①

– Remove p 5 (middle) to p. 6 — discrete

2.2.1 $u(t) = \bar{u} + \sum_{i=1}^{n} a_i(t)\,\varphi_i \qquad \bar{u}, \varphi_i \in \mathcal{U}$

choose modes φ_i (e.g. POD) <u>orthonormal</u>

f quadratic: $f(u) = d + Au + B(u,u)$

B bilinear, symmetric $\qquad d \in \mathcal{U}$

$A: \mathcal{U} \to \mathcal{U}$ lin

$B: \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ bilin

Galerkin: $\langle \varphi_j, \partial_t u \rangle = \langle \varphi_j, f(u) \rangle$ all j

(if $\bar{u} = 0$) $\dot{a}_i = \left\langle \varphi_i, d + \sum_{j=1}^{n} a_j A\varphi_j + \sum_{j,k=1}^{n} a_j a_k B(\varphi_j, \varphi_k) \right\rangle$

$= \langle \varphi_i, d \rangle + \sum_j \langle \varphi_i, A\varphi_j \rangle a_j + \sum_{j,k} \langle \varphi_i, B(\varphi_j, \varphi_k) \rangle a_j a_k$

Einstein $\sum$ → $\boxed{\dot{a}_i = d_i + A_{ij} a_j + B_{ijk} a_j a_k}$

$\dot{\underline{a}} = \underline{d} + \underline{\underline{A}}\,\underline{a} + \underline{\underline{B}}(\underline{a}, \underline{a})$

w/ $\bar{u} \neq 0$, $\dot{a}_i = \left\langle \varphi_i, f\left(\bar{u} + \sum_j a_i \varphi_i\right) \right\rangle$

$= \left\langle \varphi_i, d + A\left(\bar{u} + \sum_j a_j \varphi_j\right) + B\left(\bar{u} + \sum_j a_j \varphi_j, \bar{u} + \sum_k a_k \varphi_k\right) \right\rangle$

$= \langle \varphi_i, d + A\bar{u} + B(\bar{u}, \bar{u}) \rangle + \left\langle \varphi_i, \sum_j \left(A\varphi_j + 2B(\bar{u}, \varphi_j)\right) a_j \right\rangle$

$+ \left\langle \varphi_i, \sum_{j,k} B(\varphi_j, \varphi_k) a_j a_k \right\rangle$

$$(*) \quad \left\{ \begin{aligned} \dot{a}_i &= \underbrace{\langle \varphi_i, d + A\bar{u} + B(\bar{u}, \bar{u}) \rangle}_{\underline{d}} + \underbrace{\sum_j \langle \varphi_i, A\varphi_j + 2B(\bar{u}, \varphi_j) \rangle a_j}_{\underline{\underline{A}}\,\underline{a}} \\ &\quad + \underbrace{\sum_{j,k} \langle \varphi_i, B(\varphi_j, \varphi_k) \rangle a_j a_k}_{B(a,a)} \end{aligned} \right.$$

$$u(x,t) = \hat{u}(x-c,t) = S_c \hat{u}$$

2.3.1 $\hat{u}(t) = S_{-c(t)} u(t)$ $\qquad$ $S_c u(t,x) = \cancel{u(t,x+c)}\ u(t, x-c)$

$$\partial_t \hat{u}(t) = \partial_t \left[u(t, x+c(t)) \right]$$

$$= \partial_t u(t, x+c(t)) + \dot{c}(t)\, \partial_x u(t, x+c(t))$$

$$= S_{-c} f(u(t)) + \dot{c}\, S_{-c} \partial_x u$$

(21) $$= f(S_{-c} u(t)) + \dot{c}\, \partial_x (S_{-c} u)$$

$$\partial_t \hat{u}(t) = f(\hat{u}) + \dot{c}\, \partial_x \hat{u}$$

(22) $$c(t) = \arg\min \| S_c u(t) - u_0 \|^2$$

$$= \arg\min \langle S_c u - u_0, S_c u - u_0 \rangle$$

$$= \arg\max \langle S_c u, u_0 \rangle$$

$$= \arg\max \langle u, S_c u_0 \rangle$$

$$\frac{d}{dc} \langle u, S_c u_0 \rangle \Big|_{c=0} = \langle u, -S_c \partial_x u_0 \rangle = 0$$

$$\frac{d}{dc} S_c u = \cancel{\frac{d}{dc} u(t, x+c)} = \cancel{\partial_x u(t, x+c)} = S_c \partial_x u$$

$$\frac{d}{dc} S_c u = \frac{d}{dc} u(t, x-c) = -\partial_x u(t, x-c) = -S_c \partial_x u$$

