MATHEMATICAL PHYSICS

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1B. POWER SERIES

We follow the first half of chapter 1 in Boas,
not literally, adding some materials, and skipping others.

1. Analytic and non-analytic functions

A function $F(x)$ specifies a curve in (x, y) space by giving us the value(s) of y for every x, as $y = F(x)$. This curve is defined as a continuum although on a computer and also in real life everything ends-up being discrete, i.e., the curve is represented by a list of discrete points $(x_k, y_k) = (x, F(x_k))$.

Consider one of these points $P = (x_0, y_0) = (x_0, F(x_0))$. The curve is analytic at P if it stops wiggling while we Zoom-in (in the high density limit of nearby points) and starts to look like a straight line at that high enough resolution. A fractal curve is an object where the curve keeps wiggling at all length scales while we keep zooming-in and displays in that manner scale invariance.

A function is called analytic at P when all the local derivatives at P are well behaved and exist

$$a(k) = \left(\frac{d^k F}{dx^k}\right)_{x_0} \tag{1}$$

2. Taylor Series

A local observer at P can measure all local derivatives and then knows how the curve looks like throughout (x, y) space without ever having to travel away from its own local neighborhood P, in terms of the Taylor series

$$F(x) = F(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} a(k)(x - x_0)^k = F(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{d^k F}{dx^k}\right)_{x_0} (x - x_0)^k \tag{2}$$

(Named after Taylor who wrote this down in 1715, although others like Newton knew about this before. For the special case $x_0 = 0$ the series is also known as the MacLaurin series.) Take the derivatives of eq.(2)

$$\begin{aligned}
\frac{dF(x)}{dx} &= \sum_{k=1}^{\infty} \frac{1}{k!} k\ a(k)(x - x_0)^{k-1} \\
\frac{d^2F(x)}{dx^2} &= \sum_{k=2}^{\infty} \frac{1}{k!} k(k-1)\ a(k)(x - x_0)^{k-2} \\
\frac{d^3F(x)}{dx^3} &= \sum_{k=3}^{\infty} \frac{1}{k!} k(k-1)(k-2)\ a(k)(x - x_0)^{k-3} \\
&\text{etc}
\end{aligned} \tag{3}$$

At $x = x_0$ all terms vanish except the leading one in each case, and thus identifies the coefficients $a(k)$ with derivatives at $x = x_0$ as in eq.(1).

Knowing the curve everywhere from only local knowledge sounds too good to be true, and indeed, the Taylor series often fails beyond a specific horizon, known as the radius of convergence. We will be on the lookout for such horizons in every example below.

We proceed with several examples.

3. Exponential and Logarithmic series:

The exponential function $y = e^x - 1$ and the logarithmic function $y = \log(1+x)$ form a pair because they represent the same curves in (x, y)-space with $x \leftrightarrow y$ switched.

exponential series:

$$\left(\frac{d^k e^x}{dx^k}\right)_{x=0} = 1 \quad \to \quad e^x - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} x^k \tag{4}$$

This Taylor series expansion is valid for all x. It has an infinite radius of convergence.

The series passes the "preliminary convergence test" for all x because its coefficients, $c(k) = x^k/k!$, vanish for large k for all fixed x. The Ratio Convergence Test proves convergence for all fixed x,

$$S(\infty) = \sum_{k=1}^{\infty} c(k) \;\text{ with }\; c(k) = \frac{1}{k!} x^k \quad \to \quad \lim_{k\to\infty} \frac{c(k+1)}{c(k)} = \lim_{k\to\infty} \frac{x}{k+1} = 0 \;\text{ for all fixed } x \tag{5}$$

