Chapter 2

Exponential distribution: basic properties and usual GOF tests

This chapter is dedicated to the Exponential distribution. First, some definitions and basic properties of this distribution are given. Then, we present a quick review of GOF tests for the Exponential distribution, based on different approaches: probability plots, empirical distribution function, normalized spacings, Laplace transform, characteristic function, entropy, integrated distribution function, likelihood based tests, ... Complete and censored samples are treated. Finally, an extensive comparison study is done which leads to identify the best GOF tests for the Exponential distribution.

2.1 The Exponential distribution: definition and properties

A random variable X is from the Exponential distribution of parameter λ, denoted $\exp(\lambda)$, if and only if its cumulative distribution function (cdf) is:

$$F(x;\lambda) = 1 - \exp(-\lambda x), \quad x \geq 0,\ \lambda > 0. \tag{2.1}$$

- The probability density function (pdf) is:

$$f(x;\lambda) = \lambda \exp(-\lambda x), \quad x \geq 0,\ \lambda > 0. \tag{2.2}$$

- The reliability is $R(x) = 1 - F(x,\lambda) = \exp(-\lambda x)$.

- The expectation (or the Mean time to failure MTTF) is: $\text{MTTF} = \mathbb{E}[X] = \dfrac{1}{\lambda}$.

- The variance is $Var[X] = \dfrac{1}{\lambda^2}$.

- The hazard rate is $h(x) = \dfrac{f(x)}{R(x)} = \dfrac{\lambda \exp(-\lambda x)}{\exp(-\lambda x)} = \lambda.$

- The mean residual life is $m(x) = \mathbb{E}[X - x | X > x] = \dfrac{1}{\lambda} = \mathbb{E}[X].$

- The Laplace transform is $\psi(t) = \mathbb{E}\left[\exp(-tX)\right] = \dfrac{\lambda}{\lambda + t}.$

- The characteristic function is $\varphi(t) = \mathbb{E}\left[\exp(itX)\right] = \dfrac{\lambda}{\lambda - it}.$

- If X is from $\exp(\lambda)$, $Y = \lambda X$ follows a standard Exponential distribution $\exp(1)$.

The Exponential distribution is without memory. It means if that the system did not fail yet at time t, then it behaves as if it was new at this time. Indeed, the random variable X obeys the following relation:

$$\forall x \geq 0, \quad P(X > t + x \mid X > t) = P(X > x). \tag{2.3}$$

In reliability, it means that the Exponential distribution is suitable for systems which are not deteriorating neither improving with time.

Let $x_1, \ldots, x_n$ be realizations of independent and identically distributed (iid) random variables $X_1, \ldots, X_n$ with the $\exp(\lambda)$ distribution. The likelihood function is :

$$\mathcal{L}(\lambda; x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right). \tag{2.4}$$

Maximizing this function, we obtain that the Maximum Likelihood Estimator (MLE) of λ is:

$$\hat{\lambda}_n = \frac{n}{\sum\limits_{i=1}^{n} X_i} = \frac{1}{\bar{X}_n}. \tag{2.5}$$

After estimating λ by $\hat{\lambda}_n = \dfrac{1}{\bar{X}_n}$, we will be interested in the random variables $\hat{Y}_i = \hat{\lambda}_n X_i = \dfrac{X_i}{\bar{X}_n}$ that have a distribution that should be “close” to $\exp(1)$.

The vector $\left(\hat{Y}_1, \ldots, \hat{Y}_n\right)/n$ has the Dirichlet distribution $D(1, \ldots, 1)$. This allows to prove that asymptotically, distribution of this vector is independent of the parameter λ. Therefore, each statistic built as a function of $\left(\hat{Y}_i\right)_{1 \leq i \leq n}$ can be a GOF test statistic.

Let $X_1^* \leq \ldots \leq X_n^*$ be the order statistics of the sample $X_1, \ldots, X_n$, and $X_0^* = 0$. The distribution of the $(X_i)_{1 \leq i \leq n}$ has location and scale parameters μ and σ, if the distribution of $\frac{X_i-\mu}{\sigma}$ does not depend on μ nor on σ. For such a distribution, the normalized spacings are defined as:

$$E_i = \frac{X_i^* - X_{i-1}^*}{\mathbb{E}\left[\dfrac{X_i^* - \mu}{\sigma}\right] - \mathbb{E}\left[\dfrac{X_{i-1}^* - \mu}{\sigma}\right]}, \ \forall i \in \{1, \ldots, n\}. \tag{2.6}$$

The expectations at the denominator of E_i do not depend on μ and σ, then the E_i are observed. The normalized spacings can be written as follows:

$$E_i = \sigma \, \frac{X_i^* - X_{i-1}^*}{\mathbb{E}\left[X_i^* - X_{i-1}^*\right]} = \sigma \, \frac{\dfrac{X_i^* - \mu}{\sigma} - \dfrac{X_{i-1}^* - \mu}{\sigma}}{\mathbb{E}\left[\dfrac{X_i^* - \mu}{\sigma}\right] - \mathbb{E}\left[\dfrac{X_{i-1}^* - \mu}{\sigma}\right]}. \tag{2.7}$$

Any statistic written as $\sum_i a_i E_i / \sum_j b_j E_j$ is distributed independently of the parameters μ and σ, so it can be used to build a GOF test.

When the sample $X_1, \ldots, X_n$ comes from $\exp(\lambda)$ ($\mu = 0$ and $\sigma = \frac{1}{\lambda}$), the normalized spacings are defined in this case as:

$$E_i = (n - i + 1)(X_i^* - X_{i-1}^*), i \in \{1, \ldots, n\}. \tag{2.8}$$

Under the Exponential assumption, the $(E_i)_{1 \leq i \leq n}$ are iid with the same distribution $\exp(\lambda)$.

In the case of censored samples, when only the lowest $n-r$ failure times $x_1^* \leq \ldots \leq x_{n-r}^*$ are observed, the likelihood function in this case is:

$$\begin{aligned}
\mathcal{L}(\lambda; x_1^*, \ldots, x_{n-r}^*) &= \prod_{i=1}^{n-r} f(x_i^*) \ \left[1 - F(x_{n-r}^*)\right]^r \\
&= \lambda^{n-r} \exp\left(-\lambda \sum_{i=1}^{n-r} x_i^* - \lambda r x_{n-r}^*\right).
\end{aligned}$$

Thus, the maximum likelihood estimator of λ is:

$$\hat{\lambda}_n = \frac{n - r}{\displaystyle\sum_{i=1}^{n-r} X_i^* + r X_{n-r}^*}. \tag{2.9}$$

2.2 GOF tests for the Exponential distribution: complete samples

In this section, we present a review of GOF tests for the Exponential distribution for complete samples. There is a wide literature on GOF tests for the Exponential distribution from the 50's until now. Several review papers were published through time: Epstein [40, 41], Spurrier [117], Ascher [7], Henze-Meintanis [53], chapter 10 of D´Agostino-Stephens [31] and chapter 13 of Balakrishnan-Basu [10]. In all what follows, the studied GOF tests have the most general alternative hypothesis. There are some GOF tests that aim to test the Exponential distribution against a specific distribution such as the work of Muralidharan [91], Basu-Mitra [14] and Gatto-Jammalamadaka [46].

The GOF tests families presented are the families of tests based on the probability plot, the empirical distribution function, the normalized spacings, the likelihood, the Laplace transform, the characteristic function, the entropy, the mean residual life and the integrated distribution function.

2.2.1 Principles of GOF tests

Let $X_1, \ldots, X_n$ be iid random variables and F their cumulative distribution function. For the Exponential distribution, a GOF test is a statistical test of hypothesis H_0: "$F \in \mathcal{F}$" vs H_1: "$F \notin \mathcal{F}$", where $\mathcal{F}$ is the family of the cdfs of the Exponential distributions.

