Proof of Radon-Nikodym theorem

Theorem (Radon-Nikodym)**.** Suppose Ω is a nonempty set and $\mathscr{A}$ a σ-field on it. Suppose μ and ν are σ-finite measures on $(\Omega, \mathscr{A})$ such that for all $A \in \mathscr{A}$, $\mu(A) = 0$ implies $\nu(A) = 0$. Then

- There exists $z : \Omega \to [0, \infty)$ measurable such that for all $A \in \mathscr{A}$, $\nu(A) = \int_A z \, d\mu$.
- Such a z is unique upto a.e. equality (w.r.t. μ).
- z is integrable w.r.t. μ if and only if ν is a finite measure.

Proof. Assuming we have obtained the z, it is easy to see that z is integrable if and only if ν is a finite measure, since $\nu(\Omega) = \int z \, d\mu$.

Uniqueness

We show uniqueness first. Suppose there are two $z_1, z_2 : \Omega \to [0, \infty)$ such that for all $A \in \mathscr{A}$, $\int_A z_1 \, d\mu = \nu(A) = \int_A z_2 \, d\mu$. We would like to say $\int_A z_1 \, d\mu - \int_A z_2 \, d\mu = 0$, but both the integrals might be ∞. Both μ and ν are σ-finite. Let $\mathcal{C}$ be a countable partition of Ω such that for all $C \in \mathcal{C}$, $\mu(C) < \infty$ and $\nu(C) < \infty$ (to obtain this we can take such a partition for μ, and such a partition for ν, and take pairwise intersections). Fix a $C \in \mathcal{C}$. Let

$$B = C \cap \{\omega : z_1(\omega) - z_2(\omega) > 0\}$$

Note that $1_B(z_1 - z_2) = 1_C(z_1 - z_2)^+$ (evaluate both sides at ω: for $\omega \notin C$, both are 0; for $\omega \in C \setminus B$, LHS is 0 and since $(z_1 - z_2)(\omega) \leq 0$, RHS is 0; for $\omega \in B$ both side are $z_1(\omega) - z_2(\omega)$). $\nu(B) = \int 1_B z_1 \, d\mu = \int 1_B z_2 \, d\mu$. Since $\nu(B) \leq \nu(C) < \infty$, it is meaningful to consider $\int 1_B z_1 \, d\mu - \int 1_B z_2 \, d\mu$.

$$\begin{aligned} 0 &= \int 1_B z_1 \, d\mu - \int 1_B z_2 \, d\mu \\ &= \int 1_B (z_1 - z_2) \, d\mu \\ &= \int 1_C (z_1 - z_2)^+ \, d\mu \end{aligned}$$

If the integral of a nonnegative function is 0, it must be 0 almost everywhere. Applying to the above:

$$\mu(\{\omega : \omega \in C \text{ and } z_1(\omega) - z_2(\omega) > 0\}) = 0$$

We can interchange the roles of z_1 and z_2 in the above argument. Combining the two, we get

$$\mu(\{\omega : \omega \in C \text{ and } z_1(\omega) - z_2(\omega) \neq 0\}) = 0$$

This holds for all $C \in \mathcal{C}$. Since $\mathcal{C}$ is countable, and a countable union of null sets is a null set,

$$\mu(\{\omega : z_1(\omega) \neq z_2(\omega)\}) = 0$$

This proves that the z promised in the theorem (if it exists) is unique upto a.e. (w.r.t. μ) equality.

Existence

We now show the existence of such a z. We reduce the general case to the case when both μ and ν are finite, and then show existence for the finite case.

Reduction to the finite measure case

Suppose we know the Radon-Nikodym theorem holds for the case when the measures involved are finite. In general, assume μ and ν are σ-finite, and let $\mathcal{C}$ be a countable partition of Ω such that for all $C \in \mathcal{C}$, $\mu(C)$ and $\nu(C)$ are finite. Define finite measures μ_C and ν_C for all $C \in \mathcal{C}$,

$$\mu_C(A) = \mu(C \cap A),\ \nu_C(A) = \nu(C \cap A)$$

If $\mu_C(A) = 0$, then $\nu_C(A) = 0$. Using the finite-measure case of the theorem, let $z_C : \Omega \to [0, \infty)$ be such that for all $A \in \mathscr{A}$ and $C \in \mathcal{C}$,

$$\nu_C(A) = \int_A z_C \ d\mu_C$$

z_C and $z_C 1_C$ differ at most on $\Omega \setminus C$, which has 0 μ_C measure. So we can replace z_C by $z_C 1_C$, and so assume that for $\omega \notin C$, $z_C(\omega) = 0$. Define $z : \Omega \to [0, \infty)$, by requiring that $z|_C = (z_C)|_C$ (that is, for $\omega \in C$, $z(\omega) = z_C(\omega)$). z can also be described as $\sum_{C \in \mathcal{C}} z_C$ - this shows that z is measurable.