The Taylor series about $x = x_0$ follows from eq.(4) by translating the origin as $x = x_0 + \tilde{x}$

$$e^x = e^{x_0} e^{\tilde{x}} = e^{x_0} \sum_{k=0}^{\infty} \frac{1}{k!} \tilde{x}^k = e^{x_0} \sum_{k=0}^{\infty} \frac{1}{k!} (x - x_0)^k \tag{6}$$

logarithmic series:

$$\left(\frac{d \log(1-x)}{dx}\right) = -\frac{1}{1-x} \quad ; \quad \left(\frac{d^2 \log(1-x)}{dx^2}\right) = -\frac{1}{(1-x)^2} \quad ; \quad \left(\frac{d^3 \log(1-x)}{dx^3}\right) = -\frac{2}{(1-x)^3} \tag{7}$$

$$\left(\frac{d^k \log(1-x)}{dx^k}\right) = -\frac{(k-1)!}{(1-x)^k} = -\frac{k!}{k} \frac{1}{(1-x)^k} \tag{8}$$

such that for $x_0 = 0$ (with the $k!$ factor going into the definition of the Taylor series, eq.(2).)

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty} \frac{1}{k} x^k \quad \text{for} \quad -1 \leq x < 1 \tag{9}$$

or equivalently

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = -\sum_{k=1}^{\infty} \frac{1}{k} (-x)^k \quad \text{for} \quad -1 < x \leq 1 \tag{10}$$

This series has a finite radius of convergence.

The Taylor series for $\log(1-x)$ fails the "preliminary convergence test" for all $|x| > 1$; all its coefficients, $c(k) = x^k/k$, diverge for $|x| > 1$.

The Taylor series converges for $0 < x < 1$ because its coefficient $c(k) = x^k/k$ are positive and satisfy the ratio test,

$$\rho = \lim_{k\to\infty} \frac{c(k+1)}{c(k)} = \lim_{k\to\infty} \frac{\frac{x^{k+1}}{k+1}}{\frac{x^k}{k}} = \lim_{k\to\infty} x \frac{k+1}{k} = x < 1 \;\text{ for all } 0 < x < 1 \tag{11}$$

while for $-1 < x < 0$ the series alternates but remains absolute convergent.

The failure of the series at $x = 1$ is not a surprise, because $\log(1-x)$ is not analytic at $x = 1$; it diverges and all its derivatives diverge too. The function does not even exist for $x > 1$.

The surprise is that the series fails for $x < -1$, while nothing special seems to be going-on in the function at $x = 1$, $\log(1-x) = \log(2)$.

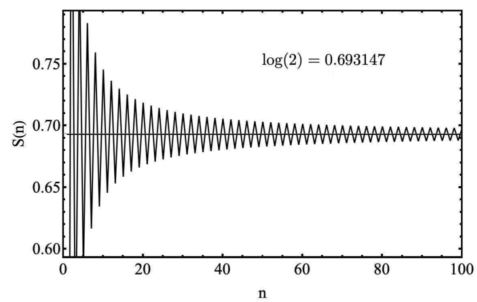
We will discuss in detail in the next chapter (Complex variables) why in general a singularity at $x = x_0$ implies also a failure of the Taylor series at equal distance in the opposite direction at $x = -x_0$, leading to the notion of "radius of convergence."

Borderline points themselves need always special attention:

At $x = 1$ the Taylor series fails the ratio test, eq.(11). Failing a convergence test does not imply a series diverges, it leaves us in limbo, but we looked at this Harmonic series in detail already and found it diverges.

The divergence of the Harmonic series at $x = 1$ implies that the Taylor series at $x = -1$ (at the $\log(2)$ border) fails the Absolute Convergence test. Again, that does not mean the series diverges.

Numerically, see figure, the $\log(1+x)$ Taylor series converges at this $x = 1$ border and reproduces correctly $S(\infty) = \log(2)$. it clearly still converges at this border.

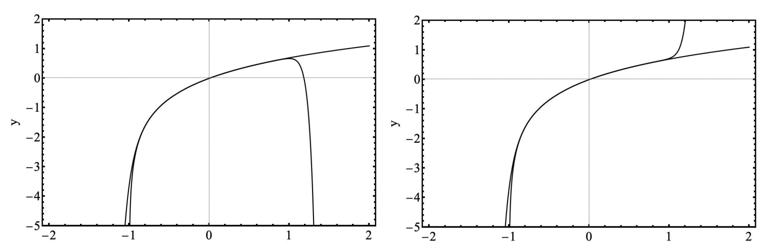


This $\log(2)$ series shows up regularly, e.g., in the earlier alternating Coulomb charges example.

