

## 1. Vector Space

### §1-3 Subspace.

- 10 Prove that  $W_1 = \{ (a_1, a_2, \dots, a_n) \in F^n \mid a_1x_1 + \dots + a_nx_n = 0 \}$  is a subspace of  $F^n$ , but  $W_2 = \{ (a_1, a_2, \dots, a_n) \in F^n \mid a_1 + \dots + a_n = 1 \}$  is not.

*Solution.* Let  $x, y \in W_1$ ,  $c \in F$ ,  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n)$

**Claim :**  $W_1$  is a subspace of  $F^n$

(a)

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \therefore x_1 + y_1 + \dots + x_n + y_n &= x_1 + x_2 + \dots + x_n + y_1 + y_2 + \dots + y_n \\ &= 0 + 0 = 0 \therefore x + y \in W_1 \end{aligned}$$

(b)

$$\begin{aligned} cx &= (cx_1 + cx_2 + \dots + cx_n)c \in F \therefore cx_1 + cx_2 + \dots + cx_n \\ &= c(x_1 + x_2 + \dots + x_n) = c \cdot 0 = 0 \therefore cx \in W_1 \end{aligned}$$

(c)

$$\therefore 0 + 0 + \dots + 0 = 0 \therefore (0, 0, \dots, 0) \in W_1$$

Concluding (a)(b)(c)  $\therefore W_1$  is a subspace of  $F$ .

**Claim**  $W_2$  is a subspace of  $F^n$

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \therefore (x_1 + y_1) + (x_2 + y_2) + \dots + (x_n + y_n) &= (x_1 + x_2 + \dots + x_n) + (y_1 + y_2 + \dots + y_n) = \\ &= 1 + 1 = 2 \\ x + y &\notin W_2 \rightarrow \leftarrow \\ \therefore W_2 &\text{ is not a subspace of } F^n \end{aligned}$$

- 13 Let  $S$  be a nonempty set and  $F$  a field. Prove that for any  $s_0 \in S, \{ f \in F(S, F) \mid f(s_0) = 0 \}$ , is a subspace of  $F(S, F)$ .

*Solution.* **Claim.**  $\{ f \in F(S, F) \mid f(s_0) = 0 \}$  is a subspace of  $F(S, F)$

(a) let  $f_a, f_b \in \{ f \in F(S, F) \mid f(s_0) = 0 \}$

$$\begin{aligned} \therefore (f_a + f_b)(s_0) &= f_a(s_0) + f_b(s_0) = 0 \\ \therefore (f_a + f_b)(s_0) &\in \{ f \in F(S, F) \mid f(s_0) = 0 \} \end{aligned}$$

(b) let  $f_a \in \{ f \in F(S, F) \mid f(s_0) = 0 \}, c \in F$

$$\begin{aligned} \therefore cf_a(s_0) &= c \cdot 0 = 0 \\ \therefore cf_a(s_0) &\in \{ f \in F(S, F) \mid f(s_0) = 0 \} \end{aligned}$$

(c) every function in  $\{ f \in F(S, F) \mid f(s_0) = 0 \}$  is zero function.

$$\therefore \{ f \in F(S, F) \mid f(s_0) = 0 \} \text{ is a subspace of } F(S, F).$$

- 14 Let  $S$  be a nonempty set and  $F$  a field. Let  $C(S, F)$  denote the set of all functions  $f \in F(S, F)$  such that  $f(s) \neq 0$  for all but a finite number of elements of  $S$ . Prove that  $C(S, F)$  is a subspace of  $F(S, F)$

