1. Vector Space

§1-3 Subspace.

10 Prove that $W_1 = \{ (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \mid a_1x_1 + \cdots + a_nx_n = 0 \}$ is a subspace of F^n, but $W_2 = \{ (a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \mid a_1 + \cdots + a_n = 1 \}$ is not.

Solution. Let $x, y \in W_1$, $c \in F$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$
Claim : W_1 is a subspace of F^n

(a)

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$
$$\because x_1 + y_1 + \cdots + x_n + y_n = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n$$
$$= 0 + 0 = 0 \therefore x + y \in W_1$$

(b)

$$cx = (cx_1 + cx_2 + \cdots + cx_n)c \in F \because cx_1 + cx_2 + \cdots + cx_n$$
$$= c(x_1 + x_2 + \cdots + x_n) = c * 0 = 0 \therefore cx \in W_1$$

(c)

$$\because 0 + 0 + \cdots + 0 = 0 \quad \therefore (0, 0, \cdots, 0) \in W_1$$

Concluding (a)(b)(c) $\therefore W_1$ is a subspace of F. ■

13 Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$,$\{ f \in F(S,F) \mid f(s_0) = 0 \}$, is a subspace of $F(S,F)$.

Solution. **Claim.** $\{ f \in F(S,F) \mid f(S_0) = 0 \}$ is a subspace of $F(S,F)$

(a) let $f_a, f_b \in \{ f \in F(S,F) \mid f(S_0) = 0 \}$
$\because (f_a + f_b)(s_0) = f_a(s_0) + f_b(s_0) = 0$
$\therefore (f_a + f_b)(s_0) \in \{ f \in F(S,F) \mid f(S_0) = 0 \}$

(b) let $f_a \in \{ f \in F(S,F) \mid f(S_0) = 0 \}, c \in \mathrm{F}$
$\because cf_a(s_0) = c * 0 = 0$
$\therefore cf_a(s_0) \in \{ f \in F(S,F) \mid f(S_0) = 0 \}$

(c) $f(s) = 0 \in \{ f \in F(S,F) \mid f(S_0) = 0 \}$

$\therefore \{ f \in F(S,F) \mid f(S_0) = 0 \}$ is a subspace of $F(S,F)$. ■

14 Let S be a nonempty set and F a field. Let $C(S,F)$ denote the set of all functions $f \in F(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $C(S,F)$ is a subspace of $F(S,F)$

Solution. **Claim.** $C(S,F)$ is a subspace of $F(S,F)$

(a) let $f, g \in C(S,F)$
$f(s) \neq 0$ when $s \in \{ s_1, s_2, \cdots, s_n \}$ $g(s) \neq 0$ when $s \in \{ s_1, s_2, \cdots, s_m \}$
$(f+g)(s)$
$= f(s) + g(s)$
$f(s) + f(s) \neq 0$ only if $s \in (\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \})$
$\because \#(\{ s_1, s_2, \cdots, s_n \} \cup \{ s'_1, s'_2, \cdots, s'_n \}) \leq n + m$ is finite
$\therefore (f+g)(s) \in C(S,F)$

(b) let $c \in$ F
$cf(s) \neq 0$ only if $s \in \{ s_1, s_2, \cdots, s_n \}$
$\because \#(\{ s_1, s_2, \cdots, s_n \}) = n$ is finite
$\therefore cf(s) \in C(S,F)$

(c) $f' \in F(S,F)$, $f'(s) = 0$ $s \in S$, $f' \in C(S,F)$

■

20 Prove that if W is a subspace of a vector space V and $w_1, w_2, \cdots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in$W for any scalars $a_1, a_2, \cdots, a_n$.

