Tracking Control for Mechanical Systems Experiencing Simultaneous Impacts and Spatial Friction

M. Bouma (0784690)
Department of Mechanical Engineering
Dynamics and Control group
Eindhoven University of Technology
DC 2017.114

Project Supervisor
PROF. DR. IR. N. VAN DE WOUW

Daily Supervisors
DR.IR A. SACCON
IR. M.W.L.M. RIJNEN

Committee
PROF. DR. IR. N. VAN DE WOUW *
DR. IR. A. SACCON *
IR. M.W.L.M. RIJNEN *
T.B.D. †

* Eindhoven University of Technology (Dynamics and Control)
† Eindhoven University of Technology (Control Systems Technology)

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Abstract

Acknowledgments

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Chapter 1

Introduction

1.1 High performance physical interaction in robotics

REWRITE A BIT THINKING ABOUT WHY, WHAT, HOW
IMAGE WITH NOMINAL VS PERTURBED CONTACT
In many applications of mechanical systems physical interaction with the environment is necessary to function, particularly in robotics. Humanoid robots, industrial robots, surgical robots are some examples of robotics that require physical interaction with their environment to function. Straightforward control strategies that are available for these applications, are limiting in performance. To avoid complexity of the controller, they often assume contact with their environment to happen at zero speed. This makes the robots less suitable for situations where speed is of significance, e.g., for industrial or walking robots. Higher performance is desirable in such cases. However, making contact with your environment at non-zero velocities complicates the control problem.

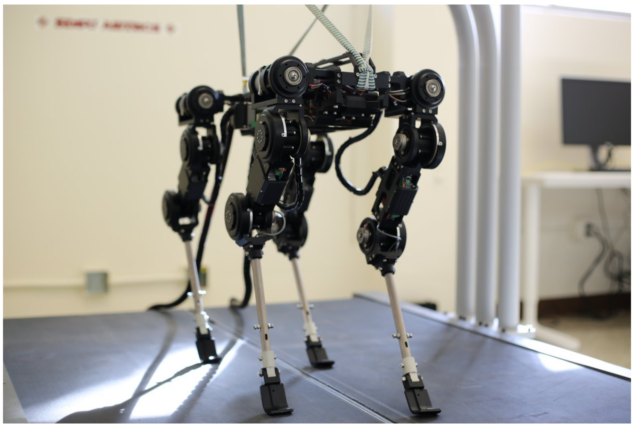


Figure 1.1: *The quadruped of Virginia Tech is an example of a robot with physical interaction that can benefit from high performance control strategies.*

When two bodies make physical contact at a considerable velocity, impact occurs. Impact is a complex physical event, that is characterized by very short durations, high force levels, high energy dissipation rates, and large accelerations and decelerations. Due to the small time scale at which impacts happen, the impact is often modeled by assuming that the change in velocity during impact happens due to an impulsive force, i.e., the velocity jump happens in zero time. Systems exhibiting such dynamics are often represented as *non-smooth*. In dynamics, the state doesn't necessarily have to be a smooth function of time, meaning that not all it's time-derivatives exist. Non-smooth mechanical systems with impact are systems that have discontinuities in their state-evolution. Mechanical systems with unilateral constraints are an example of systems that can exhibit non-smooth behavior, as they can jump in state when the unilateral constraints are closed. Control strategies for such systems are necessary to achieve better performance.

1.2 Nonsmooth modeling frameworks

Non-smooth systems can be described by several mathematical frameworks, e.g., singularly perturbed systems, hybrid systems, complementarity systems and (measure-)differential inclusions [1]. The singular perturbation framework approximates the non-smooth behaviour using a singularly perturbed smooth system. In this way, the singularly perturbed system can be evaluated numerically using a single smooth differential equation. However, due to the smooth approximation extremely small time-steps are necessary which makes the system less suitable for simulation. More suitable for this are differential inclusions, which are applicable to systems with a discontinuous right-hand side but a time-continuous state-evolution (also called Filippov-systems) [2]. A common example of Filippov-systems are systems including friction. The differential inclusion gives a description of the non-smooth dynamics in a single inclusion. However, mechanical systems with unilateral constraints and impact do not satisfy the requirement of having a time-continuous state-evolution. A measure-differential inclusion describes the continuous as well as the impulsive dynamics of a non-smooth system [3]. In this way, measure-differential inclusions are suitable for systems with time-discontinuities in their state-evolution [4, 5]. Using this approach the dynamics can be accurately integrated using the timestepping method [6]. From a control point-of-view the complementarity framework is often considered. This framework describes non-smoothness through a combination of differential equations and inequalities [7, 8]. In [9] the complementarity problem is used to describe mechanical systems with unilateral constraints. It plays a key role in mathematical programming, and several solutions for trajectory tracking using complementarity systems exist [10, 11]. In recent years the hybrid systems framework has drawn more interest for solving the trajectory tracking problem of non-smooth systems [12, 13]. A hybrid system is a dynamical system that exhibits both continuous and discrete dynamics behaviour, where it switches (discrete behaviour) between several differential equations (continuous behaviour) [14]. *Guard functions* are defined, which when activated will cause the dynamics to switch from one differential equation to another and possibly reinitialize the state. This makes it a very intuitive approach to the modeling of non-smoothness. A known difficulty with this description however, is a phenomenon called Zeno-behaviour. One speaks of Zeno-behaviour when an infinite amount of guard activations happen in a finite time. A classic example is the bouncing ball. The framework of measure-differential inclusions would be more suitable in such a situation, because the impulsive dynamics are implicit in the inclusion. An advantage of using the hybrid systems framework, is that it is a more intuitive description of the dynamics whereas measure-differential inclusions are more abstract. The stability analyses for these frameworks are well-developed, i.e., [14–16] for hybrid systems, [17–19] for measure differential inclusions, and [5, 20, 21] for linear complementarity problems.

1.3 Tracking control for non-smooth systems

Legged robotic systems account for a substantial part of the research done into the trajectory tracking control of mechanical systems with unilateral constraints. Many results in this area deal with the stability of periodic orbits of systems with impacts. In [22], a first big step has been made into modeling and controlling a one-legged hopping robot. In this research, the energy-loss during impact is modelled through damping and coupling effects are modeled as perturbations. In [23], the hybrid framework is adopted to find stable walking gaits for biped robots. Using Poincaré maps in [24], the stability of periodic orbits with discontinuities of under-actuated systems are analyzed. [25] continues on this work, generating control laws using data of walking humans and in [26] the energy-efficiency of generated walking gaits has been the focus. The analysis of stable walking gaits is applied to the MIT Cheetah in [12], where a stability analysis and controller design is presented for the trot-running of a quadrupedal robot. In [27,28], quasi-stability is introduced for systems with periodic trajectories in the so-called *Birkhoff billiard*. This notion of stability defines asymptotic stability for the periodic trajectory, except for the time-instants where impacts happen.

There has been made considerable progress in the field of walking robots and billiards, but it is easy to think of a an example where non-periodic trajectories are of interest. Under the assumption that the state trajectory jumps at the exact same time as the reference trajectory, the trajectory tracking problem for non-smooth systems had been solved multiple times. The tracking problem for Lur'e type systems has been analyzed in [29,30], using MDI's to describe non-smooth and impulsive dynamics. This work uses the convergence property to provide a solution to the tracking problem, where the solution may be time-varying and exhibit state-jumps. In [10], the passivity-based approach is used to solve the tracking control problem of complementarity Lagrangian systems. Asymptotic stability is achieved for Lagrangian systems with unilateral constraints. The same approach has been applied to the hybrid system framework in [31], which results in a control law that can guarantee stability of a trajectory with multiple impacts. By embedding the reference trajectory with discontinuities into a set, Lyapunov tools can be used to analyze stability as in [32, 33]. In [34] a stability analysis of systems with impacts and friction has been presented. The results are obtained using the measure-differential inclusion framework, resulting in a smooth control law.

In the prior discussed work, the tracking problem has often been solved under the assumption that the jump of the state occurs at the same time-instant as the jump of the desired trajectory, i.e., in [31, 33, 34]. In reality, especially in high velocity conditions, the state jump and trajectory jump do not always coincide. In this case a phenomenon called *peaking* occurs [35]. Spikes in the tracking error will arise around the jump times, which defy the notion of stability and are unfavorable when designing a stabilizing controller. Several solutions exist to solve this problem. Solutions exist for periodic orbits in [27, 28], where the results are compatible with a mismatch in reference jump-time and state jump-time. In [35, 36], a novel definition of the tracking error is introduced, using a distance function which is not sensitive to jumps of the state and the reference trajectory. Lyapunov-based conditions for the global asymptotic stability of non-smooth trajectories have been derived. Another solution for the problem of a jump-time mismatch is proposed in [37], in the form of a distance function similar to [35, 36] based on a quotient metric. When a state jump happens at different time than the reference jump, this approach applies the jump map to the reference to be able to compare the state to the reference. [38] introduces a novel controller design using gluing functions to connect the start and end point of a jump in the state space. After gluing, the system can be considered a continuous or piecewise continuous function without state jumps. In [39] a tracking error is defined by taking the smallest of two values: the ante-impact tracking error and post-impact tracking error.

1.4 Reference spreading control

In [40] a novel notion of error is defined by extending the reference trajectory segments over the entire domain of the state space, and considering the error between the trajectory and state that have encountered the same number of jumps. This notion of error, called *reference spreading*, is the basis for extending the sensitivity analysis introduced in [41] to the hybrid system framework. These results are then used to obtain a local approximation of the perturbed state dynamics. This analysis gives insight in how the system behaves under the presence of perturbations, which is useful during controller design. In [42, 43], the sensitivity analysis for hybrid systems with state triggered jumps is used for a trajectory tracking control strategy, which is named *reference spreading control*. In [44] the reference spreading control law is used in simulations with an iCub robot. The iCub robot balances on one foot and keeps himself standing upright by making and breaking contact with a wall using on of his arms. Some snapshots of the motion are depicted in Figure 1.2. As mentioned earlier, many available results work under the assumption that the exact impact time is known. This is extremely limiting during implementation of the control law. Using the reference spreading approach, the impact time of the state is allowed to deviate from that of the reference, making implementation of the control law in practical applications more viable while still allowing for state-jumps.

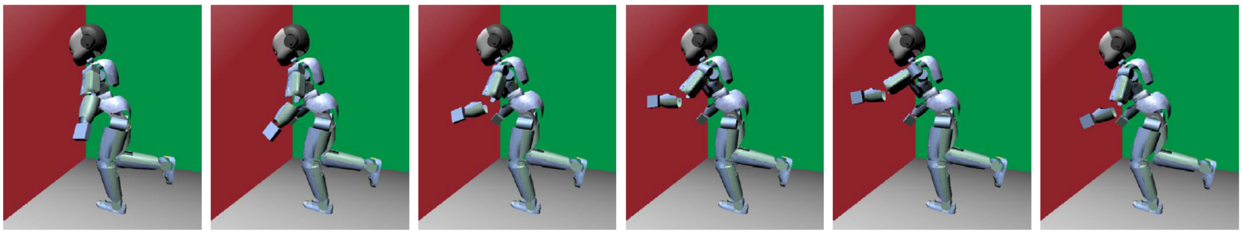


Figure 1.2: *Snapshots of the motion the iCub robot makes in the simulation done in [44] using reference spreading control. The wrist of the arm is modeled as a point-contact, which makes contact with the wall at non-zero velocities.*

As is the case in the simulations with the iCub in Figure 1.2, impacts are often modeled using a one-point contact between two bodies. However, in practice more complex geometries can make contact with each other. A more realistic contact is used in [45], in which the foot strike of a humanoid bipedal robot is modeled using multiple impacts. One strike is modeled by a heel impact, a toe impact, a heel release, and finally a toe release, resulting in a more realistic model of the walking gait of the bipedal robot. In such a movement the order of impacts is known.

When considering a trajectory with multiple contacts closing at one time-instant, the problem becomes more complex. Imagine for example the iCub in [44] having a physically realistic geometry at the end of it's arm. The arm can first make a point contact, then a line contact, and finally a surface contact to make full contact with the wall. It can also track a trajectory where the surface of the arm makes a surface contact with the wall at one time-instant. Such events can be considered as multiple impacts happening at the same time-instant and are called *simultaneous impacts*. One can imagine that at high velocities, a small perturbation can significantly change the behavior of a simultaneous impact. Instead of the expected single jump in the state, a perturbation can cause the state to have multiple jumps. Also the order of impacts of the several parts of the arm is unknown beforehand. In Figure 1.3 a 2-dimensional representation of the wrist of the iCub robot is illustrated. This image clearly visualizes the effect a perturbation has on a trajectory with simultaneous impacts. The approach used in [45] is not suitable for these situations.

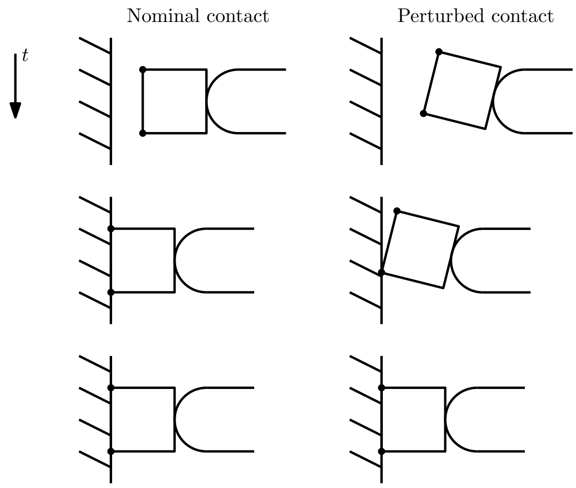


Figure 1.3: *A 2-dimensional illustration of the end effector of the wrist of the iCub robot. On the left a nominal trajectory is illustrated (from top to bottom), where a simultaneous impact happens. The wrist immediately makes a line-contact with the surface. On the right the trajectory is perturbed, which results in two impacts instead of one.*

The phenomenon of simultaneous impacts is first introduced in [46], where the first step is taken into solving the trajectory tracking problem for such impacts. A hybrid system framework is used to model the impacts, where multiple guards can be activated at once. The reference spreading control in [42,43] and the sensitivity analysis to approximate a system's behavior around trajectories with single guard activation, introduced in [40], are extended to be suitable for simultaneous guard activation. The results of the sensitivity analyses are used to find suitable gains for the reference spreading control law. However, these results do not consider friction. In addition to an impulse perpendicular to the impact-surface, taking friction into account will result in a tangential impulse. Also the release phase is not considered in the work of [46]. During the release phase there is no discontinuity in the state-evolution, only a change in the number of active constraints on the system. Therefore, the aim of this research is to investigate both stability of impacts including friction and the release phase of contact.

1.5 Contribution

Simultaneous impacts with friction

Friction elements are often disregarded when analyzing impacts. However, to set up a unifying theory friction elements should not be ignored. Tangential impulses can have a considerable effect on a system's dynamics. The impact laws used in the work of [46] can be extended using one of the several available friction laws [19]. When impact laws are supplemented with a tangential element, this is often done using a Coulomb friction law [47]. Besides the tangential impulse, also a Filippov-like discontinuity will be present due to Coulomb friction. Also in this work a Coulomb friction law is used, which can lead to more accurate models and controllers for mechanical systems with unilateral constraints.

Research Objective:
Find a model suitable to describe mechanical systems with unilateral constraints and spatial friction. The positive homogenization's notation and sensitivity analysis shall be adjusted to be compatible

with such models.

Simultaneous impacts with release

In [42] and [48], the release phase of a system with a reference spreading controller has been extensively researched. However, these results only include single guard activation. The release phase for multiple guard activation is not yet investigated. In [46], only a single contact is simulated for this reason. Extending the sensitivity analysis in [46] to be suitable for simultaneous releases would make it possible to simulate and control one trajectory with several simultaneous contacts and releases.

Research Objective:
Adjusting the positive homogenization, such that it is suitable for input-dependent guard functions. This will result in a positive homogenization suitable for trajectories that contain a releasing motion.

1.6 Report outline

First, the modeling of mechanical systems with unilateral constraints and spatial friction is discussed in Chapter 2. Here the general dynamics of a mechanical system including contact and friction laws are derived, eventually resulting in a hybrid system with impulsive effects. Then, in Chapter 3 a tracking control strategy for hybrid systems with ordered state-input-triggered events is presented. In Chapter 4 this tracking control strategy is extended to trajectories involving simultaneous impacts. After this an example of a mechanical system with unilateral constraints and spatial friction is given in Chapter 5, which is used for a numerical validation of the presented control strategy. Finally, in Chapter 6, conclusions are drawn and recommendations for future research are provided.

Chapter 2

Mechanical Systems with Unilateral Constraints and Spatial Friction

ADD ASSUMPTIONS 1-6 FROM RIJNEN2017 In this chapter several modeling approaches of mechanical systems with unilateral constraints and spatial friction are presented. First a general representation of such a system is given, which is followed by sections introducing several formulations of the contact and friction laws in these systems. First a complementarity problem formulation is given, which is a complete and often used formulation for mechanical systems with unilateral constraints. From the complementarity formulation a hybrid system formulation is derived, which although less complete is a more intuitive formulation from a control point-of-view.

2.1 General system definition

For mechanical systems a set of contact points $i_c \in \{i_1, i_2, ..., i_C\}$ is defined, with C being the number of considered contact points. A set $\mathcal{I}_{\text{cl}}$ is defined as the set of closed contact points, such that a contact $i_c \in \mathcal{I}_{\text{cl}}$ has closed a unilateral constraint. The set of contact points in open contact is defined as $\mathcal{I}_{\text{op}} := \{i_c \mid i_c \notin \mathcal{I}_{\text{cl}}\}$. When a unilateral constraint is activated at a nonzero velocity impact happens, meaning a jump in the velocity will appear. Therefore the generalized velocity $\dot{\boldsymbol{q}}$ is not continuously defined over the entire domain of a trajectory with impacts. Therefore $\boldsymbol{\xi} := \dot{\boldsymbol{q}}$ is defined, except at impact-times τ_i. Here i is a counter for unilateral constraint activations where $i \in \{1, 2, ..., N\}$, with N the amount of jumps in a trajectory. h_n is the normal contact distance and ζ_{n,i_c} and $\boldsymbol{\zeta}_{t,i_c}$ represent the relative velocities in normal and tangential direction, respectively.

In Figure 2.1 a body and a fixed world surface are illustrated. A contact point i_c is defined on the body, on which a plane tangent to the surface of the body is spanned. On the normal direction of this plane, the contact distance $h_{n,i_c}(\boldsymbol{q})$ is defined. This is the distance between the contact point and the surface that it can make contact with. By taking the time derivative of $h_{n,i_c}(\boldsymbol{q})$, the relative normal velocity of contact point i_c is found to be

$$\zeta_{n,i_c} = \frac{\partial h_{n,i_c}}{\partial \boldsymbol{q}} \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \boldsymbol{w}_{n,i_c}^T \dot{\boldsymbol{q}}, \tag{2.1}$$

with $\boldsymbol{w}_{n,i_c}^T$ representing the Jacobian of the normal velocity ζ_{n,i_c}. The relative tangential velocity $\boldsymbol{\zeta}_{t,i_c}$ is defined in the tangent plane of i_c, and thus lies perpendicular to ζ_{n,i_c}. Using the tangential velocity jacobian $\boldsymbol{W}_{t,i_c}$, the relative tangential velocity is defined as

$$\boldsymbol{\zeta}_{t,i_c} = \boldsymbol{W}_{t,i_c}^T \dot{\boldsymbol{q}}. \tag{2.2}$$

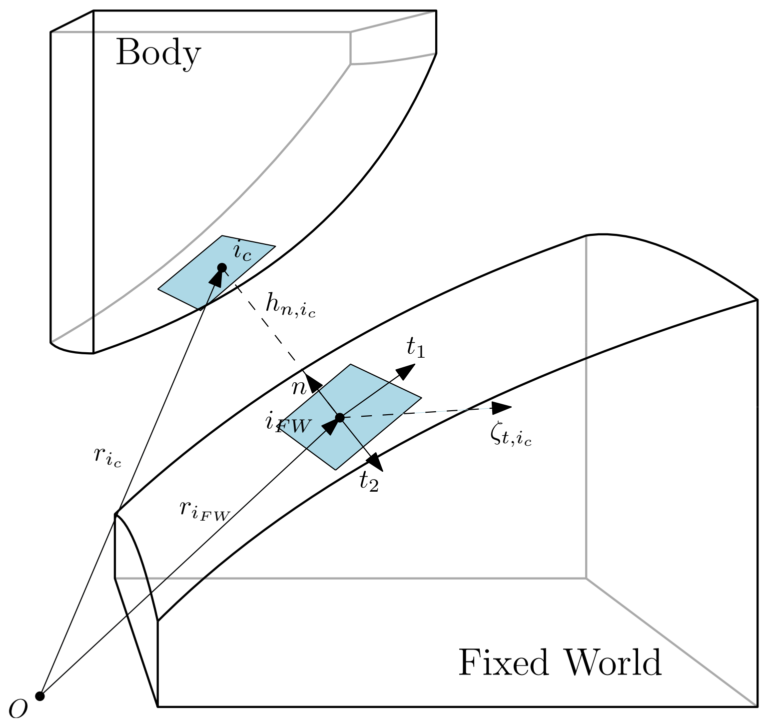


Figure 2.1: *An illustration of a body impacting the fixed world. A contact point i_c is defined on the body, which can make contact with the fixed world at i_{FW}. The points i_c and i_{FW} are connected by a line which is perpendicular to their respective bodies, which is used to define the contact distance h_{n,i_c} and normal contact velocity ζ_{n,i_c}. In the plane tangent to the body's surface, the tangential contact velocity $\boldsymbol{\zeta}_{t,i_c}$ is defined.*

Then, the continuous dynamics of a mechanical system with unilateral constraints and spatial friction are of the form

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\xi}} + \boldsymbol{H}(\boldsymbol{q},\boldsymbol{\xi}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{u} + \sum_{i_c \in \mathcal{I}_{\mathrm{cl}}} \left(\boldsymbol{w}_{n,i_c}(\boldsymbol{q})\lambda_{n,i_c} + \boldsymbol{W}_{t,i_c}(\boldsymbol{q})\boldsymbol{\lambda}_{t,i_c}\right), \tag{2.3}$$

$$\text{(Contact Law)}, \tag{2.4}$$

$$\text{(Friction Law)}, \tag{2.5}$$

with $\boldsymbol{q},\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$. Here $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the mass matrix of the system, $\boldsymbol{H}(\boldsymbol{q},\boldsymbol{\xi}) \in \mathbb{R}^n$ contains the centripetal, Coriolis and gravitational forces in the system and $\boldsymbol{S}(\boldsymbol{q}) \in \mathbb{R}^{n\times m}$ represents the generalized directions of the forces. $\lambda_{n,i_c} \in \mathbb{R}$ and $\boldsymbol{\lambda}_{t,i_c} \in \mathbb{R}^2$ are the normal and tangential reaction forces, respectively, of contact point i_c with $\boldsymbol{w}_{n,i_c} \in \mathbb{R}^n$ and $\boldsymbol{W}_{t,i_c} \in \mathbb{R}^{n\times 2}$ the corresponding reaction force jacobians. When a contact point activates a unilateral constraint, impulsive dynamics can cause the state of the system to jump. These dynamics are of the form

$$\boldsymbol{M}(\boldsymbol{q})(\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-) = \sum_{i_c \in \mathcal{I}_{\mathrm{cl}}} \left(\boldsymbol{w}_{n,i_c}(\boldsymbol{q})\Lambda_{n,i_c} + \boldsymbol{W}_{t,i_c}(\boldsymbol{q})\boldsymbol{\Lambda}_{t,i_c}\right), \tag{2.6}$$

$$\text{(Impulsive Contact Law)}, \tag{2.7}$$

$$\text{(Impulsive Friction Law)}. \tag{2.8}$$

Here Λ_{n,i_c} and $\boldsymbol{\Lambda}_{t,i_c}$ the normal and tangential impulsive reaction forces, respectively, of contact point i_c. These dynamics are impulsive, and happen at one instance in time. The $^-$ superscript indicates the ante-event state and the $^+$ superscript indicates the post-event state. In the following sections three different methods of describing the contact and friction laws are presented. First a complementarity problem formulation of mechanical systems with unilateral constraints is given, from which later a proximal point formulation and a hybrid system formulation are derived. For more information on modeling of multibody systems one can refer to [19] and [49].