If $g : \Omega \to [0, \infty)$ is 0 outside C, then $\int g \ d\mu = \int g \ d\mu_C$. This can be proven as usual by proving it for simple functions and then for all nonnegative measurable functions. This fact is applied to z_C in what follows. For any $A \in \mathscr{A}$,

$$\begin{aligned}
\nu(A) &= \sum_{C \in \mathcal{C}} \nu(A \cap C) \\
&= \sum_{C \in \mathcal{C}} \nu_C(A) \\
&= \sum_{C \in \mathcal{C}} \int_A z_C \ d\mu_C \\
&= \sum_{C \in \mathcal{C}} \int_A z_C \ d\mu \\
&= \int_A \sum_{C \in \mathcal{C}} z_C \ d\mu && \text{(by taking partial sums; MCT)} \\
&= \int_A z \ d\mu
\end{aligned}$$

It remains to prove the theorem for the finite measure case:

Finite measures - getting hold of the z:

Now assume μ and ν are finite measures. Let

$$\underline{\underline{C}} = \left\{ f : (f : \Omega \to [0, \infty]_{\bar{\mathbb{R}}}), \forall A \in \mathscr{A} (\textstyle\int_A f \ d\mu \leq \nu(A)) \right\}$$

We want to identify f such that equality holds, that is, for all $A \in \mathscr{A}$, $\int_A f \, d\mu = \nu(A)$.

$\underline{\underline{C}}$ is nonempty, because the zero function belongs to $\underline{\underline{C}}$. Intuitively, we want to choose the "largest" function in $\underline{\underline{C}}$ as a candidate for our z. Does such a "largest" function exist? We observe some properties of $\underline{\underline{C}}$ first:

- If $f, g \in \underline{\underline{C}}$, then $f \vee g$ (which is the pointwise maximum of f and g) belongs to $\underline{\underline{C}}$. Take any $A \in \mathscr{A}$. We apply the condition defining $\underline{\underline{C}}$ to the sets $A_1 := A \cap \{\omega : f(\omega) \geq g(\omega)\}$ and $A_2 := A \cap \{\omega : f(\omega) < g(\omega)\}$:
$$\int_{A_1} f \leq \nu(A_1), \int_{A_2} g \leq \nu(A_2)$$
Note that $f 1_{A_1} = (f \vee g) 1_{A_1}$ and $g 1_{A_2} = (f \vee g) 1_{A_2}$. Substituting in the above equations and adding them, we get
$$\int_A (f \vee g) \leq \nu(A)$$
- Suppose $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence in $\underline{\underline{C}}$. Let f be their supremum (equivalently limit). Then $f \in \underline{\underline{C}}$: take any $A \in \mathscr{A}$. $\{f_n 1_A\}_{n=1}^{\infty}$ increases to $f 1_A$, and so by the monotone convergence theorem the corresponding limit with integrals holds. Since each $\int f_n 1_A \leq \nu(A)$, we have $\int f 1_A \leq \nu(A)$.

We identify the "largest" function in $\underline{\underline{C}}$ by considering the function with the largest integral. For all $f \in \underline{\underline{C}}$, $\int f \, d\mu \leq \nu(\Omega) < \infty$. Let

$$\alpha = \sup \left\{ \int f \, d\mu : f \in \underline{\underline{C}} \right\}$$

$0 \leq \alpha \leq \nu(\Omega)$. In particular, $\alpha \neq \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from $\underline{\underline{C}}$ such that $\left\{ \int f_n \, d\mu \right\}_{n=1}^{\infty}$ converges to α (this can be done because of the supremum definition of α). Consider the partial maxima of $\{f_n\}_{n=1}^{\infty}$, that is, define $\{g_n\}_{n=1}^{\infty}$ by

$$g_n = \max\{f_1, f_2, \ldots, f_n\}$$

We know that $g_n \in \underline{\underline{C}}$, so $\int g_n \, d\mu \leq \alpha$. Also, $f_n \leq g_n$, so $\int f_n \, d\mu \leq \int g_n \, d\mu \leq \alpha$. Thus $\lim_{n \to \infty} \int g_n \, d\mu = \alpha$. $\{g_n\}_{n=1}^{\infty}$ is an increasing sequence; let $z = \lim_{n \to \infty} g_n$. By the monotone convergence theorem, $\int z \, d\mu = \alpha$. Since z is integrable, z is finite almost everywhere. We modify z on a set of measure 0 suitably so that z never takes the value ∞. This z is our "largest" function in $\underline{\underline{C}}$, as we show now:

Let $g \in \underline{\underline{C}}$. We will show that $g \leq z$ a.e. (w.r.t μ). Let $A = \{\omega : g(\omega) > z(\omega)\}$. Let $z \vee g = h$. $h \in \underline{\underline{C}}$. $A = \{\omega : h(\omega) > z(\omega)\} = \{\omega : h(\omega) - z(\omega) > 0\}$. $h \geq z$; $\int h \, d\mu \geq \int z \, d\mu = \alpha$, so $\int h = \alpha$. $h - z$ is a nonnegative function whose μ integral is 0, so $h - z$ is 0 almost everywhere. $\mu(A) = 0$. $g \leq z$ almost everywhere (w.r.t μ).

Finite measures - showing that the z works

Define $\lambda : \mathscr{A} \to [0, \infty)$,

$$\lambda(A) = \nu(A) - \int_A z \, d\mu$$

Note that this is well-defined, that is, the expression on the RHS indeed belongs to $[0, \infty)$. λ is a measure: clearly $\lambda(\emptyset) = 0$. Let $\mathcal{C} \subseteq \mathscr{A}$ be a countable collection of disjoint sets.

$$\sum_{C \in \mathcal{C}} \nu(C) + \sum_{C \in \mathcal{C}} \int_C z \, d\mu = \nu\left(\bigcup \mathcal{C}\right) + \int_{\bigcup \mathcal{C}} z \, d\mu < \infty$$

Since the relevant series is absolutely convergent, we can rearrange terms, and

$$\lambda\left(\bigcup \mathcal{C}\right) = \nu\left(\bigcup \mathcal{C}\right) - \int_{\bigcup \mathcal{C}} z \, d\mu = \sum_{C \in \mathcal{C}} \nu(C) - \sum_{C \in \mathcal{C}} \int_C z \, d\mu = \sum_{C \in \mathcal{C}} \left(\nu(C) - \int_C z \, d\mu\right) = \sum_{C \in \mathcal{C}} \lambda(C)$$

λ is a measure. We want to show that λ is the zero measure, and then we will be done.

If for all $A \in \mathscr{A}$ and all $k \in \mathbb{N}$, $\lambda(A) \leq \frac{1}{k}\mu(A)$ then since $\mu(A)$ is finite, $\lambda(A) = 0$, and we are done. So there exists $A \in \mathscr{A}$ and $k \in \mathbb{N}$ such that $\lambda(A) - \frac{1}{k}\mu(A) > 0$. Fix such an A and k. We will derive a contradiction.

Call a set $Z \in \mathscr{A}$ to be *good* if the following two conditions hold:

- $\lambda(Z) - \frac{1}{k}\mu(Z) > 0$.
- For all $B \subseteq Z$ with $B \in \mathscr{A}$, $\lambda(B) - \frac{1}{k}\mu(B) \geq 0$.

The first condition implies that a good set Z must have $\mu(Z) > 0$: otherwise if $\mu(Z) = 0$, then by the hypothesis, $\nu(Z) = 0$, and so $\lambda(Z) = 0$. A satisfies the first condition, but need not satisfy the second condition.

We will identify a good set - A itself may not be a good set, but we can "scrape off" some parts of it to get a good set. Once we identify a good set Z, we will use the function $z + \frac{1}{k}1_Z$ to get a contradiction.

Let $A_0 = A$. Let $\beta_0 = \inf\{\lambda(B) - \frac{1}{k}\mu(B) : B \subseteq A, B \in \mathscr{A}\}$. If $\beta_0 \geq 0$, then A itself is a good set. So assume $\beta_0 < 0$. β_0 cannot be $-\infty$, since it is at least $-(\nu(\Omega) + \int z \, d\mu + \frac{1}{k}\mu(\Omega))$. Pick $B_0 \subseteq A_0$ such that

$$\lambda(B_0) - \frac{1}{k}\mu(B_0) < \frac{1}{2}\beta_0$$

Put $A_1 = A \setminus B_0$.

$$\lambda(A_1) - \tfrac{1}{k}\mu(A_1) = \big(\lambda(A) - \lambda(B_0)\big) - \big(\tfrac{1}{k}\mu(A) - \tfrac{1}{k}\mu(B_0)\big) \geq \lambda(A) - \tfrac{1}{k}\mu(A) > 0$$

(because $\lambda(B_0) - \frac{1}{k}\mu(B_0) < \frac{1}{2}\beta_0 < 0$.)

