LOGISTIC REGRESSION

- Simple linear regression: relationship between numerical response and a numerical or categorical predictor
- Multiple regression: relationship between numerical response and multiple numerical and or categorical predictors
- What we have not seen is what to when the response is categorical
- Odds: Odds are another way of quantifying the probability of an event (commonly used in gambling (and logistic regression)
- For some event E,

$$odds(E) = P(E)/(1 - P(E)) = P(E)/P(E^c)$$

- Similarly, if we are told the odds of E are x to y, then

$$odds(E) = x/y = \frac{x/(x+y)}{y/(x+y)}$$

which implies that

$$P(E) = \frac{x}{x+y} \quad \text{and} \quad P(E^c) = \frac{y}{x+y}$$

- Logistic regression is a GLM used to model a binary categorical variable using numerical and categorical variables
- We assume a binomial distribution produced the outcome variable and we therefore want to model π the probability of success for a given set of predictors
- It turns out that there is a very general way of addressing this type of a problem and the resulting models are called generalized linear models. Logistic regression is just one example of this type of model
- All generalized linear models has the following three characteristics:
  1. A probability distribution describing the outcome variable

2. A linear model

$$\eta = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$$

3. A link function that relates the linear model to the parameter of the outcome distribution

$$g(\pi) = \eta \quad \text{or} \quad \pi = g^{-1}(\eta)$$

- Logistic regression is a GLM used to model a binary categorical variable using numerical and categorical predictors
- We assume a binomial distribution produced the outcome variable therefore we want to model π, the probability of success, as a function of some predictors.
- There are a variety of reasonable link functions to use to connect π and η, One such function that is commonly used is the logit function

$$\text{logit}(\pi) = \log\left(\frac{\pi}{1-\pi)}\right), \quad 0 < \pi < 1.$$

- The logit fuction takes a value between 0 and 1 and maps it to a value between $-\infty$ and $+\infty$.
- The inverse logit (logistic) function is

$$g^{-1}(x) = \frac{e^x}{1+e^x}$$

- The inverse logit function takes a value between $-\infty$ and ∞ and maps it to a value between 0 and 1
- This formulation also some use when it comes to interpreting the model a logit can be interpreted as a the log odds of success
- The assumptions are

$$\begin{aligned} y|x_1, x_2, \ldots, x_p &= \begin{cases} 1 & \text{with probability}\, \pi(x_1, x_2, \ldots, x_p) \\ 0 & \text{with probability}\, 1 - \pi(x_1, x_2, \ldots, x_p) \end{cases} \\ \text{logit}(\pi(x_1, x_2, \ldots, x_p)) &= \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p \end{aligned}$$

- This implies that

$$\pi(x_1, x_2, \ldots, x_p) = \frac{e^{\beta_0+\beta_1 x_1+\ldots+\beta_p x_p}}{1+e^{\beta_0+\beta_1 x_1+\ldots+\beta_p x_p}}$$

- Also

$$\frac{\pi(x_1, x_2, \ldots, x_p)}{1-\pi(x_1, x_2, \ldots, x_p)} = e^{\beta_0+\beta_1 x_1+\ldots+\beta_p x_p}$$

- Interpretation of β_i: When we increase x_i by one while holding all the other xs fixed, the odds of getting of 1 change by a multiplicative factor equal to e^{β_i}.

- In R we fit a GLM model in the same way as we did in linear regression except that we use glm instead of lm and we must specify the type of GLM to fit using the family argument.

- Example: The data include the number of students admitted, the total number of applicants broken down by gender (the variable female), and whether or not they had taken AP calculus (the variable apcalc). Since the dataset is so small, we will read it in directly.

```
Gender= 0 male 1 female, AP = 1 took AP calculus, 0 did not.
 Admit =1 admitted 0 not admitted
Gender  AP  Admit
   0  0  0
   0  0  0
   0  0  0
   0  0  0
   0  0  0
   0  0  0
   0  0  0
   0  0  1
   0  1  0
   0  1  0
   0  1  0
   0  1  1
   0  1  1
   0  1  1
```

```
0  1  1
0  1  1
0  1  1
0  1  1
1  0  0
1  0  0
1  0  0
1  0  0
1  0  0
1  0  1
1  1  0
1  1  1
1  1  1
1  1  1
1  1  1
```

```
 > glm(Admit~Gender+AP, family = binomial("logit"))

Call:  glm(formula = Admit ~ Gender + AP, family = binomial("logit"))

Coefficients:
(Intercept)       Gender           AP
    -2.0043       0.4537       2.8755

Degrees of Freedom: 28 Total (i.e. Null);  26 Residual
Null Deviance:     39.89
Residual Deviance: 28.67  AIC: 34.67
```

$\text{logit}(P(admit = 1|\text{Gender, AP})) = -2.0043{+}0.4337\text{Gender}{+}2.8755\text{AP}$

This implies that

$$P(admit = 1|\text{Gender, AP}) = \frac{e^{-2.0043+0.4337\text{Gender}+2.8755\text{AP}}}{1 + e^{-2.0043+0.4337\text{Gender}+2.8755\text{AP}}}$$

The estimated odds of a male being admitted is $e^{0.4337} = 1.543$ times the estimated odds of a female being admitted controlling for AP (i.e. holding AP fixed).