2.2 Complementarity problem formulation

2.2.1 Signorini's contact law and Poisson's impact law

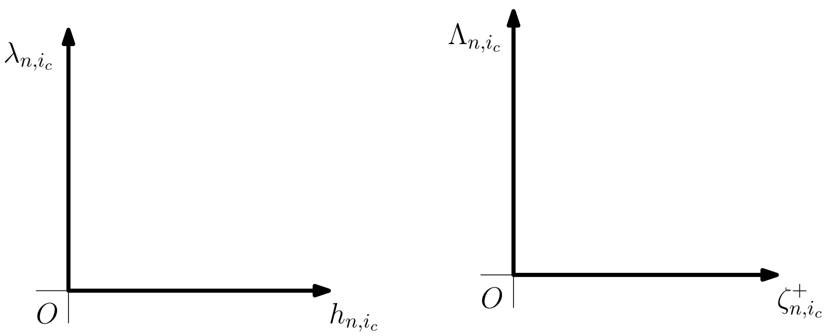
To describe the normal contact between rigid bodies Signorini's contact law is used. Since the bodies are impenetrable and reaction forces caused by contact can not prevent the bodies from seperating, both the contact distance h_{n,i_c} and λ_{n,i_c} can not become negative. Two situations are possible

1. $h_{n,i_c} = 0 \;\wedge\; \lambda_{n,i_c} \geq 0$ (closed-contact)
2. $h_{n,i_c} > 0 \;\wedge\; \lambda_{n,i_c} = 0$ (open-contact)

These situations are illustrated in Figure 2.2a, where it can be seen that the two situations are orthogonal. This behavior can be summarized in the complementarity condition

$$0 \leq h_{n,i_c} \perp \lambda_{n,i_c} \geq 0, \tag{2.9}$$

where the symbol $\perp$ is used to express the orthogonality between h_{n,i_c} and λ_{n,i_c}. The complementarity condition in (2.9) is called Signorini's contact law.



(a) *Signorini's contact law.*

(b) *Poisson's impact law without restitution.*

Figure 2.2

When contact happens at nonzero velocity, impact occurs. Newton's impact law is used to describe this impact. Newton's law of impact is defined as

$$\zeta^+_{n,i_c} = -e_{n,i_c}\zeta^-_{n,i_c}, \text{ when } h_{n,i_c} = 0,\; \dot{h}_{n,i_c} < 0. \tag{2.10}$$

In this work the coefficient of restitution e_{n,i_c} is assumed to be 0, describing a completely inelastic contact. For closed contact an impact law can be defined that relates the impulsive contact force Λ_{n,i_c} to the post-impact normal velocity ζ_{n,i_c}. When considering multi-contact systems, when a contact is closed two situations can occur:

1. $\Lambda_{n,i_c} > 0 \ \wedge \ \zeta^+_{n,i_c} = 0$ (impact)
2. $\Lambda_{n,i_c} = 0 \ \wedge \ \zeta^+_{n,i_c} \geq 0$ (no impact)

The second case can occur when a contact point other than i_c makes impact. The situations described above are illustrated in Figure 2.2b, where again the orthogonality can be observed. The behavior is written into the complementarity condition

$$0 \leq \zeta^+_{n,i_c} \perp \Lambda_{n,i_c} \geq 0, \qquad \forall i_c \in \mathcal{I}_{\mathrm{cl}}, \tag{2.11}$$

with $\mathcal{I}_{\mathrm{cl}}$ the set of closed contacts. The complementarity condition (2.11) is called Poisson's impact law. Note that the impact law is defined on velocity level, whereas the contact law is defined on position level.

2.2.2 Coulomb's friction law

Coulomb's friction law is often used to describe dry friction in mechanical systems. When considering 3-dimensional environments, Coulomb's friction law is defined as

$$||\boldsymbol{\lambda}_{t,i_c}|| \ \in \ \begin{cases} ||\boldsymbol{\lambda}_{t,i_c}|| \leq \mu\lambda_{n,i_c}, & \text{if } ||\boldsymbol{\zeta}_{t,i_c}|| = 0 \\ ||\boldsymbol{\lambda}_{t,i_c}|| = \mu\lambda_{n,i_c}, & \text{if } ||\boldsymbol{\zeta}_{t,i_c}|| > 0 \end{cases}, \tag{2.12}$$

and since friction is considered isotropic

$$\boldsymbol{\zeta}_{t,i_c} = -\kappa_{i_c}\frac{\boldsymbol{\lambda}_{t,i_c}}{||\boldsymbol{\lambda}_{t,i_c}||}. \tag{2.13}$$

with $\kappa_{i_c} > 0$.

(2.12) can be considered as a relation between the magnitude of the tangential velocity $\boldsymbol{\zeta}_{t,i_c}$ and the reaction friction force $\boldsymbol{\lambda}_{t,i_c}$. (2.13) can be considered as a relation between the direction of $\boldsymbol{\zeta}_{t,i_c}$ and $\boldsymbol{\lambda}_{t,i_c}$, namely that $\boldsymbol{\zeta}_{t,i_c}$ and $\boldsymbol{\lambda}_{t,i_c}$ are always in opposite directions. The constant κ_{i_c} can then be interpreted as the magnitude of the tangential velocity. Coulomb's friction law is illustrated in Figure 2.3a. In Figure 2.3b the same law is illustrated, but now as orthogonal vectors. As noticed earlier, this is convenient for writing the law in a complementarity form.

The complementarity formulation of the Coulomb's law is therefore defined as

$$0 \leq (\mu\lambda_{n,i_c} - ||\boldsymbol{\lambda}_{t,i_c}||) \ \perp \kappa_{i_c} \geq 0, \tag{2.14}$$

$$||\boldsymbol{\lambda}_{t,i_c}||\boldsymbol{\zeta}_{t,i_c} = -\kappa_{i_c}\boldsymbol{\lambda}_{t,i_c}. \tag{2.15}$$

(2.13) is rewritten to (2.15) to avoid singularity problems for $||\boldsymbol{\lambda}_{t,i_c}|| = 0$. For the impulsive behavior of the friction law Newton's impact law is used to define the tangential post-impact velocity as

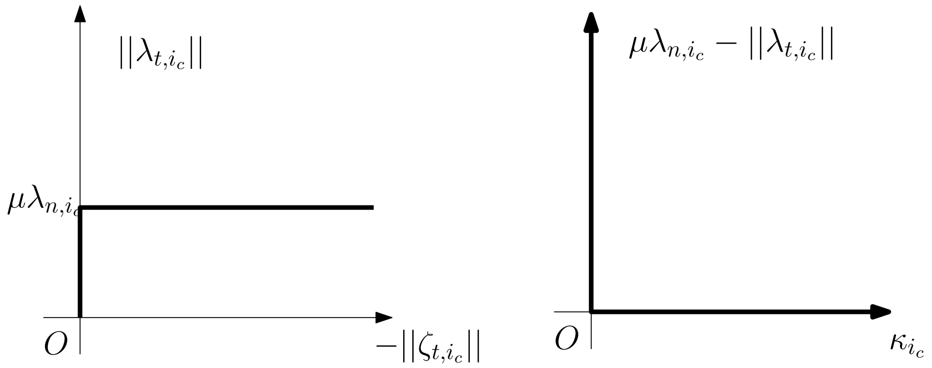
$$\boldsymbol{\zeta}^+_{t,i_c} = -e_{t,i_c}\boldsymbol{\zeta}^-_{t,i_c}, \tag{2.16}$$

where in this work $e_{t,i_c} = 0$ is assumed. Then, similarly to the non-impulsive case, the impulsive Coulomb's friction law can be defined as

$$0 \leq (\mu\Lambda_{n,i_c} - ||\boldsymbol{\Lambda}_{t,i_c}||) \perp \kappa_{i_c} \geq 0. \qquad \forall i_c \in \mathcal{I}_{\mathrm{cl}}, \tag{2.17}$$

$$||\boldsymbol{\Lambda}_{t,i}||\boldsymbol{\zeta}^+_{t,i_c} = \kappa_{i_c}\boldsymbol{\Lambda}_{t,i_c}, \qquad \forall i_c \in \mathcal{I}_{\mathrm{cl}}. \tag{2.18}$$

Note that just as the contact case, the impulsive friction law only holds for closed contacts.



(a) *Coulomb's friction law.*

(b) *Coulomb's friction law as orthogonal vectors. κ_{i_c} is defined as the magnitude of the tangential velocity $\boldsymbol{\zeta}_{t,i_c}$.*

2.2.3 System dynamics with contact and friction law

The flow dynamics are then described by

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\xi}} + \boldsymbol{H}(\boldsymbol{q},\boldsymbol{\xi}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{u} + \sum_{i_c \in \mathcal{I}_{\text{cl}}} \left(\boldsymbol{w}_{n,i_c}(\boldsymbol{q})\lambda_{n,i_c} + \boldsymbol{W}_{t,i_c}(\boldsymbol{q})\boldsymbol{\lambda}_{t,i_c}\right), \tag{2.19}$$

$$0 \leq h_{n,i_c} \perp \lambda_{n,i_c} \geq 0, \tag{2.20}$$

$$0 \leq (\mu\lambda_{n,i_c} - ||\boldsymbol{\lambda}_{t,i_c}||) \perp \kappa_{i_c} \geq 0, \tag{2.21}$$

$$||\boldsymbol{\lambda}_{t,i_c}||\boldsymbol{\zeta}_{t,i_c} = -\kappa_{i_c}\boldsymbol{\lambda}_{t,i_c}, \tag{2.22}$$

with

$$h_{n,i_c} = \boldsymbol{w}^T_{n,i_c}(\boldsymbol{q})\boldsymbol{q}, \tag{2.23}$$

$$\zeta_{n,i_c}(\boldsymbol{q}) = \boldsymbol{w}^T_{n,i_c}(\boldsymbol{q})\boldsymbol{\xi}, \tag{2.24}$$

$$\boldsymbol{\zeta}_{t,i_c}(\boldsymbol{q}) = \boldsymbol{W}^T_{t,i_c}(\boldsymbol{q})\boldsymbol{\xi}. \tag{2.25}$$

The discrete dynamics, which take place when a contact point goes through an event, are described by

$$\boldsymbol{M}(\boldsymbol{q})(\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-) = \sum_{i_c \in \mathcal{I}_{\text{cl}}} \left(\boldsymbol{w}_{n,i_c}(\boldsymbol{q})\Lambda_{n,i_c} + \boldsymbol{W}_{t,i_c}(\boldsymbol{q})\boldsymbol{\Lambda}_{t,i_c}\right), \tag{2.26}$$

$$0 \leq \zeta^+_{n,i_c} \perp \Lambda_{n,i_c} \geq 0, \qquad \forall i_c \in \mathcal{I}_{\text{cl}}, \tag{2.27}$$

$$0 \leq (\mu\Lambda_{n,i_c} - ||\boldsymbol{\Lambda}_{t,i_c}||) \perp \kappa_{i_c} \geq 0. \qquad \forall i_c \in \mathcal{I}_{\text{cl}}, \tag{2.28}$$

$$||\boldsymbol{\Lambda}_{t,i_c}||\boldsymbol{\zeta}^+_{t,i_c} = -\kappa_i\boldsymbol{\Lambda}_{t,i_c}, \qquad \forall i_c \in \mathcal{I}_{\text{cl}}, \tag{2.29}$$

with

$$\zeta^+_{n,i_c}(\boldsymbol{q}) = \boldsymbol{w}^T_{n,i_c}(\boldsymbol{q})\boldsymbol{\xi}^+, \tag{2.30}$$

$$\boldsymbol{\zeta}^+_{t,i_c}(\boldsymbol{q}) = \boldsymbol{W}^T_{t,i_c}(\boldsymbol{q})\boldsymbol{\xi}^+. \tag{2.31}$$

2.3 Hybrid system formulation

In this section the dynamics of the complementarity system defined in Section 2.2 is written to a hybrid formulation, resulting in a hybrid framework for mechanical systems with unilateral constraints and spatial friction. The hybrid system formulation only holds for trajectories where only the contact points activating a guard function are allowed to change mode, i.e. no superfluous contacts. More information on the considered trajectories in this work is given in Section E.

2.3.1 Hybrid systems with impulsive effects

According to [50] an impulsive dynamical system can be described by a hybrid system with impulsive effects. This makes it a valid framework for mechanical systems with unilateral constraints and spatial friction. The notation given in [50] is convenient from a tracking point-of-view, in that only the dynamics encountered during the trajectory need to be described. A hybrid system with impulsive effects consists of three elements:

1. Continuous dynamics, a continuous-time differential equation which defines the behavior of the system in between events
2. Discrete dynamics, which defines the way the state of the system is reset during events
3. Reset sets, which is a criterion to decide when the state of the system is to be reset

Therefore, a hybrid system with impulsive effects is given by

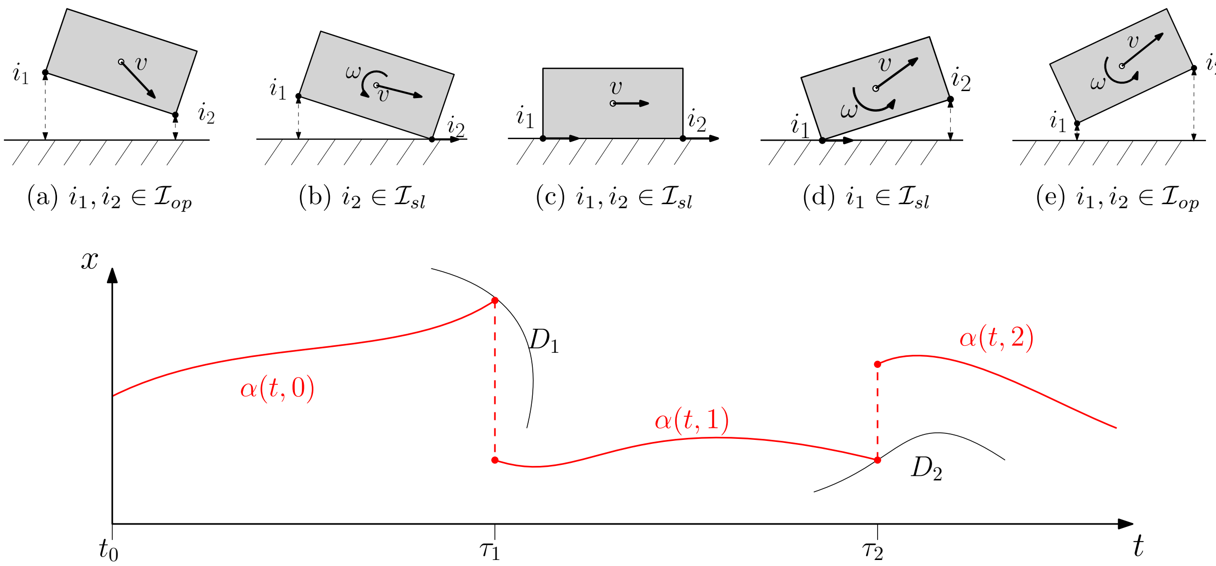
$$\begin{aligned} \dot{\boldsymbol{x}}(t,i) &= \boldsymbol{f}_i(\boldsymbol{x}(t,i), \boldsymbol{u}(t,i), t), & \boldsymbol{x}(t,i), \boldsymbol{u}(t,i) &\notin D_i \\ \boldsymbol{x}(t,i) &= \boldsymbol{g}_i(\boldsymbol{x}(t,i-1), \boldsymbol{u}(t,i-1), t), & \boldsymbol{x}(t,i-1), \boldsymbol{u}(t,i-1) &\in D_i \end{aligned} \tag{2.32}$$

with $\boldsymbol{x}(t,i) \in \mathbb{R}^{n(i)}$, $\boldsymbol{u}(t,i) \in \mathbb{R}^{m(i)}$, $\boldsymbol{f}_i(t,i) : \mathbb{R}^{n(i)} \times \mathbb{R}^{m(i)} \times \mathbb{R} \rightarrow \mathbb{R}^{n(i)}$. Note that the state dimension $n(i)$ and the input dimension $m(i)$ can vary in different modes i. For the difference equation we have $\boldsymbol{g}_i : \mathbb{R}^{n(i-1)} \times \mathbb{R}^{m(i-1)} \times \mathbb{R} \rightarrow \mathbb{R}^{n(i)}$. The set $D_i = D_i(t) := \{\boldsymbol{x}(t,i) \in \mathbb{R}^{n(i)},\ \boldsymbol{u}(t,i) \in \mathbb{R}^{m(i)} \mid \gamma_i^{=}(\boldsymbol{x}^\wedge, \boldsymbol{u}^\wedge, t) = 0,\ \gamma_i^{\geq}(\boldsymbol{x}^\wedge, \boldsymbol{u}^\wedge, t) \geq 0\}$, where $\gamma_i^{=}$ and $\gamma_i^{\geq}$ are some sets of guard functions that are activated at event i, and $\boldsymbol{x}^\wedge$,$\boldsymbol{u}^\wedge$ are some virtual state and input that are not necessarily physically realistic.

The dynamics given in (2.32) will be used to describe a tracking problem for mechanical systems with unilateral constraints and spatial friction. A nominal state-input trajectory is considered consisting of absolutely continuous segments $(\boldsymbol{\alpha}(t,i), \boldsymbol{\mu}(t,i))$, with $t \in [\tau_i, \tau_{i+1}]$ and $i \in \{0, 1, ..., N\}$ the event counter. $\boldsymbol{\alpha}(t,i)$ and $\boldsymbol{\mu}(t,i)$ are the nominal state and input, respectively, that define the nominal trajectory with N events. τ_i is referred to as the nominal event time of event i and t is referred to as regular time. Every segment of the nominal trajectory $(\boldsymbol{\alpha}(t,i), \boldsymbol{\mu}(t,i))$ and every event $i \in \{0, 1, ..., N\}$ satisfy the dynamics in (2.32), where an event happens when $(\boldsymbol{\alpha}(t,i), \boldsymbol{\mu}(t,i))$ enters D_{i+1}. Such a nominal trajectory existing of absolutely continuous segments experiencing events according to (2.32) is illustrated in Figure 2.4f.

In 2.4 an example trajectory of a block pushing towards a surface is illustrated. The block has two contact points on its edges i_1 and i_2. It starts with both contact points in open contact as can be seen in Figure 2.4a. The flow is described by $\dot{\boldsymbol{\alpha}}(t,0) = \boldsymbol{f}_0(\boldsymbol{\alpha}(t,0), \boldsymbol{\mu}(t,0), t)$ for $t \in [t_0, \tau_1]$. Then i_2 makes impact with the surface at τ_1, causing a jump in the state and a change in continuous dynamics $\dot{\boldsymbol{\alpha}}(t,1) = \boldsymbol{f}_1(\boldsymbol{\alpha}(t,1), \boldsymbol{\mu}(t,1), t)$ for $t \in [\tau_1, \tau_2]$. This is illustrated in Figure 2.4b, where i_2 is in closed contact and slipping over the contact surface. Finally, at τ_2, i_1 makes impact as well causing another jump and another change in continuous dynamics. The block now has both contact points slipping over the contact surface, which is the final mode of the trajectory.

The following sections will be used to write the complementarity system defined in Section 2.2 into a hybrid system with impulsive effects as in (2.32). In 2.3.2 the continuous dynamics $\boldsymbol{f}_i$ will be derived



(a) $i_1, i_2 \in \mathcal{I}_{op}$ (b) $i_2 \in \mathcal{I}_{sl}$ (c) $i_1, i_2 \in \mathcal{I}_{sl}$ (d) $i_1 \in \mathcal{I}_{sl}$ (e) $i_1, i_2 \in \mathcal{I}_{op}$

(f) *The state trajectory of the system.*

Figure 2.4: *An example trajectory, which satisfies the dynamics* (2.32)*, of a block moving towards a surface with velocity v. The block starts with both contact points i_1 and i_2 in open contact and through three events the block ends with both contact points in closed contact stick. At the nominal event time τ_1 the trajectory governed by $\dot{\boldsymbol{\alpha}}(t,0) = \boldsymbol{f}_0(\boldsymbol{\alpha}(t,0), \boldsymbol{\mu}(t,0), t)$ enters the set D_1, which represents contact point i_2 making impact with the surface. This leads to a state jump described by $\boldsymbol{\alpha}(\tau_1, 1) = \boldsymbol{g}_1(\boldsymbol{\alpha}(t,0), \boldsymbol{\mu}(t,0), \tau_1)$. After event 1 the flow continues according to $\dot{\boldsymbol{\alpha}}(t,1) = \boldsymbol{f}_1(\boldsymbol{\alpha}(t,1), \boldsymbol{\mu}(t,1), t)$. A similar series of events happens when contact point i_1 makes impact at τ_2. Note that the sets D_1 and D_2, besides being dependent on $\boldsymbol{x}$, are also dependent on $\boldsymbol{u}$.*

for mechanical systems with unilateral constraints and spatial friction. Then, in Section 2.3.3 the reset set D_i will be defined. Finally, in Section 2.3.4 the discrete dynamics $\boldsymbol{g}_i$ will be derived. This will fully define the hybrid system with impulsive effects formulation of mechanical systems with unilateral constraints and spatial friction.

2.3.2 Continuous dynamics

When describing mechanical systems, we take

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{bmatrix}, \tag{2.33}$$

where $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are the joint positions, velocities and accelerations respectively. As described in Section 2.2, for mechanical systems a set of contact points $i_c \in \{i_1, i_2, ..., i_C\}$ is defined. Here C is the number of considered contact points. A set $\mathcal{I}_{\text{cl}}$ is defined as the set of closed contact points, such that a contact $i \in \mathcal{I}_{\text{cl}}$ has closed a unilateral constraint. The set of contact points in open contact is defined as $\mathcal{I}_{\text{op}} := \{i_c \mid i_c \notin \mathcal{I}_{\text{cl}}\}$. The set $\mathcal{I}_{\text{cl}}$ is subdivided in two subsets $\mathcal{I}_{\text{sl}}$ and $\mathcal{I}_{\text{st}}$, where $\mathcal{I}_{\text{sl}}$ is the set of closed contact points in slip and $\mathcal{I}_{\text{st}}$ the set of closed contact points in stick. Here $\mathcal{I}_{\text{cl}} = \mathcal{I}_{\text{sl}} \cup \mathcal{I}_{\text{st}}$ and $\mathcal{I}_{\text{sl}} \cup \mathcal{I}_{\text{st}} = \emptyset$.