The type I error consists in wrongly rejecting the null hypothesis H_0. Here, it means concluding that the distribution is not Exponential while it is Exponential indeed. The significance level of the test, α, is the probability of type I error. It is generally set to $\alpha = 5\%$. The type II error consists in not rejecting the Exponential hypothesis while the distribution is indeed not Exponential. The power of the test is the probability of not committing the type II error. It measures the test ability of concluding correctly that the distribution is not Exponential.

A GOF test is generally based on a test statistic Z which is a measure of the distance between two quantities: a theoretical one which characterizes the tested hypothesis H_0 and an empirical one computed from the studied data set. The null hypothesis in this case is rejected when Z is too large. The critical region is the set of values of Z for which H_0 is rejected. If the observed value of Z, z_{obs}, belongs to the critical region, the conclusion of the test is the rejection of H_0.

The determination of the critical region is based on the distribution of the test statistic under H_0. When the rejection is done for large values of the statistic, it means that, for a fixed level α, H_0 is rejected when $z_{obs} > q_{1-\alpha}$, where $q_{1-\alpha}$ is the quantile of order $1-\alpha$ of the distribution of Z under H_0: $P_{H_0}(Z > q_{1-\alpha}) = \alpha$. The test in this case is a one-sided test. Some tests are two-sided: H_0 is rejected when Z is either larger than the quantile of order $1-\alpha/2$ or lower than the quantile of order $\alpha/2$.

In most cases, the distribution of the test statistics under H_0 is not known. Then, their quantiles are computed using simulations. We simulate a large number K of samples from the Exponential distribution. For each $k \in \{1, ..., K\}$, the value of the test statistic Z_k is computed. The quantile of order $1-\alpha$ is approximated by the $(1-\alpha)^{\text{th}}$ empirical quantile of the sample $Z_1, ..., Z_K$.

The p-value of the test is the probability under H_0 that the test statistic is greater than its observed value: $p_{obs} = P_{H_0}(Z > z_{obs})$. If the distribution of Z is not known, p_{obs} is estimated by the frequency of simulated values of Z which are greater than z_{obs}: $\hat{p}_{obs} = \frac{1}{K}\sum_{i=1}^{K} \mathbb{1}_{\{Z_i > z_{obs}\}}$.

The distribution of the test statistics under H_0 has to be known or computable. Then, it cannot depend on the parameters of the tested distribution. This is a very important point, on which we will focus in the following.

2.2.2 Test based on the probability plot

The probability plot is a graph that can be used to evaluate the fit of a distribution $F(.;\theta)$ to the observations. The principle is to look for a linear relationship such as $h_1[F(x;\theta)] = \alpha_1(\theta)h_2(x) + \alpha_2(\theta)$ where h_1 and h_2 are functions that do not depend on θ. Thus, if the real cdf is $F(;\theta)$, then $h_1[\mathbb{F}_n(x)]$ should be close to $\alpha_1(\theta)h_2(x) + \alpha_2(\theta)$ where $\mathbb{F}_n(x) = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}$ is the empirical distribution function.

Let $x_1^* < \ldots < x_n^*$ be order statistics of the observations $x_1, \ldots, x_n$. For $x = x_i^*$, $h_1[\mathbb{F}_n(x_i^*)] = h_1\left(\frac{i}{n}\right)$. When F is the real cdf, the points of the plot $\left(h_2(x_i^*), h_1\left(\frac{i}{n}\right)\right)$ should be approximately aligned. For the Exponential distribution, $F(x;\lambda) = 1 - \exp(-\lambda x)$ then, $\ln(1 - F(x;\lambda)) = -\lambda x$. Thus, the probability plot of the Exponential distribution is the plot of points [10]:

$$\left(x_i^*, \ln\left(1 - \frac{i}{n}\right)\right), \ i \in \{1, \ldots, n-1\}. \tag{2.10}$$

Patwardhan [99] worked on a variant of the probability plot based on the expectations of the order statistics of the standard Exponential distribution [99]:

$$\left(\sum_{j=1}^{i} \frac{1}{n-j+1}, x_i^*\right), \ i \in \{1, \ldots, n\}. \tag{2.11}$$

For all i, let $\delta_i = \displaystyle\sum_{j=1}^{i} \frac{1}{n-j+1}$ and $\hat{Y}_i^* = \dfrac{X_i^*}{\bar{X}_n}$. Under the Exponential assumption, these points should be approximately on the line $y = x$. Patwardhan suggested a statistic Pa_n that measures the proximity between vectors $(\delta_1, \ldots, \delta_n)$ and $(\hat{Y}_1^*, \ldots, \hat{Y}_n^*)$. This statistic can also be written as a function of the normalized spacings E_i:

$$Pa_n = n(n+1)\, \frac{\displaystyle\sum_{i=1}^{n} E_i^2}{\left[\displaystyle\sum_{i=1}^{n} E_i\right]^2}. \tag{2.12}$$

The null hypothesis H_0 is rejected for large values of Pa_n.

2.2.3 Shapiro-Wilk test

The Shapiro-Wilk test [113] is based on the ratio of two estimators of $1/\lambda$. Their procedure is applied to Exponential distribution with a location parameter and can not be applied to standard Exponential distribution. Stephens in [119] adapted Shapiro-wilk statistic for the Exponential distribution with a null location parameter. The test statistic is:

$$SW_n = \frac{\bar{X}_n^2}{(n+1)S_n^2 + \bar{X}_n^2}, \quad \text{where } \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2. \tag{2.13}$$

The rejection of the null hypothesis H_0 is done for too large or too small values of the test statistic.

2.2.4 Tests based on the empirical distribution function

These tests are based on a measure of the departure between the empirical distribution function $\mathbb{F}_n$ and the estimated theoretical distribution function $\hat{F}_0(x) = F(x; \hat{\lambda}_n) = 1 - \exp(-\hat{\lambda}_n x)$. The null hypothesis is rejected when this difference is too large.

The best known statistics are [31]:

- Kolmogorov-Smirnov statistic (KS):

$$\begin{aligned} KS_n &= \sqrt{n} \sup_{x \in \mathbb{R}} \left| \mathbb{F}_n(x) - \hat{F}_0(x) \right| \\ &= \sqrt{n} \max \left[\max\{\frac{i}{n} - U_i^*\}, \max\{U_i^* - \frac{i-1}{n}\} \right] \end{aligned} \tag{2.14}$$

- Cramer-von Mises statistic (CM):

$$\begin{aligned} CM_n &= n \int_{-\infty}^{+\infty} \left[\mathbb{F}_n(x) - \hat{F}_0(x) \right]^2 d\hat{F}_0(x) \\ &= \sum_{i=1}^{n} \left(\hat{U}_i^* - \frac{2i-1}{2n} \right)^2 + \frac{1}{12n} \end{aligned} \tag{2.15}$$

- Anderson-Darling statistic (AD):

$$\begin{aligned} AD_n &= n \int_{-\infty}^{+\infty} \frac{\left[\mathbb{F}_n(x) - \hat{F}_0(x) \right]^2}{\hat{F}_0(x) \left(1 - \hat{F}_0(x) \right)} d\hat{F}_0(x) \\ &= -n + \frac{1}{n} \sum_{i=1}^{n} \left[(2i - 1 - 2n) \ln(1 - \hat{U}_i^*)) - (2i-1) \ln(\hat{U}_i^*) \right] \end{aligned} \tag{2.16}$$

where $U_i = \hat{F}_0(X_i) = 1 - \exp\left(-X_i / \overline{X}_n\right)$.

