19  101.5
0.01  130.1
0.48  122.0
1.44   92.3
0.71  113.1
1.96   83.7
0.01  128.0
1.44   91.4
1.96   86.2

> Reduced <- lm(y ~ x, data=corrosion)
> summary(Reduced)
Coefficients:
            Estimate  Std. Error t value   Pr(>|t|)
(Intercept)   129.79  1.40         92.5     < 2e-16
  x           -24.02  1.28        -18.8     1.1e-09
Residual standard error: 3.06 on 11 degrees of freedom
Multiple R-Squared: 0.97, Adjusted R-squared: 0.967
F-statistic: 352 on 1 and 11 degrees of freedom, p-value: 1.06e-09
```

```
>Full <- lm(y~factor(x))
> summary(lm(Full)
Call:
lm(formula = y ~ factor(x))

Residuals:
    Min      1Q  Median      3Q     Max
-1.2500 -0.9667  0.0000  1.0000  1.5333

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)    128.567      0.809 158.914 4.19e-12 ***
factor(x)0.48   -5.567      1.279  -4.352  0.00481 **
factor(x)0.71  -16.617      1.279 -12.990 1.28e-05 ***
factor(x)0.95  -24.667      1.618 -15.245 5.03e-06 ***
factor(x)1.19  -27.067      1.618 -16.728 2.91e-06 ***
factor(x)1.44  -36.717      1.279 -28.703 1.18e-07 ***
factor(x)1.96  -43.617      1.279 -34.097 4.24e-08 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.401 on 6 degrees of freedom
Multiple R-squared:  0.9965,Adjusted R-squared:  0.9931
F-statistic: 287.3 on 6 and 6 DF,  p-value: 4.152e-07
```

To test for Lack of fit, we use

```
> anova(Reduced,Full)
Analysis of Variance Table
Model 1: y ~ x
Model 2: y ~ factor(x)
Res.Df Res.Sum Sq Df Sum Sq F value Pr(>F)
1   11  102.9
2    6   11.8     5    91.1  9.28   0.0086
```

$$SSE_R = 102.9, SSE_F = 11.8, df_R = 11, df_F = 6.$$

$$F = \frac{102.9 - 11.8}{11 - 6} \div \frac{11.8}{6} = \frac{91.1}{5} \div \frac{11.8}{6} = 9.28.$$

The p-value is 0.0086. Reject $H_0 : \mu_x = \beta_0 + \beta_1 x$.

- Y = volume of sales in July of some electronic store
- x = number of households in the location
- Location of the store = $\begin{cases} \text{Mall} \\ \text{Downtown} \\ \text{Street} \end{cases}$

number of household	location	sales
161	street	157.27
99	street	93.28
135	street	136.81
120	street	123.79
164	street	153.51
221	mall	241.74
179	mall	201.54
204	mall	206.71
214	mall	229.78
101	mall	135.22
231	downtown	224.71
206	downtown	195.29
248	downtown	242.16
107	downtown	115.21
205	downtown	197.82

```
> fit
Call:
lm(formula = sales ~ nhousehold + factor(location))

Coefficients:
          (Intercept)             nhousehold    factor(location)mall
              21.8415                 0.8686                 21.5100
factor(location)street
              -6.8638
```

```
> summary(fit)

Call:
lm(formula = sales ~ nhousehold + factor(location))

Residuals:
    Min      1Q  Median      3Q     Max
-13.834  -2.999   2.225   4.357   6.431

Coefficients:
                       Estimate Std. Error t value Pr(>|t|)
(Intercept)            21.84147    8.55848   2.552 0.026898 *
nhousehold              0.86859    0.04049  21.452 2.52e-10 ***
factor(location)mall   21.50998    4.06509   5.291 0.000256 ***
factor(location)street -6.86378    4.77048  -1.439 0.178047

Residual standard error: 6.349 on 11 degrees of freedom
Multiple R-squared:  0.9868,Adjusted R-squared:  0.9833
F-statistic: 275.1 on 3 and 11 DF,  p-value: 1.268e-10
```