The equations of motion, given in (2.3), can be rewritten to

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\left[\boldsymbol{S}(\boldsymbol{q})\boldsymbol{u} - \boldsymbol{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{W}_n(\boldsymbol{q})\boldsymbol{\lambda}_n + \boldsymbol{W}_t(\boldsymbol{q})\boldsymbol{\lambda}_t\right], \tag{2.34}$$

with

$$\begin{aligned}
\boldsymbol{W}_n &= \left[\boldsymbol{w}_{n,i_1}, \boldsymbol{w}_{n,i_2}, ..., \boldsymbol{w}_{n,i_c}\right] \in \mathbb{R}^{n\times C}, && (2.35)\\
\boldsymbol{W}_t &= \left[\boldsymbol{W}_{t,i_1}, \boldsymbol{W}_{t,i_2}, ..., \boldsymbol{W}_{t,i_c}\right] \in \mathbb{R}^{n\times 2C}, && (2.36)\\
\boldsymbol{\lambda}_n &= \left[\lambda_{n,i_1}; \lambda_{n,i_2}; ...; \lambda_{n,i_C}\right] \in \mathbb{R}^{C}, && (2.37)\\
\boldsymbol{\lambda}_t &= \left[\boldsymbol{\lambda}_{t,i_1}; \boldsymbol{\lambda}_{t,i_2}; ...; \boldsymbol{\lambda}_{t,i_C}\right] \in \mathbb{R}^{2C}, && (2.38)
\end{aligned}$$

Note that since $\boldsymbol{f}_i$ is only defined on $\boldsymbol{x}(t,i), \boldsymbol{u}(t,i) \notin D_i$, it is not necessary to use $\boldsymbol{\xi}$ to define the equations of motion as done in (2.3). Now $\boldsymbol{f}_i$ can be written as

$$\dot{\boldsymbol{x}}(t,i) = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}(\boldsymbol{q})\left[\boldsymbol{S}(\boldsymbol{q})\boldsymbol{u} - \boldsymbol{H}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{W}_n(\boldsymbol{q})\boldsymbol{\lambda}_n + \boldsymbol{W}_t(\boldsymbol{q})\boldsymbol{\lambda}_t\right] \end{bmatrix}. \tag{2.39}$$

The closed contact points in $\mathcal{I}_{\text{cl}}$ experience reaction forces λ_{n,i_c} and $\boldsymbol{\lambda}_{t,i_c}$, as can be seen in (2.34). Therefore for all closed contact points $i_c \in \mathcal{I}_{\text{cl}}$ constraints are given which define these reaction forces. These constraints are given by

$$\begin{array}{ll}
\boldsymbol{w}_{n,i_c}^T(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{w}}_{n,i_c}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0, & \forall i_c \in \mathcal{I}_{\text{cl}}, \\
\boldsymbol{\lambda}_{t,i_c}||\boldsymbol{W}_{t,i_c}^T\dot{\boldsymbol{q}}|| + \mu\lambda_{n,i_c}\boldsymbol{W}_{t,i_c}^T\dot{\boldsymbol{q}} = 0, & \forall i_c \in \mathcal{I}_{\text{sl}}, \\
\boldsymbol{W}_{t,i_c}^T(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{W}}_{t,i_c}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0, & \forall i_c \in \mathcal{I}_{\text{st}},
\end{array} \tag{2.40}$$

where $\mathcal{I}_{\text{sl}}$ and $\mathcal{I}_{\text{st}}$ are the sets of closed contacts in slip and closed contacts in stick respectively. Now, with (2.39) and (2.40), the continuous dynamics of the hybrid system with impulsive dynamics are correctly defined. A more thorough derivation of these dynamics can be found in Appendix A.1.

2.3.3 Discrete event sets

When the continuous dynamics enter an event set D, an event will take place. Such an event can cause a contact point to enter a different set, which will lead to different post-event continuous dynamics, and it can cause the state to reinitialize according to the discrete dynamics presented in Section 2.3.4. For each set of contact points these sets are defined differently. In this section the sets D are defined for each set of contact points. The derivation of the discrete event sets can be found in Appendix A.1.

Discrete events sets in open contact $D_{\text{op}\to}$
For all contact points $i_c \in \mathcal{I}_{\text{op}}$ the discrete event sets are defined as

$$\begin{aligned}
D_{\text{op}\to\text{sl}} &= \{\boldsymbol{q} \mid \gamma_{\text{op}\to\text{cl}} = 0,\ \dot{\gamma}_{\text{op}\to\text{cl}} < 0,\ \Gamma < 0\}, && (2.41)\\
D_{\text{op}\to\text{st}} &= \{\boldsymbol{q} \mid \gamma_{\text{op}\to\text{cl}} = 0,\ \dot{\gamma}_{\text{op}\to\text{cl}} < 0,\ \Gamma \geq 0\}, && (2.42)
\end{aligned}$$

with

$$\begin{aligned}
\gamma_{\text{op}\to\text{cl}} &= h_{n,i_c}(\boldsymbol{q}), && (2.43)\\
\Gamma &= \mu^2\underline{\Lambda}_{n,i_c}^2(\boldsymbol{q},\dot{\boldsymbol{q}}) - \underline{\boldsymbol{\Lambda}}_{t,i_c}(\boldsymbol{q},\dot{\boldsymbol{q}})\underline{\boldsymbol{\Lambda}}_{t,i_c}^T(\boldsymbol{q},\dot{\boldsymbol{q}}). && (2.44)
\end{aligned}$$

The contact distance $h_{n,i_c}(\boldsymbol{q})$ is given by

$$h_{n,i_c}(\boldsymbol{q}) = \boldsymbol{w}_{n,i_c}^T(\boldsymbol{q})\boldsymbol{q}, \tag{2.45}$$

and the virtual impulsive reaction force are determined by

$$\begin{bmatrix} \underline{\Lambda}_n \\ \underline{\boldsymbol{\Lambda}}_t \end{bmatrix} = -\boldsymbol{D}^{-1} \begin{bmatrix} \zeta_n \\ \zeta_t \end{bmatrix}, \quad \text{with } \boldsymbol{D} = \begin{bmatrix} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \\ \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \end{bmatrix}, \tag{2.46}$$

where $\boldsymbol{D}$ is often called the Delassus-matrix. For the definition of Γ the virtual reaction forces $\underline{\Lambda}_{n,i}$ and $\underline{\boldsymbol{\Lambda}}_{t,i}$ are used, because their value is not physically realistic everywhere on the manifold where Γ is defined. More information on the derivation of these virtual reaction forces can be found in Section A.1.2.

Discrete events sets in closed contact slip $D_{\text{sl}\to}$
For all contact points $i_c \in \mathcal{I}_{\text{sl}}$ the discrete event sets are defined as

$$D_{\text{sl}\to\text{st}} = \{\boldsymbol{q} \mid \gamma_{\text{sl}\to\text{st}} = 0,\ \dot{\gamma}_{\text{sl}\to\text{st}} < 0\} \tag{2.47}$$

$$D_{\text{sl}\to\text{op}} = \{\boldsymbol{q}, \boldsymbol{u} \mid \gamma_{\text{cl}\to\text{op}} = 0,\ \dot{\gamma}_{\text{cl}\to\text{op}} < 0\}, \tag{2.48}$$

with

$$\gamma_{\text{sl}\to\text{st}} = \boldsymbol{\zeta}_{t,i_c}\boldsymbol{\zeta}_{t,i_c}^T \tag{2.49}$$

$$\gamma_{\text{cl}\to\text{op}} = \lambda_{n,i_c}. \tag{2.50}$$

The tangential velocity $\boldsymbol{\zeta}_{t,i_c}$ is given by

$$\boldsymbol{\zeta}_t = \boldsymbol{W}_t^T \boldsymbol{q}, \tag{2.51}$$

and the reaction normal force in slip is given by

$$\boldsymbol{\lambda}_n = -\left[\boldsymbol{W}_n^T \boldsymbol{M}^{-1} \left(\boldsymbol{W}_n - \mu \boldsymbol{W}_t \boldsymbol{Z}_t\right)\right]^{-1} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}\right] - \dot{\boldsymbol{W}}_n^T \dot{\boldsymbol{q}}, \tag{2.52}$$

with

$$\boldsymbol{Z}_t = \begin{bmatrix} \langle \boldsymbol{\zeta}_{t,i_1} \rangle & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \langle \boldsymbol{\zeta}_{t,i_2} \rangle & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \langle \boldsymbol{\zeta}_{t,i_C} \rangle \end{bmatrix}. \quad \in \mathbb{R}^{2C\times C} \tag{2.53}$$

Discrete events sets in closed contact stick $D_{\text{st}\to}$
For all contact points $i_c \in \mathcal{I}_{\text{st}}$ the discrete event sets are defined as

$$D_{\text{st}\to\text{sl}} = \{\boldsymbol{q}, \boldsymbol{u} \mid \gamma_{\text{st}\to\text{sl}} = 0,\ \dot{\gamma}_{\text{sl}\to\text{st}} < 0\} \tag{2.54}$$

$$D_{\text{st}\to\text{op}} = \{\boldsymbol{q}, \boldsymbol{u} \mid \gamma_{\text{cl}\to\text{op}} = 0,\ \dot{\gamma}_{\text{cl}\to\text{op}} < 0\}, \tag{2.55}$$

with

$$\gamma_{\text{st}\to\text{sl}} = \mu^2 \lambda_{n,i_c}^2 - \boldsymbol{\lambda}_{t,i_c}\boldsymbol{\lambda}_{t,i_c}^T \tag{2.56}$$

$$\gamma_{\text{cl}\to\text{op}} = \lambda_{n,i_c}. \tag{2.57}$$

The tangential and normal reaction forces in stick are given by

$$\begin{bmatrix} \boldsymbol{\lambda}_n \\ \boldsymbol{\lambda}_t \end{bmatrix} = -\boldsymbol{D}^{-1} \begin{bmatrix} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}\right] - \dot{\boldsymbol{W}}_n^T \dot{\boldsymbol{q}} \\ \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}\right] - \dot{\boldsymbol{W}}_t^T \dot{\boldsymbol{q}} \end{bmatrix}, \tag{2.58}$$

with $\boldsymbol{D}$ defined as in (2.46).

2.3.4 Discrete dynamics

When a discrete event set D is entered, the discrete dynamics describe the reinitialization of the state. The discrete dynamics are defined by a jump map $\boldsymbol{g}$. Since there are three different modes for each contact point, three different discrete dynamics functions should be defined: one for a reinitialization into slip, one for a reinitialization into stick and one for a reinitialization into open-contact.

Slip post-event mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{sl}}(\dot{\boldsymbol{q}}^-)$
From the discrete dynamics, defined in (2.26)-(2.29), an expression for Λ_n and Λ_t for a slip post-event mode can be found as

$$\boldsymbol{\Lambda}_n = -\left[\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n - \mu\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\boldsymbol{Z}_t^-\right]^{-1}\boldsymbol{W}_n^T\dot{\boldsymbol{q}}^-. \tag{2.59}$$

$$\boldsymbol{\Lambda}_t = -\mu\boldsymbol{Z}_t^-\boldsymbol{\lambda}_n, \tag{2.60}$$

with

$$\boldsymbol{Z}_t = \begin{bmatrix} \langle\boldsymbol{\zeta}_{t,i_1}^-\rangle & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \langle\boldsymbol{\zeta}_{t,i_2}^-\rangle & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \langle\boldsymbol{\zeta}_{t,i_C}^-\rangle \end{bmatrix}, \quad \in \mathbb{R}^{2C\times C} \tag{2.61}$$

where $\langle\boldsymbol{\zeta}_t^-\rangle = \left[\langle\boldsymbol{\zeta}_{t,i_1}^-\rangle, \langle\boldsymbol{\zeta}_{t,i_2}^-\rangle, ..., \langle\boldsymbol{\zeta}_{t,i_C}^-\rangle\right]^T$, with $\langle a\rangle$ the unit vector of a.
The jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{sl}}(\dot{\boldsymbol{q}}^-)$ is then found by substituting (2.59) and (2.60) into (2.26),

$$\dot{\boldsymbol{q}}^+ = -\left[\boldsymbol{M}^{-1}\boldsymbol{W}_n - \mu\boldsymbol{M}^{-1}\boldsymbol{W}_t\boldsymbol{Z}_t^-\right]\left[\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n - \mu\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\boldsymbol{Z}_t^-\right]^{-1}\boldsymbol{W}_n^T\dot{\boldsymbol{q}}^- + \dot{\boldsymbol{q}}^-. \tag{2.62}$$

Note that when there is a transition from stick to slip, i.e., there is no impact, the post-event velocity is $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$. The acceleration is allowed to jump, which means that in this case the state is continuous, but not differentiable. This is called a Filippov-discontinuity [2].

Stick post-event mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{st}}(\dot{\boldsymbol{q}}^-)$
Λ_n and Λ_t for a stick post-impact mode are given by

$$\begin{bmatrix}\boldsymbol{\Lambda}_n \\ \boldsymbol{\Lambda}_t\end{bmatrix} = -\boldsymbol{D}^{-1}\begin{bmatrix}\boldsymbol{W}_n^T\dot{\boldsymbol{q}}^- \\ \boldsymbol{W}_t^T\dot{\boldsymbol{q}}^-\end{bmatrix}, \quad \text{with } \boldsymbol{D} = \begin{bmatrix}\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t \\ \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\end{bmatrix}. \tag{2.63}$$

$\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Lambda}_t$ can now be used to define the jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{st}}(\dot{\boldsymbol{q}}^-)$ as

$$\dot{\boldsymbol{q}}^+ = \boldsymbol{M}^{-1}\boldsymbol{W}_n\boldsymbol{\Lambda}_n + \boldsymbol{M}^{-1}\boldsymbol{W}_t\boldsymbol{\Lambda}_t + \dot{\boldsymbol{q}}^-. \tag{2.64}$$

Similarly to the stick to slip case, note that when the transition from slip to stick is also a Filippov-discontinuity.

Open-contact post-event mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{op}}(\dot{\boldsymbol{q}}^-)$
In the open-contact post-event mode case there are no impulsive reaction forces, and it is trivial to see from (2.26) that the jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to\mathrm{op}}(\dot{\boldsymbol{q}}^-)$ is given by

$$\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-. \tag{2.65}$$

2.4 Summary

Chapter 3

Tracking for Hybrid Systems: Ordered State-Input-Triggered Events

Use Rijnen2017 and 3.1 of Hao.

3.1 Reference spreading

- Perturbed Trajectories
- Reference Spreading (error definition, stability definition)

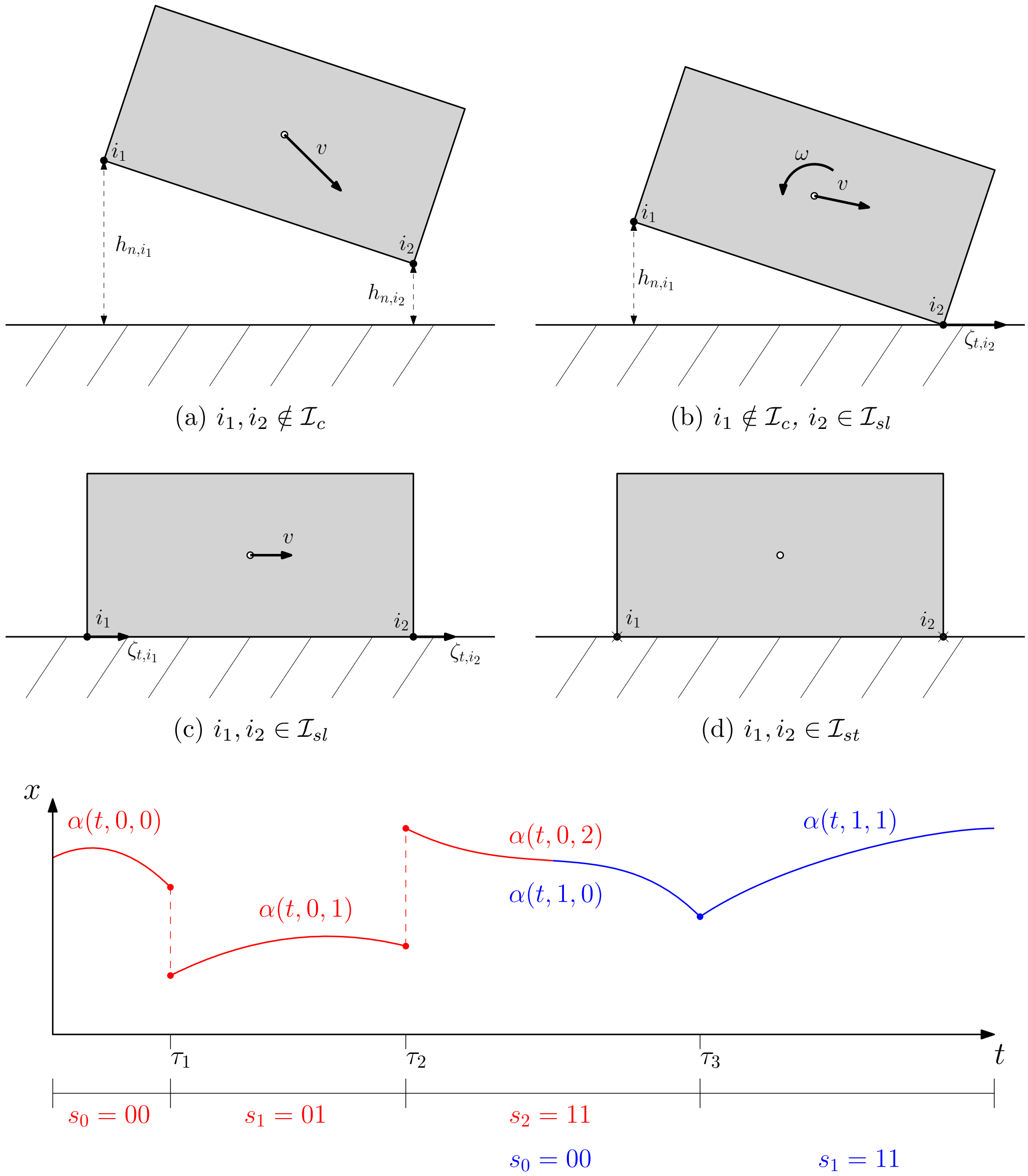
3.2 First-order approximation of trajectories with ordered guard-activations

Use Rijnen2017 alot

- LTTHS (Rijnen2017)
- First-order accuracy (Rijnen2017)

3.3 Stability analysis for linear time-triggered hybrid systems

3.4 Summary



(a) $i_1, i_2 \notin \mathcal{I}_c$

(b) $i_1 \notin \mathcal{I}_c$, $i_2 \in \mathcal{I}_{sl}$

(c) $i_1, i_2 \in \mathcal{I}_{sl}$

(d) $i_1, i_2 \in \mathcal{I}_{st}$

(e) *The state trajectory of the system.*

Figure 3.1: *An example trajectory of a block moving towards a surface with velocity v. The block starts with both contact points i_1 and i_2 in open contact and through three events the block ends with both contact points in closed contact stick.*

Chapter 4

Tracking for Hybrid Systems: Simultaneous State-Input-Triggered Events

[51]

4.1 Simultaneous guard-activation

- Simultaneous events and assumptions(same post-event mode,associativity)
- Event character, mode descriptor, micro counter, historical notation
- Unidirectional event completion

4.2 First-order approximation for trajectories with simultaneous guard-activation

- Positive homogeneity
- Positive homogeneous jump gain
- PHTTHS (Positive homogeneous time-triggered hybrid system)

4.3 Stability analysis for positively homogeneous time-triggered hybrid systems

4.4 Summary

Chapter 5

Numerical Validation

Chapter 6

Conclusions and Recommendations

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Appendix A

Nonsmooth modeling

A.1 Hybrid system formulation for mechanical systems

A.1.1 Continuous dynamics derivation

A.1.2 Discrete event set derivation

In this section the discrete event sets D are defined. When the state or input of the system enters a discrete event set, contact points can change set and a reinitialization of the state can take place. In this work the assumption is made that all contact points in closed contact are in the same set, i.e., in either $\mathcal{I}_{\text{sl}}$ or $\mathcal{I}_{\text{st}}$.

More elaboration on why the conditions that are chosen are chosen (i.e., why $||\boldsymbol{\zeta}_{t,i}|| = 0$)

Recap each paragraph using section D and mentioning γ's again.

Open to stick/slip
When a contact point is "open", it can trigger a guard function $\gamma_{\text{op}\to\text{cl}}$ to go from open to closed. This guard is defined using the contact distance h_{n,i_c}. When $h_{n,i_c} > 0$, the contact point is in open-contact. When $h_{n,i_c} = 0$ with $\dot{h}_{n,i_c} < 0$, the contact point enters the closed-contact mode with a non-zero ante-impact velocity. Therefore the guard function $\gamma_{\text{op}\to\text{cl}}$ is given by

$$\gamma_{\text{op}\to\text{cl}} = h_{n,i_c}(\boldsymbol{q}). \tag{A.1}$$

The plane that spans $\gamma = 0$ is divided in two regions: a region where the post-impact state is in slip and a region where the post-impact state is in stick. This region is defined by Γ_{i_c}, where $\Gamma_{i_c} < 0$ in the region where the contact point goes to slip and $\Gamma_{i_c} > 0$ in the region where the contact point goes to stick. When $\Gamma_{i_c} = 0$ the system is right at the border between a slip post-impact state and a stick post-impact state. This is illustrated in Figure A.1.