Solution.
$\because$ W is a subspace of V $a_1w_1, a_2w_2, \cdots, a_nw_n \in$ W by mathematical induction.
by mathematical induction
(1) $\sum_{i=1}^{1} a_iw_i \in$ W
(2) assume $\sum_{i=1}^{k} a_iw_i \in$ W
(3) $\sum_{i=1}^{k+1} a_iw_i = \sum_{i=1}^{k} a_iw_i + a_{k+1}wk + 1$
$\because \sum_{i=1}^{k} a_iw_i, a_{k+1}w_{k+1} \in$ W
$\therefore \sum_{i=1}^{k+1} a_iw_i \in$ W ■

23 Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Solution.
(a)**Claim** $W_1 + W_2$ is a subspace of V

(1) let $x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$
$(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + x_2) + (y_1 + y_2)$
$\because W_1, W_2$ is a subspace of V
$\therefore (x_1 + x_2) \in W_1, (y_1 + y_2) \in W_2 \implies (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$

(2) let $x_1 \in W_1, y_1 \in W_2$, $c \in$ F
$c(x_1 + y_1) = cx_1 + cy_1$
$\because W_1, W_2$ is a subspace of V, $cx_1 \in W_1, cy_1 \in W_2$
$\therefore cx_1 + cy_1 \in W_1 + W_2$

(3) $\because W_1, W_2$ is a subspace of V,$\therefore 0 \in W_1, 0 \in W_2$,
$0 + 0 = 0 \in W_1 + W_2 \therefore W_1 + W_2$ is a subspace of V

$W_1 = \{x + 0 \,|\, x \in W_1\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$W_2 = \{0 + y \,|\, y \in W_2\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$\therefore W_1 + W_2$ contains both W_1 and W_2
(b)let W_3 is a subspace of V, $W_3 \subseteq W_1$,$W_3 \subseteq W_2$

let $x \in W_1, y \in W_2$,$\because W_3$ is a subspace$\therefore x + y \in W_3 \implies W_1 + W_2 \in W_3$ ■

30 Let W_1 and W_2 be subspaces of a vector space V Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Solution.

($\Rightarrow$) $W_1 \cap W_2 = \{0\}, W_1 + W_2 = \mathrm{V}$
Claim. each vector in V can not be only one written as $x + y$
where $x \in W_1, y \in W_2$
let $u \in \mathrm{V}, u = x_1 + y_1 = x_2 + y_2$,
$x_1, x_2 \in W_1, y_1, y_2 \in W_2, x_1 \neq x_2, y_1 \neq y_2$
$x_1 + y_1 = x_2 + y_2 \implies x_1 - x_2 = y_2 - y_1$
$\because W_1$ is a subspace, $(x_1 - x_2) \in W_1$, W_2 is a subspace, $(y_2 - y_1) \in W_2$
$W_1 \cap W_2 = \{0\} \therefore (x_1 - x_2) = (y_2 - y_1) = 0 \implies x_1 = x_2, y_1 = y_2 \rightarrow\leftarrow$
$\therefore$ each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$and $x_2 \in W_2$

($\Leftarrow$) $\mathrm{V} = \{x + y \,|\, x \in W_1, y \in W_2\} = W$
Claim. $W_1 \cap W_2$ not only 0
$\exists\, u \in W_1 \cap W_2, u = 0 + u = u + 0 \rightarrow\leftarrow$
$\therefore W_1 \oplus W_2 = V$

∎

§1-4 Linear Combination.

13 Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$,then span(S_1) $\subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span(S_1) = V, deduce that span(S_2) = V

Solution. **Claim** span(S_1) $\subseteq$ span(S_2)
let $S_1 = \{v_1, v_2, \cdots, v_n\}, S_2 = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_m\}, x \in \mathrm{span}(S_1)$
$x = a_1v_1 + \cdots + a_nv_n, a_1, a_2, \cdots, a_n \in \mathrm{F}$
$= a_1v_1 + \cdots + a_nv_n + 0u_1 + 0u_2 + \cdots + 0u * n \in \mathrm{Span}(S_2)$
$\therefore \mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$

∎

14 Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then span($S_1 \cup S_2$)= span(S_1) + span(S_2).