The estimated odds of a person who has taken AP being admitted is $e^{2.8755} = 17.73429$ times the estimated odds of a person being admitted controlling for gender (holding gender fixed).

```
 > summary(glm(Admit~Gender+AP, family = binomial("logit")))

Call:
glm(formula = Admit ~ Gender + AP, family = binomial("logit"))

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.7667  -0.6203  -0.5028   0.8361   2.0643

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.0043     0.9170  -2.186  0.02884 *
Gender        0.4537     0.9908   0.458  0.64700
AP            2.8755     0.9898   2.905  0.00367 **
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 39.892  on 28  degrees of freedom
Residual deviance: 28.666  on 26  degrees of freedom
AIC: 34.666
Number of Fisher Scoring iterations: 4
```

Hypotheses Testing

To test $H_0 : \beta_i = 0$ against $H_a : \beta_i \neq 0$, the test statistics is

$$Z = \frac{b_i - 0}{SE(b_i}$$

and we reject H_0 if $|Z| > Z_{\alpha/2}$ or if $p - value < \alpha$. Example: Test $H_0 : \beta_{Gender} = 0$ against $H_a : \beta_{Gender} \neq 0$.. The test statistic is

$$Z = \frac{0.4537 - 0}{0.9908} = 0.458$$

If $\alpha = 0.05$ then $Z_{0.025} = 1.96$. Since $|0.458| < 1.96$, we fail to fail to reject H_0.

Confidence Intervals

The Wald method: A $100(1-\alpha)\%$ confidence interval for β_i is

$$b_i \pm Z_{\alpha/2} SE(b_i)$$

A 95% confidence interval for β_{Gender} is

$$0.4537 \pm 1.96(0.9908) = [-1.488268, 2.395668]$$

A 95% confidence for $e^{beta}\text{gender}$ is

$$[e^{-1.488268}, e^{2.395668}] = [0.2257633, 10.97553]$$

Interpretation: We are 95% confident that the odds of a male being admitted is a number between 0.226 and 10.975 times the odds of a female being admitted given they have the same status on AP.

To produce the confidence intervals for the betas using R, use the following

```
> library(MASS)
> confint.default(glm(Admit~Gender+AP, family = binomial("logit")))
```

if you use

```
> confint(glm(Admit~Gender+AP, family = binomial("logit")))
You will get
                2.5 %    97.5 %
(Intercept) -4.206356 -0.450995
Gender      -1.456204  2.605742
AP           1.115573  5.130797
```

This result is slightly different because it is based on a different method than the one use above.

Suppose we have the following model

$$\text{logit}(\pi(x_1, x_2, \ldots, x_p)) = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p$$

and wish to test

$$H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0$$

against H_a : at least one of them is not zero. We reject H_0 if

$$-2\log(L_R/L_F) = \text{Null Deviance} - \text{Residual Deviance} > \chi^2_p(1-\alpha)$$

In our example, Null deviance = 39.89, Residual Deviance = 28.67 and p= 2

$$\text{Null deviance} - \text{ Residual Deviance} = 11.22$$

If $\alpha = 0.05, \chi^2_2(0.05) = 5.99$

```
> qchisq(0.95,2) [1] 5.991465
```

Probit Model

```
Probit Model

> glm(Admit~Gender+AP, family = binomial("probit"))

Call:  glm(formula = Admit ~ Gender + AP, family = binomial("probit"))

Coefficients:
(Intercept)       Gender           AP
    -1.1848       0.2561       1.7276

Degrees of Freedom: 28 Total (i.e. Null);  26 Residual
Null Deviance:     39.89
Residual Deviance: 28.67  AIC: 34.67
```

Example 2: The following data (described in New York Times, Feb. 15, 1191) is used to study the effect of AZT in slowing the development of AIDS symptoms. In the study 338 veterans whose immune systems we beginning to falter after infection with AIDS virus were randomly assigned wither to receive AZT immediately or to wait until their T cells showed severe immune weakness. The data is a 2x2x2 cross classification of the veterans' race, whether they received AZT immediately and whether they developed AIDS symptoms during the three year study.

```
> aids<-read.csv("C:\\Users\\helbarmi\\Desktop\\deathpenalty.csv",
header=TRUE, sep=',')
> attach(aids)
> aids
  race AZTuse yes no
1    w    yes  14 93
2    w     no  32 81
3    b    yes  11 52
4    b     no  12 43
```