2.2.5 Tests based on the normalized spacings

Several statistics have been developed using the normalized spacings $E_i = (n-i+1)(X_i^* - X_{i-1}^*)$. Gnedenko in [49] suggested the following one:

$$Gn(l) = \frac{(n-l)\sum_{j=1}^{l} E_j}{l\sum_{j=l+1}^{n} E_j}. \tag{2.17}$$

The statistic Gn has, under H_0, the Fisher-Snedecor distribution $F(2l, 2(n-l))$. A second test statistic is proposed by Harris [52]:

$$Gn^*(l) = \frac{(n-2l)\left(\sum_{j=1}^{l} E_j + \sum_{j=n-l+1}^{n} E_j\right)}{2l\sum_{j=l+1}^{n-l} E_j}. \tag{2.18}$$

The test statistics $Gn(l)$ and $Gn^*(l)$ are functions of the parameter l. We will use the recommended values of the parameter l given in [52]: $l = [n/2]$ for Gn and $l = [n/4]$ for Gn^*. Gail and Gastwirth [45] proposed the Gini statistic:

$$GG_n = \frac{\sum_{i=1}^{n-1} iE_{i+1}}{(n-1)\sum_{i=1}^{n} E_i}. \tag{2.19}$$

For the previous three tests, the Exponential hypothesis is rejected for large and small values of the statistics.
Lin and Mudholkar in [77] used separately both terms of the Harris statistic $Gn^*(l)$:

$$LM_1(l) = \frac{(n-2l)\sum_{i=1}^{l} E_j}{l\sum_{j=l+1}^{n} E_j} \tag{2.20}$$

$$LM_2(l) = \frac{(n-2l)\sum_{j=l+1}^{n-l} E_j}{l\sum_{j=l+1}^{n-l} E_j}. \tag{2.21}$$

The Exponential hypothesis is rejected if at least one of the two statistics LM_1 and LM_2 is too large or too small. The test is denoted $LM(l)$. We choose $l = \lfloor \frac{(n-1)}{10} \rfloor$ as in [77].

2.2.6 Tests based on a transformation to exponentials or uniforms

Some transformations can be applied to the original sample $X_1, \ldots, X_n$. For example the normalized spacings $E_i = (n-i+1)(X_i^* - X_{i-1}^*), i \in \{1, \ldots, n-1\}$, are random variables composing a new iid sample from $\exp(\lambda)$. Stephens in [31] called it the transformation N. All the previous GOF tests for the Exponential distribution applied to $X_1, \ldots, X_n$ can also be applied to $E_1, \ldots, E_n$.

A second approach consists in transforming an iid sample from $\exp(\lambda)$ to an iid sample from the uniform distribution over $[0,1]$, $\mathcal{U}[0,1]$. Therefore, testing the exponentiality of the sample $X_1, \ldots, X_n$ is equivalent to testing the uniformity of $\dfrac{\sum_{j=1}^{i} E_j}{\sum_{j=1}^{n} E_j}$. The last transformation is called by Stephens in [31] the K transformation.

2.2.7 Likelihood based tests

The likelihood based tests consist in including the tested distribution in a larger parametric family and testing a specific value of the parameter of this family using some procedures such as the score and likelihood ratio tests. In our case, the Exponential distribution $\exp(\lambda)$ is included in the family of Weibull distributions $\mathcal{W}(1/\lambda, \beta)$. The idea is to test exponentiality by testing H_0: "$\beta = 1$" and H_1 : "$\beta \neq 1$", where β is the shape parameter of the Weibull distribution and λ is a nuisance parameter. The test proposed by Cox and Oakes [29] is the score test using the observed Fisher information instead of the exact Fisher information. The rejection of the null hypothesis H_0 is done for large values of the statistics. The likelihood based test statistics are as follows:

- Score test:

$$Sc_n = \frac{6}{n\pi^2}\left[n + \sum_{i=1}^{n} \ln X_i - \frac{1}{\bar{X}_n}\sum_{i=1}^{n} (\ln X_i) X_i\right]^2 \tag{2.22}$$

- Cox-Oakes test:

$$CO_n = \frac{\left[n + \sum_{i=1}^{n} \ln X_i - \frac{1}{\bar{X}_n}\sum_{i=1}^{n} (\ln X_i) X_i\right]^2}{n + \frac{1}{\bar{X}_n}\sum_{i=1}^{n}\left(\ln X_i/\bar{X}_n\right)^2 X_i - \frac{1}{n\bar{X}_n^2}\left[\sum_{i=1}^{n}\left(\ln X_i/\bar{X}_n\right) X_i\right]^2} \tag{2.23}$$

- Likelihood ratio test:

$$LR_n = 2n \ln \frac{\hat{\beta}_n \sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^{\hat{\beta}_n}} + 2(\hat{\beta}_n - 1)\sum_{i=1}^{n} \ln X_i \tag{2.24}$$

where $\hat{\beta}_n$ is the MLE of β defined in equation (3.5). The rejection of H_0 is done for large values of the statistics.

2.2.8 Tests based on the Laplace transform

Henze [53] proposed GOF tests for the Exponential distribution based on the Laplace transform. The building of the test is based on the measure of the difference between the empirical Laplace transform and its theoretical version.
Henze used the fact that the sample $Y_i = \lambda X_i$, $\forall i \in \{1, \ldots, n\}$ is a sample from the unit Exponential distribution. Its Laplace transform is:

$$\psi(t) = \mathbb{E}[\exp(-tY_i)] = \frac{1}{1+t}. \tag{2.25}$$

Since λ is unknown, it can be estimated by the MLE $\hat{\lambda}_n$. The distribution of $\hat{Y}_1, \ldots, \hat{Y}_n$ is independent of λ.
Henze's idea [53] is to reject the hypothesis that $X_1, \ldots, X_n$ are exponentially distributed if the empirical Laplace transform $\psi_n(t) = \frac{1}{n}\sum_{i=1}^{n} \exp(-t\hat{Y}_i)$ is too far from the theoretical Laplace transform of a standard Exponential $\psi(t)$. The closeness between both functions is measured by a test statistic of the form:

$$He_{n,a} = n \int_0^{+\infty} \left[\psi_n(t) - \frac{1}{(1+t)}\right]^2 w(t;a)dt \tag{2.26}$$

where $w(t;a) = \exp(-at)$ is a weight function. Using the integration by parts, the test statistic turns out to be:

$$He_{n,a} = \frac{1}{n}\sum_{i,j=1}^{n} \frac{1}{\hat{Y}_i + \hat{Y}_j + a} - 2\sum_{j=1}^{n} \exp(\hat{Y}_j + a)E_1(\hat{Y}_j + a) + n(1 - ae^a E_1(a)) \tag{2.27}$$

where $E_1(z) = \int_z^{+\infty} \frac{\exp(-t)}{t} dt$.
The choice of the parameter a allows to build powerful GOF tests for a large range of alternatives.
Baringhaus and Henze [12] proposed to use the fact that $\psi(t)$ is solution of the differential equation $(\lambda + t)\,\psi'(t) + \psi(t) = 0$. The corresponding test statistics is:

$$BH_{n,a} = n \int_0^{+\infty} \left[(1+t)\,\psi_n'(t) + \psi_n(t)\right]^2 \, w(t;a)\,dt. \tag{2.28}$$

The integral defining $BH_{n,a}$ can be computed and expressed as an explicit function of the $\hat{Y}_i$:

$$BH_{n,a} = \frac{1}{n}\sum_{j,k=1}^{n} \left[\frac{(1-\hat{Y}_j)(1-\hat{Y}_k)}{\hat{Y}_j + \hat{Y}_k + a} - \frac{\hat{Y}_j + \hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^2} + \frac{2\hat{Y}_j\hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^2} + \frac{2\hat{Y}_j\hat{Y}_k}{(\hat{Y}_j + \hat{Y}_k + a)^3}\right]. \tag{2.29}$$

Both tests reject the Weibull assumption for large values of the statistics.