Solution. Let $S_1 \cap S_2 = \{ v_1, v_2, \cdots, v_n \}$,
$S_1 = \{ u_1, u_2, \cdots, u_m, v_1, \cdots, v_n \}, S_2 = \{ r_1, \cdots, r_k, v_1, \cdots, v_n \}$
Claim. span(S_1)+span$(S_2) \subseteq$ span$(S_1 \cup S_2)$
let $x \in \text{span}(S_1) + \text{span}(S_2))$
$x = (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n)+$
$(b_1r_1 + \cdots + kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$= (c_1u_1 + \cdots + c_mu_m) + (c_{m+1}r_1 + \cdots + c_{m+k}) + (c_{m+k+1}v_1 + \cdots + c_{m+k+n})$
$\Rightarrow x \in \text{span}(S_1 \cup S_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$
Claim. span$(S_1 \cup S_2) \subseteq$ span(S_1) + span(S_2)
let $y \in \text{span}(S_1 \cup S_2)$
$y = (c_1u_1 + \cdots + c_mu_m) + (c_{m+1}r_1 + \cdots + c_{m+k}) + (c_{m+k+1}v_1 + \cdots + c_{m+k+n})$
$= (a_1u_1 + \cdots + a_mu_m + a_{m+1}v_1 + \cdots + a_{m+n}v_n) + (b_1r_1 + \cdots + kr_k + b_{k+1}v_1 + \cdots + b_{k+n}v_n)$
$\therefore y \in \text{span}(S_1) + \text{span}(s_2)$
$\therefore \text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2)$

■

§1-5 Linear Independent.

13 Let V be a vector space over a field of characteristic not equal to two.

Let u and v be distinct vectors in V. Prove that $\{ u, v \}$ is linearly independent if and only if $\{ u + v, u - v \}$ is linearly independent.

Solution.

($\Rightarrow$) **Claim.** $\{ u+v, u-v \}$ is linearly independent
$a_1(u+v)+a_2(u-v)=0, a_1, a_2 \in \mathrm{F}$
$\Rightarrow (a_1+a_2)u+(a_1-a_2)v=0$
$\because \{ u, v \}$ is linearly independent
$\therefore \begin{cases} a_1+a_2=0 \\ a_1-a_2=0 \end{cases} \quad \Rightarrow a_1=a_2=0$
$\therefore \{ u+v, u-v \}$ is linearly independent

($\Leftarrow$) **Claim.** $\{ u, v \}$ is linearly independent
$\Rightarrow b_1u+b_2v=0$
$\Rightarrow \frac{b_1+b_2}{2}(u+v)+\frac{b_1-b_2}{2}(u-v)=0$
$\because \{ u+v, u-v \}$ is linearly independent
$\therefore \begin{cases} \frac{b_1+b_2}{2}=0 \\ \frac{b_1-b_2}{2}=0 \end{cases} \quad \Rightarrow b_1=b_2=0$
$\therefore \{ u, v \}$ is linearly independent

∎

16 Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution. let $S=\{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\exists$ subset $S_i=\{ s'_1, s'_2, \cdots, s'_r \}, r \leq n, b_1s'_1+b_2s'_2+\cdots+b_ns'_r=0$
let $S-S_i=\left\{ S'_{r+1}, S'_{r+2}, \cdots, S'_n \right\}$
$b_1s'_1+b_2s'_2+\cdots+b_ns'_n=0$ not only $b_1=b_2=\cdots=b_n=0 \rightarrow\leftarrow$

($\Leftarrow$) by definition of linear independent, each finite subset of S is linearly independent, S is linear independent.

∎

18 Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution. let $S = \{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\exists$ subset $S_i = \{ s'_1, s'_2, \cdots, s'_r \}, r \le n, b_1 s'_1 + b_2 s'_2 + \cdots + b_n s'_r = 0$
let $S - S_i = \{ S'_{r+1}, S'_{r+2}, \cdots, S'_n \}$
$b_1 s'_1 + b_2 s'_2 + \cdots + b_n s'_n = 0$ not only $b_1 = b_2 = \cdots = b_n = 0 \rightarrow\leftarrow$

($\Leftarrow$) by definition of linear independent, each finite subset of S is linearly independent, S is linear independent.

■

20 Let $f, g \in F(R, R)$ be the functions defined by $f(t) = e^{et}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $F(R, R)$.