This series is a case where a different convergence test provides the answer. The Ratio Test for the non-alternating series, eq.(11) excludes the borderline case $\rho = 1$, but eq.(11) turns into the Ratio Test for an Alternating Series (the Leibnitz test)

$$\rho = \lim_{k\to\infty} \frac{|c(k+1)|}{|c(k)|} = \lim_{k\to\infty} |x| \frac{k+1}{k} = |x| \leq 1 \quad \text{for all } -1 \leq x < 0 \tag{12}$$

The crucial difference is that the alternating series convergence test includes the $\rho = 1$ case while the ratio test for non alternating series does not. (Boas does not give us a prove for this property of the Leibnitz test.) So the series converges at $x = -1$.



Both figures above show two curves, the full function $y = \log(1+x)$ and its Taylor series eq.(10) truncated it at $n = 20$ and $n = 21$ to illustrate the limited convergence range of the Taylor series.

The series fails for all $x > 1$. For example, the sequence $S(n)$ does not provide a convergent list of rational numbers for the irrational number log(3) while it does so correctly for log(2)

$$\begin{aligned}\{S(n)\} &= \{-\sum_{k=1}^{n} \frac{1}{k}(-1)^k\} \\ &= 1, \frac{1}{2}, \frac{5}{6}, \frac{7}{12}, \frac{47}{60}, \frac{37}{60}, \frac{319}{420}, \frac{533}{840}, \frac{1879}{2520}, \frac{1627}{2520}, \cdots \to \log(2)\end{aligned} \tag{13}$$

(A not very fast converging sequence - but there are better ones.)

This failure for $x > 1$ is rather peculiar also from another perspective. The exponential and logarithmic series are each others inverse in the sense that $y = \log(1 + x)$ and $x = e^y - 1$ list the exact same curve in $\{x, y\}$ space.

The Taylor series for $x(y)$ converges for all y but the other for $y(x)$ converges only for $-1 < x \leq 1$.

The Taylor series for the exponential function tells us correctly that the point on the curve at $y = \log(3)$ lies at $x = 2$ while the Taylor series for the logarithmic function fails to tell us that the point of the curve at $x = 2$ lies at $y = \log(3)$.

That person at point $P = (x, y) = (0, 0)$ predicting the entire curve by never leaving his couch at P, by measuring only all the local derivatives at P, is strongly advised to choose the $x = F(y)$ representation of the curve and stay away from $y = F'(x)$.

We can of course rescue the logarithmic Taylor series convergence beyond $|x| = 1$ by shifting to a different point P. Let's choose $x_0 = 1$ as expansion point. Or equivalently, do this by translating the coordinate system $x \to x + 1$, such that $\log(1 + x) \to \log(2 + x)$. The new Taylor series has a radius of convergence equal to $|x| = 2$.

Generating all the derivatives for a Taylor series is straight forward but often tedious.

Fortunately Mathematica does it quick and easy for us using the command

$$a[x\_, n\_] := Series[Log[2 + x], x, 0, n] \tag{14}$$

Noneed to use that command here because logarithmic functions have special properties:

$$\begin{aligned}\log(2 + x) = \log(2(1 + \tfrac{1}{2}x)) = \log(2) + \log(1 + \tfrac{1}{2}x) &= \log(2) - \sum_{k=1}^{\infty} \frac{1}{k}(\frac{-x}{2})^k \\ = \log(2) + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{24}x^3 - \cdots \quad \text{for} \quad --2 < x \leq 2\end{aligned} \tag{15}$$

4. The parabola and the square root:

The square root function $y = \sqrt{1 + x}$ and the parabola $y = x^2 + 2x$ form another pair that represent the identical curve in (x, y)-space with $x \leftrightarrow y$ switched.