2.2.9 Tests based on the characteristic function

The characteristic function of the Exponential distribution is

$$\varphi(t) = \mathbb{E}\left[\exp(itX)\right] = \frac{\lambda}{\lambda - it} = C(t) + iS(t) = \frac{\lambda^2}{\lambda^2 + t^2} + i\frac{\lambda t}{\lambda^2 + t^2}. \tag{2.30}$$

Epps and Pulley [39] proposed to compare the characteristic function of the standard Exponential distribution to the empirical characteristic function of the sample $\hat{Y}_1, \ldots, \hat{Y}_n$, $\varphi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(-it\hat{Y}_j) = C_n(t) + iS_n(t)$, where $C_n(t) = \frac{1}{n}\sum_{j=1}^{n} \cos(t\hat{Y}_j)$ and $S_n(t) = \frac{1}{n}\sum_{j=1}^{n} \sin(t\hat{Y}_j)$. The expression of their statistic simplifies to:

$$EP_n = \sqrt{48n}\left[\frac{1}{n}\sum_{i=1}^{n} \exp(-\hat{Y}_i) - \frac{1}{2}\right]. \tag{2.31}$$

Henze and Meintanis [54] suggested to build a test based on the equation verified by the real and the imaginary parts of the characteristic function: $S(t) - tC(t)/\lambda = 0$. This idea applied to the $\hat{Y}_j$ leads to a statistic of the form:

$$HM_{n,a} = n\int_{-\infty}^{+\infty} \left[S_n(t) - t\,C_n(t)\right]^2 \, w(t;a)\, dt \tag{2.32}$$

Two weight functions are used: $w_1(t;a) = \exp(-at)$ and $w_2(t;a) = \exp(-at^2)$. The corresponding statistics are denoted $HM^{(1)}_{n,a}$ and $HM^{(2)}_{n,a}$. The integral in (2.32) can be computed and expressed as an explicit function of the $\left(\hat{Y}_j\right)_{1\le j\le n}$:

$$\begin{aligned} HM^{(1)}_{n,a} = &\frac{a}{2n}\sum_{j,k=1}^{n}\left[\frac{1}{a^2 + (\hat{Y}_j - \hat{Y}_k)^2} - \frac{1}{a^2 + (\hat{Y}_j + \hat{Y}_k)^2} - \frac{4(\hat{Y}_j + \hat{Y}_k)}{(a^2 + (\hat{Y}_j - \hat{Y}_k)^2)^2}\right.\\ &\left. + \frac{2a^2 - 6(\hat{Y}_j - \hat{Y}_k)^2}{(a^2 + (\hat{Y}_j + \hat{Y}_k)^2)^3} + \frac{2a^2 - 6(\hat{Y}_j + \hat{Y}_k)^2}{(a^2 + (\hat{Y}_j + \hat{Y}_k)^2)^3}\right] \end{aligned} \tag{2.33}$$

$$\begin{aligned} HM^{(2)}_{n,a} = &\frac{\sqrt{\pi}}{4n\sqrt{a}}\sum_{j,k=1}^{n}\left[\left(1 + \frac{2a - (\hat{Y}_j - \hat{Y}_k)^2}{4a^2}\right)\exp\left(-\frac{(\hat{Y}_j - \hat{Y}_k)^2}{4a}\right)\right.\\ &\left. + \left(\frac{2a - (\hat{Y}_j + \hat{Y}_k)^2}{4a^2} - \frac{\hat{Y}_j + \hat{Y}_k}{a} - 1\right)\exp\left(-\frac{(\hat{Y}_j + \hat{Y}_k)^2}{4a}\right)\right]. \end{aligned} \tag{2.34}$$

Henze and Meintanis [55, 56] used a similar technique inspired by the fact, reported by Meintanis and Iliopoulos [84], that $|\varphi(t)|^2 = C(t)$. The statistic has the form:

$$MI_{n,a} = n \int_{-\infty}^{+\infty} \left[|\varphi_n(t)|^2 - C_n(t) \right]^2 \, w(t;a)\, dt. \tag{2.35}$$

As before, both weight functions $w_1(t;a) = \exp(-at)$ and $w_2(t;a) = \exp(-at^2)$ are used.

The corresponding statistics are denoted $MI_{n,a}^{(1)}$ and $MI_{n,a}^{(2)}$ and have the following explicit expressions:

$$\begin{aligned} MI_{n,a}^{(1)} =& \frac{a}{n} \sum_{j,k=1}^{n} \left[\frac{1}{a^2 + \hat{Y}_{jk-}^2} + \frac{1}{a^2 + \hat{Y}_{jk+}^2} \right] \\ & - \frac{2a}{n^2} \sum_{j,k=1}^{n} \sum_{l=1}^{n} \left[\frac{1}{a^2 + [\hat{Y}_{jk-} - \hat{Y}_l]^2} + \frac{1}{a^2 + [\hat{Y}_{jk-} + \hat{Y}_l]^2} \right] \\ & + \frac{a}{n^3} \sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \left[\frac{1}{a^2 + [\hat{Y}_{jk-} - \hat{Y}_{lm-}]^2} + \frac{1}{a^2 + [\hat{Y}_{jk-} - \hat{Y}_{lm-}]^2} \right] \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} MI_{n,a}^{(2)} =& \frac{1}{2n} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \left[\exp\left(\frac{-\hat{Y}_{jk-}^2}{4a} \right) + \exp\left(-\frac{\hat{Y}_{jk+}^2}{4a} \right) \right] \\ & - \frac{1}{n^2} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \sum_{l=1}^{n} \left[\exp\left(-\frac{[\hat{Y}_{jk-}^2 - \hat{Y}_l^2]}{4a} \right) + \exp\left(-\frac{[\hat{Y}_{jk-} + \hat{Y}_l]^2}{4a} \right) \right] \\ & + \frac{1}{2n^3} \sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \left[\exp\left(-\frac{[\hat{Y}_{jk-} - \hat{Y}_{lm-}]^2}{4a} \right) + \exp\left(-\frac{[\hat{Y}_{jk-} + \hat{Y}_{lm-}]^2}{4a} \right) \right] \end{aligned} \tag{2.37}$$

where $\hat{Y}_{jk-} = \hat{Y}_j - \hat{Y}_k$ and $\hat{Y}_{jk+} = \hat{Y}_j + \hat{Y}_k$.

For all the previous tests, H_0 is rejected for large values of the statistics.

2.2.10 Test based on the entropy

The entropy of a random variable X whose pdf is f, is defined by:

$$H(X) = \mathbb{E}[-\ln f(X)] = -\int_{-\infty}^{+\infty} f(x) \ln f(x)\, dx.$$

For all the positive random variables, $H(X) \leq 1 + \ln \mathbb{E}[X]$, which is equivalent to $\exp(H(X))/\mathbb{E}[X] \leq e$. The equality in the previous inequation is verified only for the Exponential distribution.

Grzegorzewski and Wieczorkowski [50] suggested a test that rejects the Exponential hypothesis when an estimation of $\exp(H[X])/\mathbb{E}[X]$ is too small. One of the known estimators of the entropy used in [50] is Vasicek estimator [124] defined as:

$$\hat{H}_{m,n} = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{n}{2m} (X_{i+m}^* - X_{i-m}^*) \tag{2.38}$$

where m is an integer less than $n/2$, $X_i^* = X_1^*$ for $i < 1$ and $X_i^* = X_n^*$ for $i > n$.

The corresponding statistic is:

$$GW_{m,n} = \frac{n}{2m\bar{X}_n}\left[\prod_{i=1}^{n}\left(X_{i+m}^* - X_{i-m}^*\right)\right]^{\frac{1}{n}}. \tag{2.39}$$

It can be rewritten as:

$$GW_{m,n} = \frac{n}{2m\bar{\hat{Y}}_n}\left[\prod_{i=1}^{n}\left(\hat{Y}_{i+m}^* - \hat{Y}_{i-m}^*\right)\right]^{\frac{1}{n}}. \tag{2.40}$$

Approximated formulas to compute the quantiles are given in [50].