```
> confint(fit)
                              2.5 %     97.5 %
(Intercept)               3.0043933 40.6785468
nhousehold                0.7794707  0.9577061
factor(location)mall     12.5627722 30.4571864
factor(location)street  -17.3635248  3.6359712
```

- Multicollinearity: it exists when the explanatory variables are linearly dependent.
- We use the variance inflation factor (VIF) to check whether or not multicollinearity exists
- The VIF for variable x_j is

$$VIF = \frac{1}{1 - R_j^2}$$

  where R_j^2 is the coefficient of determination when x_j is regressed on the other x_is
- As a percentage, R_j^2 is the percentage variability in x_j explained by the other x_is
- It turns out that $SE(b_j) \propto \sqrt{VIF}$, so when R_j^2 is high, the $SE(b_j)$ is also large and that leads to failing to reject $H_0 : \beta_j = 0$

Outliers and Influential Points

- An outlier is a data point whose response y does not follow the general trend of the rest of the data.
- A data point has high leverage if it has "extreme" predictor x values. With a single predictor, an extreme x value is simply one that is particularly high or low. With multiple predictors, extreme x values may be particularly high or low for one or more predictors, or may be "unusual" combinations of predictor values (e.g., with two predictors that are positively correlated, an unusual combination of predictor values might be a high value of one predictor paired with a low value of the other predictor).
- A data point is influential if it unduly influences any part of a regression analysis, such as the predicted responses, the estimated slope coefficients, or the hypothesis test results.
- Outliers and high leverage data points have the potential to be influential, but we generally have to investigate further to determine whether or not they are actually influential
- One advantage of the case in which we have only one predictor is that we can look at simple scatter plots in order to identify any outliers and influential data points.

- The hat matrix is
$$H = X(X^TX)^{-1}X^T$$
Note that $\hat{y} = H^Ty$ so that
$$\begin{aligned} \hat{y}_1 &= h_{11}y_1 + h_{12}y_2 + \ldots + h_{in}y_n \\ \hat{y}_2 &= h_{21}y_1 + h_{22}y_2 + \ldots + h_{2n}y_n \\ \vdots \quad &\vdots \quad \vdots \\ \hat{y}_n &= h_{n1}y_1 + h_{n2}y_2 + \ldots + h_{nn}y_n \end{aligned}$$
- h_{ii} is the ith element of the diagonal of H. It measures the distance of the x values of the ith case from the center of the experimental region (ignores the response)
- The leverage h_{ii}, quantifies the influence that the observed response y_i has on its predicted value $\hat{y}_i$. That is, if h_{ii} is small, then the observed response y_i plays only a small role in the value of the predicted response $\hat{y}_i$.
- On the other hand, if h_{ii} is large, then the observed response y_i plays a large role in the value of the predicted response $\hat{y}_i$. It's for this reason that the h_{ii} are called the leverages.
- In simple linear regression
$$h_{ii} = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Some important properties of the leverages:

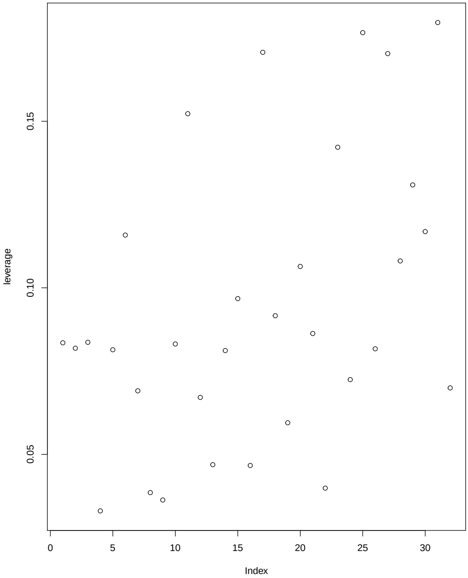
1. The leverage h_{ii} is a measure of the distance between the x value for the ith data point and the mean of the x values for all n data points.
2. The leverage h_{ii} is a number between 0 and 1, inclusive.
3. The sum of the h_{ii} equals p, the number of parameters (regression coefficients including the intercept).

The first bullet indicates that the leverage h_{ii} quantifies how far away the ith x value is from the rest of the x values. If the ith x value is far away, the leverage h_{ii} will be large; and otherwise not.

- The great thing about leverages is that they can help us identify x values that are extreme and therefore potentially influential on our regression analysis.
- How? All we need to do is determine when a leverage value should be considered large. A common rule is to flag any observation whose leverage value, h_{ii}, satisfies

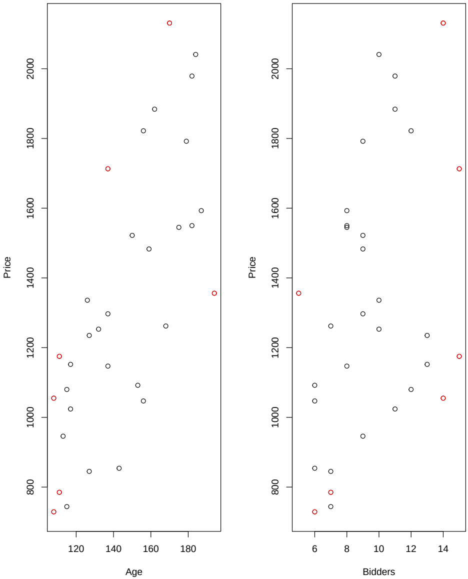
$$h_{ii} > \frac{2p}{n}$$

Example (Clocks continued)



```
> data[leverage>2*2/32,]
   Age Bidders Price
11 137      15  1713
17 170      14  2131
23 108      14  1055
25 108       6   729
27 111      15  1175
29 111       7   785
31 194       5  1356
```

Example (Clocks continued)



- Residuals: The ith residual is defined as $e_i = y_i - \hat{y}_i, i = 1, 2, \ldots, n$.
- Studentized residuals (or internally studentized residuals) are defined for each observation, $i = 1, ..., n$ as an ordinary residual divided by an estimate of its standard deviation:
$$r_i = \frac{e_i}{\sqrt{MSE(1 - h_{ii})}}$$
- An observation with an internally studentized residual that is larger than 3 (in absolute value) is generally deemed an outlier. [Sometimes, the term "outlier" is reserved for an observation with an externally studentized residual that is larger than 3 in absolute value?we consider externally studentized residuals in the next section.]

```
> r = rstudent(fit)
> data[abs(r)>3,]
[1] Age      Bidders Price
<0 rows> (or 0-length row.names)
> plot(r)
```

Example (Clocks continued)