# 1. Vector Space

## §1-3 Subspace.

10 Prove that $W_1 = \{\, (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \,|\, a_1x_1 + \cdots + a_nx_n = 0 \,\}$ is a subspace of $\mathrm{F}^n$, but $W_2 = \{\, (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \,|\, a_1 + \cdots + a_n = 1 \,\}$ is not.

*Solution.* Let $x, y \in W_1$ , $c \in F$ , $x = (x_1, x_2, \cdots, x_n)$ , $y = (y_1, y_2, \cdots, y_n)$
**Claim :** $W_1$ is a subspace of $\mathrm{F}^n$

(a)

$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$
$\because x_1 + y_1 + \cdots + x_n + y_n = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n$
$= 0 + 0 = 0 \therefore x + y \in W_1$

(b)

$cx = (cx_1 + cx_2 + \cdots + cx_n)c \in F \because cx_1 + cx_2 + \cdots + cx_n$
$= c(x_1 + x_2 + \cdots + x_n) = c * 0 = 0 \therefore cx \in W_1$

(c)

$\because 0 + 0 + \cdots + 0 = 0 \quad \therefore (0, 0, \cdots, 0) \in W_1$

Concluding (a)(b)(c) $\therefore W_1$ is a subspace of F. ■

13 Let $S$ be a nonempty set and $F$ a field. Prove that for any $s_0 \in S$,$\{\, f \in F(S, F) \,|\, f(s_0) = 0 \,\}$ , is a subspace of $F(S, F)$.

*Solution.* **Claim.** $\{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$ is a subspace of $F(S, F)$

(a) let $f_a, f_b \in \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$
$\because (f_a + f_b)(s_0) = f_a(s_0) + f_b(s_0) = 0$
$\therefore (f_a + f_b)(s_0) \in \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$

(b) let $f_a \in \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}, c \in \mathrm{F}$
$\because cf_a(s_0) = c * 0 = 0$
$\therefore cf_a(s_0) \in \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$

(c) $f(s) = 0 \in \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$

$\therefore \{\, f \in F(S, F) \,|\, f(S_0) = 0 \,\}$ is a subspace of $F(S, F)$. ■

14 Let $S$ be a nonempty set and $F$ a field. Let $C(S,F)$ denote the set of all functions $f \in F(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of $S$. Prove that $C(S,F)$ is a subspace of $F(S,F)$

*Solution.* **Claim.** $C(S,F)$ is a subspace of $F(S,F)$

(a) let $f, g \in C(S,F)$
$f(s) \neq 0$ when $s \in \{ s_1, s_2, \cdots, s_n \}$ $g(s) \neq 0$ when $s \in \{ s_1, s_2, \cdots, s_m \}$
$(f+g)(s)$
$= f(s) + g(s)$
$f(s) + f(s) \neq 0$ only if $s \in (\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \})$
$\because \#(\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \}) \leq n + m$ is finite
$\therefore (f+g)(s) \in C(S,F)$

(b) let $c \in$ F
$cf(s) \neq 0$ only if $s \in \{ s_1, s_2, \cdots, s_n \}$
$\because \#(\{ s_1, s_2, \cdots, s_n \}) = n$ is finite
$\therefore cf(s) \in C(S,F)$

(c) $f' \in F(S,F)$, $f'(s) = 0$ $s \in S$, $f' \in C(S,F)$

■

20 Prove that if W is a subspace of a vector space V and $w_1, w_2, \cdots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in$W for any scalars $a_1, a_2, \cdots, a_n$.

*Solution.*
$\because$ W is a subspace of V $a_1w_1, a_2w_2, \cdots, a_nw_n \in$ W by mathematical induction.
by mathematical induction
(1) $\sum_{i=1}^{1} a_iw_i \in$ W
(2) assume $\sum_{i=1}^{k} a_iw_i \in$ W
(3) $\sum_{i=1}^{k+1} a_iw_i = \sum_{i=1}^{k} a_iw_i + a_{k+1}wk + 1$
$\because \sum_{i=1}^{k} a_iw_i, a_{k+1}w_{k+1} \in$ W
$\therefore \sum_{i=1}^{k+1} a_iw_i \in$ W ■

23 Let $W_1$ and $W_2$ be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both $W_1$ and $W_2$.

(b) Prove that any subspace of V that contains both $W_1$ and $W_2$ must also contain $W_1 + W_2$.