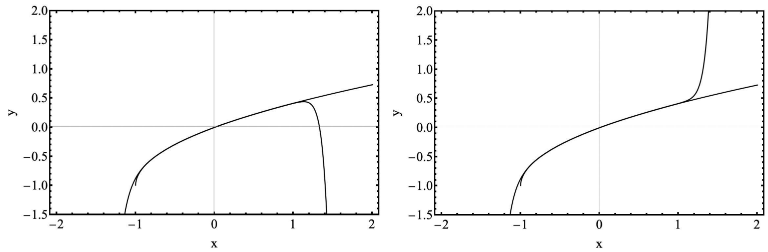
We start with looking-up the Taylor series for $\sqrt{1 + x}$ or run it in Mathematica using eq.(14),

$$\begin{aligned}\sqrt{1 + x} &= \sum_{k=0}^{n} (-1)^{k-1} \frac{(2k)!}{4^k (k!)^2 (2k - 1)} x^k \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots \quad \text{for} \quad -1 \leq x \leq 1\end{aligned} \tag{16}$$

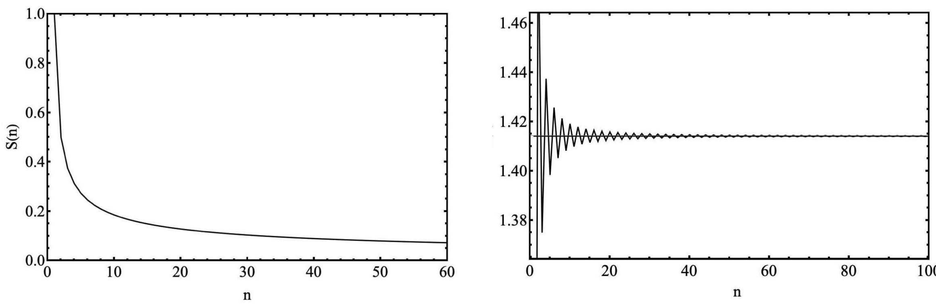
This series has a finite convergence radius similar to the logarithmic series. It converges only for $|x| \leq 1$. The reason is again that $x = -1$ is a singular point of the function and that the function does not extend to $x < -1$.

The Taylor series eq.(16) fails the “preliminary convergence test” for all $|x| > 1$t. Its coefficients, are proportional to $c(k) \sim x^k$, and diverge with k for fixed $|x| > 1$.

The series satisfies the ratio test or is absolute convergent for $|x| < 1$, just like the logarithmic series.



The difference is that this series still converges at both end points: it converges to 0 at $x = -1$ (the figure on the left) and converges to $\sqrt{2}$ at $x = 1$ (the figure on the right).



That person at point P, at $x = 0$, determining all these derivatives for the $\sqrt{1+x}$ representation of the Taylor series is really wasting a massive amount of time, because the curve is a simple parabola in (x, y) space. Rewrite $y = \sqrt{1+x} - 1$ as $x = (y+1)^2 - 1 = y^2 + 2y$.

Technically speaking: the Taylor series in the $x \leftrightarrow y$ flipped representation, $F'(y) = y^2 + 2y$, truncates at the quadratic term level.

$F'(y) = y^2 + 2y$ must be the inverse of the $\sqrt{1+x}$ series, eq.(16).

Mathematica includes a standard function that allows us to generate the inverse of a series and also to nest two series,.

$$\begin{aligned} &InverseSeries[Series[Sqrt[1+x], x, 0, n]] \\ &ComposeSeries[Series[Sqrt[1+x] - 1, x, 0, n], Series[x^2 + 2x, x, 0, n]] \end{aligned} \tag{17}$$

Check that when we nest a series with its own inverse, the answer must be $x = x$. But make sure to shift that function in both x and y so it goes through $(0, 0)$. See posted code.