*Solution.* **Claim.**  $C(S, F)$  is a subspace of  $F(S, F)$

- (a) let  $f, g \in C(S, F)$   
 $f(s) \neq 0$  when  $s \in \{s_1, s_2, \dots, s_n\}$   $g(s) \neq 0$  when  $s \in \{s'_1, s'_2, \dots, s'_m\}$   
 $(f + g)(s)$   
 $= f(s) + g(s)$   
 $f(s) + g(s) \neq 0$  only if  $s \in (\{s_1, s_2, \dots, s_n\} \cup \{s'_1, s'_2, \dots, s'_m\})$   
 $\therefore \#(\{s_1, s_2, \dots, s_n\} \cup \{s'_1, s'_2, \dots, s'_m\}) \leq n + m$  is finite  
 $\therefore (f + g)(s) \in C(S, F)$
- (b) let  $c \in F$   
 $cf(s) \neq 0$  only if  $s \in \{s_1, s_2, \dots, s_n\}$   
 $\therefore \#(\{s_1, s_2, \dots, s_n\}) = n$  is finite  
 $\therefore cf(s) \in C(S, F)$
- (c) zero function  $f_0 \in F(S, F)$ , let  $s \in S$ , 0 element of  $S$  can make  $f_0(s) \neq 0$   
 $\therefore f_0 \in C(S, F)$

■

- 20 Prove that if  $W$  is a subspace of a vector space  $V$  and  $w_1, w_2, \dots, w_n$  are in  $W$ , then  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$  for any scalars  $a_1, a_2, \dots, a_n$ .

*Solution.*

$\therefore W$  is a subspace of  $V$   $a_1w_1, a_2w_2, \dots, a_nw_n \in W$  by mathematical induction.  
 by mathematical induction

- (1)  $\sum_{i=1}^1 a_iw_i \in W$   
 (2) assume  $\sum_{i=1}^k a_iw_i \in W$   
 (3)  $\sum_{i=1}^{k+1} a_iw_i = \sum_{i=1}^k a_iw_i + a_{k+1}w_{k+1}$   
 $\therefore \sum_{i=1}^k a_iw_i, a_{k+1}w_{k+1} \in W$   
 $\therefore \sum_{i=1}^{k+1} a_iw_i \in W$

■

23 Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ .

- (a) Prove that  $W_1 + W_2$  is a subspace of  $V$  that contains both  $W_1$  and  $W_2$ .
- (b) Prove that any subspace of  $V$  that contains both  $W_1$  and  $W_2$  must also contain  $W_1 + W_2$ .

*Solution.*

(a) **Claim**  $W_1 + W_2$  is a subspace of  $V$

- (1) let  $x_1, x_2 \in W_1, y_1, y_2 \in W_2$   
 $(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + x_2) + (y_1 + y_2)$   
 $\because W_1, W_2$  is a subspace of  $V$   
 $\therefore (x_1 + x_2) \in W_1, (y_1 + y_2) \in W_2 \implies (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$
- (2) let  $x_1 \in W_1, y_1 \in W_2, c \in F$   
 $c(x_1 + y_1) = cx_1 + cy_1$   
 $\because W_1, W_2$  is a subspace of  $V, cx_1 \in W_1, cy_1 \in W_2$   
 $\therefore cx_1 + cy_1 \in W_1 + W_2$
- (3)  $\because W_1, W_2$  is a subspace of  $V, \therefore 0 \in W_1, 0 \in W_2,$   
 $0 + 0 = 0 \in W_1 + W_2 \therefore W_1 + W_2$  is a subspace of  $V$

$$W_1 = \{x + 0 \mid x \in W_1\} \subseteq \{x + y \mid x \in W_1, y \in W_2\}$$

$$W_2 = \{0 + y \mid y \in W_2\} \subseteq \{x + y \mid x \in W_1, y \in W_2\}$$

$\therefore W_1 + W_2$  contains both  $W_1$  and  $W_2$

(b) let  $W_3$  is a subspace of  $V, W_1 \subseteq W_3, W_2 \subseteq W_3$

let  $x \in W_1, y \in W_2, \because W_3$  is a subspace.  $\therefore x + y \in W_3 \implies W_1 + W_2 \subseteq W_3$  ■

- 30 Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if each vector in  $V$  can be uniquely written as  $x_1 + x_2$ , where  $x_1 \in W_1$  and  $x_2 \in W_2$ .