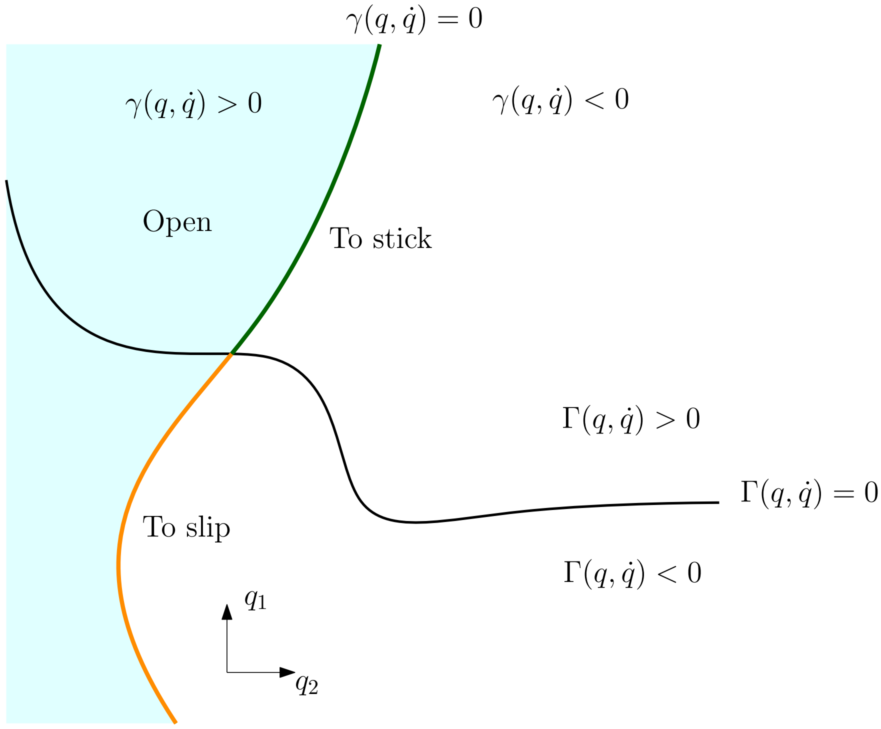

Figure A.1: *The functions $\boldsymbol{\gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{\Gamma}(\boldsymbol{q},\dot{\boldsymbol{q}})$ illustrated in the state space of $\boldsymbol{q} \in \mathbb{R}^2$. The light blue area is the state space where the contact is open, and goes the closed when it triggers $\gamma = 0$. If it triggers $\gamma = 0$ in the area where $\Gamma < 0$ (orange), then the contact will go to slip. If it triggers $\gamma = 0$ in the area where $\Gamma \geq 0$ (green), then the contact will go to stick.*

For slip, we know that $\mu\Lambda_{n,i_c} - ||\boldsymbol{\Lambda}_{t,i_c}|| = 0$ and for stick, we know that $\mu\Lambda_{n,i_c} - ||\boldsymbol{\Lambda}_{t,i_c}|| \geq 0$. From this we can derive the guard function

$$\Gamma_{i_c} = \mu^2 \underline{\Lambda}^2_{n,i_c}(\underline{\boldsymbol{q}},\underline{\dot{\boldsymbol{q}}}) - \underline{\boldsymbol{\Lambda}}_{t,i_c}(\underline{\boldsymbol{q}},\underline{\dot{\boldsymbol{q}}})\underline{\boldsymbol{\Lambda}}^T_{t,i_c}(\underline{\boldsymbol{q}},\underline{\dot{\boldsymbol{q}}}). \tag{A.2}$$

The underlined variables are virtual states, meaning that they do not necessarily have a physical meaning. The way (A.2) is given, it is defined in both open and closed contact. This map is only physically realistic during an event from open to closed contact. However, by using virtual states we obtain a differentiable guard function Γ_{i_c}. This guard function Γ_{i_c} satisfies the requirements that $\Gamma_{i_c} < 0$ in the region where the contact point goes to slip, $\Gamma_{i_c} > 0$ in the region where the contact point goes to stick and $\Gamma_{i_c} = 0$ at the border. We now find expressions for Λ_{n,i_c} and $\boldsymbol{\Lambda}_{t,i_c}$ by looking at the jump map to stick, given in (**??**) to (**??**).

We can rewrite (**??**) to

$$\dot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\boldsymbol{W}_n\underline{\boldsymbol{\Lambda}}_n + \boldsymbol{M}^{-1}\boldsymbol{W}_t\underline{\boldsymbol{\Lambda}}_t + \underline{\dot{\boldsymbol{q}}}, \tag{A.3}$$

and (**??**), (**??**) to

$$\boldsymbol{W}_n^T\underline{\dot{\boldsymbol{q}}} = 0, \tag{A.4}$$

$$\boldsymbol{W}_t^T\underline{\dot{\boldsymbol{q}}} = 0, \tag{A.5}$$

with

$$\boldsymbol{W}_n = \left[\boldsymbol{w}_{n,i_1}, \boldsymbol{w}_{n,i_2}, ..., \boldsymbol{w}_{n,i_c}\right] \in \mathbb{R}^{n\times C}, \tag{A.6}$$
$$\boldsymbol{W}_t = \left[\boldsymbol{W}_{t,i_1}, \boldsymbol{W}_{t,i_2}, ..., \boldsymbol{W}_{t,i_c}\right] \in \mathbb{R}^{n\times 2C}, \tag{A.7}$$
$$\underline{\boldsymbol{\Lambda}}_n = \left[\underline{\Lambda}_{n,i_1}; \underline{\Lambda}_{n,i_2}; ...; \underline{\Lambda}_{n,i_c}\right] \in \mathbb{R}^{C}, \tag{A.8}$$
$$\underline{\boldsymbol{\Lambda}}_t = \left[\underline{\boldsymbol{\Lambda}}_{t,i_1}; \underline{\boldsymbol{\Lambda}}_{t,i_2}; ...; \underline{\boldsymbol{\Lambda}}_{t,i_c}\right] \in \mathbb{R}^{2C}. \tag{A.9}$$

Substituting (A.3) into (A.4) and (A.5) leads to

$$\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n\underline{\boldsymbol{\Lambda}}_n + \boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\underline{\boldsymbol{\Lambda}}_t + \underline{\boldsymbol{\zeta}}_n = 0 \tag{A.10}$$
$$\boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_n\underline{\Lambda}_n + \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\underline{\boldsymbol{\Lambda}}_t + \underline{\boldsymbol{\zeta}}_t = 0, \tag{A.11}$$

respectively, with

$$\underline{\boldsymbol{\zeta}}_n = \boldsymbol{W}_n^T\underline{\dot{\boldsymbol{q}}}, \tag{A.12}$$
$$\underline{\boldsymbol{\zeta}}_t = \boldsymbol{W}_t^T\underline{\dot{\boldsymbol{q}}}. \tag{A.13}$$

This is now rewritten to

$$\begin{bmatrix}\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\\ \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\end{bmatrix}\begin{bmatrix}\underline{\boldsymbol{\Lambda}}_n\\ \underline{\boldsymbol{\Lambda}}_t\end{bmatrix} + \begin{bmatrix}\underline{\boldsymbol{\zeta}}_n\\ \underline{\boldsymbol{\zeta}}_t\end{bmatrix} = 0, \tag{A.14}$$

which is in turn rewritten to

$$\begin{bmatrix}\underline{\boldsymbol{\Lambda}}_n\\ \underline{\boldsymbol{\Lambda}}_t\end{bmatrix} = -\boldsymbol{D}^{-1}\begin{bmatrix}\underline{\boldsymbol{\zeta}}_n\\ \underline{\boldsymbol{\zeta}}_t\end{bmatrix}, \quad \text{with } \boldsymbol{D} = \begin{bmatrix}\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_n^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\\ \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_n & \boldsymbol{W}_t^T\boldsymbol{M}^{-1}\boldsymbol{W}_t\end{bmatrix}. \tag{A.15}$$

The matrix $\boldsymbol{D}$ is often called a Delassus-matrix. We now have expressions for $\underline{\boldsymbol{\Lambda}}_n$ and $\underline{\boldsymbol{\Lambda}}_t$ which are continuous and differentiable in $(\boldsymbol{q}, \dot{\boldsymbol{q}})$. It is straightforward that $\Gamma(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is continuous and differentiable as well. In this work only trajectories where all closed contacts are in the same mode are considered. This means that when (A.2) is smaller than zero, i.e., the reaction forces are infeasible for a stick post-impact mode, all contact points have a feasible slip post-impact mode. For trajectories where different contact points can be in slip and in stick at the same time, this conclusion can not be drawn. One should then iterate over all possible post-impact modes until a post-impact mode is found which has feasible reaction forces.

Slip to stick/open

When a contact point is in closed-contact slip, it can transition to closed-contact stick and it can transition to open-contact. A slipping contact transitions to sticking when the tangential velocity of the contact point is zero, i.e.,

$$||\boldsymbol{\zeta}_{t,i_c}|| = 0. \tag{A.16}$$

A guard function that can be used to describe this set is

$$\gamma_{\text{sl}\to\text{st}} = \boldsymbol{\zeta}_{t,i_c}\boldsymbol{\zeta}_{t,i_c}^T, \tag{A.17}$$

which is equal to zero when $||\boldsymbol{\zeta}_{t,i_c}|| = 0$, greater than zero when $||\boldsymbol{\zeta}_{t,i_c}|| > 0$, smaller than zero when $||\boldsymbol{\zeta}_{t,i_c}|| < 0$, and it is globally differentiable. For a slipping contact transitioning to open-contact an acceleration based guard function is necessary, since the normal velocity of the contact point is constrained in the continuous dynamics of a slipping contact. Therefore, a slipping contact point transitions to open-contact when

$$\lambda_{n,i_c} = 0. \tag{A.18}$$

Similarly to the expression found for $\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Lambda}_t$, we can use (2.34) and (2.40) to define the vector $\boldsymbol{\lambda}_n$. With

$$\begin{aligned} \underline{\boldsymbol{\lambda}}_n &= \left[\underline{\lambda}_{n,i_1}; \underline{\lambda}_{n,i_2}; ...; \underline{\lambda}_{n,i_C}\right] \in \mathbb{R}^C, &\text{(A.19)}\\ \underline{\boldsymbol{\lambda}}_t &= \left[\underline{\boldsymbol{\lambda}}_{t,i_1}; \underline{\boldsymbol{\lambda}}_{t,i_2}; ...; \underline{\boldsymbol{\lambda}}_{t,i_C}\right] \in \mathbb{R}^{2C}, &\text{(A.20)} \end{aligned}$$

the dynamics and constraints for a system with all closed contact points in slip are defined as

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{M}^{-1}\left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H} + \boldsymbol{W}_n\boldsymbol{\lambda}_n + \boldsymbol{W}_t\boldsymbol{\lambda}_t\right], &\text{(A.21)}\\ \boldsymbol{W}_n^T\ddot{\boldsymbol{q}} &+ \dot{\boldsymbol{W}}_n^T\dot{\boldsymbol{q}} = 0, &\text{(A.22)}\\ \boldsymbol{\lambda}_t &= -\mu\boldsymbol{Z}_t\boldsymbol{\lambda}_n, &\text{(A.23)} \end{aligned}$$

with

$$\boldsymbol{Z}_t = \begin{bmatrix} \langle\boldsymbol{\zeta}_{t,i_1}\rangle & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \langle\boldsymbol{\zeta}_{t,i_2}\rangle & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \langle\boldsymbol{\zeta}_{t,i_C}\rangle \end{bmatrix}, \quad \in \mathbb{R}^{2C\times C} \tag{A.24}$$

where $\langle\boldsymbol{\zeta}_t\rangle = \left[\langle\boldsymbol{\zeta}_{t,i_1}\rangle, \langle\boldsymbol{\zeta}_{t,i_2}\rangle, ..., \langle\boldsymbol{\zeta}_{t,i_C}\rangle\right]^T$, with $\langle\boldsymbol{\zeta}_{t,i_c}\rangle$ the unit vector $\boldsymbol{\zeta}_{t,i_c}$. (A.21)-(A.23) can be rewritten into

$$\boldsymbol{\lambda}_n = -\left[\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\left(\boldsymbol{W}_n - \mu\boldsymbol{W}_t\boldsymbol{Z}_t\right)\right]^{-1}\boldsymbol{W}_n^T\boldsymbol{M}^{-1}\left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}\right] - \dot{\boldsymbol{W}}_n^T\dot{\boldsymbol{q}}, \tag{A.25}$$

which can be used to define the guard function (A.18).

Stick to slip/open

When a contact point is in closed-contact stick, it can transition to closed-contact slip and it can transition to open-contact. A slipping contact transitions to sticking when the tangential reaction force becomes equal to the normal reaction force at that contact point times the friction coefficient, i.e.,

$$\mu\lambda_{n,i_c} = ||\boldsymbol{\lambda}_{t,i_c}||. \tag{A.26}$$

A guard function that can be used to describe this set is

$$\gamma_{\text{st}\to\text{sl}} = \mu^2\lambda_{n,i_c}^2 - \boldsymbol{\lambda}_{t,i_c}\boldsymbol{\lambda}_{t,i_c}^T, \tag{A.27}$$

which is equal to zero when $\mu^2\lambda_{n,i_c}^2 = ||\boldsymbol{\lambda}_{t,i_c}||$, greater than zero when $\mu^2\lambda_{n,i_c}^2 > ||\boldsymbol{\lambda}_{t,i_c}||$, and it is globally differentiable. It is physically impossible that $\mu^2\lambda_{n,i_c}^2 < ||\boldsymbol{\lambda}_{t,i_c}||$, meaning that $\gamma_{\text{st}\to\text{sl}}$ will never be smaller than zero. The dynamics and constraints of a system with all contact points in stick are defined as

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{M}^{-1}\left[\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H} + \boldsymbol{W}_n\boldsymbol{\lambda}_n + \boldsymbol{W}_t\boldsymbol{\lambda}_t\right], &\text{(A.28)}\\ \boldsymbol{W}_n^T\ddot{\boldsymbol{q}} &+ \dot{\boldsymbol{W}}_n^T\dot{\boldsymbol{q}} = 0, &\text{(A.29)}\\ \boldsymbol{W}_t^T\ddot{\boldsymbol{q}} &+ \dot{\boldsymbol{W}}_t^T\dot{\boldsymbol{q}} = 0,, &\text{(A.30)} \end{aligned}$$

Substituting (A.28) into (A.29) and (A.30) leads to

$$\boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\lambda}_n + \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\lambda}_t = \boldsymbol{W}_n^T \boldsymbol{M}^{-1} [\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}] - \dot{\boldsymbol{W}}_n^T \dot{\boldsymbol{q}}, \tag{A.31}$$

$$\boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\lambda}_n + \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\lambda}_t = \boldsymbol{W}_t^T \boldsymbol{M}^{-1} [\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}] - \dot{\boldsymbol{W}}_t^T \dot{\boldsymbol{q}}, \tag{A.32}$$

which can be rewritten to

$$\begin{bmatrix} \boldsymbol{\lambda}_n \\ \boldsymbol{\lambda}_t \end{bmatrix} = -\boldsymbol{D}^{-1} \begin{bmatrix} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} [\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}] - \dot{\boldsymbol{W}}_n^T \dot{\boldsymbol{q}} \\ \boldsymbol{W}_t^T \boldsymbol{M}^{-1} [\boldsymbol{S}\boldsymbol{u} - \boldsymbol{H}] - \dot{\boldsymbol{W}}_t^T \dot{\boldsymbol{q}} \end{bmatrix}. \tag{A.33}$$

Using (A.33) the guard function (A.27) is defined.

A.1.3 Discrete dynamics derivation

In this section the discrete dynamics are defined, describing the state reinitialization when a discrete event set is entered. These dynamics are defined by a jump map $\boldsymbol{g}$, which is defined differently for each post-event mode.

Slip post-impact mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\rightarrow \mathrm{sl}}(\dot{\boldsymbol{q}}^-)$
The discrete dynamics of a transition with a slip post-impact mode are given by

$$\boldsymbol{M}[\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-] = \boldsymbol{W}_n \boldsymbol{\Lambda}_n + \boldsymbol{W}_t \boldsymbol{\Lambda}_t, \tag{A.34}$$

$$\boldsymbol{W}_n^T \dot{\boldsymbol{q}}^+ = 0, \tag{A.35}$$

$$\boldsymbol{\Lambda}_t = -\mu \boldsymbol{Z}_t^- \boldsymbol{\Lambda}_n, \tag{A.36}$$

with $\boldsymbol{Z}_t^-$ defined as in (A.24) using the unit vector of the ante-impact tangential velocity $\langle \boldsymbol{\zeta}_t^- \rangle$. After substituting (A.36) intro (A.34) and rewriting to

$$\dot{\boldsymbol{q}}^+ = \left[\boldsymbol{M}^{-1} \boldsymbol{W}_n - \mu \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{Z}_t^-\right] \boldsymbol{\Lambda}_n + \dot{\boldsymbol{q}}^-, \tag{A.37}$$

and expression for $\boldsymbol{\Lambda}_n$ can be found by substituting this into (A.35), which is

$$\boldsymbol{\Lambda}_n = -\left[\boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n - \mu \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{Z}_t^-\right]^{-1} \boldsymbol{W}_n^T \dot{\boldsymbol{q}}^-, \tag{A.38}$$

The jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\rightarrow \mathrm{sl}}(\dot{\boldsymbol{q}}^-)$ is then found by substituting (A.38) into (A.37),

$$\dot{\boldsymbol{q}}^+ = -\left[\boldsymbol{M}^{-1} \boldsymbol{W}_n - \mu \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{Z}_t^-\right] \left[\boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n - \mu \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{Z}_t^-\right]^{-1} \boldsymbol{W}_n^T \dot{\boldsymbol{q}}^- + \dot{\boldsymbol{q}}^-. \tag{A.39}$$

Note that when there is a transition from stick to slip, i.e., there is no impact, the post-event velocity is $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$.

Stick post-impact mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\rightarrow \mathrm{st}}(\dot{\boldsymbol{q}}^-)$
The discrete dynamics of a transition with a stick post-impact mode are given by

$$\boldsymbol{M}[\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-] = \boldsymbol{W}_n \Lambda_n + \boldsymbol{W}_t \boldsymbol{\Lambda}_t, \tag{A.40}$$

$$\boldsymbol{W}_n^T \dot{\boldsymbol{q}}^+ = 0, \tag{A.41}$$

$$\boldsymbol{W}_t^T \dot{\boldsymbol{q}}^+ = 0. \tag{A.42}$$

After rewriting (A.40) into

$$\dot{\boldsymbol{q}}^+ = \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\Lambda}_n + \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\Lambda}_t + \dot{\boldsymbol{q}}^-, \tag{A.43}$$

substituting this into (A.41) and (A.42) results in

$$\boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\Lambda}_n + \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\Lambda}_t + \boldsymbol{W}_n^T \dot{\boldsymbol{q}}^- = 0 \tag{A.44}$$

$$\boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\Lambda}_n + \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\Lambda}_t + \boldsymbol{W}_t^T \dot{\boldsymbol{q}}^- = 0. \tag{A.45}$$

This can be rewritten into a definition of $\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Lambda}_t$, given by

$$\begin{bmatrix} \boldsymbol{\Lambda}_n \\ \boldsymbol{\Lambda}_t \end{bmatrix} = -\boldsymbol{D}^{-1} \begin{bmatrix} \boldsymbol{W}_n^T \dot{\boldsymbol{q}}^- \\ \boldsymbol{W}_t^T \dot{\boldsymbol{q}}^- \end{bmatrix}, \quad \text{with } \boldsymbol{D} = \begin{bmatrix} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \\ \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_t^T \boldsymbol{M}^{-1} \boldsymbol{W}_t \end{bmatrix}. \tag{A.46}$$

Substituting $\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Lambda}_t$ into (A.43) results in the jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to \text{st}}(\dot{\boldsymbol{q}}^-)$,

$$\dot{\boldsymbol{q}}^+ = \boldsymbol{M}^{-1} \boldsymbol{W}_n \boldsymbol{\Lambda}_n + \boldsymbol{M}^{-1} \boldsymbol{W}_t \boldsymbol{\Lambda}_t + \dot{\boldsymbol{q}}^-, \tag{A.47}$$

Similarly to the stick to slip case, note that when there is a transition from slip to stick the post-event velocity is $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$.

Open-contact post-impact mode $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to \text{op}}(\dot{\boldsymbol{q}}^-)$
The discrete dynamics of a transition with a slip post-impact mode are given by

$$\boldsymbol{M}(\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-) = 0. \tag{A.48}$$

In this case there are no impulsive reaction forces, and it is trivial to see that the jump map $\dot{\boldsymbol{q}}^+ = \boldsymbol{g}_{\to \text{op}}(\dot{\boldsymbol{q}}^-)$ is given by

$$\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-. \tag{A.49}$$

A.2 Proximal Point Formulation

The contact law and friction law defined in the complementarity condition formulation can be redefined to a proximal point formulation. This makes the system compatible with simulation methods as timestepping [52, Chapter 10]. More information on the definition of the proximal point formulation of contact laws and friction laws can be found in [19, Section 5.3].

A.2.1 Signorini's contact law and Poisson's impact law

In Figure A.2 a convex set C is illustrated. The normal cone $N_C(\boldsymbol{x})$ of a point $\boldsymbol{x}$ is $N_C(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \text{int}(C)$, where int(.) is the interior of a set. An example of this is point $\boldsymbol{x}_3$ in Figure A.2. Defining bd(.) as the boundary of the set, when $\boldsymbol{x} \in \text{bd}(C)$ there are two options. When $\boldsymbol{x}$ is on a smooth part of bd(C), then $N_C(\boldsymbol{x})$ is a ray normal to bd(C) at point $\boldsymbol{x}$ as depicted in at point $\boldsymbol{x}_1$. When $\boldsymbol{x}$ is on a nonsmooth part of bd(C), then $N_C(\boldsymbol{x})$ is a cone starting on the point $\boldsymbol{x}$ whose sides are normal to the left and right approximation of the point $\boldsymbol{x}$ on bd(C). This is illustrated at point $\boldsymbol{x}_2$. The proximal point $\text{prox}_C(\boldsymbol{z})$ of a point $\boldsymbol{z}$, is the point in C closest to the point $\boldsymbol{z}$. The point $\boldsymbol{x}$ is the proximal point to all points $\boldsymbol{z} \in N_C(\boldsymbol{x})$. For a point $\boldsymbol{z} \in C$, $\text{prox}_C(\boldsymbol{z}) = \boldsymbol{z}$ i.e. $\boldsymbol{x}_3$ in Figure A.2.

This formulation can be used to define Signorini's contact law, which is defined as (2.20). The normal cone formulation, as illustrated in Figure A.2, of the contact is given by

$$-h_{n,i} \in N_{C_{n,i}}(\lambda_{n,i}), \quad \text{with } C_{n,i} = (\mathbb{R}^n)^+. \tag{A.50}$$

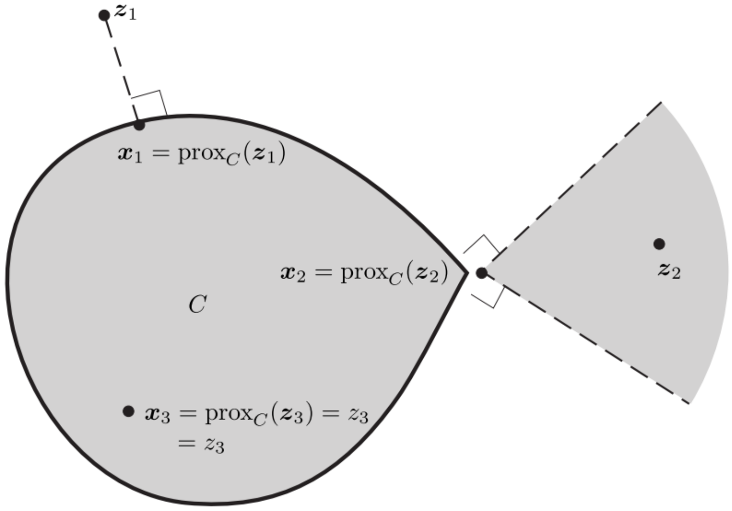

Figure A.2

The set $C_{n,i}$ is the set of admissible normal forces according to Signorini's law. See Figure 2.2a for an illustration of the set $C_{n,i}$ with $\lambda_{n,i} \in C_{n,i}$ and $h_{n,i} \in N_{C_{n,i}}(\lambda_{n,i})$. Now using the fact that

$$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{x} - r\boldsymbol{y}), r > 0 \iff -\boldsymbol{y} \in N_C(\boldsymbol{x}), \tag{A.51}$$

rewriting (A.50) to a proximal point formulation gives

$$\lambda_{n,i} = \mathrm{prox}_{C_{n,i}}(\lambda_{n,i} - rh_{n,i}), \quad \text{with } C_{n,i} = (\mathbb{R}^n)^+ \text{ and } r > 0. \tag{A.52}$$

Similarly for the Poisson's impact law illustrated in Figure 2.2b, we find the proximal point formulation

$$\Lambda_{n,i} = \mathrm{prox}_{C_{n,i}}(\Lambda_{n,i} - r\zeta^+_{n,i}), \quad \text{with } C_{n,i} = (\mathbb{R}^n)^+ \text{ and } r > 0. \tag{A.53}$$

A.2.2 Coulomb's friction law

Now we define the normal cone formulation of Coulomb's friction law

$$-\boldsymbol{\zeta}_{t,i} \in N_{C_{t,i}}(\boldsymbol{\lambda}_{t,i}) \quad \forall i \in \mathcal{I}_a, \quad \text{with } C_{t,i}(\lambda_{n,i}) = \{\boldsymbol{\lambda}_{t,i} \mid ||\boldsymbol{\lambda}_{t,i}|| \leq \mu\lambda_{n,i}\}, \tag{A.54}$$

which is illustrated in Figure A.3.