Solution. **Claim.** f, g are linearly independent in $F(R, R)$
$a_1 f(t) + a_2 g(t) = 0 \Rightarrow a_1 e^{rt} + a_2 e^{st} = 0 \Rightarrow e^{rt}(a_1 + a_2 e^{t(s-r)}) = 0$
$\Rightarrow e^{rt} = 0$(impossiable) or $(a_1 + a_2 e^{t(s-r)}) = 0 \Rightarrow a_1 = a_2 = 0$
$\therefore f, g$ are linearly independent in $F(R, R)$. ■

§1-6 Bases and Dimension.

14 Find bases for the following subspaces of F^5:

$$W_1 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_1 - a_3 - a_4 = 0 \right\}$$

and

$$W_2 = \left\{ (a_1, a_2, a_3, a_4, a_5) \in \mathrm{F}^5 \mid a_2 = a_3 = a_4 \ , \ a_1 + a_5 = 0 \right\}.$$

What are the dimensions of W_1 and W_2?

Solution. set $p, q, t, r \in \mathrm{F}$,
$W_1 = \{ (q + t, p, q, t, r) = q(1, 0, 1, 0, 0) + p(0, 1, 0, 0, 0) + t(1, 0, 0, 1, 0) + r(0, 0, 0, 0, 1) \}$
Claim. $\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is linearly independent
$c_1 \cdot (1, 0, 1, 0, 0) + c_2(0, 1, 0, 0, 0) + c_3(1, 0, 0, 1, 0) + c_4(0, 0, 0, 0, 1)$
$\Rightarrow c_1 + c_3 = 0, c_2 = 0, c_1 = 0, c_3 = 0, c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$
$\therefore \{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is linearly independent
$\therefore$ $\{ (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1) \}$ is a basis of W_1,the dimension of W_1 is 4. ■

20 Let V be a vector space having dimension n, and let S be a subset of V that generates V.

(a) Prove that there is a subset of S that is a basis for V.(Be careful not to assume that S is finite)

(b) Prove that S contains at least n vectors.

Solution. (a) if $S = \varnothing$ or $S = \{ 0 \}$
$V = \{ 0 \} \therefore$ there is a subset of S be a basis.
else pick $s_1 \neq$ from S
pick $s_{k+1} \notin \text{span}(\{ s_1, s_2, \cdots, s_k \})$, by replacement theorem, when a linearly independent set's element number equal dim(V), the set can generate V.
$\therefore$ there is a subset of S be a basis.

(b) by the definition dimension, the element number of basis is n
by replacement theory's, span(S')=V,$\#(S') \geq n$,$S' \subseteq S$,$\#(S) \geq n$. ∎

25 Let V,W, and Z be as in Exercise 21 if Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

Solution. let $Z = \{ (v, w) \mid v \in V, w \in W \}, dim(V) = m, dim(W) = n$
$Z_1 = \{ (v, 0) \mid v \in V \}$,$Z_2 = \{ (0, w) \mid w \in W \}$
$\because Z = Z_1 + Z_2$,$Z_1 \cap Z_2 = \{ (0, 0) \}$
$\therefore Z = Z_1 \bigoplus Z_2$
by Exercise 1.6.29(b), if W_1 and W_2 are finite-dimensional subspace of a vector space V, and let V = $W_1 \bigoplus W_2$. dim($W_1 + W_2$) = dim(W_1) + dim(W_2) = $m + n$ ∎

29 (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and dim($W_1 + W_2$) = dim(W_1) + dim(W_2) - dim($W_1 \cap W_2$).

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let V = $W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if dim(V) = dim(W_1) + dim(W_2).

