Homework 4

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(1). The likelihood function is

$$f_\lambda(x_1, ..., x_5) \propto e^{-5\lambda}\lambda^T$$

where $T = \sum_1^5 x_i$ is sufficient. And the prior distribution is

$$\pi(\lambda) \propto e^{-\lambda}\lambda^2$$

so the posterior $\lambda|\mathbf{x} \sim \Gamma(T+2, 6)$.

(2). I've proved in PSet3 that the Jeffery prior for Poisson distribution is

$$\pi(\lambda) \propto \lambda^{-\frac{1}{2}}$$

so the posterior $\lambda|\mathbf{x} \sim \Gamma(T - \frac{1}{2}, 5)$.

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$$\lambda \sim \Gamma(\alpha, \beta) \Rightarrow \mathbb{E}[\lambda] = \frac{\alpha}{\beta},\ Var(\lambda) = \frac{\alpha}{\beta^2}$$

From $\frac{\alpha}{\beta} = 0.0002$, $\frac{\sqrt{\alpha}}{\beta} = 0.0001$ we get

$$\lambda \sim \Gamma(4, 20000)$$

And the posterior distribution is

$$\lambda|x \sim \Gamma(5, 20000 + x)$$

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(1)&(2). Naturally $\pi(\theta, \sigma^2|\mathbf{x}) = \pi(\theta|\sigma^2, \mathbf{x}) \cdot \pi(\sigma^2|\mathbf{x})$, where

$$\begin{aligned}
\pi(\theta|\sigma^2, \mathbf{x}) &= \pi(\theta|\sigma^2, \bar{x}) \\
&\propto \pi(\bar{x}|\theta, \sigma^2)\pi(\theta|\sigma^2) \\
&\propto e^{-\frac{m(\theta-\bar{x})^2}{2\sigma^2}} \cdot e^{-\frac{k(\mu-\theta)^2}{2\sigma^2}} \\
&\propto e^{-\frac{k+m}{2\sigma^2}(\theta-(\frac{k}{k+m}\mu+\frac{m}{k+m}\bar{x}))^2} \\
&= e^{-\frac{(\theta-\mu(\mathbf{x}))^2}{2\eta^2}}
\end{aligned}$$

$$
\begin{aligned}
\pi(\sigma^2|\mathbf{x}) &\propto \pi(\sigma^2)\pi(\mathbf{x}|\sigma^2) \\
&= \pi(\sigma^2)\int_{\mathbb{R}} \pi(\mathbf{x}|\theta,\sigma^2)\pi(\theta|\sigma^2)d\theta \\
&\propto \sigma^{-2}\int_{\mathbb{R}} \frac{1}{\sigma^{m+1}} e^{-\frac{k(\mu-\theta)^2}{2\sigma^2}} \cdot e^{-\frac{m(\bar{x}-\theta)^2+\sum_1^m (x_i-\bar{x})^2}{2\sigma^2}} d\theta \\
&\propto \sigma^{-3-m} e^{-\frac{\sum_1^m (x_i-\bar{x})^2}{2\sigma^2}} e^{-\frac{(\bar{x}-\mu)^2}{2\sigma^2}\frac{mk}{m+k}} \int_{\mathbb{R}} e^{-\frac{k+m}{2\sigma^2}(\theta-(\frac{k}{k+m}\mu+\frac{m}{k+m}\bar{x}))^2} d\theta \\
&\propto (\sigma^2)^{-\frac{m}{2}-1} e^{-[\frac{\sum_1^m (x_i-\bar{x})^2}{2}+\frac{mk(\bar{x}-\mu)^2}{2(m+k)}]\frac{1}{\sigma^2}} \\
&= (\sigma^2)^{-\frac{m}{2}-1} e^{-\frac{B_m}{2}\frac{1}{\sigma^2}}
\end{aligned}
$$

So $\theta|\sigma^2,\mathbf{x} \sim N(\mu(\mathbf{x}),\eta^2)$ and $\sigma^2|\mathbf{x} \sim \Gamma^{-1}(\frac{m}{2},\frac{B_m}{2})$.

(3).

$$
\begin{aligned}
\pi(\theta,\sigma^2|\mathbf{x}) &\propto (\sigma^2)^{-\frac{m}{2}-1} e^{-\frac{B_m}{2}\frac{1}{\sigma^2}} \cdot (\eta^2)^{-1/2} e^{-\frac{(\theta-\mu(\mathbf{x}))^2}{2\eta^2}} \\
&\propto (\sigma^2)^{-\frac{m+3}{2}} e^{-\frac{B_m+(k+m)(\theta-\mu(\mathbf{x}))^2}{2\sigma^2}}
\end{aligned}
$$

$$
\begin{aligned}
\pi(\theta|\mathbf{x}) &= \int_{\mathbb{R}^+} \pi(\theta,\sigma^2|\mathbf{x})d(\sigma^2) \\
&\propto \int_{\mathbb{R}^+} s^{-\frac{m+3}{2}} e^{-\frac{B_m+(k+m)(\theta-\mu(\mathbf{x}))^2}{2s}} ds \\
&\propto [\frac{B_m}{m+k}+(\theta-\mu(\mathbf{x}))^2]^{-\frac{m+1}{2}}
\end{aligned}
$$

So $\theta|\mathbf{x} \sim \mathcal{T}_m(\mu(\mathbf{x}),\frac{B_m}{m(m+k)})$.