*Solution.*

$$(\Rightarrow) W_1 \cap W_2 = \{0\}, W_1 + W_2 = V$$

**Claim.** each vector in  $V$  can not be only one written as  $x + y$   
where  $x \in W_1, y \in W_2$

let  $u \in V, u = x_1 + y_1 = x_2 + y_2$ ,

$x_1, x_2 \in W_1, y_1, y_2 \in W_2, x_1 \neq x_2, y_1 \neq y_2$

$x_1 + y_1 = x_2 + y_2 \Rightarrow x_1 - x_2 = y_2 - y_1$

$\because W_1$  is a subspace,  $(x_1 - x_2) \in W_1, W_2$  is a subspace,  $(y_2 - y_1) \in W_2$

$W_1 \cap W_2 = \{0\} \therefore (x_1 - x_2) = (y_2 - y_1) = 0 \Rightarrow x_1 = x_2, y_1 = y_2 \rightarrow \leftarrow$

$\therefore$  each vector in  $V$  can be uniquely written as  $x_1 + x_2$ , where  $x_1 \in W_1$  and  $x_2 \in W_2$

$$(\Leftarrow) V = \{x + y \mid x \in W_1, y \in W_2\} = W$$

**Claim.**  $W_1 \cap W_2$  not only 0

$\exists u \in W_1 \cap W_2, u = 0 + u = u + 0 \rightarrow \leftarrow$

$\therefore W_1 \oplus W_2 = V$

■

#### §1-4 Linear Combination.

- 13 Show that if  $S_1$  and  $S_2$  are subsets of a vector space  $V$  such that  $S_1 \subseteq S_2$ , then  $\text{span}(S_1) \subseteq \text{span}(S_2)$ . In particular, if  $S_1 \subseteq S_2$  and  $\text{span}(S_1) = V$ , deduce that  $\text{span}(S_2) = V$ .

*Solution.* **Claim**  $\text{span}(S_1) \subseteq \text{span}(S_2)$

let  $S_1 = \{v_1, v_2, \dots, v_n\}, S_2 = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}, x \in \text{span}(S_1)$

$x = a_1 v_1 + \dots + a_n v_n, a_1, a_2, \dots, a_n \in F$

$= a_1 v_1 + \dots + a_n v_n + 0u_1 + 0u_2 + \dots + 0u_m \in \text{Span}(S_2)$

$\therefore \text{span}(S_1) \subseteq \text{span}(S_2)$

■

- 14 Show that if  $S_1$  and  $S_2$  are arbitrary subsets of a vector space  $V$ , then  $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$ .

*Solution.* Let  $S_1 \cap S_2 = \{v_1, v_2, \dots, v_n\}$ ,

$S_1 = \{u_1, u_2, \dots, u_m, v_1, \dots, v_n\}$ ,  $S_2 = \{r_1, \dots, r_k, v_1, \dots, v_n\}$

**Claim.**  $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$

let  $x \in \text{span}(S_1) + \text{span}(S_2)$

$$x = (a_1u_1 + \dots + a_mu_m + a_{m+1}v_1 + \dots + a_{m+n}v_n) +$$

$$(b_1r_1 + \dots + b_kr_k + b_{k+1}v_1 + \dots + b_{k+n}v_n)$$

$$= (a_1u_1 + \dots + a_mu_m) + (b_1r_1 + \dots + b_kr_k) + ((a_{m+1} + b_{k+1})v_1 + \dots + (a_{m+n} + b_{k+n})v_n)$$

$$\Rightarrow x \in \text{span}(S_1 \cup S_2)$$

$$\therefore \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$$

**Claim.**  $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$

let  $y \in \text{span}(S_1 \cup S_2)$

$$y = (c_1u_1 + \dots + c_mu_m) + (c_{m+1}r_1 + \dots + c_{m+k}r_k) + (c_{m+k+1}v_1 + \dots + c_{m+k+n}v_n)$$

$$= (a_1u_1 + \dots + a_mu_m + a_{m+1}v_1 + \dots + a_{m+n}v_n) + (b_1r_1 + \dots + b_kr_k + b_{k+1}v_1 + \dots + b_{k+n}v_n)$$

$$\therefore y \in \text{span}(S_1) + \text{span}(S_2)$$

$$\therefore \text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2)$$



## §1-5 Linear Independent.

13 Let  $V$  be a vector space over a field of characteristic not equal to two.

Let  $u$  and  $v$  be distinct vectors in  $V$ . Prove that  $\{u, v\}$  is linearly independent if and only if  $\{u + v, u - v\}$  is linearly independent.