C_t is the set of all admitted friction forces. The tangential velocity $\boldsymbol{\zeta}_{t,i}$ is directed opposite to the friction force $\boldsymbol{\lambda}_{t,i}$ for isotropic friction.

Now using the fact that

$$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{x} - r\boldsymbol{y}), r > 0 \iff -\boldsymbol{y} \in N_C(\boldsymbol{x}), \tag{A.55}$$

we can rewrite the normal cone to a proximal point formulation

$$\boldsymbol{\lambda}_{t,i} = \mathrm{prox}_{C_{t,i}}(\boldsymbol{\lambda}_{t,i} - r\boldsymbol{\zeta}_{t,i}) \quad \text{with } C_{t,i}(\lambda_{n,i}) = \{\boldsymbol{\lambda}_{t,i} \mid ||\boldsymbol{\lambda}_{t,i}|| \leq \mu\lambda_{n,i}\} \text{ and } r > 0. \tag{A.56}$$

Similarly, for the impact dynamics we can formulate

$$\boldsymbol{\Lambda}_{t,i} = \mathrm{prox}_{C_{t,i}}(\boldsymbol{\Lambda}_{t,i} - r\boldsymbol{\zeta}^+_{t,i}) \quad \text{with } C_{t,i}(\lambda_{n,i}) = \{\boldsymbol{\Lambda}_{t,i} \mid ||\boldsymbol{\Lambda}_{t,i}|| \leq \mu\Lambda_{n,i}\} \text{ and } r > 0. \tag{A.57}$$

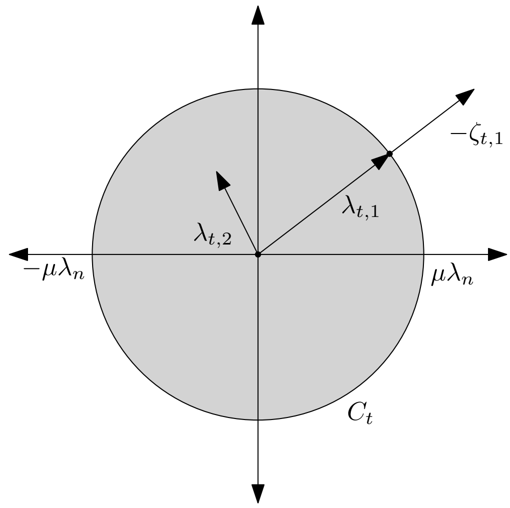

Figure A.3: *The friction disk with two seperate friction forces $\boldsymbol{\lambda}_{t,1}$ and $\boldsymbol{\lambda}_{t,2}$. $\boldsymbol{\lambda}_{t,1} = \mu\lambda_{n,1}$, resulting in a tangential velocity $\boldsymbol{\zeta}_{t,i} > 0$. $\boldsymbol{\lambda}_{t,2} < \mu\lambda_{n,2}$, leading to a tangential velocity $\boldsymbol{\zeta}_{t,i} = 0$.*

A.2.3 System dynamics with contact law and friction law

The flow dynamics is then described by

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{\xi}} + \boldsymbol{H}(\boldsymbol{q},\boldsymbol{\xi}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{u} + \sum_{i\in\mathcal{I}_c} \left(\boldsymbol{w}_{n,i}(\boldsymbol{q})\lambda_{n,i} + \boldsymbol{W}_{t,i}(\boldsymbol{q})\boldsymbol{\lambda}_{t,i}\right), \tag{A.58}$$

$$\lambda_{n,i} = \mathrm{prox}_{C_{n,i}}(\lambda_{n,i} - r h_{n,i}), \tag{A.59}$$

$$\boldsymbol{\lambda}_{t,i} = \mathrm{prox}_{C_{t,i}}(\boldsymbol{\lambda}_{t,i} - r\boldsymbol{\zeta}_{t,i}), \tag{A.60}$$

with

$$C_{n,i} = (\mathbb{R}^n)^+ \text{ and } r > 0, \tag{A.61}$$

$$C_{t,i}(\lambda_{n,i}) = \{\boldsymbol{\lambda}_{t,i} \mid ||\boldsymbol{\lambda}_{t,i}|| \leq \mu\lambda_{n,i}\} \text{ and } r > 0. \tag{A.62}$$

The impulsive dynamics that take place when a contact point opens or closes contact is described by

$$\boldsymbol{M}(\boldsymbol{q})(\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-) = \sum_{i\in\mathcal{I}_c} \left(\boldsymbol{w}_{n,i}(\boldsymbol{q})\Lambda_{n,i} + \boldsymbol{W}_{t,i}(\boldsymbol{q})\boldsymbol{\Lambda}_{t,i}\right), \tag{A.63}$$

$$\Lambda_{n,i} = \mathrm{prox}_{C_{n,i}}(\Lambda_{n,i} - r\zeta^+_{n,i}), \tag{A.64}$$

$$\boldsymbol{\Lambda}_{t,i} = \mathrm{prox}_{C_{t,i}}(\boldsymbol{\Lambda}_{t,i} - r\boldsymbol{\zeta}^+_{t,i}) \tag{A.65}$$

with

$$C_{n,i} = (\mathbb{R}^n)^+ \text{ and } r > 0, \tag{A.66}$$

$$C_{t,i}(\lambda_{n,i}) = \{\boldsymbol{\Lambda}_{t,i} \mid ||\boldsymbol{\Lambda}_{t,i}|| \leq \mu\Lambda_{n,i}\} \text{ and } r > 0. \tag{A.67}$$

Appendix B

Spatial Friction in Mechanical Systems with Unilateral Constraints

B.1 Reference trajectories with impact away from slip-stick border

Now we look at the case where a contact point goes from open to closed, away from $\boldsymbol{\Gamma} = 0$. This is illustrated in Figure B.1. The goal is to prove that for an event away from $\boldsymbol{\Gamma}$, a sufficiently small perturbation cannot cause the trajectory to hit $\gamma = 0$ at a perturbed ante-impact state $\boldsymbol{x}_{\epsilon}^{-}(t_{\epsilon})$ where $\boldsymbol{\Gamma}$ changes sign in comparison with the unperturbed ante-impact state $\boldsymbol{\alpha}^{-}(\tau)$. From [51, p. 6] we know that based on the continuity property of $\boldsymbol{\gamma}$ and $\boldsymbol{f}$, the perturbed impact state can be written as

$$\boldsymbol{x}_{\epsilon}(t_{\epsilon}) = \boldsymbol{\alpha}(\tau) + \epsilon\dot{\boldsymbol{\alpha}}(\tau)\frac{\partial t_{\epsilon}}{\partial \epsilon} + \epsilon \boldsymbol{z}(\tau) + o(\epsilon), \tag{B.1}$$

for sufficiently small ϵ. The shortest distance between $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{\alpha}(\tau))$ and $\boldsymbol{\Gamma} = 0$ on the plane where $\boldsymbol{\gamma} = 0$ is defined as the constant $\delta_{\boldsymbol{\Gamma}}$, which is also illustrated in Figure B.1.

Let's define a point in the state $\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0}$ where $\boldsymbol{\gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0}) = 0$ and $\boldsymbol{\Gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0}) = 0$. We are evaluating nominal trajectories which impact away from $\boldsymbol{\Gamma} = 0$, i.e. $\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\tau)) \neq \boldsymbol{\Gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0})$. From Section **??** we know that $\boldsymbol{\Gamma}$ is continuously differentiable, which implies that it is Lipschitz-continuous and therefore satisfies the Lipschitz-continuity condition

$$||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|| \leq \kappa ||\boldsymbol{x} - \boldsymbol{y}||, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \tag{B.2}$$

where $\kappa > 0$ [19]. By applying (B.2) to the function value of $\boldsymbol{\Gamma}$ at impact for the nominal trajectory and the perturbed trajectory, we find

$$||\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\tau)) - \boldsymbol{\Gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0})|| \leq \kappa ||\boldsymbol{\alpha}(\tau) - \boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0}||. \tag{B.3}$$

Now, since $\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\tau)) \neq \boldsymbol{\Gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0})$, we know that $||\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\tau)) - \boldsymbol{\Gamma}(\boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0})|| > 0$ and therefore from (B.3) that $||\boldsymbol{\alpha}(\tau) - \boldsymbol{x}_{\boldsymbol{\gamma}=0,\boldsymbol{\Gamma}=0}|| > 0$, i.e. $\delta_{\boldsymbol{\Gamma}} > 0$. Finally, from (B.1), we find

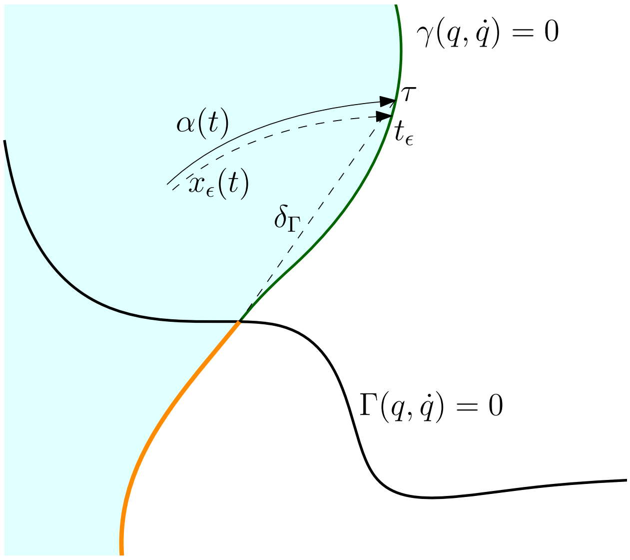

Figure B.1: *The guard functions* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\Gamma}$ *in the state space of* $\boldsymbol{q}$*. A transition from open to closed is made away from* $\boldsymbol{\Gamma}$*.* $\boldsymbol{\alpha}(t)$ *is the nominal trajectory and* $\boldsymbol{x}_\epsilon(t)$ *a perturbed trajectory of the contact point up to the transition.*

$$||\boldsymbol{x}_\epsilon(t_\epsilon) - \boldsymbol{\alpha}(\tau)|| = ||\epsilon\dot{\boldsymbol{\alpha}}(\tau)\frac{\partial t_\epsilon}{\partial \epsilon} + \epsilon \boldsymbol{z}(\tau) + o(\epsilon)||. \tag{B.4}$$

Since $\delta_{\boldsymbol{\Gamma}} > 0$ and $\lim_{\epsilon\to 0} ||\boldsymbol{x}_\epsilon(t_\epsilon) - \boldsymbol{\alpha}(\tau)|| = 0$, there always exists an ϵ such that $||\boldsymbol{x}_\epsilon(t_\epsilon) - \boldsymbol{\alpha}(\tau)|| < \delta_{\boldsymbol{\Gamma}}$. In other words, this proves that if $\boldsymbol{f}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ are continuous and the nominal trajectory makes impact away from the slip-stick post-impact mode border $\boldsymbol{\Gamma} = 0$, then there always exists a range of ϵ such that the perturbed state will have the same post-impact mode as the nominal trajectory.

B.2 Reference trajectories with impact at the slip-stick border

- Consider as simultaneous trigger of open-closed guard and slip-stick guard. However, it is not entirely the same, as the slip-stick guard cannot be triggered before the open-closed guard.
- The trigger should be transversal, in both closed-open guard and slip-stick guard.
- If we can show continuity of post-impact state, then we can define a jump gain like H, which uses one jump map for the open to stick domain and another jump map for the open to slip domain.

B.3 Post-impact accelerations in open-to-stick transitions

The mode transition from stick to slip happens when a guard is triggered at acceleration level,

$$^{\text{slip}\leftarrow\text{stick}}\gamma = \mu^2\lambda_{n,i}^2 - \boldsymbol{\lambda}_{t,i}\boldsymbol{\lambda}_{t,i}^T, \tag{B.5}$$

and the post-impact mode is determined by the guard function defined at velocity level

$$\Gamma = \mu^2\Lambda_{n,i}^2(\boldsymbol{q},\dot{\boldsymbol{q}}^-) - \boldsymbol{\Lambda}_{t,i}(\boldsymbol{q},\dot{\boldsymbol{q}}^-)\boldsymbol{\Lambda}_{t,i}^T(\boldsymbol{q},\dot{\boldsymbol{q}}^-). \tag{B.6}$$

Since the jump map from open to stick is

$$\boldsymbol{M}(\boldsymbol{q})(\dot{\boldsymbol{q}}^{+} - \dot{\boldsymbol{q}}^{-}) = \boldsymbol{w}_{n,i}(\boldsymbol{q})\Lambda_{n,i} + \boldsymbol{W}_{t,i}(\boldsymbol{q})\boldsymbol{\Lambda}_{t,i}, \tag{B.7}$$

$$\zeta_{n,i}^{+} = 0, \tag{B.8}$$

$$\zeta_{t,i}^{+} = 0, \tag{B.9}$$

which is on velocity level, the post-impact reaction forces of the open-to-stick event can be in the stick-to-slip jump set, causing an immediate transition to slip. This is demonstrated using the flow dynamics of the stick mode at the time-instant of the transition,

$$\boldsymbol{M}(\boldsymbol{q}^{+})\ddot{\boldsymbol{q}}^{+} + \boldsymbol{H}(\boldsymbol{q}^{+}, \dot{\boldsymbol{q}}^{+}) = \boldsymbol{S}(\boldsymbol{q}^{+})\boldsymbol{u}^{+} + \sum_{i \in \mathcal{I}_c} \left(\boldsymbol{w}_{n,i}(\boldsymbol{q}^{+})\lambda_{n,i}^{+} + \boldsymbol{W}_{t,i}(\boldsymbol{q}^{+})\boldsymbol{\lambda}_{t,i}^{+} \right), \tag{B.10}$$

$$\boldsymbol{w}_{n,i}^{T}(\boldsymbol{q}^{+})\ddot{\boldsymbol{q}}^{+} + \dot{\boldsymbol{w}}_{n,i}^{T}(\boldsymbol{q}^{+})\dot{\boldsymbol{q}}^{+} = 0, \tag{B.11}$$

$$\boldsymbol{W}_{t,i}^{T}(\boldsymbol{q}^{+})\ddot{\boldsymbol{q}}^{+} + \dot{\boldsymbol{W}}_{t,i}^{T}(\boldsymbol{q}^{+})\dot{\boldsymbol{q}}^{+} = 0. \tag{B.12}$$

We can deduce from (B.8)-(B.9) and (B.11)-(B.12) that the normal acceleration of the transition contact point $\boldsymbol{w}_{n,i}^{T}(\boldsymbol{q}^{+})\ddot{\boldsymbol{q}}^{+}$ and the tangential acceleration of the transitioning contact point $\boldsymbol{W}_{t,i}^{T}(\boldsymbol{q}^{+})\ddot{\boldsymbol{q}}^{+}$ are both equal to zero. From (B.10) we then notice that $\lambda_{n,i}^{+}$ and $\boldsymbol{\lambda}_{t,i}^{+}$ depend continuously on $\boldsymbol{u}^{+}$ and can therefore instantly lead to $\mu^2\lambda_{n,i}^2 - \boldsymbol{\lambda}_{t,i}\boldsymbol{\lambda}_{t,i}^T > 0$ for certain $\boldsymbol{u}^{+}$. For these inputs the contact point will immediately start slipping after the open-to-stick transition. These areas are illustrated in Figure B.2.

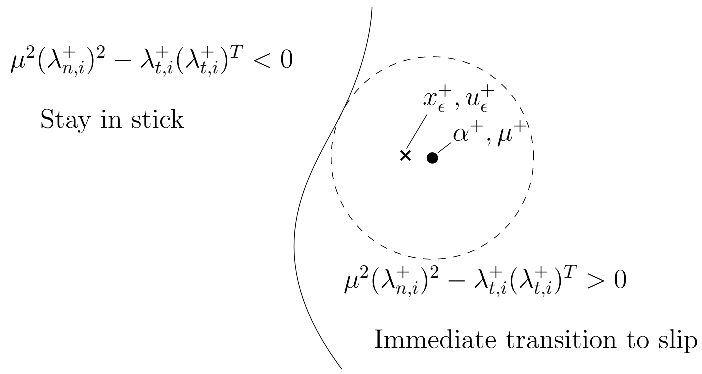

Figure B.2: *The border between an open-to-stick event that stays in stick and an open-to-stick event that immediately starts slipping is illustrated in this figure. The post event state and input indicated in the figure is in the* $\mu^2(\lambda_{n,i}^+)^2 - \boldsymbol{\lambda}_{t,i}^+(\boldsymbol{\lambda}_{t,i}^+)^T > 0$ *area, causing the contact point to immediately start slipping.*

Using the continuity of the system's flow dynamics and the function $\mu^2(\lambda_{n,i}^+)^2 - \boldsymbol{\lambda}_{t,i}^+(\boldsymbol{\lambda}_{t,i}^+)^T$, we can show that if we choose $\boldsymbol{\mu}^+$ such that $\boldsymbol{\alpha}^+, \boldsymbol{\mu}^+$ is not on $\mu^2(\lambda_{n,i}^+)^2 - \boldsymbol{\lambda}_{t,i}^+(\boldsymbol{\lambda}_{t,i}^+)^T = 0$ then there always exists a range of perturbations ϵ such that the perturbed post-impact is on the same side of $\mu^2(\lambda_{n,i}^+)^2 - \boldsymbol{\lambda}_{t,i}^+(\boldsymbol{\lambda}_{t,i}^+)^T = 0$ as the unperturbed trajectory similarly to Section B.1.

B.4 Slip-stick transition in closed-contact

Appendix C

Sensitivity Analysis for Input-Dependent Guards

C.1 Linearization for single jumps

The perturbed state is defined as

$$\boldsymbol{x}(t,\epsilon) = \boldsymbol{x}(t_0,\epsilon) + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{x}(s,\epsilon), \boldsymbol{u}(s,\epsilon), s)ds. \tag{C.1}$$

Then

$$\frac{\partial \boldsymbol{x}(t,\epsilon)}{\partial \epsilon} = \frac{\partial \boldsymbol{x}_0}{\partial \epsilon} + \int_{t_0}^{t} \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \epsilon} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}}{\partial \epsilon} \right) ds, \tag{C.2}$$

$$\frac{\partial^2 \boldsymbol{x}}{\partial t \partial \epsilon} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \epsilon} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}}{\partial \epsilon}, \tag{C.3}$$

which we can write as

$$\frac{\partial^2 \boldsymbol{x}}{\partial t \partial \epsilon} = D_1 \boldsymbol{f}(\boldsymbol{x}(t,\epsilon), \boldsymbol{u}(t,\epsilon), t) \cdot \frac{\partial \boldsymbol{x}}{\partial \epsilon} + D_2 \boldsymbol{f}(\boldsymbol{x}(t,\epsilon), \boldsymbol{u}(t,\epsilon), t) \cdot \frac{\partial \boldsymbol{u}}{\partial \epsilon}, \tag{C.4}$$

with $D_i \boldsymbol{f}$ the derivative of $\boldsymbol{f}$ wrt the ith term of $\boldsymbol{f}$. Evaluating (C.4) at $\epsilon = 0$ results in the flow dynamics of the positive homogenization

$$\dot{\boldsymbol{z}} = D_1 \boldsymbol{f}(\boldsymbol{\alpha}(t), \boldsymbol{\mu}(t), t) \cdot \boldsymbol{z}(t) + D_2 \boldsymbol{f}(\boldsymbol{\alpha}(t), \boldsymbol{\mu}(t), t) \cdot \boldsymbol{v}(t), \tag{C.5}$$

where

$$\boldsymbol{z}(t) = \left. \frac{\partial \boldsymbol{x}(t,\epsilon)}{\partial \epsilon} \right|_{\epsilon=0}, \text{ and } \boldsymbol{v}(t) = \left. \frac{\partial \boldsymbol{u}(t,\epsilon)}{\partial \epsilon} \right|_{\epsilon=0}. \tag{C.6}$$

When we consider a single jump

$$\boldsymbol{x}_\epsilon^+(t_\epsilon, \epsilon) = \boldsymbol{g}(\boldsymbol{x}_\epsilon^-(t_\epsilon,\epsilon), t_\epsilon), \tag{C.7}$$

using a Taylor approximation (**Under what conditions is this approximation valid? γ continuous, α continuous, $\boldsymbol{f}$ continuous?**) with respect to ϵ and around $\epsilon = 0$, we can write

$$\boldsymbol{x}_\epsilon^+(t_\epsilon, \epsilon) = \boldsymbol{\alpha}^+(t_\epsilon) + \epsilon \boldsymbol{z}^+(t_\epsilon) + o(\epsilon), \tag{C.8}$$

$$\boldsymbol{u}_\epsilon^+(t_\epsilon, \epsilon) = \boldsymbol{\mu}^+(t_\epsilon) + \epsilon \boldsymbol{v}^+(t_\epsilon) + o(\epsilon), \tag{C.9}$$

where $\boldsymbol{\alpha}(t)$ is a nominal reference trajectory that satisfies the dynamics of the system and $\boldsymbol{\mu}(t)$ an input that achieves this reference trajectory. Now we expand this in terms of ϵ, so with

$$\Delta = \left.\frac{\partial t_\epsilon}{\partial \epsilon}\right|_{\epsilon=0}, \tag{C.10}$$

we get

$$\begin{aligned}
\boldsymbol{\alpha}^+(t_\epsilon) &= \boldsymbol{\alpha}^+(\tau) + \epsilon\dot{\boldsymbol{\alpha}}^+(\tau)\Delta + o(\epsilon), && \text{(C.11)}\\
\boldsymbol{\mu}^+(t_\epsilon) &= \boldsymbol{\mu}^+(\tau) + \epsilon\dot{\boldsymbol{\mu}}^+(\tau)\Delta + o(\epsilon), && \text{(C.12)}\\
\boldsymbol{z}^+(t_\epsilon) &= \boldsymbol{z}^+(\tau) + \epsilon\dot{\boldsymbol{z}}^+(\tau)\Delta + o(\epsilon), && \text{(C.13)}\\
\boldsymbol{v}^+(t_\epsilon) &= \boldsymbol{v}^+(\tau) + \epsilon\dot{\boldsymbol{v}}^+(\tau)\Delta + o(\epsilon), && \text{(C.14)}
\end{aligned}$$

which when substituted into (C.8) and (C.9) gives,

$$\begin{aligned}
\boldsymbol{x}_\epsilon^+(t_\epsilon,\epsilon) &= \boldsymbol{\alpha}^+(\tau) + \epsilon\dot{\boldsymbol{\alpha}}^+(\tau)\Delta + \epsilon\boldsymbol{z}^+(\tau) + o(\epsilon). && \text{(C.15)}\\
\boldsymbol{u}_\epsilon^+(t_\epsilon,\epsilon) &= \boldsymbol{\mu}^+(\tau) + \epsilon\dot{\boldsymbol{\mu}}^+(\tau)\Delta + \epsilon\boldsymbol{v}^+(\tau) + o(\epsilon). && \text{(C.16)}
\end{aligned}$$

