1 Aaron's Proof

Claim: $\forall x \in N$, the decimal form of $\frac{x}{81} \in Q, x \in N : \frac{x}{81} = .a_0a_1a_2a_3a_4...$ has no $a_n, n \in N$ with $a_n = 9 - (xmod9)$.
First we will start by reconstructing $\frac{x}{81}$ into the following:

$$\frac{f}{3} * \frac{g}{3} * \frac{h}{3} * \frac{i}{3} = \frac{x}{81} \text{ where } f * g * h * i = x, \text{ and } f, g, h, i \in N.$$

This is an important distinction because then we can prove $x/81$ repeats by proving that two repeating numbers multiplied together is also a repeating number. This is applicable because when we separated $\frac{x}{81}$ into $\frac{f}{3} * \frac{g}{3} * \frac{h}{3} * \frac{i}{3}$, we show 4 repeating numbers that are multiplied together.
So, to prove two repeating numbers multiplied together is also repeating, we will consider two repeating decimals as fractions, showing that when their product is expressed as a fraction, it also outputs a fraction that repeats.
Consider the following repeating decimals:

$$j = .d_1d_2...d_n \text{ where } d_1, d_2, ..., d_n \text{ is the repeating part.}$$
$$k = .e_1e_2...e_n \text{ where } e_1, e_2, ..., e_n \text{ is the repeating part.}$$

We want to show $j * k$ gives us a fraction that repeats as a decimal.

$$j = \frac{d_1d_2...d_n}{10^n}$$
$$k = \frac{e_1e_2...e_m}{10^m}$$
$$j * k = \frac{d_1d_2...d_n * e_1e_2...e_m}{10^n * 10^m}$$
$$j * k = \frac{d_1d_2...d_n * e_1e_2...e_m}{10^{n+m}}$$

Here, the numerator is a product of integers, and the denominator is a power of 10. To say it repeats, we have to put it into fraction form. To do that, assume the numerator, a product of 2 integers, $d_1d_2...d_n$ and $e_1e_2...e_m$, gets you another integer. And since the denominator is 10^k, it will not change the numerator from being an integer. Therefore, since $j * k$ is an integer and 10^k is an integer, this is a fraction of an integer over an integer, meaning if there is a repeating part in the decimal, it will be preserved as a repeating number, proving $j * k$ is still a repeating. Therefore, $\frac{f}{3} * \frac{g}{3} * \frac{h}{3} * \frac{i}{3}$ will all repeat as long as $\forall f, g, h, i \neq 1$.
Now, we will prove that $\frac{xmod9}{81}$ is one of two important parts to proving that $9 - (xmod9)$ will not appear into our decimal. We first need to show that $9 - (xmod9)$ will not appear in the first 9 examples (which will act as base cases) of $\frac{x}{81}$ where $x = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. This works because afterwards, we will just add $.\overline{1}$ to $\frac{xmod9}{81}$ for however many times $9 \mid x$ evenly.

$$\frac{1}{81} = .\overline{012345679}$$
$$\frac{2}{81} = .\overline{024691358}$$
$$\frac{3}{81} = .\overline{037}$$
$$\frac{4}{81} = .\overline{049382716}$$
$$\frac{5}{81} = .\overline{061728395}$$
$$\frac{6}{81} = .\overline{074}$$
$$\frac{7}{81} = .\overline{086419753}$$
$$\frac{8}{81} = .\overline{098765432}$$
$$\frac{9}{81} = .\overline{1}$$

Now, we can represent $\frac{x}{81}$ as

$$x = 9k + r$$

where r is the remainder ($0 \leq r \leq 8$ because it's $mod9$) and k is the amount of times a number can be divided by 9 evenly. Using simple arithmetic, we can say

$$\frac{x}{81} = \frac{k}{9} + \frac{r}{81}.$$

This equation works because we can represent the $.\overline{1}$ with $\frac{k}{9}$ for however many times $9 \mid x$ evenly. Then when we add $\frac{r}{81}$, which adds the remainder, getting us $\frac{x}{81}$. This proves that all we need to show is when we add $\frac{k}{9}$ to our $\frac{r}{81}$ is that it is impossible for $9 - (xmod9)$ to appear in the decimal.

Finally, what we need to prove is that there exists a pattern to ensure that however many times you add $\frac{9}{81}$ to the base cases, $9 - (xmod9)$ will never appear. So, as we showed earlier, with the decimals of $\frac{x}{81}$ where $x = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, there does not exist a $9 - (xmod9)$ in those decimal equivalences. The reason why is when we add $.\overline{1}$ to a number, what we will look for is a 9 following $(9 - (xmod9)) - 1$. This is because when we add the 1 from $.\overline{1}$ to the 9 that comes before $(9 - (xmod9)) - 1$, then it carries a one to the digit $(9 - (xmod9)) - 1$, making it $9 - (xmod9)$ which is the number we say we cannot have. However, then we continue to add the $.\overline{1}$ to the digit that we just made, turning it then into $(9 - (xmod9)) + 1$. Here is an example:

$$\frac{1}{81} = .\overline{012345679}$$

Then we add $.\overline{1}$ to $.\overline{012345679}$ to get $.\overline{123456790}$.
This proves that when we add $.\overline{1}$ to our digit that is equivalent to $(9-(xmod9))-1$, it will always become $(9 - (xmod9)) + 1$ as long as there is a 9 following $(9 - (xmod9)) - 1$. So when we do have this scenario, we know that it can never reach the $9 - (xmod9)$ because it will loop every time you add $.\overline{1}$. Therefore, to simply prove that this scenario works, all we have to show is that there is a point in the decimal expansion of where $\frac{x}{81}$ $(x = \{1, 2, 3, 4, 5, 6, 7, 8, 9\})$ where we can see a 9 following a $9 - (xmod9) - 1$.

$$\frac{1}{81} = .\overline{012345679}$$
$$\frac{2}{81} = .\overline{024691358}$$
$\frac{3}{81} = .\overline{037}$ which when you add $.\overline{1}$ to it enough, you get $.\overline{49}$
$$\frac{4}{81} = .\overline{049382716}$$
$$\frac{5}{81} = .\overline{061728395}$$
$\frac{6}{81} = .\overline{074}$ which when you add $.\overline{1}$ to it enough, you get $.\overline{296}$
$$\frac{7}{81} = .\overline{086419753}$$
$$\frac{8}{81} = .\overline{098765432}$$
$\frac{9}{81} = .\overline{1}$ which will never contain $9 - (xmod9)$ because the closest it will get is when it gets to $x = 36$, making the impossible number 5, but because it repeats 4, it will never contain 5.

Therefore proving our claim: $\forall x \in N$, the decimal form of $\frac{x}{81} \in Q : \frac{x}{81} = .a_0a_1a_2a_3a_4...$ has no $a_n, n \in N$ with $a_n = 9 - (xmod9)$.