1. Vector Space

§1-3 Subspace.

10 Prove that $W_1 = \{ (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \mid a_1x_1 + \cdots + a_nx_n = 0 \}$ is a subspace of F^n, but $W_2 = \{ (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \mid a_1 + \cdots + a_n = 1 \}$ is not.

Solution. Let $x, y \in W_1$, $c \in F$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$
Claim : W_1 is a subspace of F^n

(a)
$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$
$$\because x_1 + y_1 + \cdots + x_n + y_n = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n$$
$$= 0 + 0 = 0 \therefore x + y \in W_1$$

(b)
$$cx = (cx_1 + cx_2 + \cdots + cx_n)c \in F \because cx_1 + cx_2 + \cdots + cx_n$$
$$= c(x_1 + x_2 + \cdots + x_n) = c * 0 = 0 \therefore cx \in W_1$$

(c)
$$\because 0 + 0 + \cdots + 0 = 0 \quad \therefore (0, 0, \cdots, 0) \in W_1$$

Concluding (a)(b)(c) $\therefore W_1$ is a subspace of F.
Claim W_2 is a subspace of F^n
$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$
$\because (x_1+y_1)+(x_2+y_2)+\cdots+(x_n+y_n) = (x_1+x_2+\cdots+x_n)+(y_1+y_2+\cdots+y_n) = 1 + 1 = 2$
$x + y \notin w_2 \rightarrow\leftarrow$
$\therefore W_2$ is not a subspace of F^n ■

13 Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$,$\{ f \in F(S, F) \mid f(s_0) = 0 \}$, is a subspace of $F(S, F)$.

Solution. **Claim.** $\{ f \in F(S, F) \mid f(S_0) = 0 \}$ is a subspace of $F(S, F)$

(a) let $f_a, f_b \in \{ f \in F(S, F) \mid f(s_0) = 0 \}$
$\because (f_a + f_b)(s_0) = f_a(s_0) + f_b(s_0) = 0$
$\therefore (f_a + f_b)(s_0) \in \{ f \in F(S, F) \mid f(s_0) = 0 \}$

(b) let $f_a \in \{ f \in F(S, F) \mid f(s_0) = 0 \}, c \in \mathrm{F}$
$\because cf_a(s_0) = c \cdot 0 = 0$
$\therefore cf_a(s_0) \in \{ f \in F(S, F) \mid f(s_0) = 0 \}$

(c) every function in $\{ f \in F(S, F) \mid f(s_0) = 0 \}$ is zero function.

$\therefore \{ f \in F(S, F) \mid f(s_0) = 0 \}$ is a subspace of $F(S, F)$. ■

14 Let S be a nonempty set and F a field. Let $C(S,F)$ denote the set of all functions $f \in F(S,F)$ such that $f(s) \neq 0$ for all but a finite number of elements of S. Prove that $C(S,F)$ is a subspace of $F(S,F)$

Solution. **Claim.** $C(S,F)$ is a subspace of $F(S,F)$

(a) let $f, g \in C(S,F)$
$f(s) \neq 0$ when $s \in \{ s_1, s_2, \cdots, s_n \}$ $g(s) \neq 0$ when $s \in \{ s'_1, s'_2, \cdots, s'_m \}$
$(f+g)(s)$
$= f(s) + g(s)$
$f(s) + f(s) \neq 0$ only if $s \in (\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \})$
$\because \#(\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \}) \leq n + m$ is finite
$\therefore (f+g)(s) \in C(S,F)$

(b) let $c \in$ F
$cf(s) \neq 0$ only if $s \in \{ s_1, s_2, \cdots, s_n \}$
$\because \#(\{ s_1, s_2, \cdots, s_n \}) = n$ is finite
$\therefore cf(s) \in C(S,F)$

(c) zero function $f_0 \in F(S,F)$, let $s \in S, 0$ element of S can make $f_0(s) \neq= 0$
$\therefore f_0 \in C(S,F)$

■

20 Prove that if W is a subspace of a vector space V and $w_1, w_2, \cdots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in$W for any scalars $a_1, a_2, \cdots, a_n$.