To find Δ, we evaluate the ante impact guard function

$$\gamma^-(\boldsymbol{x}_\epsilon^-(t_\epsilon), \boldsymbol{u}_\epsilon^-(t_\epsilon), t_\epsilon) = 0. \tag{C.17}$$

In previous work, the guard function γ was not dependent on $\boldsymbol{u}_\epsilon(t_\epsilon)$ because friction and release was not considered. We now expand $\gamma(\boldsymbol{x}_\epsilon(t_\epsilon), \boldsymbol{u}_\epsilon(t_\epsilon), t_\epsilon)$ wrt ϵ, giving

$$\begin{aligned}
\gamma(\boldsymbol{x}_\epsilon(t_\epsilon), \boldsymbol{u}_\epsilon(t_\epsilon), t_\epsilon) &= \gamma(\boldsymbol{\alpha}(\tau), \boldsymbol{\mu}(\tau), \tau) + \epsilon\left[\frac{\partial\gamma}{\partial\epsilon}(\boldsymbol{\alpha}(\tau), \boldsymbol{\mu}(\tau), \tau)\right]_{\epsilon=0} + o(\epsilon), && \text{(C.18)}\\
&= \gamma(\boldsymbol{\alpha}(\tau), \boldsymbol{\mu}(\tau), \tau) + \epsilon\left[\frac{\partial\gamma}{\partial\boldsymbol{x}}\left(\frac{\partial\boldsymbol{x}}{\partial\epsilon} + \frac{\partial\boldsymbol{x}}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right) + \frac{\partial\gamma}{\partial\boldsymbol{u}}\left(\frac{\partial\boldsymbol{u}}{\partial\epsilon} + \frac{\partial\boldsymbol{u}}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right) + \frac{\partial\gamma}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right]_{\epsilon=0} + o(\epsilon). && \text{(C.19)}
\end{aligned}$$

By definition $\gamma(\tau) = 0$, so we can rewrite (C.19) to

$$\gamma(\boldsymbol{x}_\epsilon(t_\epsilon), \boldsymbol{u}_\epsilon(t_\epsilon), t_\epsilon) = \epsilon\left[D_1\gamma\cdot(\overline{\boldsymbol{z}}(\tau) + \dot{\boldsymbol{\alpha}}(\tau)\Delta) + D_2\gamma\cdot(\overline{\boldsymbol{v}}(\tau) + \dot{\boldsymbol{\mu}}(\tau)\Delta) + D_3\cdot\gamma\Delta\right] \tag{C.20}$$

Now we can evaluate (C.19) using (C.20), which gives

$$\epsilon\left[D_1\gamma^-\cdot\big(\boldsymbol{z}^-(\tau) + \dot{\boldsymbol{\alpha}}^-(\tau)\Delta\big) + D_2\gamma^-\cdot\big(\boldsymbol{v}^-(\tau) + \dot{\boldsymbol{\mu}}^-(\tau)\Delta\big) + D_3\gamma^-\cdot\Delta\right] = 0. \tag{C.21}$$

From (C.21) we can determine the expression for Δ,

$$\Delta = -\frac{D_1\gamma^-\cdot\boldsymbol{z}^-(\tau) + D_2\gamma^-\cdot\boldsymbol{v}^-(\tau)}{\dot{\gamma}^-}, \tag{C.22}$$

with

$$\begin{aligned}
\gamma^- &= \gamma^-(\boldsymbol{\alpha}^-(\tau), \boldsymbol{\mu}^-(\tau), \tau), && \text{(C.23)}\\
\dot{\gamma}^- &= D_1\gamma^-\cdot\dot{\boldsymbol{\alpha}}^- + D_2\gamma^-\cdot\dot{\boldsymbol{\mu}}^- + D_3\gamma^-. && \text{(C.24)}
\end{aligned}$$

To find the expression for the right hand side of (C.7), we now expand $\boldsymbol{g}(\boldsymbol{x}_\epsilon^-(t_\epsilon,\epsilon), \boldsymbol{u}_\epsilon^-(t_\epsilon,\epsilon), t_\epsilon)$ with respect to ϵ as

$$\boldsymbol{g}(\boldsymbol{x}_\epsilon^-, \boldsymbol{u}_\epsilon^-, t_\epsilon) = \boldsymbol{g}(\boldsymbol{\alpha}^-(\tau), \tau) + \epsilon \left[\frac{\partial \boldsymbol{g}}{\partial \epsilon}\right] + o(\epsilon), \tag{C.25}$$

$$= \boldsymbol{\alpha}^+(\tau) + \epsilon \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{x}}{\partial \epsilon} + \frac{\partial \boldsymbol{x}}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\left(\frac{\partial \boldsymbol{u}}{\partial \epsilon} + \frac{\partial \boldsymbol{u}}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right) + \frac{\partial \boldsymbol{g}}{\partial t_\epsilon}\frac{dt_\epsilon}{d\epsilon}\right]_{\epsilon=0} + o(\epsilon), \tag{C.26}$$

$$= \boldsymbol{\alpha}^+(\tau) + \epsilon \left[D_1\boldsymbol{g} \cdot \left(\boldsymbol{z}^- + \dot{\boldsymbol{\alpha}}^-\Delta\right) + D_2\boldsymbol{g} \cdot \left(\boldsymbol{v}^- + \dot{\boldsymbol{\mu}}(\tau)\Delta\right) + D_3\boldsymbol{g} \cdot \Delta\right] + o(\epsilon). \tag{C.27}$$

Note that $\boldsymbol{g}$ does not depend on the input $\boldsymbol{u}$. Jump maps are impulsive by definition, and since impulsive inputs do not exist it is impossible for the jump map to be dependent on $\boldsymbol{u}$. For small ϵ, we can rewrite (C.7), (C.22) and (C.27) to a general jump map with counter k as

$$\boldsymbol{x}_\epsilon^k(t_\epsilon, \epsilon) = \boldsymbol{g}^k(\boldsymbol{x}_\epsilon^{k-1}, \boldsymbol{u}_\epsilon^{k-1}, t_\epsilon), \tag{C.28}$$

$$\Delta^k = -\frac{D_1\gamma^k \cdot \boldsymbol{z}^{k-1}(\tau) + D_2\gamma^k \cdot \boldsymbol{v}^{k-1}(\tau)}{\dot{\gamma}^k}, \tag{C.29}$$

$$\boldsymbol{g}^k(\boldsymbol{x}_\epsilon^{k-1}, \boldsymbol{u}_\epsilon^{k-1}, t_\epsilon) = \boldsymbol{\alpha}^k(\tau) + \epsilon \left[D_1\boldsymbol{g}^k \cdot \left(\boldsymbol{z}^{k-1}(\tau) + \dot{\boldsymbol{\alpha}}^{k-1}\Delta^k\right) + D_2\boldsymbol{g}^k \cdot \left(\boldsymbol{v}^{k-1}(\tau) + \dot{\boldsymbol{\mu}}^{k-1}\Delta^k\right) + D_3\boldsymbol{g}^k \cdot \Delta^k\right]. \tag{C.30}$$

From (C.15) we get

$$\boldsymbol{z}^k(\tau) = \frac{1}{\epsilon}\left(\boldsymbol{x}_\epsilon^k(t_\epsilon) - \boldsymbol{\alpha}^k(\tau)\right) - \dot{\boldsymbol{\alpha}}^k(\tau)\Delta^k, \tag{C.31}$$

and by equating (C.28) and (C.30) we find an expression for $\overline{\boldsymbol{x}}_\epsilon^k(t_\epsilon, \epsilon)$ which we can substitute into (C.31) resulting in

$$\boldsymbol{z}^k(\tau) = D_1\boldsymbol{g}^k \cdot \left(\boldsymbol{z}^{k-1}(\tau) + \dot{\boldsymbol{\alpha}}^{k-1}\Delta^k\right) + D_2\boldsymbol{g}^k \cdot \left(\boldsymbol{v}^{k-1}(\tau) + \dot{\boldsymbol{\mu}}^{k-1}\Delta^k\right) + D_3\boldsymbol{g}^k \cdot \Delta^k - \dot{\boldsymbol{\alpha}}^k(\tau)\Delta^k. \tag{C.32}$$

Now, by substituting (C.29) into (C.32), we get

$$\boldsymbol{z}^k(\tau) = D_1\boldsymbol{g}^k \cdot \boldsymbol{z}^{k-1} + D_2\boldsymbol{g}^k \cdot \boldsymbol{v}^{k-1}$$
$$- \left(D_1\boldsymbol{g}^k \cdot \boldsymbol{f}^{k-1} + D_2\boldsymbol{g}^k \cdot \dot{\boldsymbol{\mu}}^{k-1} + D_3\boldsymbol{g}^k \cdot 1 - \boldsymbol{f}^k\right) \frac{D_1\gamma^k \cdot \boldsymbol{z}^{k-1} + D_2\gamma^k \cdot \boldsymbol{v}^{k-1}}{\dot{\gamma}^k}, \tag{C.33}$$

$$\boldsymbol{z}^k(\tau) = \left(\frac{\boldsymbol{f}^k - \dot{\boldsymbol{g}}^k}{\dot{\gamma}^k}D_1\gamma^k + D_1\boldsymbol{g}^k\right) \cdot \boldsymbol{z}^{k-1} + \left(\frac{\boldsymbol{f}^k - \dot{\boldsymbol{g}}^k}{\dot{\gamma}^k}D_2\gamma^k + D_2\boldsymbol{g}^k\right) \cdot \boldsymbol{v}^{k-1}, \tag{C.34}$$

with

$$\dot{\boldsymbol{g}}^k = D_1\boldsymbol{g}^k \cdot \boldsymbol{f}^{k-1} + D_2\boldsymbol{g}^k \cdot \dot{\boldsymbol{\mu}}^{k-1} + D_3\boldsymbol{g}^k \cdot 1, \tag{C.35}$$

$$\boldsymbol{f}^k = {}^{s^k}\boldsymbol{f}(\boldsymbol{\alpha}^k(\tau), \boldsymbol{\mu}^k(\tau), \tau). \tag{C.36}$$

Now, using

$$\boldsymbol{G}_{\boldsymbol{z}}^k(\tau) = \frac{\boldsymbol{f}^k - \dot{\boldsymbol{g}}^k}{\dot{\gamma}^k}D_1\gamma^k \cdot 1 + D_1\boldsymbol{g}^k \cdot 1, \tag{C.37}$$

$$\boldsymbol{G}_{\boldsymbol{v}}^k(\tau) = \frac{\boldsymbol{f}^k - \dot{\boldsymbol{g}}^k}{\dot{\gamma}^k}D_2\gamma^k \cdot 1 + D_2\boldsymbol{g}^k \cdot 1, \tag{C.38}$$

we can write

$$\boldsymbol{z}^k(\tau) = \boldsymbol{G}^k_{\boldsymbol{z}} \boldsymbol{z}^{k-1} + \boldsymbol{G}^k_{\boldsymbol{v}} \boldsymbol{v}^{k-1}. \tag{C.39}$$

C.2 Linearization for multiple jumps

With $k+1 = k^+$ and $k-1 = k^-$, we now assume that we find the first order approximation of the perturbed post-impact state of two simultaneous jumps, by considering these jumps after each other as

$$\boldsymbol{z}^{k^+}(\tau) = \boldsymbol{G}^{k^+}_{\boldsymbol{z}} \boldsymbol{z}^k + \boldsymbol{G}^{k^+}_{\boldsymbol{v}} \boldsymbol{v}^k \tag{C.40}$$

$$\begin{aligned}
\boldsymbol{z}^{k^+}(\tau) &= \boldsymbol{G}^{k^+}_{\boldsymbol{z}} \left(\boldsymbol{G}^k_{\boldsymbol{z}} \boldsymbol{z}^{k^-} + \boldsymbol{G}^k_{\boldsymbol{v}} \boldsymbol{v}^{k^-} \right) + \boldsymbol{G}^{k^+}_{\boldsymbol{z}} \boldsymbol{v}^k, && \text{(C.41)} \\
&= \boldsymbol{G}^{k^+}_{\boldsymbol{z}} \boldsymbol{G}^k_{\boldsymbol{z}} \boldsymbol{z}^{k^-} + \boldsymbol{G}^{k^+}_{\boldsymbol{z}} \boldsymbol{G}^k_{\boldsymbol{v}} \boldsymbol{v}^{k^-} + \boldsymbol{G}^{k^+}_{\boldsymbol{v}} \boldsymbol{v}^k. && \text{(C.42)}
\end{aligned}$$

We prove that this is true by deriving an expression for the post-impact state of two simultaneous jumps, and comparing it with (C.42). Now we evaluate the jump map of two jumps at the same time instant τ,

$$^{s^{k^+} \leftarrow s^k \leftarrow s^{k^-}} \boldsymbol{x}_\epsilon(t^{k^+}_\epsilon) = \boldsymbol{g}^{k^+}(\boldsymbol{x}^k_\epsilon(t^{k^+}_\epsilon), \boldsymbol{u}^k_\epsilon(t^{k^+}_\epsilon), t^{k^+}_\epsilon), \tag{C.43}$$

with

$$\boldsymbol{x}^k_\epsilon(t^{k^+}_\epsilon) = \int_{t^k_\epsilon}^{t^{k^+}_\epsilon} \left[{}^{s^k}\boldsymbol{f}\left(\boldsymbol{x}^k_\epsilon(t), \boldsymbol{u}^k_\epsilon(t) \right) \right] dt + \boldsymbol{g}^k(\boldsymbol{x}^{k^-}_\epsilon(t^k_\epsilon), \boldsymbol{u}^{k^-}_\epsilon(t^k_\epsilon), t^k_\epsilon). \tag{C.44}$$

We rewrite the integral in (C.44) to

$$\int_{t^k_\epsilon}^{t^{k^+}_\epsilon} {}^{s^k}\boldsymbol{f}(t,\epsilon) dt = \boldsymbol{F}(t^{k^+}_\epsilon, \epsilon) - \boldsymbol{F}(t^k_\epsilon, \epsilon) = \boldsymbol{\Phi}(t^k_\epsilon, t^{k^+}_\epsilon, \epsilon), \tag{C.45}$$

where ${}^{s^k}\boldsymbol{f}\left(\boldsymbol{x}^k_\epsilon(t), \boldsymbol{u}^k_\epsilon(t) \right)$ can be written as ${}^{s^k}\boldsymbol{f}(t,\epsilon)$, because $\boldsymbol{x}_\epsilon$ and $\boldsymbol{u}$ depend solely on t and ϵ.

We now expand $\boldsymbol{\Phi}$ with respect to ϵ, which results in

$$\begin{aligned}
\boldsymbol{\Phi}(t^k_\epsilon, t^{k^+}_\epsilon, \epsilon) &= \boldsymbol{\Phi}(t^k_0, t^{k^+}_0, \epsilon) + \epsilon \left. \frac{\partial \boldsymbol{\Phi}}{\partial \epsilon} \right|_{\epsilon=0} + o(\epsilon), && \text{(C.46)} \\
&= \boldsymbol{F}(\tau, 0) - \boldsymbol{F}(\tau, 0) + \left[{}^{s^k}\boldsymbol{f}(t^{k^+}_\epsilon, \epsilon) \frac{dt^{k^+}_\epsilon}{d\epsilon} - {}^{s^k}\boldsymbol{f}(t^{k^+}_\epsilon, \epsilon) \frac{dt^k_\epsilon}{d\epsilon} + \int_{t^k_\epsilon}^{t^{k^+}_\epsilon} \frac{\partial\, {}^{s^k}\boldsymbol{f}(t,\epsilon)}{\partial \epsilon} dt \right]_{\epsilon=0}, && \text{(C.47)} \\
&= \boldsymbol{f}^k(\Delta^{k^+} - \Delta^k), && \text{(C.48)}
\end{aligned}$$

since $\int_{t^k_\epsilon}^{t^{k^+}_\epsilon} \frac{\partial\, {}^{s^k}\boldsymbol{f}(t,\epsilon)}{\partial \epsilon} dt_{\epsilon=0} = 0$. Note that ϵ is assumed sufficiently small, such that we can write t as a function of ϵ.

By expanding (C.43) with respect to ϵ, we find

$$
{}^{s^{k^+} \leftarrow s^k \leftarrow s^{k^-}} \boldsymbol{x}_\epsilon(t_\epsilon^{k^+}) = \boldsymbol{\alpha}^{k^+}(\tau) + \epsilon \left. \frac{\partial \boldsymbol{g}^{k^+}(\boldsymbol{x}_\epsilon^k(t_\epsilon^{k^+}), \boldsymbol{u}_\epsilon^k(t_\epsilon^{k^+}), t_\epsilon^{k^+})}{\partial \epsilon} \right|_{\epsilon=0} + o(\epsilon), \tag{C.49}
$$

where

$$
\begin{aligned}
\left. \frac{\partial \boldsymbol{g}^{k^+}(\boldsymbol{x}_\epsilon^k(t_\epsilon^{k^+}), \boldsymbol{u}_\epsilon^k(t_\epsilon^{k^+}), t_\epsilon^{k^+})}{\partial \epsilon} \right|_{\epsilon=0} = \Bigg[& \frac{\partial \boldsymbol{g}^{k^+}}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{\Phi}}{\partial \epsilon} + \frac{\partial \boldsymbol{g}^k}{\partial \boldsymbol{x}} \left(\frac{\partial \boldsymbol{x}^{k^-}}{\partial \epsilon} + \frac{\partial \boldsymbol{x}^{k^-}}{\partial t} \frac{d t_\epsilon^k}{d\epsilon} \right) + \right. \\
& \left. \left. \frac{\partial \boldsymbol{g}^k}{\partial \boldsymbol{u}} \left(\frac{\partial \boldsymbol{u}^{k^-}}{\partial \epsilon} + \frac{\partial \boldsymbol{u}^{k^-}}{\partial t} \frac{d t_\epsilon^k}{d\epsilon} \right) + \frac{\partial \boldsymbol{g}^k}{\partial t} \frac{d t_\epsilon^k}{d\epsilon} \right) + \frac{\partial \boldsymbol{g}^{k^+}}{\partial t} \frac{d t_\epsilon^{k^+}}{d\epsilon} \right]_{\epsilon=0},
\end{aligned} \tag{C.50}
$$

$$
\begin{aligned}
\left. \frac{\partial \boldsymbol{g}^{k^+}(\boldsymbol{x}_\epsilon^k(t_\epsilon^{k^+}), \boldsymbol{u}_\epsilon^k(t_\epsilon^{k^+}), t_\epsilon^{k^+})}{\partial \epsilon} \right|_{\epsilon=0} &= D_1 \boldsymbol{g}^{k^+} \cdot \Big(\boldsymbol{f}^k (\Delta^{k^+} - \Delta^k) + D_1 \boldsymbol{g}^k \cdot \Big(\boldsymbol{z}^{k^-} + \boldsymbol{f}^{k^-} \Delta^k \Big) \\
&\quad + D_2 \boldsymbol{g}^k \cdot \Big(\boldsymbol{v}^{k^-} + \dot{\boldsymbol{\mu}}^{k^-} \Delta^k \Big) + D_3 \boldsymbol{g}^k \cdot \Delta^k \Big) + D_3 \boldsymbol{g}^{k^+} \cdot \Delta^{k^+}.
\end{aligned} \tag{C.51}
$$

We now substitute (C.51) into (C.49), which we in turn substitute into (C.31) to get

$$
\begin{aligned}
\boldsymbol{z}^{k^+}(\tau) = D_1 \boldsymbol{g}^{k^+} \cdot \boldsymbol{f}^k \Delta^{k^+} - D_1 \boldsymbol{g}^{k^+} \cdot \boldsymbol{f}^{k^+} \Delta^k + D_1 \boldsymbol{g}^{k^+} \cdot \Big(& D_1 \boldsymbol{g}^k \cdot \Big(\boldsymbol{z}^{k^-} + \boldsymbol{f}^{k^-} \Delta^k \Big) \\
& + D_2 \boldsymbol{g}^k \cdot \Big(\boldsymbol{v}^{k^-} + \dot{\boldsymbol{\mu}}^{k^-} \Delta^k \Big) + D_3 \boldsymbol{g}^k \cdot \Delta^k \Big) + \Big(D_3 \boldsymbol{g}^{k^+} \cdot 1 - \boldsymbol{f}^{k^+} \Big) \Delta^{k^+},
\end{aligned} \tag{C.52}
$$

under the assumption that ϵ is small. The four terms in (C.52) can be rewritten into

$$
D_1 \boldsymbol{g}^{k^+} \cdot \boldsymbol{f}^k \Delta^{k^+} = \frac{-D_1 \boldsymbol{g}^{k^+} \cdot \boldsymbol{f}^{k^+}}{\dot{\gamma}^{k^+}} \Big(D_1 \gamma^{k^+} \cdot \Big(\boldsymbol{G}_{\boldsymbol{z}}^k \boldsymbol{z}^{k^-} + \boldsymbol{G}_{\boldsymbol{v}}^k \boldsymbol{v}^{k^-} \Big) + D_2 \gamma^{k^+} \cdot \boldsymbol{v}^k \Big), \tag{C.53}
$$

$$
D_1 \boldsymbol{g}^{k^+} \cdot \Big(-\boldsymbol{f}^k \Delta^k \Big) = \frac{D_1 \boldsymbol{g}^{k^+} \cdot \boldsymbol{f}^k}{\dot{\gamma}^k} \Big(D_1 \gamma^k \cdot \boldsymbol{z}^{k^-} + D_2 \gamma^k \cdot \boldsymbol{v}^{k^-} \Big), \tag{C.54}
$$

$$
\begin{aligned}
& D_1 \boldsymbol{g}^{k^+} \cdot \Big(D_1 \boldsymbol{g}^k \cdot \Big(\boldsymbol{z}^{k^-} + \boldsymbol{f}^{k^-} \Delta^k \Big) + D_2 \boldsymbol{g}^k \cdot \Big(\boldsymbol{v}^{k^-} + \dot{\boldsymbol{\mu}}^{k^-} \Delta^k \Big) + D_3 \boldsymbol{g}^k \cdot \Delta^k \Big) = \\
& \quad D_1 \boldsymbol{g}^{k^+} \cdot \left(\frac{-D_1 \boldsymbol{g}^k \cdot \boldsymbol{f}^{k^-}}{\dot{\gamma}^k} D_1 \gamma^k + \frac{-D_2 \boldsymbol{g}^k \cdot \dot{\boldsymbol{\mu}}^{k^-}}{\dot{\gamma}^k} D_1 \gamma^k + \frac{-D_3 \boldsymbol{g}^k \cdot 1}{\dot{\gamma}^k} D_1 \gamma^k + D_1 \boldsymbol{g}^k \right) \cdot \boldsymbol{z}^{k^-} \\
& \quad + D_1 \boldsymbol{g}^{k^+} \cdot \left(\frac{-D_1 \boldsymbol{g}^k \cdot \boldsymbol{f}^{k^-}}{\dot{\gamma}^k} D_2 \gamma^k + \frac{-D_2 \boldsymbol{g}^k \cdot \dot{\boldsymbol{\mu}}^{k^-}}{\dot{\gamma}^k} D_2 \gamma^k + \frac{-D_3 \boldsymbol{g}^k \cdot 1}{\dot{\gamma}^k} D_2 \gamma^k + D_2 \boldsymbol{g}^k \right) \cdot \boldsymbol{v}^{k^-},
\end{aligned} \tag{C.55}
$$

$$
\Big(D_3 \boldsymbol{g}^{k^+} \cdot 1 - \boldsymbol{f}^{k^+} \Big) \Delta^{k^+} = \frac{-D_3 \boldsymbol{g}^{k^+} \cdot 1 + \boldsymbol{f}^{k^+}}{\dot{\gamma}^{k^+}} \Big(D_1 \gamma^{k^+} \cdot \Big(\boldsymbol{G}_{\boldsymbol{z}}^k \boldsymbol{z}^{k^-} + \boldsymbol{G}_{\boldsymbol{v}}^k \boldsymbol{v}^{k^-} \Big) + D_2 \gamma^{k^+} \cdot \boldsymbol{v}^k \Big). \tag{C.56}
$$