*Solution.*

( $\Rightarrow$ ) **Claim.**  $\{u + v, u - v\}$  is linearly independent

$$a_1(u + v) + a_2(u - v) = 0, a_1, a_2 \in F$$

$$\Rightarrow (a_1 + a_2)u + (a_1 - a_2)v = 0$$

$$\because \{u, v\} \text{ is linearly independent}$$

$$\therefore \begin{cases} a_1 + a_2 = 0 \\ a_1 - a_2 = 0 \end{cases} \Rightarrow a_1 = a_2 = 0$$

$$\therefore \{u + v, u - v\} \text{ is linearly independent}$$

( $\Leftarrow$ ) **Claim.**  $\{u, v\}$  is linearly independent

$$\Rightarrow b_1u + b_2v = 0$$

$$\Rightarrow \frac{b_1+b_2}{2}(u + v) + \frac{b_1-b_2}{2}(u - v) = 0$$

$$\because \{u + v, u - v\} \text{ is linearly independent}$$

$$\therefore \begin{cases} \frac{b_1+b_2}{2} = 0 \\ \frac{b_1-b_2}{2} = 0 \end{cases} \Rightarrow b_1 = b_2 = 0$$

$$\therefore \{u, v\} \text{ is linearly independent}$$

■

16 Prove that a set  $S$  of vectors is linearly independent if and only if each finite subset of  $S$  is linearly independent.

*Solution.* let  $S = \{s_1, s_2, \dots, s_n\}$

( $\Rightarrow$ ) **Claim.**  $\exists$  subset  $S_i = \{s'_1, s'_2, \dots, s'_r\}, r \leq n$ ,

$$b_1s'_1 + b_2s'_2 + \dots + b_ns'_r = 0, \text{ not all } b_i = 0, 1 \leq i \leq r$$

$$\text{let } S - S_i = \{s'_{r+1}, s'_{r+2}, \dots, s'_n\}$$

$$b_1s'_1 + b_2s'_2 + \dots + b_ns'_n = 0 \text{ not only } b_1 = b_2 = \dots = b_n = 0 \rightarrow \leftarrow$$

( $\Leftarrow$ ) by definition of linear independent, each finite subset of  $S$  is linearly independent,  $S$  is linear independent.

■

- 18 Let  $S$  be a set of non zero polynomials in  $P(F)$  such that no two have the same degree. Prove that  $S$  is linearly independent.

*Solution.* ■

- 20 Let  $f, g \in F(R, R)$  be the functions defined by  $f(t) = e^{rt}$  and  $g(t) = e^{st}$ , where  $r \neq s$ . Prove that  $f$  and  $g$  are linearly independent in  $F(R, R)$ .

*Solution. Claim.*  $f, g$  are linearly independent in  $F(R, R)$   
 $a_1 f(t) + a_2 g(t) = 0 \Rightarrow a_1 e^{rt} + a_2 e^{st} = 0 \Rightarrow e^{rt}(a_1 + a_2 e^{t(s-r)}) = 0$   
 $\Rightarrow e^{rt} = 0$  (impossible) or  $(a_1 + a_2 e^{t(s-r)}) = 0 \Rightarrow a_1 = a_2 = 0$   
 $\therefore f, g$  are linearly independent in  $F(R, R)$ . ■

### §1-6 Bases and Dimension.

- 14 Find bases for the following subspaces of  $F^5$ :

$$W_1 = \{ (a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_1 - a_3 - a_4 = 0 \}$$

and

$$W_2 = \{ (a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4, a_1 + a_5 = 0 \}.$$

What are the dimensions of  $W_1$  and  $W_2$ ?

