Homework 8

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P[265] 2.

For likelihood $\mathbf{x}|\mu \sim N(\mu, 1)$ and different prior

$$\pi_1(\mu) \propto e^{-\frac{\mu^2}{8}}$$

$$\pi_2(\mu) \propto e^{-\frac{(\mu-1)^2}{8}}$$

$$\pi_3(\mu) \propto \delta_{[-3,3]}$$

Respectively the posteriors are

$$\begin{aligned}\pi_1(\mu|\mathbf{x}) &\propto e^{-\frac{\mu^2}{8}-\frac{30(\bar{x}-\mu)^2}{2}} \\ &\propto e^{-\frac{(\mu-\frac{120}{121}\bar{x})^2}{\frac{8}{121}}}\end{aligned}$$

i.e. $\mu|\mathbf{x} \overset{1}{\sim} N(\frac{120}{121}\bar{x}, (\frac{2}{11})^2)$;

$$\begin{aligned}\pi_2(\mu|\mathbf{x}) &\propto e^{-\frac{(\mu-1)^2}{8}-\frac{30(\bar{x}-\mu)^2}{2}} \\ &\propto e^{-\frac{(\mu-\frac{120}{121}\bar{x}-\frac{1}{121})^2}{\frac{8}{121}}}\end{aligned}$$

i.e. $\mu|\mathbf{x} \overset{2}{\sim} N(\frac{120}{121}\bar{x}+\frac{1}{121}, (\frac{2}{11})^2)$;

$$\pi_3(\mu|\mathbf{x}) \propto e^{-\frac{30(\bar{x}-\mu)^2}{2}}\delta_{[-3,3]}$$

If we use the fact that the posterior is asymptotically normal around the true value $\mu = 0$, that is, $\sqrt{n} \cdot \mu|\mathbf{x} \to N(0, 1)$, after computing

$$I(0) = \mathbb{E}_0\Big[-\partial_\mu^2\Big|_0 (-\frac{(x-\mu)^2}{2})\Big] = 1$$

And if 30 is regarded as a number "large enough", we can approximately draw the posterior distribution as

$$\mu|\mathbf{x} \overset{approx}{\sim} N(0, \frac{1}{30})$$

And the posterior probability $\mathbb{P}(\mu \in (-0.5, 0.5)|\mathbf{x})$ and $\mathbb{P}(\mu \in (-0.2, 0.6)|\mathbf{x})$ is respectively $\Phi(2.73)-\Phi(-2.73)$ and $\Phi(3.29) - \Phi(-1.09)$, *i.e.* 99.36 and 86.16.

We use Monte-Carlo method to compute the exact required probability:

```
M <- 100
p11 <- numeric(M)
p12 <- numeric(M)
p21 <- numeric(M)
p22 <- numeric(M)
p31 <- numeric(M)
p32 <- numeric(M)

f3 <- function(mu, xbar){
  if (mu > 3)
    return(1)
  else if (mu < -3)
    return(0)
  else{
    preg <- integrate(function(mu1)exp(-15*(xbar-mu1)^2), lower = -3, upper = 3)$value
    p <- integrate(function(mu1)exp(-15*(xbar-mu1)^2), lower = -3, upper = mu)$value
    p <- p/preg
    return(p)
  }
}

for (i in 1:M){
  xbar <- rnorm(1, 0, 1/sqrt(30))
  p11[i] <- pnorm (0.5, (120/121)*xbar, 2/11) - pnorm (-0.5, (120/121)*xbar, 2/11)
  p12[i] <- pnorm (0.6, (120/121)*xbar, 2/11) - pnorm (-0.2, (120/121)*xbar, 2/11)
  p21[i] <- pnorm (0.5, (120/121)*xbar+1/121, 2/11) - pnorm (-0.5, (120/121)*xbar+1/121, 2/11)
  p22[i] <- pnorm (0.6, (120/121)*xbar+1/121, 2/11) - pnorm (-0.2, (120/121)*xbar+1/121, 2/11)
  p31[i] <- f3(0.5, xbar) - f3(-0.5, xbar)
  p32[i] <- f3(0.6, xbar) - f3(-0.2, xbar)
}

pa1 <- pnorm (0.5, 0, 1/sqrt(30)) - pnorm (-0.5, 0, 1/sqrt(30))
pa2 <- pnorm (0.6, 0, 1/sqrt(30)) - pnorm (-0.2, 0, 1/sqrt(30))
print(round(c(mean(p11), mean(p12), mean(p21), mean(p22), mean(p31), mean(p32), pa1, pa2),4))
```

```
## [1] 0.9417 0.7394 0.9424 0.7487 0.9401 0.7375 0.9938 0.8628
```

There's a significant difference.

p[265] 5.

The likelihood function is

$$p(\mathbf{x}|\theta_i) = \prod_{k=1}^{n} f(x_k|\theta_i)$$

while the prior is

$$\pi(\theta_i) = \pi_i$$

The posterior distribution can be expressed as

$$\begin{aligned} p(\theta_i|\mathbf{x}) &= \frac{\pi_i \prod_{k=1}^n f(x_k|\theta_i)}{\sum_{r=1}^n (\pi_r \prod_{k=1}^n f(x_k|\theta_r))} \\ &= \frac{\pi_i}{\sum_{r=1}^n (\pi_r \prod_{k=1}^n \frac{f(x_k|\theta_r)}{f(x_k|\theta_i)})} \\ &= \frac{\pi_i}{\sum_{r=1}^n (\pi_r {P_{r,n}}^n)} \end{aligned}$$

where

$$log\ P_{r,n} = \frac{1}{n} \sum_{k=1}^n log\ \frac{f(x_k|\theta_r)}{f(x_k|\theta_i)}$$

we have $P_{i,n} = 1$ and, suppose $\mathbf{x}$ is generated according to f_{θ_i}, by weak LLN,

$$log\ P_{r,n} \xrightarrow{P} -KL(f_{\theta_i}||f_{\theta_r}) < 0$$

since each f_{θ_r} are different essentially. So $P_{r,n} \xrightarrow{P} P_r < 1$ and

$$p(\theta_i|\mathbf{x}) \xrightarrow{P} 1$$

That is the weak consistency of θ_i.