Let’s run this specific inversion by hand. This is a nested iterative process.

$$\begin{aligned} x = y^2 + 2y \quad &\rightarrow \quad y = \tfrac{1}{2}(x - y^2) \\ &\rightarrow \quad y = \frac{1}{2}(x - [\tfrac{1}{2}(x - y^2)]^2) = \frac{1}{2}x - \frac{1}{8}(x^2 - 2xy^2 + y^4) \\ &\rightarrow \quad y = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{4}x[\tfrac{1}{2}(x - y^2)]^2 - \frac{1}{8}[\tfrac{1}{2}(x - y^2)]^4 \\ &\qquad\;\; = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x(x^2 - 2xy^2 + y^4) - \frac{1}{104}(x^2 - 2xy^2 + y^4)^2 \\ &\rightarrow \quad y = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \mathcal{O}(x^4) \end{aligned} \tag{18}$$

We reproduce already the 3 leading terms of eq.(16), and the rest will fall into place too.

But this bootstrapping process looks very fishy. We keep pushing the remaining y dependency into higher-and-higher powers n in y^n. It resembles a Ponzi scheme.

This procedure works for small values of y and x, such that the remainder vanishes, $y^n \to 0$. Note that at every step the remaining y on the right hand side is of order x.

This works in the local neighborhood of P, because we made sure that point P coincides with the origin $(x, y) = (0, 0)$. But there can be a horizon where the inversion process fails. Which is of course exactly the origin of the finite radius of convergence, at $|x| = 1$, of the $\sqrt{1+x}$ series.

5. More examples:

```
ClearAll["Global`*"];
Print["\nSin[x]=" Normal[Series[ Sin[x], {x, 0, 15}]]]
Print["\nCos[x]=" Normal[Series[ Cos[x], {x, 0, 15}]]]
Print["\nTan[x]=" Normal[Series[ Cos[x], {x, 0, 15}]]]
Print["\nArcTan[x]=" Normal[Series[ ArcTan[x], {x, 0, 15}]]]
```

$$\mathrm{Sin}[x] = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362\,880} - \frac{x^{11}}{39\,916\,800} + \frac{x^{13}}{6\,227\,020\,800} - \frac{x^{15}}{1\,307\,674\,368\,000}\right)$$

$$\mathrm{Cos}[x] = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40\,320} - \frac{x^{10}}{3\,628\,800} + \frac{x^{12}}{479\,001\,600} - \frac{x^{14}}{87\,178\,291\,200}\right)$$

$$\mathrm{Tan}[x] = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40\,320} - \frac{x^{10}}{3\,628\,800} + \frac{x^{12}}{479\,001\,600} - \frac{x^{14}}{87\,178\,291\,200}\right)$$

$$\mathrm{ArcTan}[x] = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15}\right) \tag{19}$$

$\tan[x]$ and $\arctan[x]$ are each others inverse and have finite convergence radii. The other examples in this list have an infinite radius of convergence.

The Taylor series for the so-called Gaussian function follows by nesting $\tilde{x} = x^2$ with the exponential series for $e^{\tilde{x}}$,

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = 1 - ax^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots \tag{20}$$

We can do the same for $\exp(-\frac{1}{x^2})$, using as substitution $\tilde{x} = 1/x^2$,

$$\exp(-\frac{1}{x^2}) = 1 - \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x^4} - \frac{1}{3!}\frac{1}{x^6} + \cdots \tag{21}$$

But keep in mind that tihs amounts to performing a Taylor type series expansion starting from a point P at $x = \infty$, not from $x = 0$. This series diverges at $x = 0$.

This function is rather esoteric at $x = 0$, but one we want to know about this function because it appears in Physics. It has a so-called essential singularity at $x = 0$, meaning that all its derivatives at $x = 0$ are equal to zero, e.g.,

$$F(x) = \exp(-\frac{1}{x^2}) \quad \to \quad \frac{dF}{dx} = -\exp(-\frac{1}{x^2})\ \frac{d}{dx}(\frac{1}{x^2}) = \frac{2}{x^3}\exp(-\frac{1}{x^2}) \to 0 \ \text{ for } \ x \to 0 \tag{22}$$

6. Binomial Expansion

The binomial expansion is not a Taylor series but is important and often used. So Boas parked it at the end of this chapter and so we do too.