*Solution.* set  $p, q, t, r \in F$ ,  
 $W_1 = \{ (q + t, p, q, t, r) = q(1, 0, 1, 0, 0) + p(0, 1, 0, 0, 0) + t(1, 0, 0, 1, 0) + r(0, 0, 0, 0, 1) \}$   
**Claim.**  $\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$  is linearly independent  
 $c_1(1, 0, 1, 0, 0) + c_2(0, 1, 0, 0, 0) + c_3(1, 0, 0, 1, 0) + c_4(0, 0, 0, 0, 1)$   
 $\Rightarrow c_1 + c_3 = 0, c_2 = 0, c_1 = 0, c_3 = 0, c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$   
 $\therefore \{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$  is linearly independent  
**Claim**  $\text{span}(\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}) = W_1$   
let  $x \in W_1$ ,  $x = q(1, 0, 1, 0, 0) + p(0, 1, 0, 0, 0) + t(1, 0, 0, 1, 0) + r(0, 0, 0, 0, 1)$   
 $x \in \text{span}(\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \})$   
 $\therefore W_1 \subseteq \text{span}(\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \})$   
 $\therefore W_1$  is a subspace of  $V$ , any linearly combination of  $W_1$ 's subset is in  $W_1$   
 $\therefore \text{span}(\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}) \in W_1$   
 $\therefore \text{span}(\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}) = W_1$   
 $\therefore \{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$  is a basis of  $W_1$ , the dimension of  $W_1$  is 4. ■

- 20 Let  $V$  be a vector space having dimension  $n$ , and let  $S$  be a subset of  $V$  that generates  $V$ .

- (a) Prove that there is a subset of  $S$  that is a basis for  $V$ . (Be careful not to assume that  $S$  is finite)
- (b) Prove that  $S$  contains at least  $n$  vectors.

*Solution.* (a) if  $S = \emptyset$  or  $S = \{0\}$   
 $V = \{0\} \therefore$  there is a subset of  $S$  be a basis.  
 else pick  $s_1 \neq 0$  from  $S$   
 pick  $s_{k+1} \notin \text{span}(\{s_1, s_2, \dots, s_k\})$ , by replacement theorem, when a linearly independent set's element number equal  $\dim(V)$ , the set can generate  $V$ .  
 $\therefore$  there is a subset of  $S$  be a basis.

(b) by the definition dimension, the element number of basis is  $n$   
 by replacement theory's,  $\text{span}(S') = V, \#(S') \geq n, S' \subseteq S, \#(S) \geq n$ . ■

- 25 Let  $V, W$ , and  $Z$  be as in Exercise 21 if Section 1.2. If  $V$  and  $W$  are vector spaces over  $F$  of dimensions  $m$  and  $n$ , determine the dimension of  $Z$ .

*Solution.* let  $Z = \{(v, w) \mid v \in V, w \in W\}$ ,  $\dim(V) = m, \dim(W) = n$   
 $Z_1 = \{(v, 0) \mid v \in V\}, Z_2 = \{(0, w) \mid w \in W\}$   
**Claim**  $Z \subseteq Z_1 + Z_2$ , let  $x \in Z, x = (v, w) v \in V, w \in W$   
 $x = (v, 0) + (0, w) \in Z_1 + Z_2$   
 $\therefore Z \subseteq Z_1 + Z_2$   
**Claim**  $Z_1 + Z_2 \subseteq Z$ , let  $x \in Z_1 + Z_2$   
 $x = (v, 0) + (0, w) v \in V, w \in W = (v, w) \in Z$   
 $\therefore Z_1 + Z_2 \subseteq Z$   
 $\therefore Z = Z_1 + Z_2$   
 $\therefore Z = Z_1 + Z_2, Z_1 \cap Z_2 = \{(0, 0)\}$   
 $\therefore Z = Z_1 \oplus Z_2$   
 by Exercise 1.6.29(b), if  $W_1$  and  $W_2$  are finite-dimensional subspace of a vector space  $V$ , and let  $V = W_1 \oplus W_2$ .  $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) = m + n$  ■



- 29 (a) Prove that if  $W_1$  and  $W_2$  are finite-dimensional subspaces of a vector space  $V$ , then the subspace  $W_1 + W_2$  is finite-dimensional, and  $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ .
- (b) Let  $W_1$  and  $W_2$  be finite-dimensional subspaces of a vector space  $V$ , and let  $V = W_1 + W_2$ . Deduce that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if  $\dim(V) = \dim(W_1) + \dim(W_2)$ .