2.2.11 Tests based on the mean residual life

The mean residual life of the Exponential distribution is:

$$m(t) = \mathbb{E}[X - t | X > t] = \mathbb{E}[X] = 1/\lambda,\ \forall t \geq 0. \tag{2.41}$$

This property is equivalent to $\mathbb{E}[\min(X,t)] = F(t)\mathbb{E}[X], \forall t \geq 0$. Then, Baringhaus and Henze [13] proposed to build a GOF test based on the comparison between an estimator of $\mathbb{E}[\min(X,t)]$ and an estimator of $F(t)\mathbb{E}[X]$.
Two statistics are suggested, using Kolmogorov-Smirnov and Cramer-Von Mises type metrics:

$$BHK_n = \sqrt{n}\sup_{t\geq 0}\left|\frac{1}{n}\sum_{i=1}^{n}\min(\hat{Y}_i, t) - \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\hat{Y}_i \leq t\}}\right| \tag{2.42}$$

$$BHC_n = n\int_0^{\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\min(\hat{Y}_i, t) - \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\hat{Y}_i \leq t\}}\right]^2 \exp(-t)dt. \tag{2.43}$$

2.2.12 Tests based on the integrated distribution function

The integrated distribution function of the standard Exponential distribution is:

$$\Psi(t,\lambda) = \mathbb{E}\left[\max(X - t, 0)\right] = \int_t^{+\infty} R(x)\,dx = \frac{e^{-\lambda t}}{\lambda}. \tag{2.44}$$

Klar [65] proposed to build a GOF test based on the Cramer-Von Mises distance between the estimated $\Psi(t;\hat{\lambda}_n)$ and the empirical integrated distribution function $\Psi_n(t) = \frac{1}{n}\sum_{j=1}^{n}\max(X_j - t, 0)$. The statistic has the expression:

$$Kl_n = n\hat{\lambda}_n^3\int_0^{+\infty}\left(\Psi_n(t) - \Psi(t;\hat{\lambda}_n)\right)^2 dt \tag{2.45}$$

The statistic Kl_n can be written as a function of $\left(\hat{Y}_i\right)_{1\leq i\leq n}$ which proves the fact that the null distribution of Kl_n does not depend on the parameter λ:

$$Kl_n = n\int_0^{+\infty}\left(\frac{1}{n}\sum_{i=1}^{n}(\hat{Y}_i-u)\mathbb{1}_{\{\hat{Y}_i>u\}}-\exp(-u)\right)^2 du. \tag{2.46}$$

The use of a weight function usually allows to increase the power of the test. The statistic will have the form:

$$Kl_{a,n} = na^3\hat{\lambda}_n^3\int_0^{\infty}[\Psi_n(t)-\Psi(t;\hat{\lambda}_n)]^2\exp(-a\hat{\lambda}_n t)\,dt. \tag{2.47}$$

The statistic $Kl_{n,a}$ can be written using the sample $\left(\hat{Y}_i\right)_{1\leq i\leq n}$:

$$\begin{aligned} Kl_{n,a} = \frac{2(3a+2)n}{(2+a)(1+a)^2} &- 2a^3\sum_{i=1}^{n}\frac{\exp(-(1+a)\hat{Y}_i)}{(1+a)^2}-\frac{2}{n}\sum_{i=1}^{n}\exp(-a\hat{Y}_i)\\ &+\frac{2}{n}\sum_{i<j}\left[a(\hat{Y}_j^*-\hat{Y}_i^*)-2\right]\,\exp(-a\hat{Y}_i^*). \end{aligned} \tag{2.48}$$

The Exponential hypothesis is rejected for large values of the statistic $Kl_{a,n}$.

2.3 GOF tests for the Exponential distribution: censored samples

In this section we give a short bibliographical review of some GOF tests for the Exponential distribution in the case of simply type II censored samples. s and r denote respectively the number of the left and right censored observations. Let us remind that it means that only $X_{s+1}^*,\ldots,X_{n-r}^*$ are observed.

2.3.1 Tests based on the normalized spacings

In the case of censored observations $X_{s+1}^*,\ldots,X_{n-r}^*$, the observed normalized spacings are $E_{s+2},\ldots,E_{n-r}$. They constitute a sample of size $n-r-s-1$ of the $\exp(\lambda)$ distribution. So all the previous GOF tests for the Exponential distribution can be applied to this sample. In the simulations presented in section 2.4.1, we apply the GOF tests Gn, Gn^*, LM and CO to the spacings $E_{s+2},\ldots,E_{n-r}$.

2.3.2 Tests based on the lack of trend

Two test statistics were suggested by Brain and Shapiro in the case of doubly censored samples [19].

Under the Exponential assumption, the E_i are iid, so they do not exhibit a trend. This lack of trend can be tested using the Laplace test statistic:

$$BS_1 = \frac{\sum_{i=1}^{m-1}(i - m/2)(E_{i+s+1} - \overline{E})}{\sum_{i=1}^{m-1} E_{s+i+1} \left\{(i - m/2)^2/m(m-1)\right\}^{1/2}} \tag{2.49}$$

where $\overline{E} = \sum_{i=1}^{m-1} E_{s+i+1}/(m-1)$ and $m = n - r + 1$. The Exponentiality assumption is rejected for large and small values of the statistics. The distribution of BS_1 under H_0 converges to the standard normal distribution when m goes to infinity. The statistic can be rewritten as:

$$BS_1 = [12(m-2)]^{1/2}(\overline{U} - 1/2)$$

where $T_i = \sum_{j=1}^{i} E_{s+j+1}, i = 1, \ldots, m-1$, $U_i = T_i/T_{m-1}, i = 1, \ldots, m-2$, $\overline{U} = \sum_{i=1}^{m-2} U_i/(m-2)$.

The last expression of the statistic BS_1 is the usual expression of the Laplace test statistic applied to the uniform order statistics $U_i, i = 1, \ldots, m-2$.

A second statistic BS^* is introduced. It is built as the sum of squares of two components, the first one associated to BS_1 and the second one to BS_2 obtained by replacing in the previous expression $(i - m/2)$ by $(i - m/2)^2 - m(m-2)/12$. The aim is to build a test sensitive to non-monotonic hazard functions.

$$BS_2 = [5(m-2)(m+1)(m-3)]^{1/2}\big(m - 3 + 6(m-1)\overline{U} - 12\sum_{i=1}^{m-2} iU_i/(m-2)\big) \tag{2.50}$$

The combined statistic is BS^* as follows:

$$BS^* = BS_1^2 + BS_2^2. \tag{2.51}$$

The distribution of BS^* under H_0 can be approximated by the χ^2 distribution. The null hypothesis H_0 is rejected when the statistic is too large. This idea of combining two test statistics will be used later in section 6.2.

2.3.3 Tests based on the empirical distribution function

Pettitt and Stephens [100] introduced versions of the Cramer-von Mises, Watson and Anderson-Darling statistics in the case of simple right censoring. The statistics are obtained by modifying the upper limit of integration in their definitions in subsection 2.2.4. After simplification, the statistics have the following expressions [31]:

- Cramer-von-Mises statistic (CM):

$$CM = \sum_{i=1}^{n-r}\left(U_i^* - \frac{2i-1}{2n}\right)^2 + \frac{n-r}{12n^2} + \frac{n}{3}\left(U_{n-r}^* - \frac{n-r}{n}\right)^3 \tag{2.52}$$

- Watson statistic (W):

$$W = CM - nU_{n-r}^*\left[\frac{n-r}{n} - \frac{U_{n-r}^*}{2} - \frac{(n-r)\bar{U}}{nU_{n-r}^*}\right]^2 \tag{2.53}$$

- Anderson-Darling statistic (AD):

$$\begin{aligned} AD = &-\frac{1}{n}\sum_{i=1}^{n-r}(2i-1)[\ln U_i^* - \ln(1-U_i^*)] - 2\sum_{i=1}^{n-r}\ln(1-U_i^*) \\ &-\frac{1}{n}[r^2\ln(1-U_{n-r}^*) - (n-r)^2\ln(U_{n-r}^*) + n^2U_{n-r}^*] \end{aligned} \tag{2.54}$$

- Kolmogorov-Smirnov statistic (KS) can also be adapted for censored data:

$$KS = \sup_{1\le i\le n-r}\left|\frac{i-0.5}{n} - U_i^*\right| + \frac{0.5}{n} \tag{2.55}$$

where $U_i = 1 - e^{-\hat{\lambda}_n X_i}$, $\hat{\lambda}_n = \dfrac{n-r}{\sum_{i=1}^{n-r} X_i^* + rX_{n-r}^*}$ and $U_1^*, \ldots, U_{n-r}^*$ are the order statistics of the sample $U_1, \ldots, U_{n-r}$.