Solution.
$\because$ W is a subspace of V $a_1w_1, a_2w_2, \cdots, a_nw_n \in$ W by mathematical induction.
by mathematical induction
(1) $\sum_{i=1}^{1} a_iw_i \in$ W
(2) assume $\sum_{i=1}^{k} a_iw_i \in$ W
(3) $\sum_{i=1}^{k+1} a_iw_i = \sum_{i=1}^{k} a_iw_i + a_{k+1}w_{k+1}$
$\because \sum_{i=1}^{k} a_iw_i, a_{k+1}w_{k+1} \in$ W
$\therefore \sum_{i=1}^{k+1} a_iw_i \in$ W ■

23 Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Solution.
(a)**Claim** $W_1 + W_2$ is a subspace of V
let $u_1, u_2 \in W_1 + W_2$, $u_1 = x_1 + y_1$, $u_2 = x_2 + y_2$
$x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$

(1) $u_1 + u_2$

$\Rightarrow (x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + x_2) + (y_1 + y_2)$

$\because W_1, W_2$ is a subspace of V

$\therefore (x_1 + x_2) \in W_1, (y_1 + y_2) \in W_2 \implies (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$

(2) let $c \in$ F

$cu_1 = c(x_1 + y_1) = cx_1 + cy_1$

$\because W_1, W_2$ is a subspace of V, $cx_1 \in W_1, cy_1 \in W_2$

$\therefore cx_1 + cy_1 \in W_1 + W_2$

(3) $\because W_1, W_2$ is a subspace of V,$\therefore 0 \in W_1, 0 \in W_2$,

$0 + 0 = 0 \in W_1 + W_2$

$\therefore W_1 + W_2$ is a subspace of V

$W_1 = \{x + 0 \,|\, x \in W_1\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$W_2 = \{0 + y \,|\, y \in W_2\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$\therefore W_1 + W_2$ contains both W_1 and W_2
(b)let W_3 is a subspace of V, $W_1 \subseteq W_3$,$W_2 \subseteq W_3$

let $x \in W_1, y \in W_2$,$\because W_3$ is a subspace$\therefore x + y \in W_3 \implies W_1 + W_2 \subseteq W_3$ ■

30 Let W_1 and W_2 be subspaces of a vector space V Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Solution.

($\Rightarrow$) $W_1 \cap W_2 = \{ 0 \}, W_1 + W_2 = \mathrm{V}$
Claim. each vector in V can not be only one written as $x + y$
where $x \in W_1, y \in W_2$
let $u \in \mathrm{V}, u = x_1 + y_1 = x_2 + y_2$,
$x_1, x_2 \in W_1, y_1, y_2 \in W_2, x_1 \neq x_2, y_1 \neq y_2$
$x_1 + y_1 = x_2 + y_2 \implies x_1 - x_2 = y_2 - y_1$
$\because$ W_1 is a subspace, $(x_1 - x_2) \in W_1$, W_2 is a subspace, $(y_2 - y_1) \in W_2$
$W_1 \cap W_2 = \{ 0 \} \therefore (x_1 - x_2) = (y_2 - y_1) = 0 \implies x_1 = x_2, y_1 = y_2 \rightarrow\leftarrow$
$\therefore$ each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$and $x_2 \in W_2$

($\Leftarrow$) $\mathrm{V} = \{ x + y \,|\, x \in W_1, y \in W_2 \} = W$
Claim. $W_1 \cap W_2$ not only 0
$\exists\ u \in W_1 \cap W_2, u = 0 + u = u + 0 \rightarrow\leftarrow$
$\therefore W_1 \oplus W_2 = V$

■

§1-4 Linear Combination.

