Homework 5

PkGu

11/1/2021

P[184] 8.

The likelihood function of this model is

$$p(x|M) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

Given prior $\pi(M) = \frac{1}{N+1}$, the posterior distribution for x is

$$\pi(M|x) \propto \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

the corresponding posterior risk function is

$$\begin{aligned} R(\delta(x)|x) &\propto \sum_{M=0}^{N} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}(M-\delta(x))^2 \\ &= \frac{\binom{N+1}{n+1}}{\binom{N}{n}}\delta^2(x) - 2\frac{(x+1)\binom{N+2}{n+2} - \binom{N+1}{n+1}}{\binom{N}{n}}\delta(x) + C \end{aligned}$$

So the Bayesian estimate of $p = M/N$ is $\delta(x) = [\frac{N+2}{n+2}(x+1) - 1]/N$.

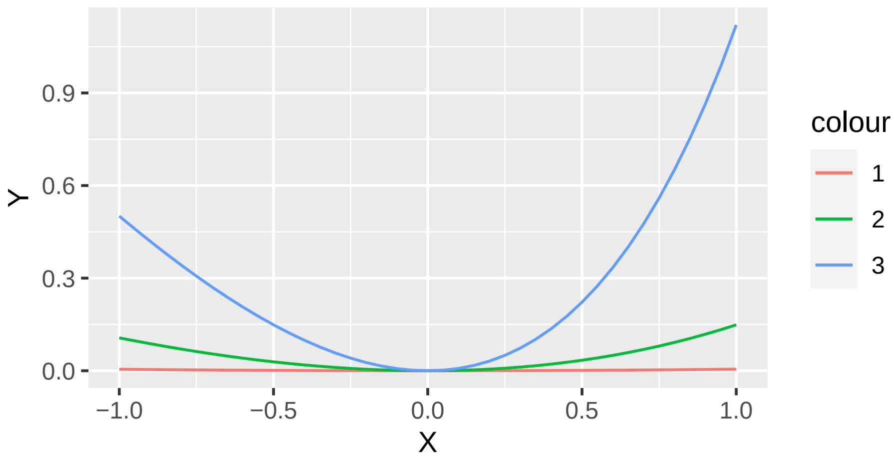
P[185] 19.

(1).Obvious.

(2).

```
library(ggplot2)
c=c(0.1,0.5,1.2)
f=function(X,c)
{
  Y=exp(c*X)-c*X-1
  return(Y)
}
x=seq(-1,1,0.05)
y1=f(x,c[1])
y2=f(x,c[2])
y3=f(x,c[3])
d1=data.frame(X=x,Y=y1)
```

```
d2=data.frame(X=x,Y=y2)
d3=data.frame(X=x,Y=y3)
ggplot()+
  geom_line(d1,mapping=aes(x=X,y=Y,color="1"))+
  geom_line(d2,mapping=aes(x=X,y=Y,color="2"))+
  geom_line(d3,mapping=aes(x=X,y=Y,color="3"))+
  coord_fixed()
```



(3). For prior $\pi(\theta)$ and likelihood function $p(\mathbf{x}|\theta)$, the posterior $\pi(\theta|\mathbf{x})$ can be computed. Given data $\mathbf{x}$, the posterior risk

$$R(\delta(\mathbf{x})|\mathbf{x}) = \int (e^{c(\theta-\delta(\mathbf{x}))} - c(\theta - \delta(\mathbf{x})) - 1)\pi(\theta|\mathbf{x})d\theta$$

$$\begin{aligned}\partial_\delta R(\delta(\mathbf{x})|\mathbf{x}) &= \int c(1 - e^{c(\theta-\delta(\mathbf{x}))})\pi(\theta|\mathbf{x})d\theta \\ &= \mathbb{E}_{\theta|\mathbf{x}}[c(1 - e^{c(\theta-\delta(\mathbf{x}))})]\end{aligned}$$

To minimize $R(\delta(\mathbf{x})|\mathbf{x})$, it requires

$$\mathbb{E}_{\theta|\mathbf{x}}[e^{c\theta}] = e^{c\delta(\mathbf{x})}$$

i.e. $\delta(\mathbf{x}) = \frac{1}{c}log(\mathbb{E}_{\theta|\mathbf{x}}[e^{c\theta}])$.

(4). For $\pi(\theta) = 1$ and $\mathbf{x}|\theta \sim N(\theta, \mathbf{I})$, *i.e.* using sufficient statistic $T(x) = \bar{x}$,

$$p(\bar{x}|\theta) = (\frac{n}{2\pi})^{\frac{1}{2}} e^{-\frac{n}{2}(\bar{x}-\theta)^2}$$

it yields

$$p(\bar{x}) = \int (\frac{n}{2\pi})^{\frac{1}{2}} e^{-\frac{n}{2}(\bar{x}-\theta)^2} d\theta = 1$$

and

$$\pi(\theta|\mathbf{x}) = (\frac{n}{2\pi})^{\frac{1}{2}} e^{-\frac{n}{2}(\bar{x}-\theta)^2}$$

so the Bayesian estimate minimizing the previous risk is

$$\begin{aligned}\delta(\mathbf{x}) &= \frac{1}{c} log \int e^{c\theta} \cdot (\frac{n}{2\pi})^{\frac{1}{2}} e^{-\frac{n}{2}(\bar{x}-\theta)^2} d\theta \\ &= \frac{c}{2n} + \bar{x}\end{aligned}$$

P[185] 20.

$$\begin{aligned}L_{Ent}(\theta, a) &= h(p_\theta || p_a) \\ &= \int log \left[\frac{p(\mathbf{x}|\theta)}{p(\mathbf{x}|a)}\right] p(\mathbf{x}|\theta) d\mathbf{x} \\ &= \int \frac{n}{2}((\bar{x}-a)^2 - (\bar{x}-\theta)^2) \cdot \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{n}{2}(\bar{x}-\theta)^2} d\bar{x} \\ &= \frac{n}{2}(\theta - a)^2\end{aligned}$$

$$\begin{aligned}L_{Hel}(\theta, a) &= \frac{1}{2} \int \left[\sqrt{\frac{p(x|a)}{p(x|\theta)}} - 1\right]^2 p(x|\theta) dx \\ &= \int \frac{1}{2} (e^{\frac{1}{4}((x-\theta)^2 - (x-a)^2)} - 1)^2 \cdot (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\theta)^2} dx \\ &= 1 - \int (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{4}((x-\theta)^2 + (x-a)^2)} dx \\ &= 1 - e^{-\frac{1}{8}(\theta - a)^2}\end{aligned}$$