Homework 2

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P[141] 10.

The posterior distribution $\pi(\theta|x)$ is (up to a constant) :

$$\pi(\theta|x) \propto \theta^{-11}e^{-\frac{100}{\theta}} \cdot \theta^{-5}e^{-\frac{53}{\theta}} \propto \theta^{-16}e^{-\frac{153}{\theta}}$$

i.e. the posterior distribution is of $\Gamma^{-1}(15, 153)$.

(1). The expectation is

$$\hat{\theta}_E = \mathbb{E}[\pi(\theta|x)] = \frac{153}{14} \approx 10.929$$

and the PMSE is

$$PMSE(\hat{\theta}_E) = Var(\pi(\theta|x)) = \frac{153^2}{14^2 \cdot 13} \approx 9.187$$

(2).

$$\begin{aligned}\frac{\partial}{\partial\theta}(p(\theta|x)) &\propto \frac{\partial}{\partial\theta}(\theta^{-16}e^{-\frac{153}{\theta}}) \\ &= -16 \cdot \theta^{-17}e^{-\frac{153}{\theta}} + 153 \cdot \theta^{-18}e^{-\frac{153}{\theta}}\end{aligned}$$

So the MAP estimation is

$$\hat{\theta}_{MD} = \frac{153}{16} \approx 9.563$$

and the PMSE

$$PMSE(\hat{\theta}_{MD}) = Var(\pi(\theta|x)) + (\hat{\theta}_{MD} - \hat{\theta}_E)^2 \approx 11.054$$

P[142] 22.

(1). The standardized variance of prior distribution is $\tau = 1.481$ and $\mu = 0$, *i.e.* $\theta \sim \mathcal{N}(0, 1.481^2)$

And the posterior distribution for $x = 6$

$$\begin{aligned}\pi(\theta|x) &\propto e^{-\frac{(6-\theta)^2}{2}} \cdot e^{-\frac{\theta^2}{2\cdot\tau^2}} \\ &\propto e^{-\frac{(\theta-4.121)^2}{2*0.829^2}}\end{aligned}$$

i.e. $\theta|x \sim \mathcal{N}(4.121, 0.829^2)$ and the 90% HPD is $(2.757, 5.485)$.

(2). By computation

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Solve[Integrate[(a/(t^2 + a^2))/Pi, {t, -Infinity, 1}] == 3/4]
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we find $\theta \sim C(0,1)$, *i.e.*

$$\pi(\theta) = \frac{1}{\pi \cdot (1+\theta^2)}$$

the posterior distribution for $x = 6$

$$\pi(\theta|x) \propto \frac{e^{-\frac{1}{2}(\theta-6)^2}}{(1+\theta^2)}$$

and by computing

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NIntegrate[Exp[-0.5*(x - 6)^2]/(1 + x^2), {x, -Infinity, Infinity}]
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the marginal term for the posterior distribution is about 0.07385, so

$$\pi(\theta|x) \approx \frac{e^{-\frac{1}{2}(\theta-6)^2}}{(1+\theta^2)} \Big/ 0.07385$$

The 90% HPD is computed by

```
f[t_?NumericQ] := Exp[-0.5*(t - 6)^2]/((t^2 + 1)*0.07385);
X = Table[i/100.0, {i, 1, 1000}];
Table[{NIntegrate[f[t], {t, -Infinity, X[[i]]}], X[[i]]}, {i, 1, 1000}]
```

and the result is $(3.61, 7.66)$

P[143] 32.

By the conjugate relation between Pareto distributions and uniform distributions, the posterior distribution is of form $Pa(5, 14)$ where

$$\pi(\theta|x) = \frac{2689120}{\theta^6}, \theta > 14$$

Hence

$$\mathbf{P}(H_1) = \int_{14}^{15} \frac{2689120}{\theta^6} dx \approx 0.292$$
$$\mathbf{P}(H_2) = \int_{15}^{20} \frac{2689120}{\theta^6} dx \approx 0.540$$
$$\mathbf{P}(H_3) = \int_{20}^{\infty} \frac{2689120}{\theta^6} dx \approx 0.168$$

So H_2 is accepted.

P[143] 33.

For data x=5, the posterior distribution is

$$\pi(\theta|x) \propto \theta^5(1-\theta)^3\delta_{(0,1)}$$

If we denote the the number of wins by variable Z, for z in {0,1,2,3,4}

$$\begin{aligned}
p(z|x) &= \int_{\boldsymbol{\Theta}} p(z|\theta)p(\theta|x)d\theta \\
&\propto \int_0^1 \binom{4}{z}\theta^z(1-\theta)^{(4-z)} \cdot \theta^5(1-\theta)^3 d\theta \\
&= \int_0^1 \binom{4}{z}\theta^{(z+5)}(1-\theta)^{(7-z)}d\theta \\
&= \binom{4}{z}\mathbf{B}(z+6, 8-z)
\end{aligned}$$

And the most probable prediction is $Z = 3$.

P[143] 34.

Given

$$x = (9.45, 10.62, 9.40, 10.12, 9.85, 10.92, 10.93, 9.85, 9.81, 10.28)$$

where

$$X|\theta \sim N(\theta, 0.5^2\mathbf{I})$$

and the prior

$$\theta \sim N(0, 1)$$

we get that the posterior distribution

$$\begin{aligned}
\pi(\theta|x) &\propto e^{-2\sum_1^n (x_i-\theta)^2 - \frac{1}{2}(\theta-10)^2} \\
&\propto e^{-20(\bar{x}-\theta)^2 - \frac{1}{2}(\theta-10)^2} \\
&= e^{-20(\theta-10.123)^2 - \frac{1}{2}(\theta-10)^2} \\
&\propto e^{-\frac{(\theta-10.12)^2}{2\cdot(1/41)}}
\end{aligned}$$

i.e.

$$\theta|x \sim N(10.120, 0.156^2)$$

The new variable Z over parameter θ is given by

$$Z|\theta \sim N(\theta, 0.5^2)$$

so

$$\begin{aligned}
p(z|x) &= \int_{\boldsymbol{\Theta}} p(z|\theta)p(\theta|x)d\theta \\
&\propto \int_{\mathbb{R}} e^{-20.5(\theta-10.120)^2-2(z-\theta)^2}d\theta \\
&= \int_{\mathbb{R}} e^{-20.5\theta^2-2((z-10.120)-\theta)^2}d\theta \\
&= \int_{\mathbb{R}} e^{-22.5(\theta-0.089(z-10.120))^2-1.822(z-10.120)^2}d\theta \\
&\propto e^{-1.822(z-10.120)^2}
\end{aligned}$$

i.e.

$$Z|x \sim N(10.120, 0.524^2)$$

And the 95% HPD is $(9.093, 11.147)$