The same statistics can be applied in the case of left-censored samples. We use the transformation $V_i^* = 1 - U_{n+1-i}^*$, $i = 1, \ldots, n-s$, where $s = r$ is the number of censored observations. The exponentiality hypothesis is rejected for large values of the statistics.

2.3.4 Test based on the Kullback-Leibler information

This test is based on the Kullback-Leibler information. It was proposed in order to test the exponentiality in the case of progressively censored samples of type II [11]. It can be applied to the special case of simply right-censored samples:

$$KL = -H(w,m,n) + \frac{m-1}{n}\left[\ln\left(\frac{1}{m-1}\sum_{i=1}^{m-1}X_i\right) + 1\right]^2 \tag{2.56}$$

where $m = n - r$ and

$$H(w,m,n) = \frac{1}{n}\sum_{i=1}^{m}\ln\frac{(n+1)(X_{\min(i+w,m-1)}^* - X_{\max(i-w,1)}^*)}{\min(i+w,m-1) - \max(i-w,1)} + \left(1-\frac{m-1}{n}\right)\ln\left(1-\frac{m-1}{n}\right).$$

The choice of w is given as a function of the sample size. We will use the value recommended in [38]. The rejection of the Exponential hypothesis is done for large values of the test statistic.

2.4 Comparison of the GOF tests for the Exponential distribution

In this section, we make an exhaustive comparison of all the previous GOF tests for the Exponential distribution. The comparisons are based on Monte-Carlo simulations. Some reviews were already done for complete samples, by Henze-Meintanis [55], Spurrier [117] and Ascher [7]. The review presented here is more complete with more compared GOF tests, more alternatives with various hazard rates shapes and more sample sizes. All the GOF tests studied in this section have been implemented in the R package EWGoF that we have developed.

2.4.1 Complete samples

For complete sample the comparison includes the following test statistics:

- Pa: Patwardhan test statistic defined in (2.12).
- SW: Shapiro-Wilk test statistic defined in (2.13).
- KS: Kolmogorov-Smirnov test statistic defined in (2.14).
- CM: Cramer-von-Mises test statistic defined in (2.15).
- AD: Anderson-Darling test statistic defined in (2.16).
- Gn: Gnedenko test statistic defined in (2.17).
- Gn^*: Harris test statistic defined in (2.18).
- GG: Gini test statistic defined in (2.19).
- LM_1 and LM_2: Lin-Mudholkar test statistics defined respectively in (2.20) and (2.21).
- Sc: Score test statistic defined in (2.22).
- CO: Cox and Oakes test statistic defined in (2.23).
- LR: Likelihood ratio test statistic defined in (2.24).
- He: Henze test statistic defined in (2.27).
- BH: Baringhaus-Henze test statistic defined in (2.29).
- EP: Epps-Pulley test statistic defined in (2.31).
- $HM^{(1)}$ and $HM^{(2)}$: Henze and Meintanis test statistics defined respectively in equations (2.33) and (2.34).
- $MI^{(1)}$ and $MI^{(2)}$: Meintanis and Iliopoulos test statistics defined respectively in equations (2.36) and (2.37).

- GW: Grzegorzewski and Wieczorkowski test statistic defined in (2.40).
- BHK and BHC: Baringhaus and Henze test statistics based on the mean residual life defined in (2.42).
- Kl: Klar test statistic defined in (2.48).

We first simulate iid exponentially distributed samples to verify that the rejection percentage of the Exponential distribution is close to the theoretical significance level. Then, we simulate samples with the following alternative distributions. For each distribution we give their pdfs $f(x)$ and hazard rate $h(x)$ when it has an explicit expression:

- The Gamma distribution $\mathcal{G}(\alpha, \lambda)$:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp(-\lambda x)\, x^{\alpha-1}$$

- The Lognormal distribution $\mathcal{LN}(m, \sigma^2)$:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(- \frac{1}{2\sigma^2} (\ln x - m)^2 \right)$$

- The Uniform distribution $\mathcal{U}[0, a]$:

$$f(x) = \frac{1}{a} \mathbb{1}_{[0,a]}(x)$$

$$h(x) = \frac{1}{a - x} \mathbb{1}_{[0,a]}(x)$$

- The Inverse-Gamma distribution $\mathcal{IG}(\alpha, \beta)$:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\frac{\beta}{x}\right).$$

For the sake of simplicity, we adopt the following conventions: scale parameters of the Weibull, Gamma and Inverse-Gamma distribution (respectively η, λ and β) are arbitrarily set to 1 and the parameter m of the Lognormal distribution is set to 0. The corresponding distributions are denoted $\mathcal{W}(1, \beta) \equiv \mathcal{W}(\beta)$, $\mathcal{G}(\alpha, 1) \equiv \mathcal{G}(\alpha)$, $\mathcal{IG}(\alpha, 1) \equiv \mathcal{IG}(\alpha)$ and $\mathcal{LN}(0, \sigma^2) \equiv \mathcal{LN}(\sigma^2)$. Parameters of the simulated distributions are selected to obtain different shapes of the hazard rate:

- IHR: increasing hazard rate
- DHR: decreasing hazard rate
- BT: bathtub-shaped hazard rate
- UBT: upside-down bathtub-shaped hazard rate.

Table 2.1: Simulated distributions

Exponential	exp(0.2)	exp(1)	exp(2)	exp(42)
IHR	$\mathcal{W}(1.5)$	$\mathcal{W}(3)$	$\mathcal{U}[0,2]$	$\mathcal{G}(2)$
DHR	$\mathcal{W}(0.5)$	$\mathcal{W}(0.8)$	$\mathcal{W}(0.98)$	$\mathcal{G}(0.5)$
UBT	$\mathcal{LN}(0.6)$	$\mathcal{LN}(0.8)$	$\mathcal{LN}(1.4)$	

For the Exponential case, we use only UBT alternatives. BT alternatives will be also used for the Weibull case in the following chapter. Table 2.1 gives the values of the parameters and the notations used for all the simulated distributions:

For a given alternative with fixed parameters and a fixed sample size, we simulate 50000 samples of size $n \in \{5, 10, 20, 50\}$.

All the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis. The quantiles of the distribution of the test statistics under H_0 are obtained mainly by simulations. For instance, let us consider the Cox-Oakes test, the test statistic is given by (2.23):

$$CO_n = \frac{\left[n + \sum_{i=1}^{n} \ln X_i - \frac{1}{\bar{X}_n} \sum_{i=1}^{n} (\ln X_i) X_i\right]^2}{n + \frac{1}{\bar{X}_n} \sum_{i=1}^{n} \left(\ln X_i/\bar{X}_n\right)^2 X_i - \frac{1}{n\bar{X}_n^2} \left[\sum_{i=1}^{n} \left(\ln X_i/\bar{X}_n\right) X_i\right]^2}.$$

For a given sample size n, we simulate $X_1, \ldots, X_n$ from exp(1), then we compute the corresponding value of CO_n. This process is done $m = 100000$ times. The quantiles of the distribution of CO_n under H_0 are given by the empirical quantiles of the m values of CO_n. Table 2.2 gives some quantiles for several values of n. We observe that, for small n, the distribution of CO_n under H_0 may be quite far from the χ_1^2 distribution. So it is important to be able to apply these GOF tests without using the chi-square approximation especially for small samples.