The model we want to use here is

$$\text{logit}(P(yes|\text{race, AZTuse})) = \beta_0 + \beta_1\text{race} + \beta_2\text{AZTuse}$$

To fit this model in R, we use

```
> logit1<-glm(cbind(yes, no)~factor(race)+factor(AZTuse), family=binomial)
> logit1

Call:  glm(formula = cbind(yes, no) ~ factor(race) + factor(AZTuse),
    family = binomial)

Coefficients:
      (Intercept)      factor(race)w  factor(AZTuse)yes
         -1.07357            0.05548           -0.71946

Degrees of Freedom: 3 Total (i.e. Null);  1 Residual
Null Deviance:     8.35
Residual Deviance: 1.384  AIC: 24.86
```

Interpretation of the result:

1. Interpretation of b_1 the estimate of β_1.: If we hold AZTuse fixed (i.e controlling for AZT use), we estimate the odds that a white person develops AIDS symptoms to be $e^{0.05548} = 1.057$ times the odds that a back person does (a 5.7% increase roughly)

2. Interpretation of b_2 the estimate of β_2.: If we hold race fixed (i.e controlling for race), we estimate the odds that a person who takes AZT develops AIDS symptoms to be $e^{-0.71946} = 0.49$ times the odds that a person does who does not(a 50% decrease roughly)

You can compute these numbers using

```
> OR=exp(coef(logit1))
> OR
  (Intercept)     factor(race)w factor(AZTuse)yes
    0.3417849         1.0570527         0.4870152
```

Likelihood Ratio Test

- Log likelihoods can be used to test the hypotheses of nested models (similar to F and partial F in regression)

- Say we want to test the null hypothesis H_0 about one or more coefficients, then we have a fall and reduced models

- Then the likelihood ratio is the ratio likelihood of imposing H_0 over the unrestricted likelihood

- If H_0 is true, the ratio should be near 1

- Under general H_0

  $$-2 \text{ (log of the likelihood ratio)} = -2[\log(L(R)) - \log(L(F))] \sim \chi^2_k$$

  where k is the number of parameters set equal to zero to get the reduced model.

- Reject H_0 if

  $$-2 \text{ (log of the likelihood ratio)} > \chi^2_k(1-\alpha)$$

- Under general H_0, $-2\log$(log of the likelihood ratio) can be computed using the deviances in the output

Example (AIDS exm): suppose we want to test

$$H_0 : \beta_1 = \beta_2 = 0$$

against H_a : Note H_0. The deviances are

```
Null deviance: 8.3499  on 3  degrees of freedom
Residual deviance: 1.3835  on 1  degrees of freedom
```

The test statistics $= 8.3499 - 1.3835 = 6.9674$. Since $p = 2$ we reject H_0 since $6.9674 > \chi^2_2(0.05) = 5.99$.

Next we look at each individual β

```
> summary(logit1)

Call:
glm(formula = cbind(yes, no) ~ factor(race) + factor(AZTuse),
    family = binomial)

Deviance Residuals:
      1        2        3        4
-0.5547   0.4253   0.7035  -0.6326

Coefficients:
                  Estimate Std. Error z value Pr(>|z|)
(Intercept)       -1.07357    0.26294  -4.083 4.45e-05 ***
factor(race)w      0.05548    0.28861   0.192  0.84755
factor(AZTuse)yes -0.71946    0.27898  -2.579  0.00991 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 8.3499  on 3  degrees of freedom
Residual deviance: 1.3835  on 1  degrees of freedom
AIC: 24.86

Number of Fisher Scoring iterations: 4
```

From this output we see that the p-value for testing that $H_0 : \beta_1 = 0$ against $H_a : \beta_1 \neq 0$ is 0.84755. Therefore, we fail to reject H_0

