Exercises
Computational Intelligence Lab
SS 2016

Machine Learning Institute
Dept. of Computer Science, ETH Zürich
Prof. Dr. Thomas Hofmann
Web http://cil.inf.ethz.ch/

Series 10 Solutions (Sparse Coding and Wavelets)

Problem 1

$$\begin{aligned}
\left\|\mathbf{x}-\hat{\mathbf{x}}_\sigma\right\|_2^2 &= \langle \mathbf{x}-\hat{\mathbf{x}}_\sigma, \mathbf{x}-\hat{\mathbf{x}}_\sigma\rangle = \\
&= \langle \sum_{k=1}^{K} z_k \mathbf{u}_k - \sum_{k=1}^{\tilde{K}} z_{\sigma(k)}\mathbf{u}_{\sigma(k)}, \sum_{l=1}^{K} z_l \mathbf{u}_l - \sum_{l=1}^{\tilde{K}} z_{\sigma(l)}\mathbf{u}_{\sigma(l)}\rangle = \\
&= \langle \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)}\mathbf{u}_{\sigma(k)}, \sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)}\mathbf{u}_{\sigma(l)}\rangle = \qquad \text{(because of the linearity of the inner product)} \\
&= \sum_{k=\tilde{K}+1}^{K}\sum_{l=\tilde{K}+1}^{K} z_{\sigma(k)} z_{\sigma(l)} \langle \mathbf{u}_{\sigma(k)}, \mathbf{u}_{\sigma(l)}\rangle = \qquad \text{(because of the orthonormality of the basis)} \\
&= \sum_{k=\tilde{K}+1}^{K} \left(z_{\sigma(k)}\right)^2
\end{aligned}$$

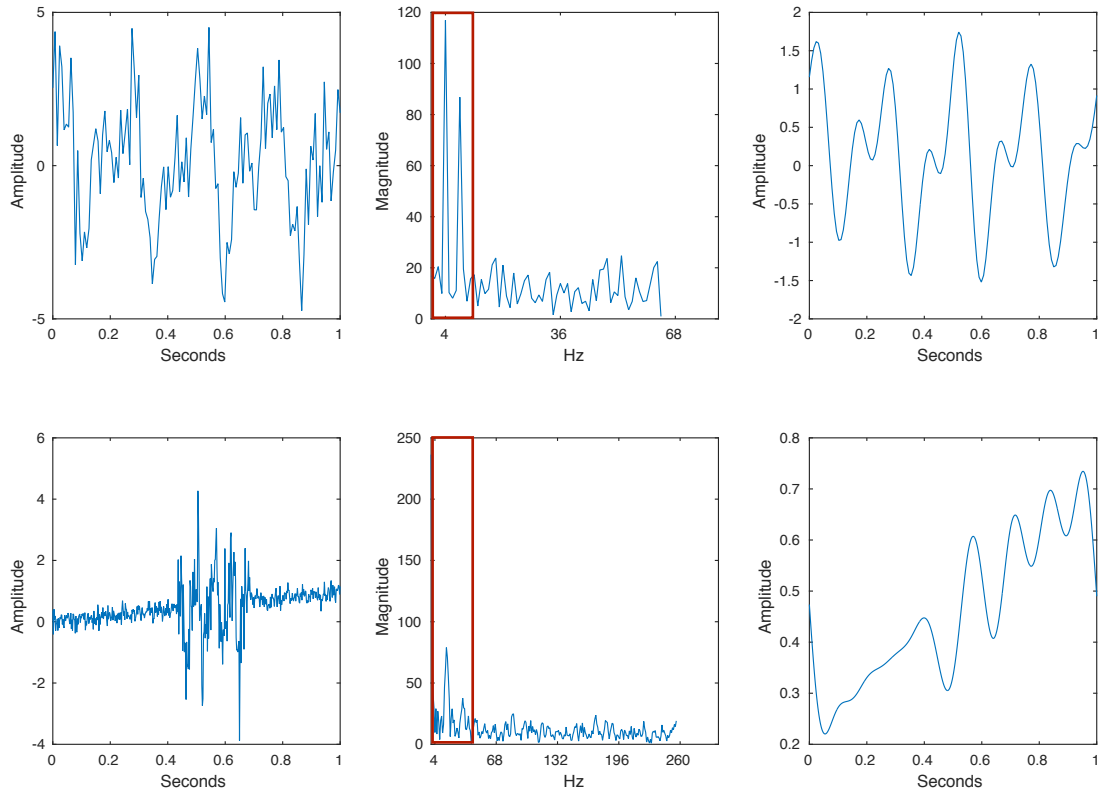
Thus:

$$\sigma^{min} = \underset{\sigma}{\operatorname{argmin}} \left\{ \sum_{k=\tilde{K}+1}^{K} \left(z_{\sigma(k)}\right)^2 \right\}$$

It is obvious that the above expression is minimized when the permutation sorts the coefficients in order of decreasing magnitude.

Problem 2

- (1) Write down the formula to obtain the spectrum in the middle column of the previous figure, in terms of linear transformation or change of basis (assuming a given basis $\mathbf{U}$) applied to the original signal $\mathbf{x}$.
  Solution:
  $$\mathbf{z} = \mathbf{U}^T\mathbf{x} \tag{1}$$
  (2) Write down the inverse formula to obtain the reconstructed signal in the right column in terms of linear transformation (change of basis) applied to the filtered spectrum $\hat{\mathbf{z}}$.
  Solution:
  $$\hat{\mathbf{x}} = \mathbf{U}\hat{\mathbf{z}} \tag{2}$$
- What part of the signal would you discard to obtain the reconstructed signal? Draw a rectangle on each spectrum in the middle column where everything inside the rectangle is kept for the reconstruction.
  Solution:



- **false** The Wavelet transform is a better choice than Fourier for the first signal (top row).

- **true** The Wavelet transform is a better choice than Fourier for the second signal (bottom row).

- Looking at the middle figure in the top row, what do the first peaks in the spectrum correspond to?
  Solution: The coefficient of low frequency that will be preserved for sparse coding.