*Solution.*

(a)**Claim** $W_1 + W_2$ is a subspace of V

(1) let $x_1, x_2 \in W_1$ , $y_1, y_2 \in W_2$
$(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + x_2) + (y_1 + y_2)$
$\because W_1, W_2$ is a subspace of V
$\therefore (x_1 + x_2) \in W_1, (y_1 + y_2) \in W_2 \implies (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$

(2) let $x_1 \in W_1, y_1 \in W_2$ , $c \in$ F
$c(x_1 + y_1) = cx_1 + cy_1$
$\because W_1, W_2$ is a subspace of V, $cx_1 \in W_1, cy_1 \in W_2$
$\therefore cx_1 + cy_1 \in W_1 + W_2$

(3) $\because W_1, W_2$ is a subspace of V,$\therefore 0 \in W_1, 0 \in W_2$,
$0 + 0 = 0 \in W_1 + W_2 \therefore W_1 + W_2$ is a subspace of V

$W_1 = \{x + 0 \,|\, x \in W_1\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$W_2 = \{0 + y \,|\, y \in W_2\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$\therefore W_1 + W_2$ contains both $W_1$ and $W_2$
(b)let $W_3$ is a subspace of $V$, $W_3 \subseteq W_1$,$W_3 \subseteq W_2$

let $x \in W_1, y \in W_2$,$\because W_3$ is a subspace.$\therefore x + y \in W_3 \implies W_1 + W_2 \in W_3$ ■

30 Let $W_1$ and $W_2$ be subspaces of a vector space V Prove that V is the direct sum of $W_1$ and $W_2$ if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

*Solution.*

($\Rightarrow$) $W_1 \cap W_2 = \{0\}, W_1 + W_2 = \text{V}$
**Claim.** each vector in V can not be only one written as $x + y$
where $x \in W_1, y \in W_2$
let $u \in \text{V}, u = x_1 + y_1 = x_2 + y_2$,
$x_1, x_2 \in W_1, y_1, y_2 \in W_2, x_1 \neq x_2, y_1 \neq y_2$
$x_1 + y_1 = x_2 + y_2 \implies x_1 - x_2 = y_2 - y_1$
$\because$ $W_1$ is a subspace, $(x_1 - x_2) \in W_1$, $W_2$ is a subspace, $(y_2 - y_1) \in W_2$
$W_1 \cap W_2 = \{0\} \therefore (x_1 - x_2) = (y_2 - y_1) = 0 \implies x_1 = x_2, y_1 = y_2 \rightarrow\leftarrow$
$\therefore$ each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$and $x_2 \in W_2$

($\Leftarrow$) $\text{V} = \{x + y \mid x \in W_1, y \in W_2\} = W$
**Claim.** $W_1 \cap W_2$ not only 0
$\exists\ u \in W_1 \cap W_2, u = 0 + u = u + 0 \rightarrow\leftarrow$
$\therefore W_1 \oplus W_2 = V$

■

## §1-4 Linear Combination.

13 Show that if $S_1$ and $S_2$ are subsets of a vector space V such that $S_1 \subseteq S_2$,then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = \text{V}$, deduce that $\text{span}(S_2) = \text{V}$

*Solution.* **Claim** $\text{span}(S_1) \subseteq \text{span}(S_2)$
let $S_1 = \{v_1, v_2, \cdots, v_n\}, S_2 = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_m\}, x \in \text{span}(S_1)$
$x = a_1v_1 + \cdots + a_nv_n, a_1, a_2, \cdots, a_n \in \text{F}$
$= a_1v_1 + \cdots + a_nv_n + 0u_1 + 0u_2 + \cdots + 0u * n \in \text{Span}(S_2)$
$\therefore \text{span}(S_1) \subseteq \text{span}(S_2)$

■

14 Show that if $S_1$ and $S_2$ are arbitrary subsets of a vector space V, then span($S_1 \cup S_2$ )= span($S_1$) + span($S_2$).

