University of Colorado **Boulder**

Department of Computer Science

CSCI 5622: Machine Learning

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Lecture 5: Perceptron II

Slides adapted from Chris Ketelsen, Jordan Boyd-Graber, and Noah Smith

Administrivia

- HW 1 due on Friday
- Minor change to HW1

Learning Objectives

- Understand the perceptron algorithm

Perceptron algorithm

- Vanilla perceptron algorithm
- Interpretation of weight values
- Convergence of the perceptron algorithm

Perceptron learning algorithm

Data: $D = \langle (\boldsymbol{x}_n, y_n) \rangle_{n=1}^{N}$, number of epochs E

Result: weights $\boldsymbol{w}$ and bias b

initialize: $\boldsymbol{w} = \boldsymbol{0}$ and $b = 0$;

for $e \in \{1, \ldots, E\}$ **do**

  for $n \in \{1, \ldots, N\}$, *in random order* **do**

    # predict

    $a = (\boldsymbol{w} \cdot \boldsymbol{x}_n + b)$;

    if $a y_n \leq 0$ **then**

      # update

      $\boldsymbol{w} \leftarrow \boldsymbol{w} + y_n \cdot \boldsymbol{x}_n$;

      $b \leftarrow b + y_n$;

    end

  end

end

$y \in \{-1, +1\}$

$$\hat{y} = \begin{cases} 1 & wx+b > 0 \\ -1 & wx+b \leq 0 \end{cases}$$

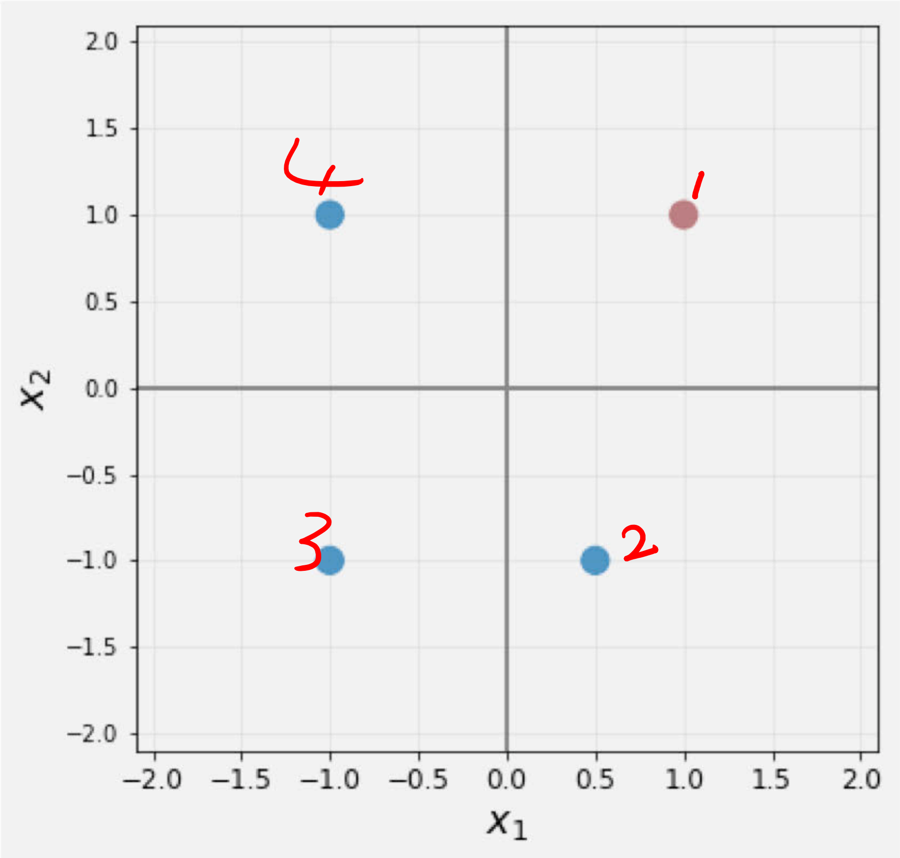
Example

- Start with $\boldsymbol{w} = [1, 0], b = 0$
- Process points in order (red for $+1$, blue for -1):

$$(1, 1), (0.5, -1), (-1. -1), (-1, 1)$$

$a = w \cdot x + b = 1, \quad y = 1$

$ay = 1 > 0$



Example

- Start with $\boldsymbol{w} = [1, 0], b = 0$
- Process points in order (red for $+1$, blue for -1):