The binomial expansion tells us how to expand out the product

$$(a+b)^N = (a+b)(a+b)\cdots(a+b) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} a^n b^{N-n} \tag{23}$$

All possible combinations of powers of a and b appear on the right hand side because while walking along the $(a+b)(a+b)(a+b)\cdots$ chain you make N times a choice between a or b.

The prefactor for each is the so-called combinatorial factor

$$C(N,n) = \frac{N!}{n!(N-n)!} \tag{24}$$

This $C(N,n)$ counts how many choices lead to the same $a^n b^{N-n}$ power.

Imagine bag filled with two types of marbles A and B. You draw one marble at random N times. In that process $C(N,n)$ is the probability you draw n marbles of type A and $N-n$ of type B.

Probabilities need to be normalized

$$\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} = 1 \tag{25}$$

which is true according to eq.(21) by setting $a=b=1$.

I prefer the following more symmetric notation

$$(a+b)^N = (a+b)(a+b)\cdots(a+b) = \sum_{n_a,n_b;\star} C(N;n_a,n_b) a^{n_a} b^{n_b} \quad \text{with} \quad C(N;n_a,n_b) = \frac{N!}{n_a! n_b!} \tag{26}$$

where the summation runs over all possible numbers n_a and n_b with as constraint that $n_a + n_b = N$ (the star $\star$ denotes the presence of the constaint. One advantage of this is that the generalization to more than 2 types of marbles looks natural , e.g.,

$$(a+b_c)^N = (a+b+c)(a+b+c)\cdots(a+b+c) = \sum_{n_a,n_b,n_c;\star} \frac{N!}{n_a! n_b! n_c} a^{n_a} b^{n_b} c^{n_c} \tag{27}$$

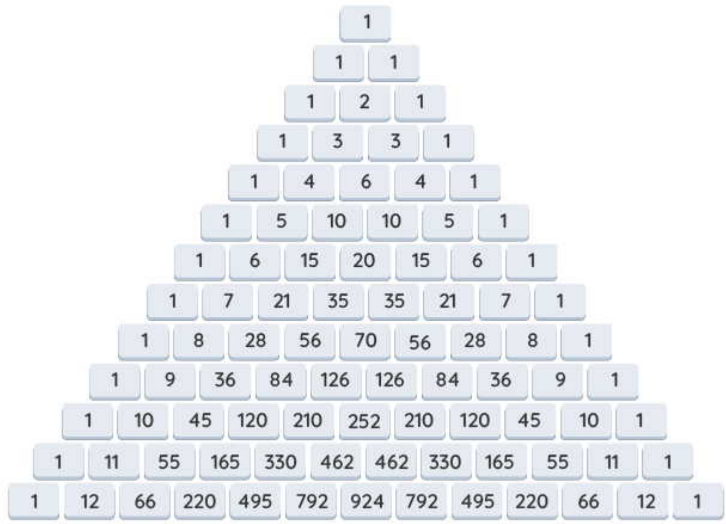
where the summation now runs over all possible numbers n_a, n_b, and n_c with as constraint that $n_a + n_b + n_c = N$.

Eq.(26) can be proven in several ways. Let's use the following Pascal Triangle iterative version

The Pascal triangle is shown below as a brick wall with each brick carrying a number. All bricks carrying a zero are left invisible. The numbers in the Pascal triangle are constructed line-by-line by adding for each brick the two numbers directly above it, starting from the top line where all bricks carry a zero (the unseen ones) except for one.

$$C(N;n_a,n_b) = C(N-1;n_a,n_r-1) + C(N-1;n_a-1,n_b) \tag{28}$$

We see by inspection that each horizontal line reads correctly the binomial coefficient, e.g., for the fourth line, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.



We see also that each brick number counts how many ways we can descend from the top brick down by making left and right steps. For example there are 2 paths from the top to the brick carrying a 2 in the second. There are 6 paths to reach the brick with a 6 in the center of line of line 5. All these paths to the same brick make the same number of left and right turns, but do so in different orders.