*Solution.* Let $S_1 \cap S_2 = \{ v_1, v_2, \cdots, v_n \}$,
$S_1 = \{ u_1, u_2, \cdots, u_m, v_1, \cdots, v_n \}, S_2 = \{ r_1, \cdots, r_k, v_1, \cdots, v_n \}$
**Claim.** span($S_1$)+span($S_2$) $\subseteq$ span($S_1 \cup S_2$)
let $x \in \text{span}(S_1) + \text{span}(S_2))$
$x = (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n)+$
$(b_1r_1 + \cdots + b_kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$= (c_1u_1 + \cdots + c_mu_m) + (c_{m+1}r_1 + \cdots + c_{m+k}) + (c_{m+k+1}v_1 + \cdots + c_{m+k+n})$
$\Rightarrow x \in \text{span}(S_1 \cup S_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$
**Claim.** $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$
let $y \in \text{span}(S_1 \cup S_2)$
$y = (c_1u_1 + \cdots + c_mu_m) + (c_{m+1}r_1 + \cdots + c_{m+k}) + (c_{m+k+1}v_1 + \cdots + c_{m+k+n})$
$= (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) + (b_1r_1 + \cdots + b_kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$\therefore y \in \text{span}(S_1) + \text{span}(s_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2)$

■

## §1-5 Linear Independent.

13 Let V be a vector space over a field of characteristic not equal to two.

Let $u$ and $v$ be distinct vectors in V. Prove that $\{ u, v \}$ is linearly independent if and only if $\{ u+v, u-v \}$ is linearly independent.

*Solution.*

($\Rightarrow$) **Claim.** $\{ u+v, u-v \}$ is linearly independent
$a_1(u+v) + a_2(u-v) = 0, a_1, a_2 \in \mathrm{F}$
$\Rightarrow (a_1+a_2)u + (a_1-a_2)v = 0$
$\because \{ u, v \}$ is linearly independent
$\therefore \begin{cases} a_1+a_2=0 \\ a_1-a_2=0 \end{cases} \quad \Rightarrow a_1 = a_2 = 0$
$\therefore \{ u+v, u-v \}$ is linearly independent

($\Leftarrow$) **Claim.** $\{ u, v \}$ is linearly independent
$\Rightarrow b_1u + b_2v = 0$
$\Rightarrow \frac{b_1+b_2}{2}(u+v) + \frac{b_1-b_2}{2}(u-v) = 0$
$\because \{ u+v, u-v \}$ is linearly independent
$\therefore \begin{cases} \frac{b_1+b_2}{2}=0 \\ \frac{b_1-b_2}{2}=0 \end{cases} \quad \Rightarrow b_1 = b_2 = 0$
$\therefore \{ u, v \}$ is linearly independent

■

16 Prove that a set $S$ of vectors is linearly independent if and only if each finite subset of $S$ is linearly independent.

*Solution.* let $S = \{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\exists$ subset $S_i = \{ s'_1, s'_2, \cdots, s'_r \}, r \le n, b_1s'_1 + b_2s'_2 + \cdots + b_ns'_r = 0$
let $S - S_i = \{ S'_{r+1}, S'_{r+2}, \cdots, S'_n \}$
$b_1s'_1 + b_2s'_2 + \cdots + b_ns'_n = 0$ not only $b_1 = b_2 = \cdots = b_n = 0 \rightarrow\leftarrow$

($\Leftarrow$) by definition of linear independent, each finite subset of $S$ is linearly independent, $S$ is linear independent.

■

18 Prove that a set $S$ of vectors is linearly independent if and only if each finite subset of $S$ is linearly independent.

*Solution.* let $S = \{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\exists$ subset $S_i = \{ s'_1, s'_2, \cdots, s'_r \}, r \leq n, b_1 s'_1 + b_2 s'_2 + \cdots + b_n s'_r = 0$
let $S - S_i = \left\{ S'_{r+1}, S'_{r+2}, \cdots, S'_n \right\}$
$b_1 s'_1 + b_2 s'_2 + \cdots + b_n s'_n = 0$ not only $b_1 = b_2 = \cdots = b_n = 0 \rightarrow\leftarrow$

($\Leftarrow$) by definition of linear independent, each finite subset of $S$ is linearly independent, $S$ is linear independent.

$\blacksquare$

20 Let $f, g \in F(R, R)$ be the functions defined by $f(t) = e^{et}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that $f$ and $g$ are linearly independent in $F(R, R)$.

*Solution.* **Claim.** $f, g$ are linearly independent in $F(R, R)$
$a_1 f(t) + a_2 g(t) = 0 \Rightarrow a_1 e^{rt} + a_2 e^{st} = 0 \Rightarrow e^{rt}(a_1 + a_2 e^{t(s-r)}) = 0$
$\Rightarrow e^{rt} = 0$(impossiable) or $(a_1 + a_2 e^{t(s-r)}) = 0 \Rightarrow a_1 = a_2 = 0$
$\therefore f, g$ are linearly independent in $F(R, R)$. $\blacksquare$

## §1-6 Bases and Dimension.

14 Find bases for the following subspaces of $\mathrm{F}^5$:

$$W_1 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_1 - a_3 - a_4 = 0 \right\}$$

and

$$W_2 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_2 = a_3 = a_4 \ , \ a_1 + a_5 = 0 \right\}.$$

What are the dimensions of $W_1$ and $W_2$?