13 Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$,then span(S_1) $\subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span$(S_1) = \mathrm{V}$, deduce that span$(S_2) = \mathrm{V}$

Solution. **Claim** span$(S_1) \subseteq$ span(S_2)
let $S_1 = \{ v_1, v_2, \cdots, v_n \}, S_2 = \{ v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_m \}, x \in \mathrm{span}(S_1)$
$x = a_1 v_1 + \cdots + a_n v_n, a_1, a_2, \cdots, a_n \in \mathrm{F}$
$= a_1 v_1 + \cdots + a_n v_n + 0u_1 + 0u_2 + \cdots + 0u * n \in \mathrm{Span}(S_2)$
$\therefore \mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$

■

14 Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then span($S_1 \cup S_2$)= span(S_1) + span(S_2).

Solution. Let $S_1 \cap S_2 = \{ v_1, v_2, \cdots, v_n \}$,
$S_1 = \{ u_1, u_2, \cdots, u_m, v_1, \cdots, v_n \}, S_2 = \{ r_1, \cdots, r_k, v_1, \cdots, v_n \}$
Claim. span(S_1)+span(S_2) $\subseteq$ span($S_1 \cup S_2$)
let $x \in \text{span}(S_1) + \text{span}(S_2))$
$x = (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n)+$
$(b_1r_1 + \cdots + b_kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$= (a_1u_1+\cdots+a_mu_m)+(b_1r_1+\cdots+b_kr_k)+((a_{m+1}+b_{k+1})v_1+\cdots+(a_{m+n}+b_{k+n})v_n)$
$\Rightarrow x \in \text{span}(S_1 \cup S_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$
Claim. span($S_1 \cup S_2$) $\subseteq$ span(S_1) + span(S_2)
let $y \in \text{span}(S_1 \cup S_2)$
$y = (c_1u_1 + \cdots + c_mu_m) + (c_{m+1}r_1 + \cdots + c_{m+k}r_k) + (c_{m+k+1}v_1 + \cdots + c_{m+k+n}v_n)$
$= (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) + (b_1r_1 + \cdots + b_kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$\therefore y \in \text{span}(S_1) + \text{span}(s_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2)$

■

§1-5 Linear Independent.

13 Let V be a vector space over a field of characteristic not equal to two.

Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

Solution.

($\Rightarrow$) **Claim.** $\{u+v, u-v\}$ is linearly independent

$a_1(u+v) + a_2(u-v) = 0, a_1, a_2 \in \mathrm{F}$

$\Rightarrow (a_1+a_2)u + (a_1-a_2)v = 0$

$\because \{u, v\}$ is linearly independent

$$\therefore \begin{cases} a_1+a_2=0 \\ a_1-a_2=0 \end{cases} \Rightarrow a_1=a_2=0$$

$\therefore \{u+v, u-v\}$ is linearly independent

($\Leftarrow$) **Claim.** $\{u, v\}$ is linearly independent

$\Rightarrow b_1u + b_2v = 0$

$\Rightarrow \frac{b_1+b_2}{2}(u+v) + \frac{b_1-b_2}{2}(u-v) = 0$

$\because \{u+v, u-v\}$ is linearly independent

$$\therefore \begin{cases} \frac{b_1+b_2}{2}=0 \\ \frac{b_1-b_2}{2}=0 \end{cases} \Rightarrow b_1=b_2=0$$

$\therefore \{u, v\}$ is linearly independent

■

16 Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution. let $S = \{s_1, s_2, \cdots, s_n\}$

($\Rightarrow$) **Claim.** $\exists$ subset $S_i = \{s'_1, s'_2, \cdots, s'_r\}, r \leq n$,

$b_1s'_1 + b_2s'_2 + \cdots + b_ns'_r = 0$, not all $b_i = 0, 1 \leq i \leq r$

let $S - S_i = \left\{ s'_{r+1}, s'_{r+2}, \cdots, s'_n \right\}$

$b_1s'_1 + b_2s'_2 + \cdots + b_ns'_n = 0$ not only $b_1 = b_2 = \cdots = b_n = 0 \rightarrow\leftarrow$

($\Leftarrow$) by definition of linear independent, each finite subset of S is linearly independent, S is linear independent.