$$\Rightarrow \langle S_{-c(t)} u(t), \partial_x u_0 \rangle = 0$$

$$\Rightarrow \langle \hat{u}(t), \partial_x u_0 \rangle = 0 \quad \text{all } t.$$

$$\partial_t \langle \hat{u}(t), \partial_x u_0 \rangle = 0$$

$$\Rightarrow \langle f(\hat{u}) + \dot{c}\, \partial_x \hat{u}, \partial_x u_0 \rangle = 0$$

$$\Rightarrow \dot{c} = \frac{-\langle f(\hat{u}), \partial_x u_0 \rangle}{\langle \partial_x \hat{u}, \partial_x u_0 \rangle} \qquad (26)$$

2.3.2 SR-Galerkin for quadratic dynamics

$$\hat{u}(t) = \bar{u} + \sum_{j=1}^{n} \varphi_j a_j(t)$$

(21) $\Rightarrow$ $\partial_t \hat{u}(t) = f(\hat{u}(t)) + \dot{c}\, \partial_x \hat{u}$ ($\dot{c}$ from (26))

$$\Rightarrow \langle \varphi_i, \partial_t \hat{u}(t) \rangle = \langle \varphi_i, f(\hat{u}(t)) + \dot{c}\, \partial_x \hat{u} \rangle$$

$$\dot{a}_i = [(*) \text{ from p. 1}] + \dot{c} \left(\underbrace{\langle \varphi_i, \partial_x \bar{u} \rangle}_{b_i} + \sum_j \underbrace{\langle \varphi_i, \partial_x \varphi_j \rangle}_{C_{ij}} a_j \right)$$

with $\dot{c} = - \dfrac{\langle f(\bar{u} + \sum_j \varphi_j a_j), \partial_x u_0 \rangle}{\langle \partial_x (\bar{u} + \sum_j \varphi_j a_j), \partial_x u_0 \rangle}$

$$= \frac{\langle d + A\bar{u} + \sum_j A\varphi_j a_j + B(\bar{u},\bar{u}) + 2\sum_j B(\bar{u},\varphi_j) a_j + \sum_{j,k} B(\varphi_j,\varphi_k) a_j a_k, \partial_x u_0 \rangle}{\langle \partial_x \bar{u}, \partial_x u_0 \rangle + \sum_j \langle \partial_x \varphi_j, \partial_x u_0 \rangle a_j}$$

Einstein $\longrightarrow$ $\dot{c} = \dfrac{e + \sum_j p_j a_j + \sum_{j,k} q_{jk} a_j a_k}{w + \sum_j s_j a_j}$

$e = \langle d + A\bar{u} + B(\bar{u},\bar{u}), \partial_x u_0 \rangle$

$p_j = \langle A\varphi_j + 2B(\bar{u},\varphi_j), \partial_x u_0 \rangle$

$q_{jk} = \langle B(\varphi_j, \varphi_k), \partial_x u_0 \rangle$

depend on f

$w = \langle \partial_x \bar{u}, \partial_x u_0 \rangle$

$s_j = \langle \partial_x \varphi_j, \partial_x u_0 \rangle$

$b_i = \langle \varphi_i, \partial_x \bar{u} \rangle$

$C_{ij} = \langle \varphi_i, \partial_x \varphi_j \rangle$

(28) $\dot{a}_i = d_i + A_{ij} a_j + B_{ijk} a_j a_k - \dot{c}(a) [b_i + C_{ij} a_j]$

2.2.2 ~~What are $\tilde{d}, \tilde{B}, \tilde{H}$ in (18)?~~

$$\dot{a}_i = d_i + \sum_j A_{ij} a_j + \sum_{j,k} B_{ijk} a_j a_k \qquad (17a)$$

fit d_i, A_{ij}, B_{ijk} to data:

$$f_i^{(t_m)} = \langle \varphi_i, f(u(t_m))\rangle \qquad f_i(t_m) \quad i = 1, \dots, n$$

$$\min_{d_i, A_{ij}, B_{ijk}} \sum_m \| \underline{d} + \underline{A}\, a(t_m) + \underline{\underline{B}}(a(t_m), a(t_m)) - \underline{f}(t_m) \|^2 + \text{Reg}$$

Einstein notation

$$\text{or} \quad \min \sum_m \sum_i \left(d_i + A_{ij} a_j(t_m) + B_{ijk} a_j(t_m) a_k(t_m) - f_i(t_m) \right)^2 + \text{Reg}$$

3.1

$$f_i(a) = d_i + A_{ij} a_j(t) + B_{ijk} a_j(t) a_k(t) \qquad (31)$$

Idea: $\tilde{f}(a) = \sum_{i=1}^n \varphi_i f_i(a)$ $\qquad u(a) = \sum_{i=1}^n \varphi_i a_i$

Then $\dot{c}(a) = - \dfrac{\langle \tilde{f}(a), \partial_x u_0 \rangle}{\langle \partial_x u(a), \partial_x u_0 \rangle}$ (26) $\qquad$ maybe leave it at that

(num. quadratic in a, depends on coefs d_i, A_{ij}, B_{ijk})

~~the~~ den. linear in a, known coefs.