*Solution.* set $p, q, t, r \in \mathrm{F}$,
$W_1 = \{(q + t, p, q, t, r) = q(1, 0, 1, 0, 0) + p(0, 1, 0, 0, 0)$
$+t(1, 0, 0, 1, 0) + r(0, 0, 0, 0, 1)\}$
**Claim.** $\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is linearly independent
$c_1 \cdot (1, 0, 1, 0, 0) + c_2(0, 1, 0, 0, 0) + c_3(1, 0, 0, 1, 0) + c_4(0, 0, 0, 0, 1)$
$\Rightarrow c_1 + c_3 = 0, c_2 = 0, c_1 = 0, c_3 = 0, c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$
$\therefore \{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is linearly independent
$\therefore \{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is a basis of $W_1$,the dimension of $W_1$ is 4. $\blacksquare$

20 Let V be a vector space having dimension $n$, and let $S$ be a subset of V that generates V.

(a) Prove that there is a subset of $S$ that is a basis for V.(Be careful not to assume that $S$ is finite)

(b) Prove that $S$ contains at least $n$ vectors.

*Solution.* (a) if $S = \varnothing$ or $S = \{0\}$
$V = \{0\}$ $\therefore$ there is a subset of $S$ be a basis.
else pick $s_1 \neq$ from $S$
pick $s_{k+1} \notin \text{span}(\{s_1, s_2, \cdots, s_k\})$, by replacement theorem, when a linearly independent set's element number equal dim(V), the set can generate V.
$\therefore$ there is a subset of $S$ be a basis.

(b) by the definition dimension, the element number of basis is $n$
by replacement theory's, span($S'$)=V,$\#(S') \geq n$,$S' \subseteq S$,$\#(S) \geq n$.

■

25 Let V,W, and Z be as in Exercise 21 if Section 1.2. If V and W are vector spaces over $F$ of dimensions $m$ and $n$, determine the dimension of Z.

*Solution.* let $Z = \{(v,w) \mid v \in V, w \in W\}, dim(V) = m, dim(W) = n$
$Z_1 = \{(v,0) \mid v \in V\}$,$Z_2 = \{(0,w) \mid w \in W\}$
$\because Z = Z_1 + Z_2$,$Z_1 \cap Z_2 = \{(0,0)\}$
$\therefore Z = Z_1 \bigoplus Z_2$
by Exercise 1.6.29(b), if $W_1$ and $W_2$ are finite-dimensional subspace of a vector space V, and let V $= W_1 \bigoplus W_2$. $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) = m + n$

■

29 (a) Prove that if $W_1$ and $W_2$ are finite-dimensional subspaces of a vector space $V$, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$.

(b) Let $W_1$ and $W_2$ be finite-dimensional subspaces of a vector space V, and let V = $W_1 + W_2$. Deduce that V is the direct sum of $W_1$ and $W_2$ if and only if dim(V) = $\dim(W_1)$ + $\dim(W_2)$.

*Solution.* (a) let $\beta$ is a basis of $W_1 \cap W_2$ $\dim(W_1) = k + m$,

$\dim(W_2) = k+n$, $\dim(W_1 \cap W_2) = k$, $k, m, n \in \mathbb{Z}^{\geq 0}$ $\beta \in = \{ u_1, u_2, \cdots, u_k \}$, $u_1, \cdots, u_k \in W_1 \cap W_2 \because \beta \in W_1, \beta \in W_2$

by Replacement Theorem, every linearly independent subset of V can be extended to a basis for V.