■

18 Let S be a set of non zero polynomials in $P(F)$ such that no two have the same degree. Prove that S is linearly independent.

Solution. let $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$, $a_1, \cdots, a_n \in \mathrm{F}$, $u_1, \cdots, u_n \in S$
$u_1 = c_{10} + c_{11}x + c_{12}x^2 + \cdots + c_{1k}x^k$
$u_22 = c_{20} + c_{21}x + c_{22}x^2 + \cdots + c_{2k}x^k$
$\vdots$
$u_n = c_{n0} + c_{n1}x + c_{n2}x^2 + \cdots + c_{nk}x^k$
d_i is the degree of u_i
$d_1 < d_2 < \cdots < d_n, d_n = k$
$a_1u_1 + a_2u_2 + \cdots + a_nu_n$
$= (a_1c_{10} + \cdots + a_nc_{n0}) + \cdots + (a_1c_{1k} + \cdots + a_nc_{nk})x^k$
$= 0$

$$\Rightarrow \begin{cases} a_1c_{10} + \cdots + a_nc_{n0} = 0 \\ a_1c_{11} + \cdots + a_nc_{n1} = 0 \\ \vdots \\ a_1c_{1k} + \cdots + a_nc_{nk} = 0 \end{cases}$$

$\because$ the element in S no two have same degree
$\therefore$ only u_n contain x^k
$\Rightarrow c_{1k} = c_{2k} = \cdots = c_{n-1k} = 0c_{nk} \neq 0$
$\Rightarrow a_1c_{1k} + \cdots + a_nc_{nk} = 0$ only $a_n = 0$
(1) at most u_{n-1}, u_n contains $x^{d_{n-1}}$
$\Rightarrow c_{1d_{n-1}} = c_{2d_{n-1}} = \cdots = c_{(n-2)(d_{n-1})} = 0, c_{n-1d_{n-1}} \neq 0, c_{nd_{n-1}} \neq 0$
$\because a_n = 0$
$\therefore a_1c_{1d_{d_{n-1}}} + \cdots + a_nc_{nd_{n-1}} = 0$, only $a_{n-1} = 0$
(2) $u_{n-i}, \cdots, u_n$ contains $x^{d_{n-i}}$
$\Rightarrow c_{1d_{n-i}} = c_{2d_{n-i}} = \cdots = c_{(n-i-1)(d_{n-i})} = 0$
assume $a_1c_{1d_{n-i}} + \cdots + a_nc_{nd_{n-i}} = 0$
only $a_nn - i, a_{n-i+1}, \cdots, a_n = 0$
(3)$u_{n-(i+1)}, \cdots, u_n$ contains $x^{d_{n-(i+1)}}$
$\Rightarrow c_{1d_{(n-i-1)}} = c_{2d_{(n-i-1)}} = \cdots = c_{(n-i-2)d_{(n-i-1)}}$
$a_1c_{1d_{(n-i-1)}} + \cdots + a_nc_{nd_{(n-i-1)}} = 0$
$\Rightarrow a_{n-i-1}c_{(n-i-1)d_{(n-i-1)}} + \cdots + a_nc_{nd_{(n-i-1)}} = 0$
$\because a_{n-i}, \cdots, a_n = 0$
$\Rightarrow a_{n-i-1}c_{(n-i-1)d_{(n-i-1)}} = 0$
$\because u_{n-i-1}$ contains $x^{d_{(n-i-1)}}$
$\therefore c_{(n-i-1)d_{(n-i-1)}} \neq 0$
$a_{n-i-1} = 0$
by mathematical induction $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$ only $a_1 = a_2 = \cdots = a_n = 0$
S is Linearly independent ■

20 Let $f, g \in F(R, R)$ be the functions defined by $f(t) = e^{et}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $F(R, R)$.