Limitation: φ_i even (in example) $\Rightarrow \tilde{f}(a)$ even

u_0 even $\Rightarrow \partial_x u_0$ odd

so $\dot{c}(a) = 0$.

true f not even ($\partial_x u$ term odd). $\dot{c} = -\dfrac{\langle f(u), \partial_x u_0 \rangle}{\langle \partial_x u, \partial_x u_0 \rangle}$ +a

~~Sign error in (36)?~~

$$\cancel{\partial_t u = -\partial_x u + \partial_x^2 u}$$

$$\cancel{u(x,t) = g(x-t)}$$

$$\cancel{\frac{\partial}{\partial t} g(x-t) = -\partial_x g(x-t)} \checkmark$$

$$\cancel{\frac{\partial}{\partial x} g(x-t) = \partial_x g(x-t)}$$

$$\cancel{u_t + a u_x = 0} \qquad \cancel{u(x,t) = g(x-at)}$$

3.2 $\quad \dot{c}(a) = \dfrac{e + p_j a_j + q_{jk} a_j a_k}{w + s_j a_j}$ $\qquad e,\ p_j,\ q_{jk}$ learned

$w,\ s_j$ known

$$\cancel{(41)\quad \min \sum_{m=0}^{N_t-1} \sum_{i=1}^{n} \left[d_i + A_{ij} a_j(t_m) + B_{ijk} a_j(t_m) a_k(t_m) - f_i(t_m) \right]^2 + \sum_{i=1}^{n} \left(\frac{e + p_j a_j + q_{jk} a_j a_k}{w + s_j a_j} (b_i + C_{ij} a_j) - \dot{c}(t_m)\, r_i(t_m) \right)^2}$$

$$\boxed{r_i^{(t_m)} = \langle \varphi_i, \partial_x \hat{u}(t_m) \rangle}.$$

$$(41) \quad \min_{\theta} \sum_{m=0}^{N_t-1} \sum_{i=1}^{n} \left\{ \left[d_i + A_{ij} a_j(t_m) + B_{ijk} a_j(t_m) a_k(t_m) - f_i(t_m) \right]^2 + \left[\frac{e + p_j a_j + q_{jk} a_j a_k}{w + s_j a_j} (b_i + C_{ij} a_j(t_m)) - \dot{c}(t_m)\, r_i(t_m) \right]^2 \right\} + \mathrm{Reg}(\theta)$$

$$\theta = (d_i, A_{ij}, B_{ijk}, e, p_j, q_{jk})$$

symm

w/ $B_{ikj} = B_{ijk}$, $Q_{kj} = Q_{jk}$

Sign convention

p. 5 $\cancel{u(x,t) = \hat{u}(x - c(t), t)}$

$\cancel{\partial_t u + a\, \partial_x u = 0 \qquad u(x,t) = w(x - c(t)) \quad \dot{c}(t) = a.}$

3.3 $\tilde{\hat{u}}_{RP}(t) = \bar{u} + \sum_j \varphi_j\, a_j(t)$

$\tilde{u}_{RP}(t) = S_{c(t)}\, \tilde{\hat{u}}(t)$

$\cancel{u^+(t + \delta t) = u_{RP}(t) + \int_t^{t+\delta t} f(u_{RP}(\tau))}$

FOM: timestepper $F_{\delta t} : u(t) \mapsto u(t + \delta t)$

$\cancel{u^+ = F_{\delta t}\, u_{RP}(t)}$

$$f^{RP}(\tilde{u}) = \frac{F_{\delta t}(\tilde{u}_{RP}) - \tilde{u}_{RP}}{\delta t}$$

$\cancel{\tilde{f}_i(t) = \langle \varphi_i, \tilde{f}(t) \rangle}$

$$\dot{c}(\tilde{u}) = -\frac{\langle f^{RP}(\tilde{u}), \partial_x u_0 \rangle}{\langle \partial_x \tilde{u}, \partial_x u_0 \rangle}$$