*Solution.* (a) let  $\beta$  is a basis of  $W_1 \cap W_2$   $\dim(W_1) = k + m$ ,

$$\dim(W_2) = k + n, \dim(W_1 \cap W_2) = k, k, m, n \in \mathbb{Z}^{\geq 0}$$

$$\beta \in \{u_1, u_2, \dots, u_k\}, u_1, \dots, u_k \in W_1 \cap W_2$$

$$\because \beta \in W_1, \beta \in W_2$$

by Replacement Theorem, every linearly independent subset of  $V$  can be extended to a basis for  $V$ .

$$\exists \beta_1 \text{ is a basis of } W_1 \quad \beta_1 = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m\} \quad v_1, v_2, \dots, v_m \in W_1$$

$$\exists \beta_2 \text{ is a basis of } W_2$$

$$\beta_2 = \{u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_n\} \quad w_1, w_2, \dots, w_n \in W_2$$

$$\text{let } x \in W_1 + W_2$$

$$\textbf{Claim.} \quad \text{span}(\{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\}) = W_1 + W_2$$

$$\begin{aligned} x &= (a_1u_1 + a_2u_2 + \dots + a_{k+1}v_1 + a_{k+2}v_2 + \dots + a_{k+m}v_m) + (b_1u_1 + b_2u_2 + \dots + \\ &b_ku_k + b_{k+1}w_1 + b_{k+2}w_2 + \dots + b_{k+n}w_n), a_1, a_2, \dots, a_{k+m}, b_1, b_2, \dots, b_{k+n} \in F \\ &= c_1u_1 + c_2u_2 + \dots + c_ku_k + a_{k+1}v_1 + a_{k+2}v_2 + \dots + a_{k+m}v_m + b_{k+1}w_1 + \\ &b_{k+2}w_2 + \dots + b_{k+n}w_n, c_1, c_2, \dots, c_k \in F \end{aligned}$$

$$x \in W_1 + W_2 \therefore W_1 + W_2 \subseteq \text{span}(\{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\})$$

$\because W_1 + W_2$  is a subspace, any linear combination of  $W_1 + W_2$ 's subset are in  $W_1 + W_2$

$$\therefore \text{span}(\{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\}) \in W_1 + W_2$$

$$\therefore \text{span}(\{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\}) = W_1 + W_2$$

$$\because v_1, v_2, \dots, v_m \notin W_2, w_1, w_2, \dots, w_n \notin W_1$$

$$\therefore \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\} \text{ is linearly independent}$$

$$\therefore \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_n\} \text{ is a basis of } W_1 + W_2$$

$$\therefore \dim(W_1 + W_2) = k + m + n$$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$(b) \quad W_1 \cap W_2 = \{0\}$$

by Exercise 1.16.29(a), if  $W_1$  and  $W_2$  are finite-dimensional subspace of a vector space  $V$ ,  $\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(\{0\}) = \dim(W_1) + \dim(W_2)$

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31 Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  having dimensions  $m$  and  $n$ , respectively, where  $m \geq n$ .

- (a) Prove that  $\dim(W_1 \cap W_2) \leq n$ .
- (b) Prove that  $\dim(W_1 + W_2) \leq m + n$ .

*Solution.* (a) let  $\beta_1$  is a basis of  $W_2$

$\beta$  is a basis of  $W_1 \cap W_2$

$$\#(\beta_1) = n$$

by Replacement Theorem,  $V$  be a vector space is generated by a set  $G$ ,  $\#(G) = n$ , a linearly independent set  $L \in V$ ,  $\#(L) = m$

$$\because W_1 \cap W_2 \subseteq W_2$$

$\therefore L = \beta$  is a linearly independent set of  $W_2$ ,  $G = \beta_1$  can generated  $W_2$

**by Replacement Theorem**  $\Rightarrow \#(\beta) \leq \#(\beta_1)$

$$\Rightarrow \dim(W_1 \cap W_2) \leq n$$

- (b) by Exercise 29,  $W_1, W_2$  are finite-dimensional subspaces of a vector space  $V$ , then  $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$   
 $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = m + n - \dim(W_1 \cap W_2)$   
 $\leq m + n$