Solution. **Claim.** f, g are linearly independent in $F(R, R)$
$a_1 f(t) + a_2 g(t) = 0 \Rightarrow a_1 e^{rt} + a_2 e^{st} = 0 \Rightarrow e^{rt}(a_1 + a_2 e^{t(s-r)}) = 0$
$\Rightarrow e^{rt} = 0$(impossiable) or $(a_1 + a_2 e^{t(s-r)}) = 0 \Rightarrow a_1 = a_2 = 0$
$\therefore f, g$ are linearly independent in $F(R, R)$. ■

§1-6 Bases and Dimension.

14 Find bases for the following subspaces of F^5:

$$W_1 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_1 - a_3 - a_4 = 0 \right\}$$

and

$$W_2 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_2 = a_3 = a_4 \ , \ a_1 + a_5 = 0 \right\}.$$

What are the dimensions of W_1 and W_2?

Solution. set $p, q, t, r \in \mathrm{F}$,
$W_1 = \{(q+t, p, q, t, r) = q(1,0,1,0,0) + p(0,1,0,0,0)$
$+t(1,0,0,1,0) + r(0,0,0,0,1)\}$
Claim. $\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}$ is linearly independent
$c_1(1,0,1,0,0) + c_2(0,1,0,0,0) + c_3(1,0,0,1,0) + c_4(0,0,0,0,1)$
$\Rightarrow c_1 + c_3 = 0, c_2 = 0, c_1 = 0, c_3 = 0, c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$
$\therefore \{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}$ is linearly independent
Claim $\mathit{span}(\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}) = W_1$
let $x \in W_1$, x $= q(1,0,1,0,0) + p(0,1,0,0,0) + t(1,0,0,1,0) + r(0,0,0,0,1)$
$x \in \mathit{span}(\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \})$
$\therefore W_1 \subseteq \mathit{span}(\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \})$
$\because W_1$ is a subspace of V, any linearly combination of W_1's subset is in W_1
$\therefore \mathit{span}(\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}) \in W_1$
$\therefore \mathit{span}(\{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}) = W_1$
$\therefore \{ (1,0,1,0,0), (0,1,0,0,0), (1,0,0,1,0), (0,0,0,0,1) \}$ is a basis of W_1,the dimension of W_1 is 4. ■

20 Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a) Prove that there is a subset of S that is a basis for V.(Be careful not to assume that S is finite)

(b) Prove that S contains at least n vectors.

Solution. (a) if $S = \varnothing$ or $S = \{ 0 \}$
$V = \{ 0 \}$ $\therefore$ there is a subset of S be a basis.
else pick $s_1 \neq$ from S
pick $s_{k+1} \notin \text{span}(\{ s_1, s_2, \cdots, s_k \})$, by replacement theorem, when a linearly independent set's element number equal dim(V), the set can generate V.
$\therefore$ there is a subset of S be a basis.

(b) by the definition dimension, the element number of basis is n
by replacement theory's, span(S')=V,$\#(S') \geq n$,$S' \subseteq S$,$\#(S) \geq n$. ■

25 Let V,W, and Z be as in Exercise 21 if Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

Solution. let $Z = \{ (v, w) \mid v \in V, w \in W \}, dim(V) = m, dim(W) = n$
$Z_1 = \{ (v, 0) \mid v \in V \}$,$Z_2 = \{ (0, w) \mid w \in W \}$
Claim $Z \subseteq Z_1 + Z_2$, let $x \in Z, x = (v, w)v \in V, w \in W$
$x = (v, 0) + (0, w) \in Z_1 + Z_2$
$\therefore Z \subseteq Z_1 + Z_2$
Claim $Z_1 + Z_2 \subseteq Z$, let $x \in Z_1 + Z_2$
$x = (v, 0) + (0, w)v \in V, w \in W = (v, w) \in Z$
$\therefore Z_1 + Z_2 \subseteq Z$
$\therefore Z = Z_1 + Z_2$
$\because Z = Z_1 + Z_2$,$Z_1 \cap Z_2 = \{ (0, 0) \}$
$\therefore Z = Z_1 \bigoplus Z_2$
by Exercise 1.6.29(b), if W_1 and W_2 are finite-dimensional subspace of a vector space V, and let V = $W_1 \bigoplus W_2$. $\text{dim}(W_1 + W_2) = \text{dim}(W_1) + \text{dim}(W_2) = m + n$ ■

29 (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and dim$(W_1 + W_2) =$ dim(W_1) + dim(W_2) - dim$(W_1 \cap W_2)$.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let V = $W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if dim(V) = dim(W_1) + dim(W_2).