$\exists$ $\beta_1$ is a basis of $W_1$ $\beta_1 = \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m \}$ $v_1, v_2, \cdots, v_m \in W_1$ $\exists \beta_2$ is a basis of $W_2$ $\beta_2 = \{ u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_n \}$ $w_1, w_2, \cdots, w_m \in W_2$
let $x \in W_1 + W_2$
$x = (a_1 u_1 + a_2 u_2 + \cdots + a_{k+1} v_1 + a_{k+2} v_2 + \cdots + a_{k+m} v_m) + (b_1 u_1 + b_2 u_2 + \cdots + b_k u_k + b_{k+1} w_1 + b_{k+2} w_2 + \cdots + b_{k+n} w_n), a_1, a_2, \cdots, a_{k+m}, b_1, b_2, \cdots, b_{k+n} \in$ F
$= c_1 u_1 + c_2 u_2 + \cdots + c_k u_k + a_{k+1} v_1 + a_{k+2} v_2 + \cdots + a_{k+m} v_m + b_{k+1} w_1 + b_{k+2} + \cdots + b_{k+n} w_n, c_1, c_2, \cdots, c_k \in$ F
$\because V_1, V_2, \cdots, v_m \notin W_2, w_1, w_2, \cdots w_n \notin W_1$
$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is linearly independent
$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is a basis of $W_1 + W_2$
$\therefore \dim(W_1 + W_2) = k + m + n$
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2)$

(b) $W_1 \cap W_2 = \{ 0 \}$
by Exercise 1.16.29(a), if $W_1$ and $W_2$ are finite-dimensional subspace of a vector space V, dim(V) = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$ = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(\{ 0 \})$ = $\dim(W_1)$ + $\dim(W_2)$

■

31 Let $W_1$ and $W_2$ be subspaces of a vector space V having dimensions $m$ and $n$, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

*Solution.* (a) let $\beta_1$ is a basis of $W_2$
$\beta$ is a basis of $W_1 \cap W_2$
$\#(\beta_1) = n$

by Replacement Theorem, V be a vector space is generated by a set $G$, $\#(G) = n$
a linearly independent set $L \in$ V,$\#(L) = m$
$\because W_1 \cap W_2 \subseteq W_2$
$\therefore \beta$ is a linearly independent set of $W_2$, $\beta_1$ can generated $W_2$
$\Rightarrow \#(\beta) \leq \#(\beta_1)$
$\Rightarrow$dim$(W_1 \cap W_2) \leq n$

(b) by Exercise 29,$W_1, W_2$ are finite-dimensional subspaces of a vector space V,then dim$(W_1 + W_2)$=dim$(W_1)$ + dim$(W_2)$ - dim$(W_1 \cap W_2)$
dim$(W_1 + W_2)$ = dim$(W_1)$+dim$(S_2)$-dim$(W_1 \cap W_2)$ = m+n - dim$(W_1 \cap W_2)$ $\leq m + n$

■

33 (a) Let $W_1$ and $W_2$ be subspaces of a vector space V such that V $= W_1 \bigoplus W_2$. If $\beta_1$ and $\beta_2$ are bases for $W_1$ and $W_2$, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let $\beta_1$ and $\beta_2$ be disjoint bases for subspaces $W_1$ and $W_2$, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then V $= W_1 \bigoplus W_2$.

*Solution.* let $\beta_1 = \{ v_1, v_2, \cdots, v_n \}, v_1, v_2, \cdots, v_n \in W_1$,
$\beta_2 = \{ u_1, u_2, \cdots, u_m \}, u_1, u_2, \cdots, u_m \in W_2$
$W_1 + W_2 = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_1 + b_2u_2 + \cdots + b_mu_m \mid$
$a_1, \cdots, a_n, b_1, \cdots, b_m \in \mathrm{F}\} \implies \mathrm{span}(\beta_1 \cup \beta_2) = W_1 + W_2$
**Claim** $\beta_1 \cup \beta_2$ is linearly dependent
$\exists\ a_iv_i$ or $b_iu_i$ can be express by
$-a_iv_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$\because v_i \notin \mathrm{span}(\beta_1 - \{ v_i \}), v_i \notin \mathrm{span}(\beta_2)$
$-b_iu_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$u_i \notin \mathrm{span}(\beta_2 - \{ u_i \}), u_i \notin \mathrm{span}(\beta_1)$
$\therefore \rightarrow\leftarrow$

■

34 Prove that if $W_1$ is any subspace of a finite-dimensional vector space V, then there exists a subspace $W_2$ of V such that V $= W_1 \bigoplus W_2$

*Solution.* let $\beta_1 = \{ v_1, v_2, \cdots, v_m \}$,dim(V) $= n$
by Corollary of Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V
$\Rightarrow \exists \beta = \{ v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_{n-m} \}$ is a basis of V
**Claim.** $\text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}) \bigoplus W_1 = V$ let $v' \in W_1, u' \in \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \})$
$\because \beta$ is a linearly independent set $\therefore v' \notin \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}), u' \notin W_1$
$\Rightarrow W_1 \cap W_2 = \{ 0 \} \therefore \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}) \bigoplus W_1 = V$ ■