$$(1, 1), \underline{(0.5, -1)}, (-1. -1), (-1, 1)$$

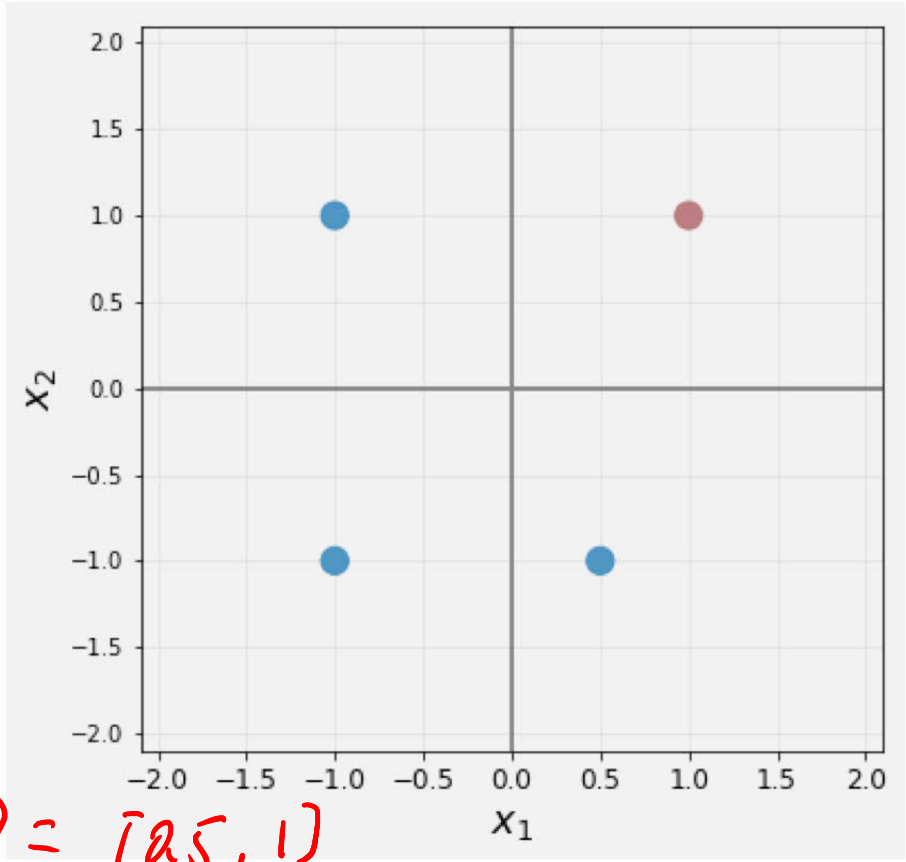
- $(1, 1) : a = 1, y = 1$: no update

$a = wx + b = 0.5 \quad y = -1$

$ay = -0.5$

$w \leftarrow [1, 0] + (-1) \cdot (0.5, -1) = [0.5, 1]$

$b \leftarrow 0 + (-1) = -1$



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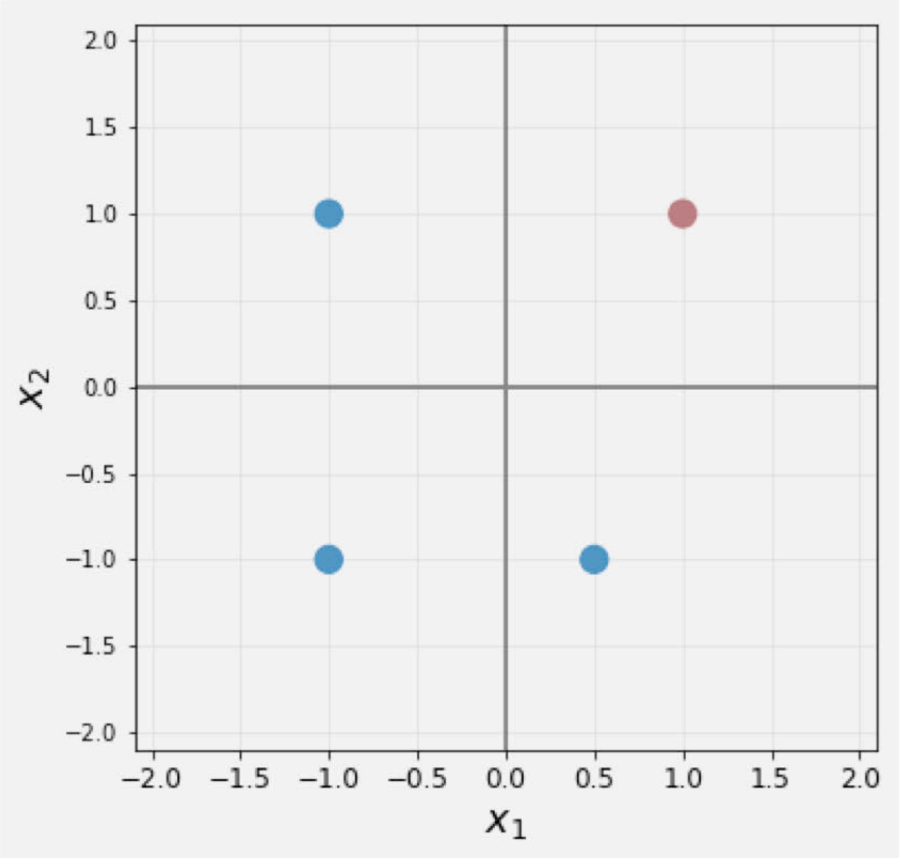
- Process points in order (red for $+1$, blue for -1):