Solution. (a) let β is a basis of $W_1 \cap W_2$ $\dim(W_1) = k + m$,

$\dim(W_2) = k + n$, $\dim(W_1 \cap W_2) = k$, $k, m, n \in \mathbb{Z}^{\geq 0}$

$\beta \in = \{ u_1, u_2, \cdots, u_k \}$, $u_1, \cdots, u_k \in W_1 \cap W_2$

$\because \beta \in W_1, \beta \in W_2$

by Replacement Theorem, every linearly independent subset of V can be extended to a basis for V.

$\exists$ β_1 is a basis of W_1 $\beta_1 = \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m \}$ $v_1, v_2, \cdots, v_m \in W_1$

$\exists \beta_2$ is a basis of W_2

$\beta_2 = \{ u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_n \} w_1, w_2, \cdots, w_m \in W_2$

let $x \in W_1 + W_2$

Claim. *span*$(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) = W_1 + W_2$

$x = (a_1u_1 + a_2u_2 + \cdots + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m) + (b_1u_1 + b_2u_2 + \cdots + b_ku_k + b_{k+1}w_1 + b_{k+2}w_2 + \cdots + b_{k+n}w_n), a_1, a_2, \cdots, a_{k+m}, b_1, b_2, \cdots, b_{k+n} \in \mathrm{F}$

$= c_1u_1 + c_2u_2 + \cdots + c_ku_k + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m + b_{k+1}w_1 + b_{k+2} + \cdots + b_{k+n}w_n, c_1, c_2, \cdots, c_k \in \mathrm{F}$

$x \in W_1 + W_2 \therefore W_1 + W_2 \subseteq span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \})$

$\because W_1 + W_2$ is a subspace, any linear combination of $W_1 + W_2$'s subset are in $W_1 + W_2$

$\therefore span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) \in W_1 + W_2$

$\therefore span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) = W_1 + W_2$

$\because v_1, v_2, \cdots, v_m \notin W_2, w_1, w_2, \cdots w_n \notin W_1$

$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is linearly independent

$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is a basis of $W_1 + W_2$

$\therefore \dim(W_1 + W_2) = k + m + n$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2)$

(b) $W_1 \cap W_2 = \{ 0 \}$

by Exercise 1.16.29(a), if W_1 and W_2 are finite-dimensional subspace of a vector space V, dim(V) = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$ = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(\{ 0 \})$ = $\dim(W_1)$ + $\dim(W_2)$

■

31 Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Solution. (a) let β_1 is a basis of W_2

β is a basis of $W_1 \cap W_2$

$\#(\beta_1) = n$

by Replacement Theorem, V be a vector space is generated by a set G, $\#(G) = n$, a linearly independent set $L \in$ V,$\#(L) = m$

$\because W_1 \cap W_2 \subseteq W_2$

$\therefore L = \beta$ is a linearly independent set of W_2, $G = \beta_1$ can generated W_2

by Replacement Theorem$\Rightarrow \#(\beta) \leq \#(\beta_1)$

$\Rightarrow \dim(W_1 \cap W_2) \leq n$

(b) by Exercise 29,W_1, W_2 are finite-dimensional subspaces of a vector space V,then $\dim(W_1 + W_2)$=$\dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$
$\dim(W_1 + W_2) = \dim(W_1)$+$\dim(S_2)$-$\dim(W_1 \cap W_2)$ = m+n - $\dim(W_1 \cap W_2)$
$\leq m + n$

■

33 (a) Let W_1 and W_2 be subspaces of a vector space V such that V $= W_1 \bigoplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then V $= W_1 \bigoplus W_2$.