We know refit the model without race.

```
> logit2

Call:  glm(formula = cbind(yes, no) ~ factor(AZTuse), family = binomial)

Coefficients:
      (Intercept)  factor(AZTuse)yes
          -1.0361            -0.7218

Degrees of Freedom: 3 Total (i.e. Null);  2 Residual
Null Deviance:     8.35
Residual Deviance: 1.421  AIC: 22.9


> summary(logit2)

Call:
glm(formula = cbind(yes, no) ~ factor(AZTuse), family = binomial)

Deviance Residuals:
      1        2        3        4
-0.4813   0.5102   0.6026  -0.7521

Coefficients:
                  Estimate Std. Error z value Pr(>|z|)
(Intercept)        -1.0361     0.1755  -5.904 3.54e-09 ***
factor(AZTuse)yes  -0.7218     0.2787  -2.590  0.00961 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Polytomous (multicategory) logistic regression

- Suppose the response has J categories

- The response of the ith individual is $(Y_{i1}, Y_{i2}, \ldots, Y_{iJ})$ where

$$Y_{ij} = \begin{cases} 1 & \text{if the the response if j} \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum_{j=1}^{J} Y_{ij} = 1.$$

- Let

$$\pi_{ij} \equiv \pi_{ij}(x_{1i}, x_{2i}, \ldots, x_{pi}) = P[Y_{ij} = 1 | x_{1i}, x_{2i}, \ldots, x_{pi}].$$

- Choose a baseline or reference response category, the Jth say and let

$$\log \left(\frac{\pi_{ij}(x_{1i}, x_{2i}, \ldots, x_{pi})}{\pi_{iJ}(x_{i1}, x_{i2}, \ldots, x_{ip})} \right) = \beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}$$

- This is equivalent to

$$\pi_{ij}(x_{1i}, x_{2i}, \ldots, x_{pi}) = \pi_{iJ}(x_{1i}, x_{2i}, \ldots, x_{pi}) e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{i2} + \ldots + \beta_{pj}x_{pi}}$$

or

$$\pi_{ij} = \pi_{iJ} e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}}$$

- But

$$1 = \sum_{j=1}^{J} \pi_{ij} = \pi_{iJ}(1 + \sum_{j=1}^{J-1} e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}})$$

therefore

$$\pi_{iJ}(x_{1i}, x_{2i}, \ldots, x_{pi}) = \frac{1}{1 + \sum_{j=1}^{J-1} e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}}}$$

and

$$\pi_{ij}(x_{1i}, x_{2i}, \ldots, x_{pi}) = \frac{e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}}}{1 + \sum_{j=1}^{J-1} e^{\beta_{0j} + \beta_{1j}x_{1i} + \beta_{2j}x_{2i} + \ldots + \beta_{pj}x_{pi}}}$$

- We estimate $\beta_{0j}, \beta_{1j}, \ldots, \beta_{pj}, j = 1, 2, \ldots, J-1$, by $b_{0j}, b_{1j}, \ldots, b_{pj}, j = 1, 2, \ldots, J-1$. The estimation technique is the maximum likelihood approach.

- Interpretation of the $\beta_{\ell j}$: Given that the response is either j or J, is we increase $x_{i\ell}$ by 1 while holding everything elsefixed, the odds of the response is j change by a multiplicative factor equal to $e^{\beta_{\ell j}}$

Example: To illustrate, we analyze the data below. The response $Y =$ belief in afterlife and it has three categories (yes, undecided and no). The predictor variables are $X_1 =$ race and $X_2 =$ gender. We use no as the baseline. The model is (Yes =1, Undecided=2 and No=3)

$$\log\left(\frac{\pi_j}{\pi_3}\right) = \beta_{0j} + \beta_{1j}X_1 + \beta_{2j}X_2.$$

The SAS program to fit this model is

```
data polylogistic;
input Race $ Gender $ afterlife $ count ;
datalines;
w f yes 371
w f und 49
w f no 74
w m yes 250
w m und 45
w m no 71
b f yes 64
b f und 9
b f no 15
b m yes 25
b m und 5
b m no 13
;
proc logistic desceding;
class Race(ref='b') Gender(ref='m');
freq count;
model afterlife= Race Gender/link=glogit;
run;
```

The estimations resulted in

$$\log\left(\frac{\hat{\pi}_1}{\hat{\pi}_3}\right) = 0.883 + 0.342X1 + 0.419X_2$$

and

$$\log\left(\frac{\hat{\pi}_2}{\hat{\pi}_3}\right) = -0.758 + 0.271X1 + 0.105X_2$$

This gives

$$\hat{\pi}_3(x_1, x_2) = \frac{1}{1 + e^{0.883+0.342x1+0.419x_2} + e^{-0.758+0.271X1+0.105X_2}}$$

$$\hat{\pi}_1(x_1, x_2) = \frac{e^{0.883+0.342x1+0.419x_2}}{1 + e^{0.883+0.342x1+0.419x_2} + e^{-0.758+0.271X1+0.105X_2}}$$

$$\hat{\pi}_3(x_1, x_2) = \frac{e^{-0.758+0.271x1+0.105x_2}}{1 + e^{0.883+0.342x1+0.419x_2} + e^{-0.758+0.271x1+0.105x_2}}$$