- 33 (a) Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  such that  $V = W_1 \oplus W_2$ . If  $\beta_1$  and  $\beta_2$  are bases for  $W_1$  and  $W_2$ , respectively, show that  $\beta_1 \cap \beta_2 = \emptyset$  and  $\beta_1 \cup \beta_2$  is a basis for  $V$ .
- (b) Conversely, let  $\beta_1$  and  $\beta_2$  be disjoint bases for subspaces  $W_1$  and  $W_2$ , respectively, of a vector space  $V$ . Prove that if  $\beta_1 \cup \beta_2$  is a basis for  $V$ , then  $V = W_1 \oplus W_2$ .

*Solution.* let  $\beta_1 = \{v_1, v_2, \dots, v_n\}, v_1, v_2, \dots, v_n \in W_1$ ,  
 $\beta_2 = \{u_1, u_2, \dots, u_m\}, u_1, u_2, \dots, u_m \in W_2$   
 $W_1 + W_2 = \{a_1v_1 + a_2v_2 + \dots + a_nv_n + b_1u_1 + b_2u_2 + \dots + b_mu_m \mid$   
 $a_1, \dots, a_n, b_1, \dots, b_m \in F\}$   
**Claim**  $\text{span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$   
let  $x \in \text{span}(\beta_1 \cup \beta_2)$   $x = a_1v_1 + a_2v_2 + \dots + b_1u_1 + b_2u_2 + \dots + b_mu_m$   
 $\therefore W_1, W_2$  is a subspace of  $V$   
 $\therefore \sum_{i=1}^n a_i v_i \in W_1, \sum_{i=1}^m b_i u_i \in W_2$   
 $\therefore x \in W_1 + W_2, \text{span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$   
**Claim.**  $W_1 + W_2 \subseteq \text{span}(\beta_1 \cup \beta_2)$   
let  $x \in W_1 + W_2, x = (a_1v_1 + \dots + a_nv_n) + (b_1u_1 + \dots + b_mu_m)$   
 $\therefore x \in \text{span}(\beta_1 \cup \beta_2)$   
 $\therefore W_1 + W_2 \subseteq \text{span}(\beta_1 \cup \beta_2)$   
 $\therefore \text{span}(\beta_1 \cup \beta_2) = W_1 + W_2$   
**Claim**  $\beta_1 \cup \beta_2$  is linearly dependent  
 $\exists a_i v_i$  or  $b_i u_i$  can be express by  
 $-a_i v_i = (a_1v_1 + \dots + a_nv_n) + (b_1u_1 + \dots + b_mu_m)$   
 $\therefore v_i \notin \text{span}(\beta_1 - \{v_i\}), v_i \notin \text{span}(\beta_2)$   
 $-b_i u_i = (a_1v_1 + \dots + a_nv_n) + (b_1u_1 + \dots + b_mu_m)$   
 $u_i \notin \text{span}(\beta_2 - \{u_i\}), u_i \notin \text{span}(\beta_1)$   
 $\therefore \rightarrow \leftarrow$

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- 34 Prove that if  $W_1$  is any subspace of a finite-dimensional vector space  $V$ , then there exists a subspace  $W_2$  of  $V$  such that  $V = W_1 \oplus W_2$

*Solution.* let  $\beta_1 = \{v_1, v_2, \dots, v_m\}, \dim(V) = n$   
by Corollary of Replacement Theorem, Every linearly independent subset of  $V$  can be extended to a basis for  $V$   
 $\Rightarrow \exists \beta = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_{n-m}\}$  is a basis of  $V$   
**Claim.**  $\text{span}(\{u_1, u_2, \dots, u_{n-m}\}) \oplus W_1 = V$  let  $v' \in W_1, u' \in \text{span}(\{u_1, u_2, \dots, u_{n-m}\})$   
 $\therefore \beta$  is a linearly independent set  $\therefore v' \notin \text{span}(\{u_1, u_2, \dots, u_{n-m}\}), u' \notin W_1$   
 $\Rightarrow W_1 \cap W_2 = \{0\} \therefore \text{span}(\{u_1, u_2, \dots, u_{n-m}\}) \oplus W_1 = V$

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