When we substitute the equations above into (C.52), after reordering the expression we get

$$\begin{aligned}
\boldsymbol{z}^{k^+}(\tau) = &\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_1\gamma^{k^+}\cdot\boldsymbol{G}_{\boldsymbol{z}}^k \\
&+D_1\boldsymbol{g}^{k^+}\cdot\frac{\boldsymbol{f}^k - D_1\boldsymbol{g}^k\cdot\boldsymbol{f}^{k^-} - D_2\boldsymbol{g}^k\cdot\dot{\boldsymbol{\mu}}^{k^-} - D_3\boldsymbol{g}^k\cdot 1}{\dot{\gamma}^k} D_1\gamma^{k^+} + D_1\boldsymbol{g}^{k^+}D_1\boldsymbol{g}^k\cdot 1\Bigg)\boldsymbol{z}^{k^-} \\
&+\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_1\gamma^{k^+}\cdot\boldsymbol{G}_{\boldsymbol{v}}^k \\
&+D_1\boldsymbol{g}^{k^+}\cdot\frac{\boldsymbol{f}^k - D_1\boldsymbol{g}^k\cdot\boldsymbol{f}^{k^-} - D_2\boldsymbol{g}^k\cdot\dot{\boldsymbol{\mu}}^{k^-} - D_3\boldsymbol{g}^k\cdot 1}{\dot{\gamma}^k} D_2\gamma^{k^+} + D_1\boldsymbol{g}^{k^+}D_2\boldsymbol{g}^k\cdot 1\Bigg)\boldsymbol{v}^{k^-} \\
&+\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_2\gamma^{k^+}\Bigg)\boldsymbol{v}^k,
\end{aligned} \tag{C.57}$$

from which we can isolate $\boldsymbol{G}_{\boldsymbol{z}}^k$, and $\boldsymbol{G}_{\boldsymbol{v}}^k$ resulting in

$$\begin{aligned}
\boldsymbol{z}^{k^+}(\tau) = &\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_1\gamma^{k^+} + D_1\boldsymbol{g}^{k^+}\cdot 1\Bigg)\boldsymbol{G}_{\boldsymbol{z}}^k\boldsymbol{z}^{k^-} \\
&+\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_1\gamma^{k^+} + D_1\boldsymbol{g}^{k^+}\cdot 1\Bigg)\boldsymbol{G}_{\boldsymbol{v}}^k\boldsymbol{v}^{k^-} \\
&+\Bigg(\frac{\boldsymbol{f}^{k^+} - D_1\boldsymbol{g}^{k^+}\cdot\boldsymbol{f}^k - D_2\boldsymbol{g}^{k^+}\cdot\dot{\boldsymbol{\mu}}^k - D_3\boldsymbol{g}^{k^+}\cdot 1}{\dot{\gamma}^{k^+}} D_2\gamma^{k^+}\Bigg)\boldsymbol{v}^k,
\end{aligned} \tag{C.58}$$

which is equal to

$$\boldsymbol{z}^{k^+}(\tau) = \boldsymbol{G}_{\boldsymbol{z}}^{k^+}\boldsymbol{G}_{\boldsymbol{z}}^k\boldsymbol{z}^{k^-} + \boldsymbol{G}_{\boldsymbol{z}}^{k^+}\boldsymbol{G}_{\boldsymbol{v}}^k\boldsymbol{v}^{k^-} + \boldsymbol{G}_{\boldsymbol{v}}^{k^+}\boldsymbol{v}^k. \tag{C.59}$$

Here we see that (C.59) is equal to (C.42). This proves that for any k, the first-order approximation of the post-impact state of two simultaneous jumps at τ can be found by evaluating the two jumps separately. Since k is a variable in this proof, using an induction-like proof, this holds for any amount of jumps as well. This is illustrated in Figure C.1.

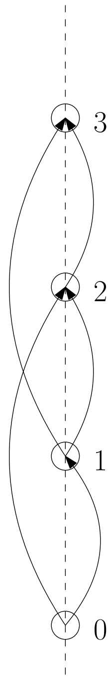

Figure C.1

When we take $k = 1$, we show that the jumps from 0 to 2 can be described by evaluating the jump from 0 to 1 and the jump 1 to 2 in succession. This also holds for $k = 2$. The jump from 0 to 3 can be described by evaluating the jump from 0 to 2 and the jump from 2 to 3 in succession. This way we proved that we can use (C.59) to find an expression for the first-order approximation of the post-impact state for l simultaneous jumps, with $l \in \mathbb{Z}$. Also, we show that multiple constant jump gains will result in a total jump which can also be described by a constant jump gain.

For l simultaneous jumps, we can find the first-order approximation of the post-impact state using

$$ {}^{l\leftarrow 0}\boldsymbol{z}(\tau) = {}^{l\leftarrow 0}\boldsymbol{G}(\boldsymbol{z}^0(\tau), \boldsymbol{v}(\tau), \tau) = {}^{l\leftarrow 0}\boldsymbol{G_z}\boldsymbol{z}^0(\tau) + \sum_{i=0}^{l-1} \left({}^{l\leftarrow i+1}\boldsymbol{G_z}\ {}^{i+1\leftarrow i}\boldsymbol{G_v}\boldsymbol{v}^i(\tau) \right), \tag{C.60} $$

where the superscript

$$ b \leftarrow a = b \leftarrow (b-1) \leftarrow \cdots \leftarrow (a+1) \leftarrow a, \tag{C.61} $$

and

$$ {}^{b\leftarrow a}\boldsymbol{G_z} = {}^{b\leftarrow (b-1)}\boldsymbol{G_z} \cdots {}^{(a+2)\leftarrow (a+1)}\boldsymbol{G_z^{(a+1)\leftarrow a}}\boldsymbol{G_z}, \tag{C.62} $$

$$ {}^{b\leftarrow a}\boldsymbol{G_v} = {}^{b\leftarrow (b-1)}\boldsymbol{G_v} \cdots {}^{(a+2)\leftarrow (a+1)}\boldsymbol{G_v^{(a+1)\leftarrow a}}\boldsymbol{G_v}. \tag{C.63} $$

Appendix D

Positive Homogenization for Input-Dependent Guards

D.1 Conewise constant jump gain

This section is written under the assumption that in one macro event, only two modes are possible per contact point. The general form of a jump map for simultaneous impacts with is

$$ {}^{s^l \leftarrow s^0}\boldsymbol{z}(\tau) = {}^{s^l \leftarrow s^0}\boldsymbol{H}(\boldsymbol{z}^0(\tau), \boldsymbol{v}(\tau), \tau) = \begin{cases} \boldsymbol{G}^1(\boldsymbol{z}^0, \boldsymbol{v}^0, \tau), & \text{if condition 1 is true,} \\ \boldsymbol{G}^2(\boldsymbol{z}^0, \boldsymbol{v}^0, \tau), & \text{if condition 1 is true,} \\ \vdots & \vdots \\ \boldsymbol{G}^{p^{c_i}}(\boldsymbol{z}^0, \boldsymbol{v}^0, \tau), & \text{if condition } p^{c_i} \text{ is true,} \end{cases} \tag{D.1} $$

where $\boldsymbol{G}$ is in the form of (C.60). We will now derive the jump maps and associated conditions in (D.1) to make the expression explicit.

During a macro event for a certain perturbation, only a single order of micro events is feasible. This order can be found by determining the perturbed jump time of all possible micro events, and selecting the micro event with the earliest impact time as the next event. Mathematically, this is written as

$$ s^{k+1} = \underset{s^{k+1}}{\operatorname{argmin}} \left({}^{s^{k+1} \leftarrow S^k} t_\epsilon \right). \tag{D.2} $$

The impact time of the next micro event ${}^{s^{k+1} \leftarrow S^k} t_\epsilon$ can be approximated using the first order approximation

$$ {}^{s^{k+1} \leftarrow S^k} t_\epsilon = \tau + \epsilon \Delta^{k+1}. \tag{D.3} $$

Now, since τ and ϵ are equal for each impact time, we can rewrite (D.2) to

$$ s^{k+1} = \underset{s^{k+1}}{\operatorname{argmin}} \left(\Delta^{k+1} \right), \tag{D.4} $$

which can be written as

$$ s^{k+1} = s^k + \underset{\eta^* \in \chi}{\operatorname{argmin}} \left(- \frac{D_1 \gamma^{\eta^*} \left({}^{s^k}\alpha, {}^{s^k}\mu, \tau \right) \; {}^{S^k}\overline{\boldsymbol{z}} + D_2 \gamma^{\eta^*} \left({}^{s^k}\alpha, {}^{s^k}\mu, \tau \right) \; {}^{S^k}\overline{\boldsymbol{v}}}{\dot{\gamma}^{\eta^*}} \right), \tag{D.5} $$

with χ the set of guard identifiers that are still open. Then

$$\eta^{k+1} = \underset{\eta^* \in \chi}{\operatorname{argmin}} \left(- \frac{D_1 \gamma^{\eta^*} \left({}^{s^k}\alpha, {}^{s^k}\mu, \tau \right) \ {}^{S^k}\overline{\boldsymbol{z}} + D_2 \gamma^{\eta^*} \left({}^{s^k}\alpha, {}^{s^k}\mu, \tau \right) \ {}^{S^k}\overline{\boldsymbol{v}}}{\dot{\gamma}^{\eta^*}} \right), \tag{D.6}$$

with $s^{k+1} = s^k + \eta^{k+1}$ and η^{k+1} a guard function identifier. Finally, with

$${}^{S^k}\boldsymbol{a} = - \frac{D_1 \gamma^{\eta} \left({}^{S^k}\alpha, {}^{s^k}\mu, \tau \right)}{\dot{\gamma}^{\eta}}, \quad {}^{S^k}\boldsymbol{b} = - \frac{D_2 \gamma^{\eta} \left({}^{S^k}\alpha, {}^{s^k}\mu, \tau \right)}{\dot{\gamma}^{\eta}}, \tag{D.7}$$

(D.6) can be rewritten as

$$\eta^{k+1} = \underset{\eta^* \in \chi}{\operatorname{argmin}} \left({}^{S^k}\boldsymbol{a}^T \ {}^{S^k}\overline{\boldsymbol{z}} + \ {}^{s^k}\boldsymbol{b}^T \ {}^{S^k}\overline{\boldsymbol{v}} \right). \tag{D.8}$$

Now, by checking (D.6) for every micro event, we know which jump gains we should take to substitute into (C.60). For example a system with $c_i = 2$ and $p = 3$, with a macro event starting in $s_0 = 00$ and ending in $s_l = 22$, this gives

$$\begin{aligned} &{}^{S^{2\leftarrow 0}}\boldsymbol{H}(\ {}^{s^0}\boldsymbol{z}(\tau), \ {}^{S^2}\boldsymbol{v}(\tau), \tau) = \\ &\begin{cases} {}^{(12)}{}^{S^{2\leftarrow 0}}\boldsymbol{G_z} \ {}^{s^0}\boldsymbol{z} + \ {}^{(12)}{}^{S^{2\leftarrow 0}}\boldsymbol{G_v} \ {}^{s^0}\boldsymbol{v}, & \text{if (I)}, \\ {}^{12}{}^{S^{2\leftarrow 1}}\boldsymbol{G_z} \ {}^{1}{}^{S^{1\leftarrow 0}}\boldsymbol{G_z} \ {}^{s^0}\boldsymbol{z} + \ {}^{12}{}^{S^{2\leftarrow 1}}\boldsymbol{G_z} \ {}^{1}{}^{S^{1\leftarrow 0}}\boldsymbol{G_v} \ {}^{s^0}\boldsymbol{v} + \ {}^{12}{}^{S^{2\leftarrow 1}}\boldsymbol{G_v} \ {}^{1}{}^{S^1}\boldsymbol{v}, & \text{if (II)}, \\ {}^{21}{}^{S^{2\leftarrow 1}}\boldsymbol{G_z} \ {}^{2}{}^{S^{1\leftarrow 0}}\boldsymbol{G_z} \ {}^{s^0}\boldsymbol{z} + \ {}^{21}{}^{S^{2\leftarrow 1}}\boldsymbol{G_z} \ {}^{2}{}^{S^{1\leftarrow 0}}\boldsymbol{G_v} \ {}^{s^0}\boldsymbol{v} + \ {}^{21}{}^{S^{2\leftarrow 1}}\boldsymbol{G_v} \ {}^{2}{}^{S^1}\boldsymbol{v}, & \text{if (III)}, \end{cases} \end{aligned} \tag{D.9}$$

with

$$\text{(I)}: \ {}^{2}{}^{S^1}\boldsymbol{a}^T \ {}^{2}{}^{S^1}\boldsymbol{z} + \ {}^{2}{}^{S^1}\boldsymbol{b}^T \ {}^{2}{}^{S^1}\boldsymbol{v} = \ {}^{1}{}^{S^1}\boldsymbol{a}^T \ {}^{1}{}^{S^1}\boldsymbol{z} + \ {}^{1}{}^{S^1}\boldsymbol{b}^T \ {}^{1}{}^{S^1}\boldsymbol{v}, \tag{D.10}$$

$$\text{(II)}: \ {}^{2}{}^{S^1}\boldsymbol{a}^T \ {}^{2}{}^{S^1}\boldsymbol{z} + \ {}^{2}{}^{S^1}\boldsymbol{b}^T \ {}^{2}{}^{S^1}\boldsymbol{v} > \ {}^{1}{}^{S^1}\boldsymbol{a}^T \ {}^{1}{}^{S^1}\boldsymbol{z} + \ {}^{1}{}^{S^1}\boldsymbol{b}^T \ {}^{1}{}^{S^1}\boldsymbol{v}, \tag{D.11}$$

$$\text{(III)}: \ {}^{2}{}^{S^1}\boldsymbol{a}^T \ {}^{2}{}^{S^1}\boldsymbol{z} + \ {}^{2}{}^{S^1}\boldsymbol{b}^T \ {}^{2}{}^{S^1}\boldsymbol{v} < \ {}^{1}{}^{S^1}\boldsymbol{a}^T \ {}^{1}{}^{S^1}\boldsymbol{z} + \ {}^{1}{}^{S^1}\boldsymbol{b}^T \ {}^{1}{}^{S^1}\boldsymbol{v}. \tag{D.12}$$

These conditions are illustrated in Figure D.1. Since the conditions are linear in $\boldsymbol{z}$ and $\boldsymbol{v}$, they appear as lines in the state space of $\boldsymbol{z}$ and $\boldsymbol{v}$. When we introduce more conditions, we will find several cones that relate a certain jump gain to $\boldsymbol{z}, \boldsymbol{v}$ pair. When we look at the vector $r(\boldsymbol{z}, \boldsymbol{v})$ in Figure D.1, we notice that when r is multiplied with a constant α we will always stay in the same cone, i.e. use the same jump gain. Hence the name, conewise constant jump gain.

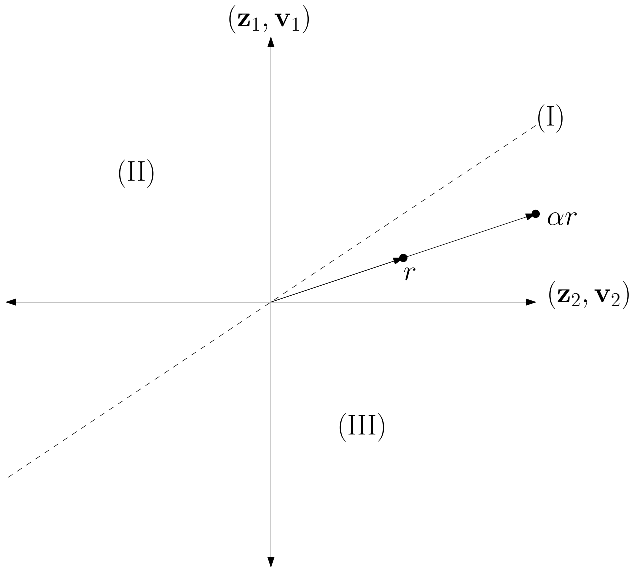

Figure D.1

D.2 Positive homogeneity

The first order approximation of the perturbed trajectory $\boldsymbol{x}_\epsilon$ can be found using $\boldsymbol{\alpha} + \epsilon \boldsymbol{z}$, with

$$
\begin{aligned}
{}^{s_{k-1}}\dot{\boldsymbol{z}} &= {}^{s_{k-1}}\boldsymbol{A}(t)\ {}^{s_{k-1}}\boldsymbol{z} + {}^{s_{k-1}}\boldsymbol{B}(t)\ {}^{s_{k-1}}\boldsymbol{v}, \\
{}^{s_k}\boldsymbol{z} &= {}^{s_k \leftarrow s_{k-1}}\boldsymbol{H}\left({}^{s_{k-1}}\boldsymbol{z}, t\right), \\
{}^{s_k}\dot{\boldsymbol{z}} &= {}^{s_k}\boldsymbol{A}(t)\ {}^{s_k}\boldsymbol{z} + {}^{s_k}\boldsymbol{B}(t)\ {}^{s_{k-1}}\boldsymbol{v},
\end{aligned}
\tag{D.13}
$$

where

$$
{}^{s_k}\boldsymbol{A}(t) = D_1\ {}^{s_k}\boldsymbol{f}\left({}^{s_k}\boldsymbol{\alpha}(t),\ {}^{s_k}\boldsymbol{\mu}(t)\right), \tag{D.14}
$$

$$
{}^{s_k}\boldsymbol{B}(t) = D_2\ {}^{s_k}\boldsymbol{f}\left({}^{s_k}\boldsymbol{\alpha}(t),\ {}^{s_k}\boldsymbol{\mu}(t)\right). \tag{D.15}
$$

When we look at (D.13), the continuous dynamics of the system are linear. Because of the conewise constant jump gain however, this linearity property is lost. We can see this by looking at the general solution of (D.13). A system $f(x,u)$ with state x and input u is linear if $f(x_1, v_1) + f(x_2, v_2) = f(x_1 + x_2, v_1 + v_2)$. Two solutions of (D.13) before jump are

$$
{}^{s_{k-1}}\boldsymbol{z}_1(\tau) = {}^{s_{k-1}}\boldsymbol{\phi}(t, t_0)\ {}^{s_{k-1}}\boldsymbol{z}_1(t_0) + \int_{t_0}^{\tau} \left[{}^{s_{k-1}}\boldsymbol{\phi}(t, s)\ {}^{s_{k-1}}\boldsymbol{B}(s)\boldsymbol{v}_1(s) \right] ds, \tag{D.16}
$$

$$
{}^{s_{k-1}}\boldsymbol{z}_2(\tau) = {}^{s_{k-1}}\boldsymbol{\phi}(t, t_0)\ {}^{s_{k-1}}\boldsymbol{z}_2(t_0) + \int_{t_0}^{\tau} \left[{}^{s_{k-1}}\boldsymbol{\phi}(t, s)\ {}^{s_{k-1}}\boldsymbol{B}(s)\boldsymbol{v}_2(s) \right] ds, \tag{D.17}
$$

with t_0 the initial time and τ the jump time. When we add these solutions together we find

$$
\begin{aligned}
{}^{s_{k-1}}\boldsymbol{z}_1(\tau) + {}^{s_{k-1}}\boldsymbol{z}_2(\tau) = {}^{s_{k-1}}\boldsymbol{\phi}(t, t_0) &\left({}^{s_{k-1}}\boldsymbol{z}_1(t_0) + {}^{s_{k-1}}\boldsymbol{z}_2(t_0) \right) \\
&+ \int_{t_0}^{\tau} \left[{}^{s_{k-1}}\boldsymbol{\phi}(t, s)\ {}^{s_{k-1}}\boldsymbol{B}(s) \left(\boldsymbol{v}_1(s) + \boldsymbol{v}_2(s)\right) \right] ds,
\end{aligned}
\tag{D.18}
$$

which is equal to the solution of ${}^{s^{k-1}}\boldsymbol{z}_3(t_0) ={}^{s^{k-1}} \boldsymbol{z}_1(t_0) +{}^{s^{k-1}} \boldsymbol{z}_2(t_0)$ with $\boldsymbol{v}_3(t) = \boldsymbol{v}_1(t) + \boldsymbol{v}_2(t)$. When ${}^{s^{k-1}}\boldsymbol{z}_1$ jumps with $\boldsymbol{G}^1(\boldsymbol{z},\tau)$ and ${}^{s^{k-1}}\boldsymbol{z}_1(\tau)$ jumps with $\boldsymbol{G}^2(\boldsymbol{z},\tau)$ we find the solutions post jump to be

$$ {}^{s^k}\boldsymbol{z}_1(\tau) = {}^{s^k}\boldsymbol{\phi}(t,t_0)\ \boldsymbol{G}^1({}^{s^{k-1}}\boldsymbol{z}_1(t_0),\tau) + \int_{t_0}^{\tau} \left[{}^{s^k}\boldsymbol{\phi}(t,s)\ {}^{s^k}\boldsymbol{B}(s)\boldsymbol{v}_1(s)\right] ds, \tag{D.19} $$

$$ {}^{s^k}\boldsymbol{z}_2(\tau) = {}^{s^k}\boldsymbol{\phi}(t,t_0)\ \boldsymbol{G}^2({}^{s^{k-1}}\boldsymbol{z}_2(t_0),\tau) + \int_{t_0}^{\tau} \left[{}^{s^k}\boldsymbol{\phi}(t,s)\ {}^{s^k}\boldsymbol{B}(s)\boldsymbol{v}_2(s)\right] ds, \tag{D.20} $$

which when added together results in

$$ \begin{aligned} {}^{s^k}\boldsymbol{z}_1(\tau) + {}^{s^k}\boldsymbol{z}_2(\tau) = {}^{s^k}\boldsymbol{\phi}(t,t_0) \Big(\boldsymbol{G}^1_{\boldsymbol{z}}\ {}^{s^{k-1}}\boldsymbol{z}_1 + \boldsymbol{G}^1_{\boldsymbol{v}}\ {}^{s^{k-1}}\boldsymbol{v}_1 + \boldsymbol{G}^2_{\boldsymbol{z}}\ {}^{s^{k-1}}\boldsymbol{z}_2 + \boldsymbol{G}^2_{\boldsymbol{v}}\ {}^{s^{k-1}}\boldsymbol{v}_2\Big) \\ + \int_{t_0}^{\tau} \left[{}^{s^k}\boldsymbol{\phi}(t,s)\ {}^{s^k}\boldsymbol{B}(s)\left(\boldsymbol{v}_1(s) + \boldsymbol{v}_2(s)\right)\right] ds. \end{aligned} \tag{D.21} $$

Here we see that the solution of ${}^{s^{k-1}}\boldsymbol{z}_3(t_0) ={}^{s^{k-1}} \boldsymbol{z}_1(t_0) +{}^{s^{k-1}} \boldsymbol{z}_2(t_0)$ with $\boldsymbol{v}_3(t) = \boldsymbol{v}_1(t) + \boldsymbol{v}_2(t)$, which jumps with $\boldsymbol{G}_3(\boldsymbol{z},\tau)$, is only the equal to (D.21) if $\boldsymbol{G}_1(\boldsymbol{z},\tau) = \boldsymbol{G}_2(\boldsymbol{z},\tau) = \boldsymbol{G}_3(\boldsymbol{z},\tau)$. In other words, the system only maintains its linearity after jump if the jump maps are equal for each ante jump state. Because this is generally not true, we show that the system is positive homogeneous for any jump gains. A system $f(x,u)$ with state x and input u is called positive homogeneous, when $\alpha f(x,u) = f(\alpha x, \alpha u)$. If we multiply (D.19) with a constant α, we find

$$ {}^{s^k}\boldsymbol{z}_1(\tau) = \alpha\ {}^{s^k}\boldsymbol{\phi}(t,t_0)\ \left(\boldsymbol{G}^1_{\boldsymbol{z}}\ {}^{s^{k-1}}\boldsymbol{z}_1 + \boldsymbol{G}^1_{\boldsymbol{v}}\ {}^{s^{k-1}}\boldsymbol{v}_1\right) + \alpha \int_{t_0}^{\tau} \left[{}^{s^k}\boldsymbol{\phi}(t,s)\ {}^{s^{k-1}}\boldsymbol{B}(s)\boldsymbol{v}_1(s)\right] ds. \tag{D.22} $$

If we now look at the solution for $\boldsymbol{z}_4(t_0) = \alpha\boldsymbol{z}_1(t_0)$ with $\boldsymbol{v}_4(t) = \alpha\boldsymbol{v}_1(t)$ jumping with $\boldsymbol{G}^4(\tau)$, and using the fact that $\boldsymbol{G}^4(\tau) = \boldsymbol{G}^1(\tau)$ since the gains are conewise constant as illustrated in Figure D.1, we find the same solution as (D.22). This shows that (D.13) is positive homogeneous for any conewise constant jump gain ${}^{s^k \leftarrow s^{k-1}}\boldsymbol{H}\left({}^{s^{k-1}}\boldsymbol{z}, t\right)$. Hence the name, positive homogenization.