$$(1,1), (0.5, -1), (-1. -1), (-1, 1)$$

- $(1,1) : a = 1, y = 1$: no update
- $(0.5, -1) : a = 0.5, y = -1$: $w = [0.5, 1], b = -1$

$a = w \cdot x + b = -1.5 - 1 = -2.5$ $\quad y = -1$

$ay > 0$



Example

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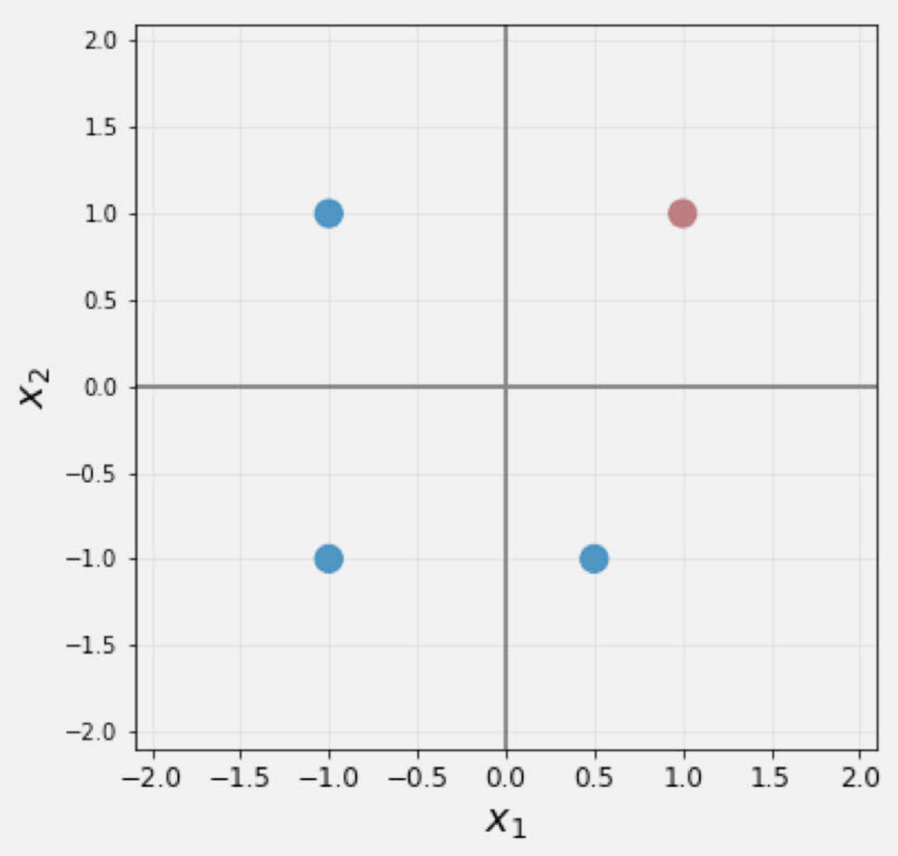
- Process points in order (red for $+1$, blue for -1):

$$(1, 1), (0.5, -1), (-1. - 1), (-1, 1)$$

- $(1, 1) : a = 1, y = 1$: no update
- $(0.5, -1) : a = 0.5, y = -1$: $w = [0.5, 1], b = -1$
- $(-1, -1) : a = -2.5, y = -1$: no update
- $(-1, 1) : a = -0.5, y = -1$: no update



Why does this algorithm work?