We need to prove that the combinatoric factors in eq.(28) satisfy the recursive formula eq.(28). We do this proof by inspection (showing our guess is correct)

$$\frac{(N-1)!}{n_a!(n_b-1)!} + \frac{(N-1)!}{(n_a-1)!n_b!} = \frac{(N-1)!}{n_a!n_b!}(n_a+n_b) = \frac{(N-1)!}{n_a!n_b!}N = \frac{N!}{n_a!n_b!} \tag{29}$$

So we are done.

The combinatoric factor is also linked to the number of ways to draw a subset of n objects out of a total set of N objects.

The number of choices to draw n distinguishable (individually labeled) objects is equal to

$$C^{(D)}(N,n) = N(N-1)(N-2)....(N-n+1) = \frac{N!}{(N-n)!} \tag{30}$$

because after each draw the remaining pool is reduced by one.

Assume the objects are indistinguishable. Then, we over counted by a factor equal to the number of permutations of n objects. The latter is the number of choices to draw n distinguishable objects out of a set of n; i.e., equal to $C^{(D)}(n,n) = n!$, and therefore

$$C(N, n_R) = \frac{N!}{(N-1)!n!} \tag{31}$$

Appendix

Timid tower: $r \to 1$ limit of the stability lines

In the main text of part 1A of the lecture notes, we derived the stability lines of the timid towers, eq.(28).

$$\tfrac{1}{2} = \frac{a}{1-r}\left(1 - \frac{r}{n}\,\frac{1-r^n}{1-r}\right) \tag{A1}$$

We wanted to know their end points in the limit $r = 1$. We did this by redoing the entire calculation explicitly at $r = 1$.

It would seem more natural to set simply $r = 1$ in the formulas of the lines, but this is an example of a function where this is a delicate issue. We encounter ∞ in both the nominator and denominator of the second term between the brackets.

Define a new variable Δ to denote the deviation from $r = 1$, as $r = 1 - \Delta$, and expand out the nominator using the Binomial expansion

$$\begin{aligned}
\tfrac{1}{2} &= \frac{a}{1-r}\left(1 - \frac{r}{n}\,\frac{1-r^n}{1-r}\right)\\
&= \frac{a}{\Delta}\left(1 - \frac{(1-\Delta)}{n}\,\frac{1-(1-\Delta)^n}{\Delta}\right)\\
&= \frac{a}{\Delta}\left(1 - \frac{(1-\Delta)}{n}\,\frac{1-(1-n\Delta+\frac{1}{2}n(n-1)\Delta^2+\mathcal{O}(\Delta^3))}{\Delta}\right)\\
&= \frac{a}{\Delta}\left(1 - \frac{(1-\Delta)}{n}\,\frac{n\Delta-\frac{1}{2}n(n-1)\Delta^2+\mathcal{O}(\Delta^3))}{\Delta}\right)\\
&= \frac{a}{\Delta}\left(1-(1-\Delta)\,(1-\tfrac{1}{2}(n-1)\Delta+\mathcal{O}(\Delta^2))\right)\\
&= \frac{a}{\Delta}\left(1-(1-\Delta-\tfrac{1}{2}(n-1)\Delta+\mathcal{O}(\Delta^2))\right)\\
&= \frac{a}{\Delta}\left(\Delta+\tfrac{1}{2}(n-1)\Delta+\mathcal{O}(\Delta^2))\right)\\
&= a\,(1+\tfrac{1}{2}(n-1)+\mathcal{O}(\Delta)))\\
&= \tfrac{1}{2}a(n+1)+\mathcal{O}(\Delta) \quad\to\quad a = 1/(n+1)
\end{aligned} \tag{A2}$$

The so-called l'Hospital Rule does exactly the same but phrased in terms of derivatives. It says when both the nominator and denominator function $f(x)$ and $g(x)$ go to zero at $r = 1$, we can replace them by their first derivatives

$$\lim_{r\to 1}\frac{f(r)}{g(r)} = \lim_{r\to 1}\frac{f'(r)}{g'(r)} \tag{A3}$$

This is equivalent to saying that we should Taylor expand both functions at $r = 1$ and approximate the functions locally by there first derivatives since the leading terms in the expansion vanish. In case also the first derivatives vanish, we keep only the second derivatives, and so on.