P[103] 23.

(1)&(2). Naturally $\pi(\theta,\sigma^2|\mathbf{x}) = \pi(\theta|\sigma^2,\mathbf{x})\cdot\pi(\sigma^2|\mathbf{x})$, where

$$
\begin{aligned}
\pi(\theta|\sigma^2,\mathbf{x}) &= \pi(\theta|\sigma^2,\bar{x}) \\
&\propto \pi(\bar{x}|\theta,\sigma^2)\pi(\theta|\sigma^2) \\
&\propto e^{-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}} \cdot e^{-\frac{(\theta-\mu)^2}{2\tau\sigma^2}} \\
&\propto e^{-\frac{(\theta-\mu(\mathbf{x}))^2}{2\eta^2}}
\end{aligned}
$$

where

$$
\mu(\mathbf{x}) = \frac{\mu+n\tau\bar{x}}{n\tau+1}\ ,\ \eta^2 = \frac{\tau\sigma^2}{1+n\tau}
$$

And

$$
\begin{aligned}
\pi(\sigma^2|\mathbf{x}) &\propto \pi(\sigma^2)\pi(\mathbf{x}|\sigma^2) \\
&= \pi(\sigma^2)\int_{\mathbb{R}} \pi(\mathbf{x}|\theta,\sigma^2)\pi(\theta|\sigma^2)d\theta \\
&\propto (\sigma^2)^{-\alpha-1}e^{-\frac{\beta}{\sigma^2}}\int_{\mathbb{R}}(\sigma^2)^{-\frac{n+1}{2}}e^{-\frac{n(\bar{x}-\theta)^2+\sum_1^n(x_i-\bar{x})^2}{2\sigma^2}}\cdot e^{-\frac{(\theta-\mu)^2}{2\tau\sigma^2}}d\theta \\
&\propto (\sigma^2)^{-\alpha-\frac{n+3}{2}}e^{-\frac{\beta+\frac{1}{2}\sum_1^n(x_i-\bar{x})^2}{\sigma^2}}\int_{\mathbb{R}} e^{-\frac{n\tau(\theta-\bar{x})^2+(\theta-\mu)^2}{2\tau\sigma^2}}d\theta \\
&\propto (\sigma^2)^{-\alpha-\frac{n}{2}-1}e^{-\frac{\beta+\frac{1}{2}\sum_1^n(x_i-\bar{x})^2+\frac{n(\bar{x}-\mu)^2}{2(1+n\tau)}}{\sigma^2}} \\
&= (\sigma^2)^{-\alpha-\frac{n}{2}-1}e^{-\frac{\tilde{\beta}}{\sigma^2}}
\end{aligned}
$$

So $\theta|\sigma^2,\mathbf{x} \sim N(\mu(\mathbf{x}),\eta^2)$ and $\sigma^2|\mathbf{x} \sim \Gamma^{-1}(\alpha+\frac{n}{2},\tilde{\beta})$.

(3).

$$
\begin{aligned}
\pi(\theta,\sigma^2|\mathbf{x}) &\propto (\sigma^2)^{-\alpha-\frac{n}{2}-1}e^{-\frac{\tilde{\beta}}{\sigma^2}}\cdot(\eta^2)^{-\frac{1}{2}}e^{-\frac{(\theta-\mu(\mathbf{x}))^2}{2\eta^2}} \\
&= (\sigma^2)^{-\alpha-\frac{n+3}{2}}e^{-\frac{\tilde{\beta}+\frac{1+n\tau}{2\tau}(\theta-\mu(\mathbf{x}))^2}{\sigma^2}}
\end{aligned}
$$

$$
\begin{aligned}
\pi(\theta|\mathbf{x}) &= \int_{\mathbb{R}^+}\pi(\theta,\sigma^2|\mathbf{x})d(\sigma^2) \\
&\propto \int_{\mathbb{R}^+} s^{-(\alpha+\frac{n+3}{2})}e^{-\frac{\tilde{\beta}+\frac{1+n\tau}{2\tau}(\theta-\mu(\mathbf{x}))^2}{s}}ds \\
&\propto [\frac{2\tau\tilde{\beta}}{1+n\tau}+(\theta-\mu(\mathbf{x}))^2]^{\frac{2\alpha+n+1}{2}}
\end{aligned}
$$

So $\theta|\mathbf{x} \sim \mathcal{T}_{2\alpha+n}(\mu(\mathbf{x}),\frac{2\tau\tilde{\beta}}{(1+n\tau)(2\alpha+n)})$.