Assume that we have just misclassified a point $(\boldsymbol{x}, y)$, it means

$$\underline{a = \boldsymbol{w} \cdot \boldsymbol{x} + b}, \underline{ay \leq 0}$$

$\sum_{i=1}^{d} w_i x_i$

After the update: $\boldsymbol{w}' = \boldsymbol{w} + y\boldsymbol{x}, b' = b + y$

$ay \leq 0$

$y = 1$ $\quad a > 0$

$a \leq 0 \rightarrow$ wrong

$y = -1$ $\quad a < 0 \rightarrow$ Right

$a \geq 0 \rightarrow$ wrong

$$a' = w'x + b'$$
$$= (w + yx) \cdot x + b + y$$
$$= \underline{w \cdot x} + y(x \cdot x) \underline{+ b} + y$$
$$= a + y(1 + (x \cdot x))$$
$$a'y = ay + y^2(1 + (x \cdot x))$$
$$= ay + \underline{(1 + (x \cdot x))} > ay$$

$x \cdot x \geq 0$

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$$a = \boldsymbol{w} \cdot \boldsymbol{x} + b, ay \leq 0$$

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$$\begin{aligned} a' &= \boldsymbol{w}' \cdot \boldsymbol{x} + b' \\ &= \boldsymbol{w} \cdot \boldsymbol{x} + y||x||^2 + b + y \\ &= a + y||x||^2 + y \end{aligned}$$

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Perceptron learning algorithm

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Result: weights $\boldsymbol{w}$ and bias b
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        $a = (\boldsymbol{w} \cdot \boldsymbol{x}_n + b)$;
        if $a y_n \leq 0$ **then**
            # update
            $\boldsymbol{w} \leftarrow \boldsymbol{w} + y_n \cdot \boldsymbol{x}_n$;
            $b \leftarrow b + y_n$;
        end
    end
end

- why $a y_n \leq 0$ rather than $a y_n < 0$?
- why random order?

Parameters and Hyperparameters

This is the first supervised algorithm we've seen that has **parameters** that are numerical values (w and b).

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The perceptron learning algorithm's sole hyperparameter is E, the number of epochs (passes over the training data).

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What does it mean when . . .

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Interpretation of Weight Values

What does it mean when ...

- $w_{12} = 100$?
- $w_{12} = -1$?
- $w_{12} = 0$?

$w_{13} = 100$ $x_3 \in (-0.0001, 0.0001)$

$w_{12} = -1$ $x_2 \in [-100, 100]$

Interpretation of Weight Values

What does it mean when . . .

- $w_{12} = 100$?
- $w_{12} = -1$?
- $w_{12} = 0$?

What if we multiply w by 2?

$wx \quad +b$

$2wx \quad +2b$

Interpretation of Weight Values

In other words, how sensitive is the final classification to changes in individual features?

$$y = \text{sign}(\boldsymbol{w} \cdot \boldsymbol{x} + b) \begin{cases} +1 & \text{if } \boldsymbol{w} \cdot \boldsymbol{x} + b \geq 0 \\ -1 & \text{if } \boldsymbol{w} \cdot \boldsymbol{x} + b < 0 \end{cases}$$

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If features are similar size then large weights indicate important features

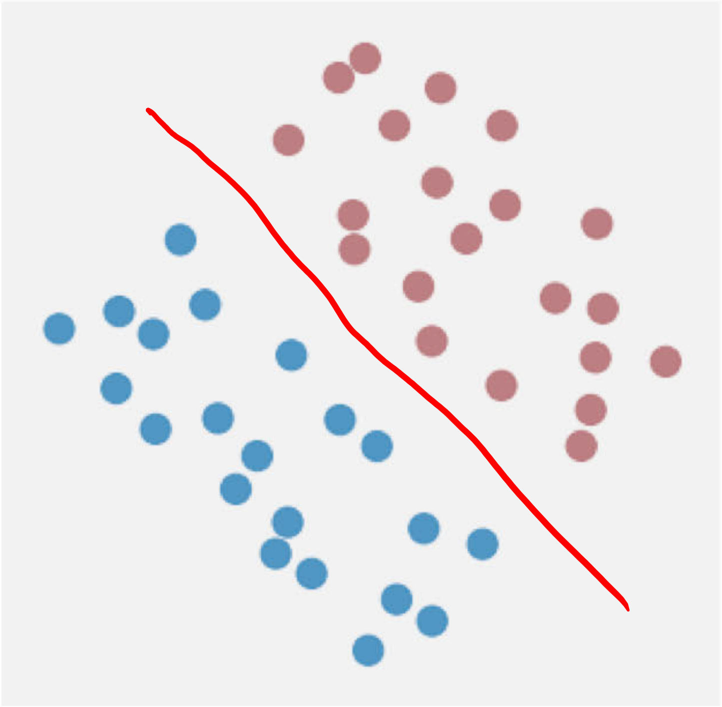
Interpretability of machine learning

- Example definitions
  - Can you understand how a prediction is made?
  - Can you explain how the model works as a whole?
  - Can you make better decisions with assistance of the model?

