Homework 6

PkGu

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P[244] 1.

The data given can be reconstructed by $\mathbf{Y} = (y_1, ..., y_n; i_1, ..., i_m)$, where i_k indicates the occurrence of a bulb kep lightning after time=t. Since the measure on the sample space $\mathcal{Y}$ is neither continuous nor discrete, we consider a random variable $\mathbf{Z} = (z_1, ..., z_n)$ where z_k represents the lifetime of a bulb if the experiment had continued after t. The joint distribution $(\mathbf{Y}, \mathbf{Z})|\theta$ is

$$f(\mathbf{Y}, \mathbf{Z}|\theta; \forall k(i_k \sim z_k)) \propto \theta^{m+n} e^{-\theta(\sum_1^n y_j + \sum_1^m z_k)} \delta_{\forall k, i_k \sim z_k}$$

and the posterior is

$$log\ \pi(\theta|\mathbf{Y}, \mathbf{Z}; \forall k(i_k \sim z_k)) = -\theta\ (\sum_1^n y_j + \sum_1^m z_k) + (m+n-1)\ log\ \theta + C$$

and the target to be optimized is

$$\begin{aligned}
&\mathbb{E}_{\mathbf{Z}|\mathbf{Y},\theta;\forall k(i_k \sim z_k)}(-\theta\ (\sum_1^n y_j + \sum_1^m z_k) + (m+n-1)\ log\ \theta) \\
&= -\theta \sum_1^n y_j + (m+n-1)\ log\ \theta - \theta \sum_1^m [(t + \frac{1}{\theta}) \cdot \delta_{i_k=1} + (\frac{1}{\theta} - \frac{t}{e^{t\theta} - 1}) \cdot \delta_{i_k=0}] \\
&= -\theta\ (\sum_1^n y_j + \sum_1^m (t \cdot \delta_{i_k=1} - \frac{t}{e^{t\theta} - 1} \cdot \delta_{i_k=0})) + (m+n-1)\ log\ \theta - m
\end{aligned}$$

which means we should maximize (denoting $M_1 := \#\{k|i_k = 1\}$ and M_0 similarly)

$$-\theta \cdot (\sum_1^n y_j + t \cdot M_1 - \frac{t}{e^{t\theta} - 1} \cdot M_0) + (m+n-1)\ log\ \theta$$

which is hard to compute.

If we use E-M method: after getting $\hat{\theta_{(r)}}$,

Step (r+1):

1. Expectation (up to $+C$)

$$E(\theta; \hat{\theta_{(r)}}) = -\theta \cdot (\sum_1^n y_j + t \cdot M_1 - \frac{t}{e^{t \cdot \hat{\theta_{(r)}}} - 1} \cdot M_0 + \frac{m}{\hat{\theta_{(r)}}}) + (m+n-1)\ log\ \theta$$

2. Maximization

$$\theta_{(r+1)} = \frac{\sum_1^n y_j + t \cdot M_1 - \frac{t}{e^{t \cdot \hat{\theta_{(r)}}} - 1} \cdot M_0 + \frac{m}{\hat{\theta_{(r)}}}}{m+n-1}$$

P[244] 2.

The joint distribution $(\mathbf{Y}, \mathbf{Z})|\theta$ is

$$f(\mathbf{Y}, \mathbf{Z}|\theta; \forall k(i_k \sim z_k)) \propto \frac{1}{\theta^{m+n}} \delta_{(0,\theta)^{m+n}}$$

and the posterior is

$$\pi(\theta|\mathbf{Y}, \mathbf{Z}; \forall k(i_k \sim z_k)) \propto \theta^{-(m+n)} \delta_{\theta > max\{y_j; z_k\}}$$

If we use E-M method: after getting $\hat{\theta_{(r)}}$, assuming $\hat{\theta_{(r)}} > t$, $\hat{\theta_{(r)}} > \forall y_j$ and $\theta > t$,

Step (r+1)

1. Expectation (up to $\cdot C$)

$$\begin{aligned}
E(\theta; \hat{\theta_{(r)}}) &= \mathbb{E}_{\mathbf{Z}|\mathbf{Y}, \hat{\theta_{(r)}}; \forall k(i_k \sim z_k)}(\theta^{-(m+n)} \delta_{\theta > max\{y_j; z_k\}}) \\
&= P_{\mathbf{Z}|\mathbf{Y}, \hat{\theta_{(r)}}; \forall k(i_k \sim z_k)}(\theta > max\{y_j; z_k\}) \cdot \theta^{-(m+n)} \\
&= \theta^{-(m+n)} (P_{\hat{\theta_{(r)}}}(z_k < \theta | z_k \in (t, \theta_{(r)})))^{M_1} (P_{\hat{\theta_{(r)}}}(z_k < \theta | z_k \in (0, t)))^{M_0} \delta_{\theta > max\{y_j\}} \\
&= \theta^{-(m+n)} \left[min\{ \frac{\theta - t}{\hat{\theta_{(r)}} - t}, 1\} \right]^{M_1} \delta_{\theta > max\{y_j\}}
\end{aligned}$$

2. Maximization :

$$\hat{\theta_{(r+1)}} = max\{y_j,\ min\{\frac{m+n}{m+n-M_1} \cdot t,\ \hat{\theta_{(r)}}\}\}$$

So it's useless.

But if we try the usual method: the optimal solution is

$$\hat{\theta} = max\{y_j, \frac{m+n}{m+n-M_1} \cdot t\}$$

So it is in a sense useless, especially when $\hat{\theta_{(r)}} < \frac{m+n}{m+n-M_1} \cdot t$, it will never converge to the optimal point.