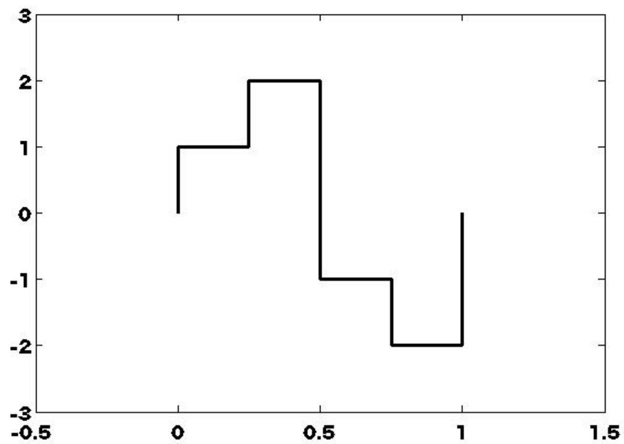
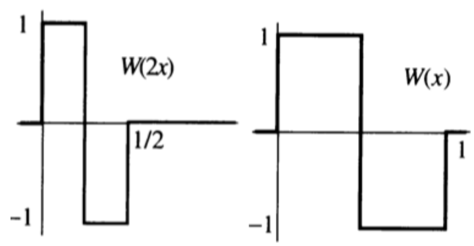
Problem: Reconstruct the following signal with shifted and scaled Haar wavelets





Mathematical approach: Vectorize the Haar wavelets as following:

$$
\begin{aligned}
\mathbf{w}_0 &= [1,1,1,1]^T \\
\mathbf{w}(x) &= [1,1,-1,-1]^T \\
\mathbf{w}(2x) &= [1,-1,0,0]^T \\
\mathbf{w}(2x-1) &= [0,0,1,-1]^T
\end{aligned}
$$

Compose matrix $\mathbf{W}$ whose columns are non-normalized Haar wavelets:

$$
\begin{aligned}
\mathbf{W} &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \\
&= [\mathbf{w}_0, \mathbf{w}(x), \mathbf{w}(2x), \mathbf{w}(2x-1)]
\end{aligned}
$$

The orthogonal matrix $\mathbf{U}$ consists of columns that are normalized Haar wavelets:

$$\begin{aligned}\mathbf{U} &= \frac{1}{2}\begin{bmatrix}1 & 1 & \sqrt{2} & 0\\ 1 & 1 & -\sqrt{2} & 0\\ 1 & -1 & 0 & \sqrt{2}\\ 1 & -1 & 0 & -\sqrt{2}\end{bmatrix}\\ &= [\frac{1}{2}\mathbf{w}_0, \frac{1}{2}\mathbf{w}(x), \frac{\sqrt{2}}{2}\mathbf{w}(2x), \frac{\sqrt{2}}{2}\mathbf{w}(2x-1)]\\ &= \mathbf{WS},\end{aligned}$$

where $\mathbf{S}$ is a diagonal scaling matrix:

$$\mathbf{S} = \begin{bmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 0 & \frac{\sqrt{2}}{2}\end{bmatrix}$$

And we have $\mathbf{W} = \mathbf{US}^{-1}$ Denote the signal as:

$$\mathbf{x} = [1, 2, -1, -2]^T$$

The wavelet coefficents $\mathbf{a}$ can be computed from the following equation*:

$$\mathbf{x} = \mathbf{Wa} = \mathbf{US}^{-1}\mathbf{a}$$

Finally we have:

$$\begin{aligned}\mathbf{a} &= \mathbf{SU}^T\mathbf{x}\\ &= \frac{1}{2}\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 0 & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ \sqrt{2} & -\sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{2} & -\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ -1\\ -2\end{bmatrix}\\ &= [0, 3/2, -1/2, 1/2]^T\end{aligned}$$

Therefore,

$$\mathbf{x} = \frac{3}{2}\mathbf{w}(x) - \frac{1}{2}\mathbf{w}(2x) + \frac{1}{2}\mathbf{w}(2x-1)$$