- K-nearest neighbors Yes, Yes, Unclear
- Perceptron Yes, Yes, Unclear

Convergence of the perceptron algorithm

- If possible for a linear classifier to separate data, Perceptron will find it



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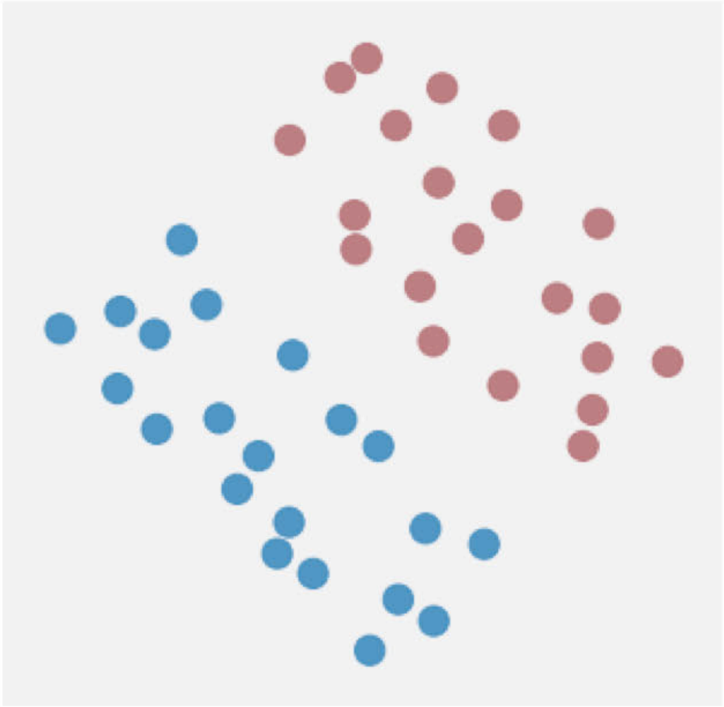

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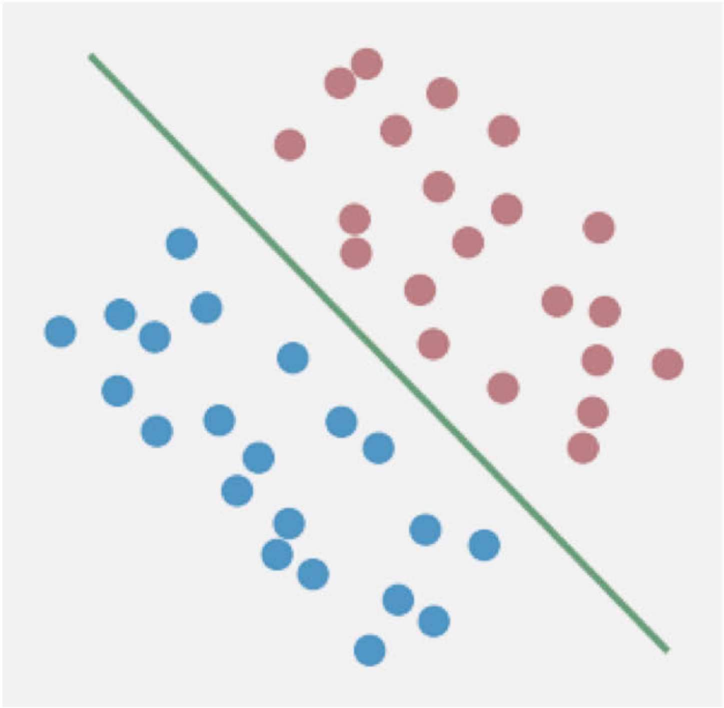

