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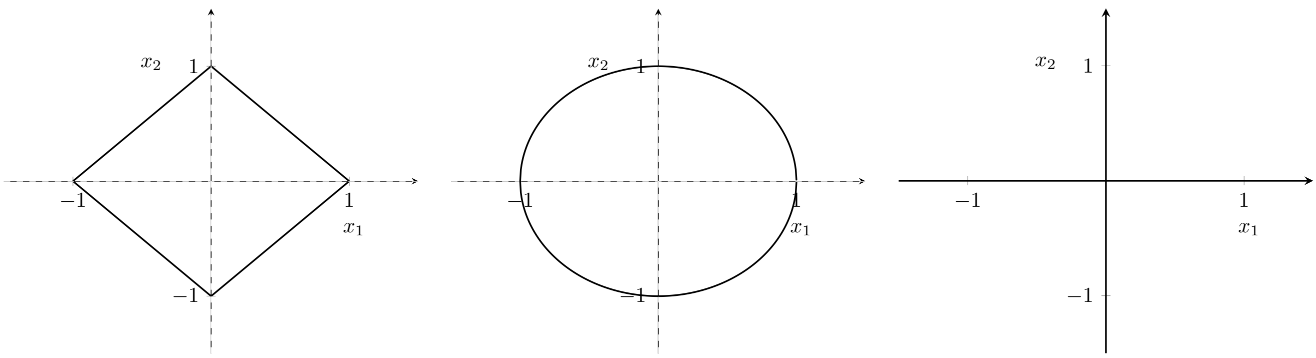
Series 11 Solutions (Dictionary Learning and Compressed Sensing)

Problem 2 (Compressed Sensing):

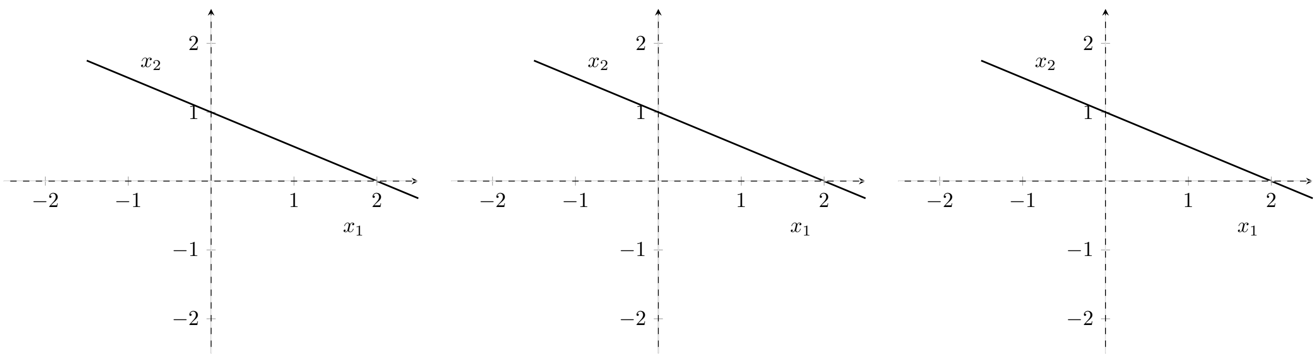
a. Map each of the three equations $\|\mathbf{x}\|_2 = 1$, $\|\mathbf{x}\|_1 = 1$, and $\|\mathbf{x}\|_0 = 1$ to a plot among a., b., or c. on the following figure. Note that $\mathbf{x}$ is s 2D vector with coordinates x_1 and x_2 (i.e. $\mathbf{x} = \left[x_1, x_2\right]$).



a. $\|\mathbf{x}\|_1$ b.$\|\mathbf{x}\|_2$ c. $\|\mathbf{x}\|_0$

b. Show the solution of each optimization problem on plots a., b., and c. of the following figure.

$\min \|\mathbf{x}\|_2$ Subject to $\frac{1}{2}x_1 + x_2 = 1$ $\min \|\mathbf{x}\|_1$ Subject to $\frac{1}{2}x_1 + x_2 = 1$ $\min \|\mathbf{x}\|_0$ Subject to $\frac{1}{2}x_1 + x_2 = 1$



a.   b.   c.

Solution:
a. The answer will be the closest point on the line to the origin, i.e.

$$\frac{1}{2}\mathbf{x}_1 + \mathbf{x}_2 = 1 \leftrightarrow \mathbf{x}_1 = 2 - 2\mathbf{x}_2 \tag{1}$$

$$\min\{\mathbf{x}_1^2 + \mathbf{x}_2^2\} = \min\{(2 - 2\mathbf{x}_2)^2 + \mathbf{x}_2^2\} \tag{2}$$

$$\frac{d}{d\mathbf{x}_2}\left[(2 - 2\mathbf{x}_2)^2 + \mathbf{x}_2^2\right] \overset{!}{=} 0 \leftrightarrow \mathbf{x}_2 = 0.8, \mathbf{x}_1 = 0.4 \tag{3}$$

b. $\mathbf{x}_1 = 0, \mathbf{x}_2 = 1$ c. two solutions $[\mathbf{x}_1 = 0, \mathbf{x}_2 = 1], [\mathbf{x}_1 = 2, \mathbf{x}_2 = 0]$

c. We can formulate the above three optimization problem as

$$\min \|\mathbf{x}\|_p$$
$$\text{subject to } \frac{1}{2}x_1 + x_2 = 1,$$

where $p \in \{0, 1, 2\}$. Mark the right sentence using your previous answers.

[ ] Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.

[ ] Solutions of the constrained problems have intersection for $p = 2$ and $p = 0$.

Solution:
Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.