Solution. let $\beta_1 = \{ v_1, v_2, \cdots, v_n \}, v_1, v_2, \cdots, v_n \in W_1$,
$\beta_2 = \{ u_1, u_2, \cdots, u_m \}, u_1, u_2, \cdots, u_m \in W_2$
$W_1 + W_2 = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_1 + b_2u_2 + \cdots + b_mu_m \mid$
$a_1, \cdots, a_n, b_1, \cdots, b_m \in \mathrm{F}\}$
Claim $\mathrm{span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$
let $x \in \mathrm{span}(\beta_1 \cup \beta_2)$, $x = a_1v_1 + a_2V_2 + \cdots + b_1u_1 + b_2u_2 + \cdots + b_mu_m$
$\because W_1, W_2$ is a subspace of V
$\therefore \sum_{i=1}^{n} a_iv_i \in W_1$, $\sum_{i=1}^{m} b_iu_i \in W_2$
$\therefore x \in W_1 + W_2, \mathrm{span}(\beta_1 \cup beta_2) \subseteq W_1 + W_2$
Claim. $W_1 + W_2 \subseteq \mathrm{span}(\beta_1 \cup \beta_2)$
let $x \in W_1 + W_2, x = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$\therefore x \in \mathrm{span}(\beta_1 \cup \beta_2)$
$\therefore W_1 + W_2 \subseteq \mathrm{span}(\beta_1 \cup \beta_2)$
$\therefore \mathrm{span}(\beta_1 \cup \beta_2) = W_1 + W_2$
Claim $\beta_1 \cup \beta_2$ is linearly dependent
$\exists$ a_iv_i or b_iu_i can be express by
$-a_iv_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$\because v_i \notin \mathrm{span}(\beta_1 - \{ v_i \}), v_i \notin \mathrm{span}(\beta_2)$
$-b_iu_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$u_i \notin \mathrm{span}(\beta_2 - \{ u_i \}), u_i \notin \mathrm{span}(\beta_1) \rightarrow\leftarrow$

■

34 Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that V $= W_1 \bigoplus W_2$

Solution. let $\beta_1 = \{ v_1, v_2, \cdots, v_m \}$,dim(V) $= n$
by Corollary of Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V
$\Rightarrow \exists \beta = \{ v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_{n-m} \}$ is a basis of V
let $W_2 = \text{span}(\{u_1, u_2, \cdots, u_{n-m}\})$
$\because u_1, u_2, \cdots, u_{n-m} \in V, V$ is a vector space
by Thm 1.5, the span of any subset S of a vector space V is a subspace.
$\therefore W_2$ is a subspace of V
Claim. $W_1 \cap W_2 = \{0\}$
$\because W_1, W_2$ is a subspace of V
$\therefore 0 \in W_1, W_2$
assume $\exists$ vector $r \in V$, $r \in W_1$, $r \in W_2$, $r \neq 0$
$r = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$
$= b_1 u_1 + b_2 u_2 + \cdots + b_{n-m} u_{n-m}$
$= c_1 v_1 + \cdots + c_m v_m + d_1 u_1 + \cdots + d_{n-m} u_{n-m}$

$$\Rightarrow \begin{cases} (c_1 - a_1)v_1 + \cdots + (c_m - a_m)v_m + d_1 u_1 + \cdots + d_{n-m} u_{n-m} = 0 \\ c_1 v_1 + \cdots + c_m v_m + (d_1 - b_1)u_1 + \cdots + (d_{n-m} - b_{n-m}) \end{cases}$$

$\Rightarrow c_1 = a_1, c_2 = a_2, \cdots, c_m = a_m, d_1 = b_1, \cdots, d_{n-m} = b_{n-m}$
$\Rightarrow r = r + r \Rightarrow r = 0 \rightarrow\leftarrow$
$\therefore W_1 \cap W_2 = \{0\}$ ■