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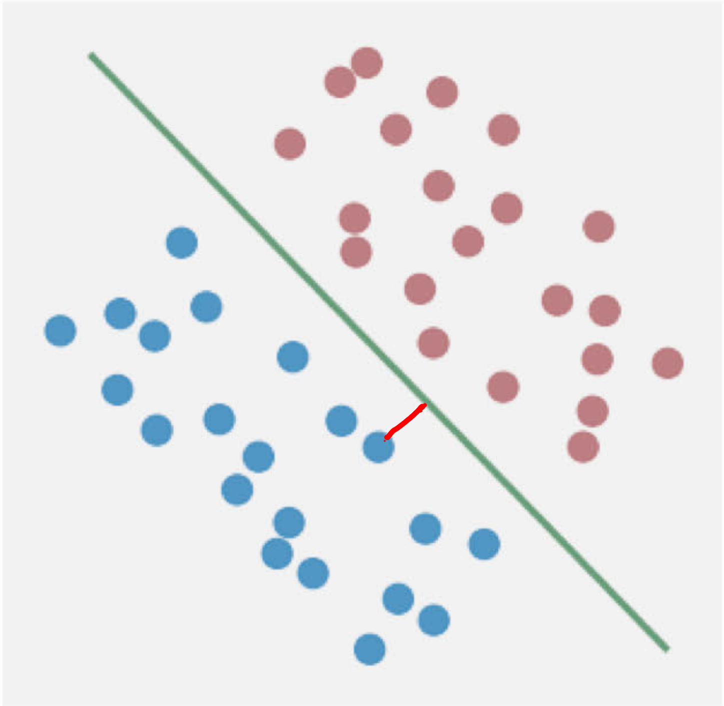
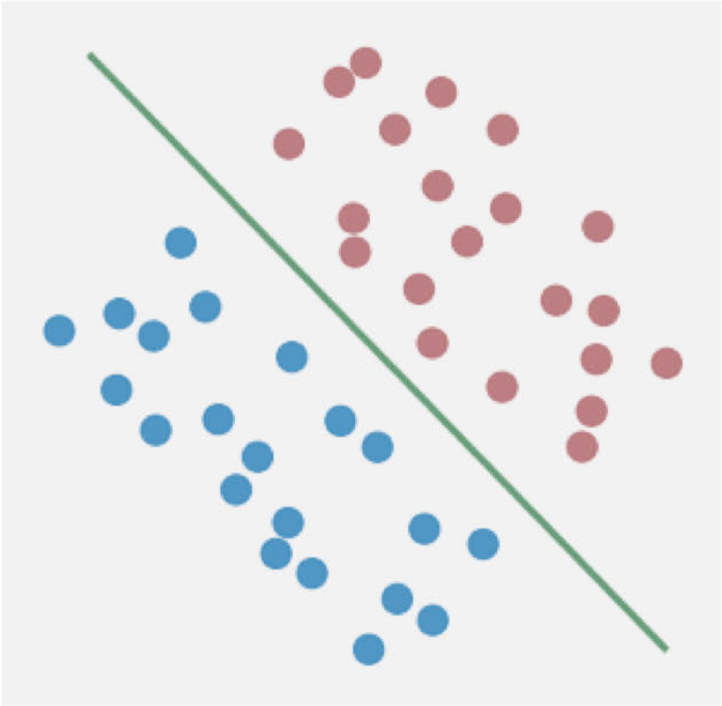


Convergence of the perceptron algorithm

- If possible for a linear classifier to separate data, Perceptron will find it
- Such training sets are called linearly separable
- *Margin* characterizes how separable a dataset is
- How long it takes to converge depend on the margin



Convergence of the perceptron algorithm

Due to Rosenblatt [1958].

If D is linearly separable with margin $\gamma > 0$ and for all $n \in \{1, \ldots, N\}$, $\|\boldsymbol{x}_n\|_2 \leq 1$, then the perceptron algorithm will converge in at most $\frac{1}{\gamma^2}$ updates.

$$\gamma = \text{margin}(D, \boldsymbol{w}, b) = \begin{cases} \min_n y_n \cdot (\boldsymbol{w} \cdot \boldsymbol{x}_n + b) & \text{if } \boldsymbol{w} \text{ and } b \text{ separate } D \\ -\infty & \text{otherwise} \end{cases}$$

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- Proof can be found in Daume [2017], pp. 50–51.
- The theorem does not guarantee that the perceptron's classifier will achieve margin γ.

Perceptron Wrap-up

- The perceptron is a simple classifier that sometimes works very well
- Linear classifiers in general will pop up again and again, e.g., Logistic Regression will be pretty similar
- The idea of margins will show up again when we talk about Support Vector Machines
- Neural Networks are essentially generalizations of the perceptron