§ Linear Transformations and Matrices

2-1 Linear Transformations, Null spaces, and Ranges.

13 Let V and W be vector spaces, let T: V $\to$ W be linear, and let $\{ w_1, \cdots, w_k \}$ be a linearly independent subset of R(T). Prove that if $S = \{ v_1, \cdots, v_k \}$ is chosen so that $\mathrm{T}(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent.

Solution. **Claim.** S is linearly independent

$$\sum_{i=1}^{k} a_i v_i = 0 \implies \mathrm{T}(\sum_{i=1}^{k} a_i v_i) = 0 \implies \sum_{i=1}^{k} a_i \mathrm{T}(v_i) = 0 \implies \sum_{i=1}^{k} a_i w_i = 0$$

$\because \{ w_1, \cdots, w_k \}$ is linearly independent $\therefore \sum_{i=1}^{k} a_i v_i = 0$ only $a_i = 0, i = 1, \cdots, k$

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14 Let V and W be vector spaces and T:V $\to$ W be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.

(c) Suppose $\beta = \{ v_1, v_2, \cdots, v_n \}$ is a basis for V and T is one-to-one and onto. Prove that T(β) = {T(v_1),T(v_2),$\cdots$,T(v_n)} is a basis for W.

Solution. (a) ($\Rightarrow$) let $\{ s_1, \cdots, s_n \}$ be a linearly independent subset of S

Claim. $\{ T(s_1, \cdots, T(s_n)) \}$ is a linearly independent subset of W

$$\sum_{i=1}^{n} a_i T(s_i) = 0 \implies \sum_{i=1}^{n} T(a_i s_i) = 0 \implies T(\sum_{i=1}^{n} a_i s_i) = 0$$

$\because$ T is one-to-one $\therefore \sum_{i=1}^{n} a_i s_i = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is a linearly independent subset of W

($\Leftarrow$) let $x, y \in$V, β is a basis of V, $\beta = \{ v_1, v_2, \cdots, v_n \}$

$x = a_1 v_1 + \cdots + a_n v_n, y = b_1 v_1 + \cdots + b_n v_n$

T(x) = T(y) $\implies$ T(x)-T(y) = 0 $\implies$ T($x - y$) = 0 $\implies$ T(($a_1 - b_1$)$v_1 + \cdots + (a_n - b_n)v_n$) = 0

$\implies (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0 \because \{ T(v_1), \cdots, T(v_n) \}$ is linearly independent

$\therefore (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0$ only $a_1 - b_1 = \cdots = a_n - b_n = 0$

$\implies a_1 = b_1, \cdots, a_n = b_n \implies T(x) = T(y)$ only $x = y$

T is one-to-one

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(b) let S = $\{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\{ T(s_1, \cdots, T(s_n)) \}$ is linearly independent

$a_1T(s_1) + \cdots + a_nT(s_n) = 0 \implies T(a_1s_1) + \cdots + T(a_ns_n) = 0 \implies T(a_1s_1 + \cdots + a_ns_n) = 0$

$\because$ S is linearly independent, T is one-to-one $\therefore T(a_1s_1 + \cdots + a_ns_n) = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is linearly independent

($\Leftarrow$) **Claim.** $\{ s_1, \cdots, s_n \}$ is linearly independent

$a_1s_1 + \cdots + a_ns_n = 0 \implies T(a_1s_1 + \cdots + a_ns_n = 0) \implies a_1T(s_1) + \cdots + a_nT(s_n) = 0$

$\because \{ T(s_1), \cdots, T(s_n) \}$ is linearly independent, $\therefore a_1 = \cdots = a_n = 0$

$\{ s_1, \cdots, s_n \}$ is linearly independent

(c) **Claim.** T(β) can generate W

$\because$ T is onto, by Thm 2.2, T : V $\to$ W be linear, β is a basis of V, then R(T) = span(T(β)) = W

T(β) can generate W

Claim T(β) is a linearly independent set of W

by (b), T is one-to-one,S is a linearly independent subset of V, T(S) is linearly independent

T(S) is a basis of W

16 Let T:P($\mathbb{R}$) $\to$ P($\mathbb{R}$) be defined by T($f(x)$) = $f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

Solution. **test**
test ■

17 Let V and W be finite-dimensional vector spaces and T:V $\to$ W be linear.

(a) Prove that if dim(V) $<$ dim(W), then T cannot be onto.

(b) Prove that if dim(V) $>$ dim(W), then T cannot be one-to-one.

Solution. ■

21 Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions T,U:V $\to$ V by

$$T(a_1, a_2, \cdots) = (a_2, a_3, \cdots) \text{ and } U(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Solution. ■

26 Using the notation in the definition above, assume that T: V → V is the projection on W_1 along W_2.

(a) Prove that T is linear and $W_1 = \{ x \in V \mid T(x) = x \}$.

(b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

(c) Describe T if $W_1 = V$.

(d) Describe T if W_1 is the zero subspace.

Solution. ■

35 Let V be a finite-dimensional vector space and T: V → V be linear.

(a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \bigoplus N(T)$.

(b) Suppose that $R(T) \cap N(T) = \{ 0 \}$. Prove that $V = R(T) \bigoplus N(T)$.

Solution. ■