Appendix E

Considered Trajectories

E.1 Associativity

E.2 Transversality

E.3 Superfluous Contacts

E.4 Nominal Guard-Activations

For impacts, the theory is valid for impacts away from a simultaneous impacts and for simultaneous impacts, but not for impacts very close to simultaneous impacts. In this case the nominal impact is not a simultaneous impact, but a perturbation can cause the order of impacts to change. I believe the jump gain is not conewise constant anymore in this case.

The same is true for impacts close to the border of stick and slip.

E.5 Non-impacting contact points can not switch modes

E.6 All closed contact points are in the same mode

If I do not assume this, then every combination of Γ_{i_C} has to be iterated until a feasible solution is found. If we do assume this, we can just check if all Γ_{i_C} go to stick. If not, then all Γ_{i_C} go to slip.

In this work only trajectories where all closed contacts are in the same mode are considered. This means that when (A.2) is smaller than zero, i.e. the reaction forces are infeasible for a stick post-impact mode, all contact points have a feasible slip post-impact mode. For trajectories where different contact points can be in slip and in stick at the same time, this conclusion can not be drawn. One should then iterate over all possible post-impact modes until a post-impact mode is found which has feasible reaction forces.

Appendix F

Simulation Design

F.1 Plank box dynamics

In Figure F.1 the plank-box model is illustrated. The block is fully actuated and the plank is attached to the solid environment with a spring and damper. The line contact is for now modeled using two contact-points C_L and C_R.

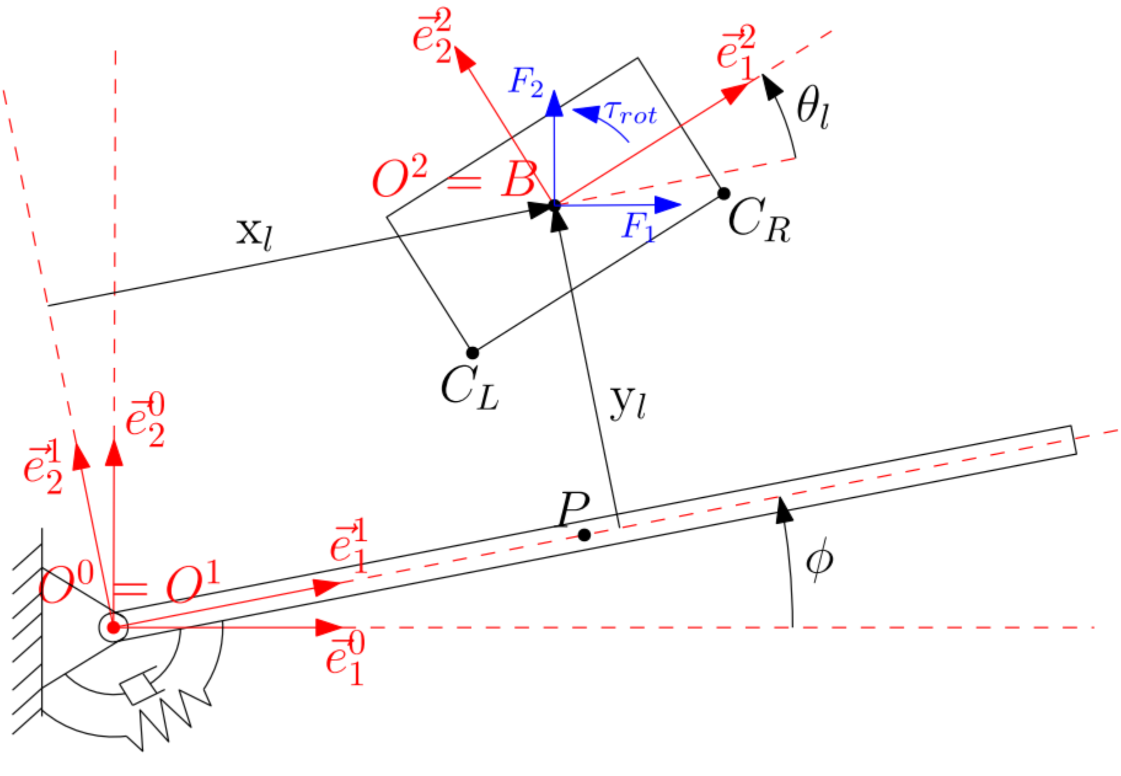

Figure F.1: *Block-plank model*

Global unconstrained dynamics

The global generalized coordinates are defined as

$$q_g = \begin{bmatrix} x_g & y_g & \theta_g & \varphi \end{bmatrix}, \tag{F.1}$$

$$v_g = \begin{bmatrix} \dot{x}_g & \dot{y}_g & \dot{\theta}_g & \dot{\varphi} \end{bmatrix}. \tag{F.2}$$

The equations of motion for the global unconstrained dynamics are then described by

$$\boldsymbol{M}_g(\boldsymbol{q}_g)\dot{\boldsymbol{v}}_g - \boldsymbol{H}_g(\boldsymbol{q}_g, \boldsymbol{v}_g) = \boldsymbol{S}_g(\boldsymbol{q}_g)\boldsymbol{u}, \tag{F.3}$$

with

$$\boldsymbol{M}_g(\boldsymbol{q}_g) = \begin{bmatrix} m_B & 0 & 0 & 0 \\ 0 & m_B & 0 & 0 \\ 0 & 0 & J_B & 0 \\ 0 & 0 & 0 & \frac{m_P {L_p}^2}{4} + J_P \end{bmatrix} \tag{F.4}$$

$$\boldsymbol{H}_g(\boldsymbol{q}_g, \boldsymbol{v}_g) = \begin{bmatrix} 0 \\ -g m_B \\ 0 \\ k_P \varphi - b_P \dot{\varphi} - \frac{L_p g m_P \cos(\varphi)}{2} \end{bmatrix} \tag{F.5}$$

$$\boldsymbol{S}_g(\boldsymbol{q}_g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{F.6}$$

Local unconstrained dynamics

The global generalized coordinates $\boldsymbol{q}_g$ can be rewritten to a set of local coordinates $\boldsymbol{q}_l$. In the plank box case they are related via

$$\boldsymbol{q}_g(\boldsymbol{q}_l) = \begin{bmatrix} \cos(\varphi)x_l - \sin(\varphi)y_l \\ \sin(\varphi)x_l + \cos(\varphi)y_l \\ \theta_l + \varphi \\ \varphi \end{bmatrix}. \tag{F.7}$$

The local unconstrained equations of motion are then defined as

$$\boldsymbol{M}_l(\boldsymbol{q}_l)\dot{\boldsymbol{v}}_l - \boldsymbol{H}_l(\boldsymbol{q}_l, \boldsymbol{v}_l) = \boldsymbol{S}_l(\boldsymbol{q}_l)\boldsymbol{u}, \tag{F.8}$$

with

$$\boldsymbol{M}_l(\boldsymbol{q}_l) = \begin{bmatrix} m_B & 0 & 0 & -m_B y_l \\ 0 & m_B & 0 & m_B x_l \\ 0 & 0 & J_B & J_B \\ -m_B y_l & m_B x_l & J_B & \frac{m_P {L_p}^2}{4} + m_B {x_l}^2 + m_B {y_l}^2 + J_B + J_P \end{bmatrix} \tag{F.9}$$

$$\boldsymbol{H}_l(\boldsymbol{q}_l, \boldsymbol{v}_l) = \begin{bmatrix} m_B \left(x_l \dot{\varphi}^2 + 2\dot{y}_l \dot{\varphi} - g \sin(\varphi)\right) \\ -m_B \left(-y_l \dot{\varphi}^2 + 2\dot{x}_l \dot{\varphi} + g \cos(\varphi)\right) \\ 0 \\ k_P \varphi - b_P \dot{\varphi} - 2m_B \dot{\varphi} x_l \dot{x}_l - 2m_B \dot{\varphi} y_l \dot{y}_l - \frac{L_p g m_P \cos(\varphi)}{2} - g m_B x_l \cos(\varphi) + g m_B y_l \sin(\varphi) \end{bmatrix} \tag{F.10}$$

$$\boldsymbol{S}_l(\boldsymbol{q}_l) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \\ -y_l \cos(\varphi) - x_l \sin(\varphi) & x_l \cos(\varphi) - y_l \sin(\varphi) & 1 \end{bmatrix}. \tag{F.11}$$

Local constrained dynamics

First we have to determine the position vectors of the points C_L and C_R using Figure F.2. First we define the position vector of C_R,

$$r_{C_R} = r_B + r_{BC_R} \text{ with,} \tag{F.12}$$

$$r_B = \begin{bmatrix} x_l & y_l & 0 \end{bmatrix} \vec{\boldsymbol{e}}^1, \tag{F.13}$$

$$r_{BC_R} = \begin{bmatrix} BH & HC_R & 0 \end{bmatrix} \vec{\boldsymbol{e}}^1, \tag{F.14}$$

using the axis systems defined in the report of Hao. From Figure F.2 and that $\triangle BFG \cong \triangle C_R HG$ we can say

$$HC_R = \cos(\theta_l) C_R G, \tag{F.15}$$

$$C_R G = FG - FC_R, \tag{F.16}$$

$$FG = \tan(\theta_l) BF, \tag{F.17}$$

and with $FC_R = \frac{l_B}{2}$ and $BF = \frac{L_B}{2}$ we find

$$HC_R = \sin(\theta_l) \frac{L_B}{2} - \cos(\theta_l) \frac{l_B}{2}. \tag{F.18}$$

For BH we say

$$BH = BG - HG, \tag{F.19}$$

$$BG = \frac{1}{\sin(\theta_l)} FG, \tag{F.20}$$

$$HG = \sin(\theta_l) C_R G, \tag{F.21}$$

which gives us

$$BH = \cos(\theta_l) \frac{L_B}{2} - \sin(\theta_l) \frac{l_B}{2}. \tag{F.22}$$

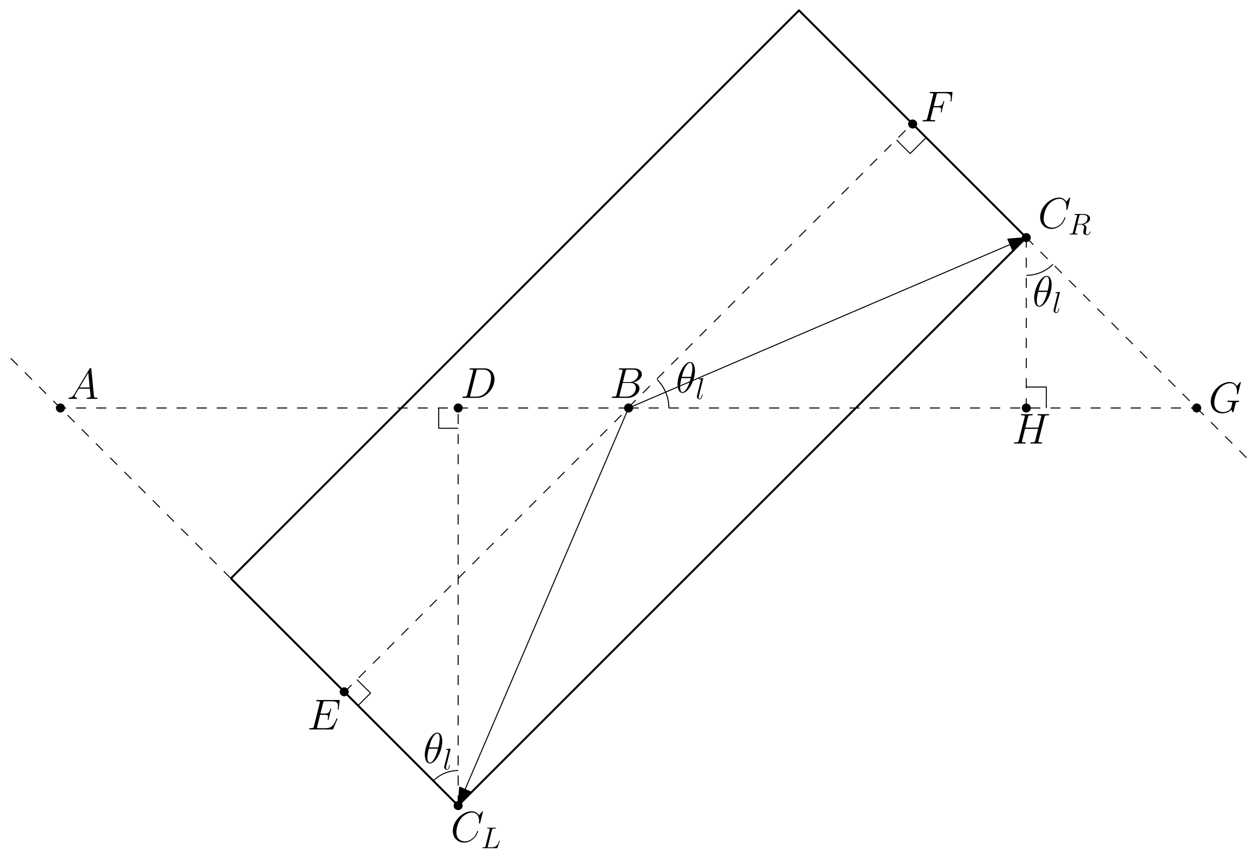

Figure F.2: *Geometry of the box, used to determine the position vectors of contactpoints C_L and C_R.*

With HC_R and BH known, the position vector of C_R is

$$r_{C_R} = \begin{bmatrix} x_l + \cos(\theta_l)\frac{L_B}{2} - \sin(\theta_l)\frac{l_B}{2} \\ y_l + \sin(\theta_l)\frac{L_B}{2} - \cos(\theta_l)\frac{l_B}{2} \\ 0 \end{bmatrix}^T \vec{\boldsymbol{e}}^1. \tag{F.23}$$

Using a similar approach for C_L we find

$$r_{C_L} = \begin{bmatrix} x_l + \cos(\theta_l)\frac{L_B}{2} + \sin(\theta_l)\frac{l_B}{2} \\ y_l - \sin(\theta_l)\frac{L_B}{2} - \cos(\theta_l)\frac{l_B}{2} \\ 0 \end{bmatrix}^T \vec{\boldsymbol{e}}^1. \tag{F.24}$$

The guard functions g_{N1} and g_{N2} are defined as

$$g_{N1} = \vec{\boldsymbol{r}}^1_{C_L}\vec{\boldsymbol{e}}^1_2 - l_p = y_l - \frac{l_P}{2} - \sin(\theta_l)\frac{L_B}{2} - \cos(\theta_l)\frac{l_B}{2} = 0, \tag{F.25}$$

$$g_{N2} = \vec{\boldsymbol{r}}^1_{C_R}\vec{\boldsymbol{e}}^1_2 - l_p = y_l - \frac{l_P}{2} + \sin(\theta_l)\frac{L_B}{2} - \cos(\theta_l)\frac{l_B}{2} = 0, \tag{F.26}$$

Since we are in a 2D-environment, the tangential reaction forces have the same dimensions as the normal reaction-forces. Therefore Equations (**??**), (**??**) and (**??**) can now be considered as, respectively,

$$\boldsymbol{g}_T = \begin{bmatrix} g_{Ti_1}; g_{Ti_2}; ...; g_{Ti_c} \end{bmatrix} \in \mathbb{R}^c \tag{F.27}$$

$$\boldsymbol{\Lambda}_T = \begin{bmatrix} \Lambda_{Ti_1}; \Lambda_{Ti_2}; ...; \Lambda_{Ti_c} \end{bmatrix} \in \mathbb{R}^c \tag{F.28}$$

$$\boldsymbol{W}_T = \begin{bmatrix} \boldsymbol{w}_{Ti_1}; \boldsymbol{w}_{Ti_2}; ...; \boldsymbol{w}_{Ti_c} \end{bmatrix}. \in \mathbb{R}^{n\times c} \tag{F.29}$$

The velocity vectors $\dot{r}_{C_L}$ and $\dot{r}_{C_R}$ are found by taking the time-derivative of r_{C_L} and r_{C_R}, and can be written as

$$\dot{r}_{C_L} = \begin{bmatrix} \dot{x}_l - \dot{\theta}_l \sin(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \cos(\theta_l)\frac{l_B}{2} \\ \dot{y}_l - \dot{\theta}_l \cos(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \sin(\theta_l)\frac{l_B}{2} \\ 0 \end{bmatrix} =: \begin{bmatrix} \dot{g}_{T1} \\ \dot{g}_{N1} \\ 0 \end{bmatrix}, \tag{F.30}$$

$$\dot{r}_{C_R} = \begin{bmatrix} \dot{x}_l - \dot{\theta}_l \sin(\theta_l)\frac{L_B}{2} - \dot{\theta}_l \cos(\theta_l)\frac{l_B}{2} \\ \dot{y}_l + \dot{\theta}_l \cos(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \sin(\theta_l)\frac{l_B}{2} \\ 0 \end{bmatrix} =: \begin{bmatrix} \dot{g}_{T2} \\ \dot{g}_{N2} \\ 0 \end{bmatrix}, \tag{F.31}$$

with $\dot{g}_{Ni}$ and $\dot{g}_{Ti}$ the normal and tangential relative velocities. From (**??**) and (**??**) we can write

$$\dot{\boldsymbol{g}}_N = \boldsymbol{W}_N^T \boldsymbol{v} = \begin{bmatrix} \dot{y}_l - \dot{\theta}_l \cos(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \sin(\theta_l)\frac{l_B}{2} \\ \dot{y}_l + \dot{\theta}_l \cos(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \sin(\theta_l)\frac{l_B}{2} \end{bmatrix}, \tag{F.32}$$

$$\dot{\boldsymbol{g}}_T = \boldsymbol{W}_T^T \boldsymbol{v} = \begin{bmatrix} \dot{x}_l - \dot{\theta}_l \sin(\theta_l)\frac{L_B}{2} + \dot{\theta}_l \cos(\theta_l)\frac{l_B}{2} \\ \dot{x}_l - \dot{\theta}_l \sin(\theta_l)\frac{L_B}{2} - \dot{\theta}_l \cos(\theta_l)\frac{l_B}{2} \end{bmatrix}, \tag{F.33}$$

from which we can deduce

$$\boldsymbol{W}_N^T = \begin{bmatrix} 0 & 1 & -\cos(\theta_l)\frac{L_B}{2} + \sin(\theta_l)\frac{l_B}{2} & 0 \\ 0 & 1 & \cos(\theta_l)\frac{L_B}{2} + \sin(\theta_l)\frac{l_B}{2} & 0 \end{bmatrix}, \tag{F.34}$$

$$\boldsymbol{W}_T^T = \begin{bmatrix} 0 & 1 & -\sin(\theta_l)\frac{L_B}{2} + \cos(\theta_l)\frac{l_B}{2} & 0 \\ 0 & 1 & -\sin(\theta_l)\frac{L_B}{2} - \cos(\theta_l)\frac{l_B}{2} & 0 \end{bmatrix}. \tag{F.35}$$

Now we have all the information to write down the local constrained dynamics

$$\boldsymbol{M}_l(\boldsymbol{q}_l)\dot{\boldsymbol{v}}_l - \boldsymbol{H}_l(\boldsymbol{q}_l, \boldsymbol{v}_l) = \boldsymbol{S}_l(\boldsymbol{q}_l)\boldsymbol{u} + \boldsymbol{W}(\boldsymbol{q}_l)\boldsymbol{\Lambda}, \tag{F.36}$$

with

$$\boldsymbol{W}(\boldsymbol{q}_l) := \begin{bmatrix} \boldsymbol{W}_N & \boldsymbol{W}_T \end{bmatrix} \text{ and } \boldsymbol{\Lambda} := \begin{bmatrix} \boldsymbol{\Lambda}_N \\ \boldsymbol{\Lambda}_T \end{bmatrix}. \tag{F.37}$$

F.2 Reference Trajectory Design