Table 2.2: Quantiles of the distribution of CO_n under H_0

n	75%	80%	85%	90%	95%	99%
5	1.548	1.847	2.214	2.697	3.422	5.079
10	1.460	1.762	2.164	2.728	3.658	5.853
20	1.379	1.702	2.128	2.729	3.777	6.318
50	1.335	1.656	2.079	2.690	3.810	6.537
χ_1^2	1.323	1.642	2.072	2.705	3.841	6.635

For the power study, we simulate a sample $X_1, \ldots, X_n$ of size n of a given distribution. For $n = 50$, the Exponential assumption is rejected at the level 5% if $CO_n > 3.810$. This process is done $K = 50000$ times. The percentage of rejection of H_0 is an estimation of the power of the test for this alternative. For instance, we see in Table 2.4 that the power of the CO_n test for simulated $\mathcal{LN}(0, 0.8)$ samples and $n = 50$ is estimated at 63.8%. The higher the rejection percentage is, the better the test is. We will observe that the results are tightly linked to the tested alternatives. In order to evaluate globally the power of the tests, we compute for each test the mean value of the rejection percentage for all the alternatives. The power tables of the studied GOF tests are given in Appendix A in order to avoid a complex and long dissertation in this chapter.

In a first step, the tests are compared inside each family. The choice of parameters such as a and m is discussed. In a second step, the best tests of each family are compared.

Tables A.1 and A.2 present the power results for the GOF tests based on the empirical distribution function (KS, CM and AD) with and without the application of the K transformation. AD is the best and KS is the worst of the three. The use of the K transformation gives better results for some special cases such as the Weibull, $\mathcal{LN}(1.4)$ and uniform distributions.

Tables from A.3 to A.5 present the power results of the tests based on the normalized spacings. GG has the best performance followed by LM.

Tables A.6 and A.7 compare the power results for the three likelihood based tests (Sc, CO and LR). It seems clearly that the score test Sc is more appropriate for the DHR alternatives and the test LR based on the likelihood ratio is powerful for the IHR alternatives. The test CO has never been the best one for specific alternatives, but it represents an excellent compromise by giving generally good results.

Tables from A.8 to A.11 present the power results of Henze test based on the Laplace transform. Small values of the parameter a are appropriate for DHR alternatives ($\mathcal{W}(0.5)$, $\mathcal{W}(0.8)$, $\mathcal{G}(0.5)$), while moderately large values of a, are appropriate to IHR alternatives ($\mathcal{W}(1.5)$, $\mathcal{W}(3)$, $\mathcal{G}(2)$, uniform). The best compromise is made for $a = 1$.

Tables from A.12 to A.15, present the power results of Baringhaus-Henze test based on the Laplace transform. The conclusions are similar to those of the previous test. We recommend also the value $a = 1$. Baringhaus-Henze test is slightly more powerful than the test of Henze.

Tables from A.16 to A.19 present the power results of Henze-Meintanis tests based on the characteristic function. The power difference between the two tests $HM^{(1)}_{n,a}$ and $HM^{(2)}_{n,a}$ can be very important for some alternatives, for instance: 82.3% and 28.9% for $\mathcal{LN}(0.8)$ distribution with $n = 50$. Generally, the test $HM^{(1)}_{n,a}$ is more powerful than $HM^{(2)}_{n,a}$. But for DHR alternatives, we recommend the use of $HM^{(2)}_{n,a}$ with large value of the parameter a. If nothing is known about the tested alternatives, a good compromise is to choose the test $HM^{(1)}_{n,a}$ with $a = 1.5$ for $n \leq 10$ and $a = 1$ for $n > 10$.

Tables from A.20 to A.22 present the power results of Meintanis-Iliopoulos tests based on the characteristic function. The fact that the statistics $MI^{(1)}_{n,a}$ and $MI^{(2)}_{n,a}$ have more complex expression than the previous ones, slows down the simulations. These tests present the characteristic to have extremely weak powers for DHR alternatives and good ones for IHR alternatives. There is no significant difference between $MI^{(1)}_{n,a}$ and $MI^{(2)}_{n,a}$. We choose $a = 2.5$, even if the choice of the parameter a has no significant effect on the results.

Table A.23 presents the power results of Grzegorzewski-Wieczorkowski test based on the entropy. The choice of the parameter m depends slightly on the tested alternatives. We recommend $m = 4$ to have the best compromise. This test and Pa test are not very powerful that is why they will not be presented later in the comparison tables.

Tables A.24 to A.27 present the power results of the test of Klar. For small size samples, we should absolutely avoid to choose large values of the parameter a which give null rejection percentages for some alternatives. For $n \geq 20$, the best suitable values of the parameter a depend on the used alternatives. The best compromise is obtained for $a = 5$.

After finding the best GOF tests within each family, tables A.28 to A.33 of the appendix are given to compare all the selected GOF tests for the sample sizes $n \in \{5, 10, 20\}$. The following tables 2.3 and 2.4 give the power results for $n = 50$. Our first conclusion is that none of these tests is always powerful. The performances of the tests depend strongly on the alternatives used in the simulations. Secondly, the family of the likelihood based tests gives globally the best results. The test Sc is recommended for the DHR alternatives and LR test is rather recommended for the IHR alternatives. The test CO gives a good compromise and can be recommended in all cases. Besides their good performances, the likelihood based GOF test statistics do not require any parameter to be chosen and have simple expressions.

altern.	KS	CM	AD	GG	SW	BHK	BHC	$K_{5,n}$	$BH_{n,0.1}$	$He_{n,1}$
exp(0.2)	5.1	5.7	4.9	4.9	4.8	4.5	5.2	5.1	5.1	5.1
exp(1)	4.7	4.9	5.6	5	4.6	5.4	5.4	5.1	4.9	4.8
exp(2)	5.3	5.3	5.1	4.9	4.8	4.8	5.1	5	5.2	5.1
exp(42)	4.9	4.8	5.2	5	4.7	4.8	4.9	5	5.1	5
$\mathcal{W}(0.5)$	99.9	99.9	100	100	98.6	99.9	99.9	100	100	100
$\mathcal{W}(0.8)$	36.4	41.8	50.8	48.2	31.5	34.5	46.3	49.1	53.6	50
$\mathcal{W}(0.98)$	5.3	5.5	5.9	5.6	5	4.8	4.3	5.5	6	5.4
$\mathcal{G}(0.5)$	83.3	89.7	90.5	89.2	63.3	86.4	84.9	94.5	97.6	93
$\mathcal{W}(1.5)$	79.4	89.5	91.4	92.6	89.1	88.1	87.5	91.2	87.7	93.9
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100	100	100
$\mathcal{U}[0, 2]$	92.1	98.5	98.2	99	99.8	99.4	98.6	86.2	80.4	96.1
$\mathcal{G}(2)$	81.7	90.4	91.4	89.2	79.9	81.2	82.1	94.5	93.6	93.6
$\mathcal{LN}(0.6)$	99.2	99.6	99.1	99.2	88.8	100	98.2	100	100	99.9
$\mathcal{LN}(0.8)$	71.1	75.9	85.5	46.5	28.2	64.5	61.7	82.1	93.6	61.9
$\mathcal{LN}(1.4)$	81.5	85.1	87.4	88.4	84.8	83.6	87.2	80.7	77.6	86.9
Mean	75.4	79.6	81.8	78	69.9	76.6	77.3	80.3	80.9	80.1

Table 2.3: Exponential distribution - Tests comparison, $n = 50$ - 1

Without any information about the tested alternative, we recommend the test CO followed by the tests AD and Kl. For the IHR alternatives, the test LR is the best followed by MI and CO. For the DHR alternatives, the test Sc is the best followed by BH, AD and CO. For the alternatives with upside-down bathtub shaped hazard rate, the two tests Kl and BH are powerful. Even though the comparison study presented here is larger than those of Ascher [7] and Henze-Meintanis [55], the conclusions are globally similar.