An important observation is this: if $B \subseteq A_1$, then $\lambda(B) - \frac{1}{k}\mu(B) \geq \frac{1}{2}\beta_0$. Otherwise, we could take a B which violates this, and $B \cup B_0$ would violate the infimum definition of β_0.

If A_1 is a good set, we have found a good set. Otherwise, repeat with A_1 what we did with A_0. The above observation says that the β_1 so obtained will satisfy $\beta_1 \geq \frac{1}{2}\beta_0$. Inductively, we construct a sequences of sets $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, and a sequence of real numbers $\{\beta_n\}_{n=1}^{\infty}$, such that the following holds: some A_n is good, or for all $n \in \mathbb{N}$,

- $B_n \subseteq A_n$.
- $A_{n+1} = A_n \setminus B_n$.
- $\beta_n = \inf\left\{\lambda(B) - \frac{1}{k}\mu(B) : B \subseteq A_n, B \in \mathscr{A}\right\}$, $\beta_n < 0$.
- $\lambda(A_{n+1}) - \frac{1}{k}\mu(A_{n+1}) \geq \lambda(A_n) - \frac{1}{k}\mu(A_n)$.
- $\beta_{n+1} \geq \frac{1}{2}\beta_n$.

This immediately implies that for all n, $\beta_n \geq \frac{1}{2^n}\beta_0$ and $\lambda(A_n) - \frac{1}{k}\mu(A_n) \geq \lambda(A) - \frac{1}{k}\mu(A) > 0$.
If no A_n was good, define $A_\infty = \bigcap_{n\in\mathbb{N}} A_n$. Since λ and μ are finite measures,

$$\lambda(A_\infty) = \lim_{n\to\infty} \lambda(A_n) \text{ and } \mu(A_\infty) = \lim_{n\to\infty} \mu(A_n)$$

So $\lambda(A_\infty) - \frac{1}{k}\mu(A_\infty) \geq \lambda(A) - \frac{1}{k}\mu(A) > 0$. If $B \subseteq A_\infty$ and $B \in \mathscr{A}$, then since $B \subseteq A_n$, $\lambda(B) - \frac{1}{k}\mu(B) \geq \frac{1}{2^n}\beta_0$. Since this holds for all n, $\lambda(B) - \frac{1}{k}\mu(B) \geq 0$. A_∞ is a good set! If some A_n was good, let A_∞ be that good set.
We will show that $z + \frac{1}{k}1_{A_\infty} \in \underline{\underline{C}}$. For any $S \in \mathscr{A}$, if $S \subseteq \Omega \setminus A_\infty$, then

$$\int_S \left(z + \frac{1}{k}1_{A_\infty}\right)\, d\mu = \int_S z\, d\mu \leq \nu(S)$$

If $S \subseteq A_\infty$, then

$$\begin{aligned} \textstyle\int_S \left(z + \frac{1}{k}1_{A_\infty}\right)\, d\mu &= \textstyle\int_S z\, d\mu + \frac{1}{k}\mu(S) \\ &\leq \textstyle\int_S z\, d\mu + \lambda(S) && \text{(since } A_\infty \text{ is a good set)} \\ &= \nu(S) && \text{(by definition of } \lambda) \end{aligned}$$

For general S, we just combine the part of S in A and the part in $\Omega \setminus A$:

$$\int_S \left(z + \frac{1}{k}1_{A_\infty}\right)\, d\mu = \int_{S\cap A} \left(z + \frac{1}{k}1_{A_\infty}\right)\, d\mu + \int_{S\setminus A} \left(z + \frac{1}{k}1_{A_\infty}\right)\, d\mu \leq \nu(S\cap A) + \nu(S\setminus A) = \nu(S)$$

We have shown that $z + \frac{1}{k}1_{A_\infty} \in \underline{\underline{C}}$. Since A_∞ is a good set, $\mu(A_\infty) > 0$. It is not the case that $z + \frac{1}{k}1_{A_\infty} \leq z$ a.e. (w.r.t. μ), which contradicts what we have shown earlier.
λ must be the zero measure, and so for all $D \in \mathscr{A}$,

$$\nu(D) = \int_D z\, d\mu$$

$\square$