2.4.2 Censored samples

For censored samples, to our knowledge similar reviews have never been done. In this subsection, we compare the following tests:

altern.	$MI^{(1)}_{n,2.5}$	$MI^{(2)}_{n,2.5}$	$HM^{(1)}_{n,1.5}$	$HM^{(2)}_{n,0.5}$	EP	Sc	CO	LR
exp(0.2)	5.2	5	4.9	5.3	5	5	5	5
exp(1)	5.4	5.2	5.1	5.2	5.1	5	4.9	5
exp(2)	5.1	5.2	5	5.2	5.2	4.9	5.1	5
exp(42)	5.5	5.2	5.1	5	5.4	4.9	4.8	5.1
$\mathcal{W}(0.5)$	98.5	98.8	99.8	99.6	99.9	100	100	100
$\mathcal{W}(0.8)$	24.8	31.2	38.1	36	47.8	56.4	52.8	48.4
$\mathcal{W}(0.98)$	4	4.2	5.2	5.3	5.6	5.9	5.8	5.1
$\mathcal{G}(0.5)$	71.8	74	83.4	79.5	89.3	97.4	97.2	96.5
$\mathcal{W}(1.5)$	93.8	92.6	88.5	87.9	92.9	93.4	94.9	96.3
$\mathcal{W}(3)$	100	100	100	100	100	100	100	100
$\mathcal{U}[0,2]$	99.8	99.9	99.5	99.7	98.2	90.1	93.7	95
$\mathcal{G}(2)$	90	85.6	86.8	83.4	90.4	94.8	95.7	96.5
$\mathcal{LN}(0.6)$	99.3	95.7	99.8	98.4	98.9	99.7	99.6	99.7
$\mathcal{LN}(0.8)$	49.1	32.2	56.6	39.9	45.1	64.6	63.8	66
$\mathcal{LN}(1.4)$	60.7	72.8	82.7	82.5	88.7	82.3	78.1	76.1
Mean	72	71.6	76.4	73.9	77.9	80.4	80.1	79.9

Table 2.4: Exponential distribution - Tests comparison, $n = 50$ - 2

- Gn: Gnedenko test statistic defined in (2.17) applied to the normalized spacings.
- Gn^*: Harris test statistic defined in (2.18) applied to the normalized spacings.
- LM_1 and LM_2: Lin-Mudholkar test statistics defined respectively in (2.20) and (2.21) applied to the normalized spacings.
- CO: Cox and Oakes test statistic defined in (2.23) applied to the normalized spacings.
- BS_1 and BS^*: Brain and Shapiro test statistics defined respectively in (2.49) and (2.51).
- CM: Cramer-von-Mises test statistic defined in (2.52).
- W: Watson test statistic defined in (2.53).
- AD: Anderson-Darling test statistic defined in (2.54).
- KS: Kolmogorov-Smirnov test statistic defined in (2.55).
- KL: Test based on the Kullback-Leibler information defined in (3.38).

As previously, we first simulate iid exponentially distributed samples to verify that the rejection percentage of the Exponential distribution is close to the theoretical significance level. Then, we simulate samples with the alternatives given in table 2.5.

For a given alternative with fixed parameters and a fixed sample size, we simulate 50000 samples of size $n \in \{10, 20, 50\}$ and we consider only simple type II right-censoring where $r \in \{\lfloor \frac{n}{8} \rfloor, \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor\}$.

Table 2.5: Simulated distributions for the censored samples

Exponential				
Exponential	exp(0.2)	exp(1)	exp(2)	exp(42)
IHR	$\mathcal{W}(1.5)$	$\mathcal{W}(3)$	$\mathcal{U}[0,2]$	$\mathcal{G}(2)$
DHR	$\mathcal{W}(0.5)$	$\mathcal{W}(0.8)$	$\mathcal{W}(0.98)$	$\mathcal{G}(0.5)$
UBT	$\mathcal{LN}(0.6)$	$\mathcal{LN}(0.8)$	$\mathcal{LN}(1.4)$	$\mathcal{LN}(2.4)$
	$\mathcal{IG}(0.5)$	$\mathcal{IG}(1.5)$	$\mathcal{IG}(2)$	$\mathcal{IG}(3)$

As before, all the GOF tests are applied with a significance level set to 5%. The power of the tests is assessed by the percentage of rejection of the null hypothesis. Table 2.6 shows the power results for $n = 50$ and $r = 25$.

altern.	BS_1	BS^*	CM	W	AD	Gn	Gn^*	LM	KL	CO
exp(0.2)	5.1	5.1	4.9	4.9	5	5	5	4.9	5.1	5
exp(1)	5.1	5	5	4.9	5	5.1	5	4.8	5.1	4.8
exp(2)	5	5	5	4.9	5	5.1	5.1	4.9	5.1	5.1
exp(42)	5	5	5.1	5	5	5	4.9	4.8	5.3	5.1
$\mathcal{W}(0.5)$	80.4	75.2	83.9	0	94.1	79.8	74.3	71	44.4	53.8
$\mathcal{W}(0.8)$	15.8	12.4	15.4	0	23.6	16.7	14.7	12.2	5.1	6.7
$\mathcal{W}(0.98)$	5	5.1	4.9	4.1	5.2	5.2	5.2	5.2	5	5.2
$\mathcal{G}(0.5)$	60	53.4	64.4	0	83.4	61.3	61.9	58.2	24.1	36.1
$\mathcal{W}(1.5)$	28.1	24.5	47.9	58.3	46.5	28.4	25.8	23.7	29.1	6.7
$\mathcal{W}(3)$	84.7	79	100	100	100	98.9	97.3	96.4	99.4	73.9
$\mathcal{U}[0,2]$	14	10.7	17.2	23.8	13.2	12.6	7.8	7.3	8.9	5.1
$\mathcal{G}(1.5)$	15.5	14.4	27.1	35.7	26	15	15.7	15.7	19.2	5.3
$\mathcal{G}(2)$	31.8	29.1	63.9	72.5	65.5	36.4	37.8	39.7	44.2	8.8
$\mathcal{G}(3)$	53.7	49.6	96.6	97.8	97.5	74.5	76.6	80.8	84.3	27.6
$\mathcal{LN}(0.6)$	46.5	45	99.8	99.8	99.9	86.7	93.8	98.4	98.2	53.2
$\mathcal{LN}(0.8)$	24.8	26.2	84.3	86.5	89.8	44.4	61	76.1	72.7	16.1
$\mathcal{LN}(1.4)$	7.5	8.4	5.8	4.2	5	4.1	3.9	7.1	11.6	5
$\mathcal{LN}(1.8)$	29.5	23.7	23.3	0	24	23.4	6.2	9.5	11.5	6.3
$\mathcal{LN}(2.4)$	71.3	62.4	70.5	0	78.1	68.4	39.3	28.7	31.6	23.2
$\mathcal{IG}(0.5)$	26.5	25.6	15.4	3.5	14.5	7.8	6.4	19	24.4	6.1
$\mathcal{IG}(1.5)$	13.8	18.7	91.9	88.6	96.7	44.4	76.4	95.1	19.2	27.6
$\mathcal{IG}(2)$	22.6	26.5	99.1	98.1	99.8	68.8	91.5	99.4	43.9	47
$\mathcal{IG}(3)$	37.3	38.6	100	100	100	92.9	99.1	100	83.8	77.7
Mean	35.2	33.1	58.5	45.9	61.2	45.8	47.1	49.7	40	25.9

Table 2.6: Exponential distribution - Tests comparison, $n = 50$ and $r = 25$

Tables from A.34 to A.41 present other power results. Mostly the same results come out whatever the size and the rate of the censoring. For DHR alternatives, AD followed by CM are the best tests; for IHR alternatives, the test W is recommended and for UBT alternatives, CM is the best test. The CO test applied to the normalized spacings $E_1, \ldots, E_{n-r}$, is the worst test. Generally, the two tests based on the empirical distribution function AD and CM have the best performances, unlike the test of Watson W that is biased in some cases.

To sum up, for the censored samples, Anderson-Darling test has the best performances among all the studied ones. For the complete samples, the GOF tests of Anderson-Darling AD, Cox-Oakes CO and the tests based on the empirical characteristic function BH seem to have the best performances. The comparisons were done among 60 GOF tests for complete samples and 10 GOF tests for censored samples. All the previous GOF tests for censored samples are implemented in the R package we have developed EWGoF. A part of this work has been presented in ESREL 2012 conference [70].

The good performance of Cox-Oakes CO test has attracted our attention. That is why we have developed new GOF tests based on the likelihood for the Weibull distribution (chapter 4).