Solution. (a) let β is a basis of $W_1 \cap W_2$ $\dim(W_1) = k+m$, $\dim(W_2) = k+n$, $\dim(W_1 \cap W_2) = k$, $k, m, n \in \mathbb{Z}^{\geq 0}$
$\beta \in = \{ u_1, u_2, \cdots, u_k \}$, $u_1, \cdots, u_k \in W_1 \cap W_2$
$\because \beta \in W_1, \beta \in W_2$
by Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V.
$\exists \beta_1$ is a basis of W_1
$\beta_1 = \{ u_1, u_2, \cdots, k, v_1, v_2, \cdots, v_m \}\, v_1, v_2, \cdots, v_m \in W_1$
$\exists \beta_2$ is a basis of W_2
$\beta_2 = \{ u_1, u_2, \cdots, k, w_1, w_2, \cdots, w_n \}\, w_1, w_2, \cdots, w_m \in W_2$
let $x \in W_1 + W_2$
$x = (a_1u_1 + a_2u_2 + \cdots + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m) + (b_1u_1 + b_2u_2 + \cdots + b_ku_k + b_{k+1}w_1 + b_{k+2}w_2 + \cdots + b_{k+n}w_n), a_1, a_2, \cdots, a_{k+m}, b_1, b_2, \cdots, b_{k+n} \in \mathrm{F}$
$= c_1u_1 + c_2u_2 + \cdots + c_ku_k + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m + b_{k+1}w_1 + b_{k+2} + \cdots + b_{k+n}w_n, c_1, c_2, \cdots, c_k \in \mathrm{F}$
$\because V_1, V_2, \cdots, v_m \notin W_2, w_1, w_2, \cdots w_n \notin W_1$
$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is linearly independent
$\therefore \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \}$ is a basis of $W_1 + W_2$
$\therefore \dim(W_1 + W_2) = k + m + n$
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2)$

(b) $W_1 \cap W_2 = \{ 0 \}$
by Exercise 1.16.29(a), if W_1 and W_2 are finite-dimensional subspace of a vector space V, dim(V) = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$ = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(\{ 0 \})$ = $\dim(W_1)$ + $\dim(W_2)$

■

31 Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

Solution. (a) let β_1 is a basis of W_2
β is a basis of $W_1 \cap W_2$
$\#(\beta_1) = n$

by Replacement Theorem, V be a vector space is generated by a set G, $\#(G) = n$
a linearly independent set $L \in$ V,$\#(L) = m$
$\because W_1 \cap W_2 \subseteq W_2$
$\therefore \beta$ is a linearly independent set of W_2, β_1 can generated W_2
$\Rightarrow \#(\beta) \leq \#(\beta_1)$
$\Rightarrow$dim$(W_1 \cap W_2) \leq n$

(b) by Exercise 29,W_1, W_2 are finite-dimensional subspaces of a vector space V,then dim$(W_1 + W_2)$=dim(W_1) + dim(W_2) - dim$(W_1 \cap W_2)$
dim(W_1+W_2) = dim(W_1)+dim(S_2)-dim$(W_1 \cap W_2)$ = m+n - dim$(W_1 \cap W_2)$
$\leq m + n$

■

33 (a) Let W_1 and W_2 be subspaces of a vector space V such that V $= W_1 \bigoplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then V $= W_1 \bigoplus W_2$.

Solution. let $\beta_1 = \{\, v_1, v_2, \cdots, v_n \,\}, v_1, v_2, \cdots, v_n \in W_1$,
$\beta_2 = \{\, u_1, u_2, \cdots, u_m \,\}, u_1, u_2, \cdots, u_m \in W_2$
$W_1 + W_2 = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_1 + b_2u_2 + \cdots + b_mu_m \mid$
$a_1, \cdots, a_n, b_1, \cdots, b_m \in \mathrm{F}\}$
$\Rightarrow$ span$(\beta_1 \cup \beta_2) = W_1 + W_2$
Claim $\beta_1 \cup \beta_2$ is linearly dependent
$\exists a_iv_i$ or b_iu_i can be express by
$-a_iv_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$\because v_i \notin \text{span}(\beta_1 - \{\, v_i \,\}), v_i \notin \text{span}(\beta_2)$
$-b_iu_i = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$u_i \notin \text{span}(\beta_2 - \{\, u_i \,\}), u_i \notin \text{span}(\beta_1)$
$\therefore \rightarrow\leftarrow$

■

34 Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that V $= W_1 \bigoplus W_2$

Solution. let $\beta_1 = \{ v_1, v_2, \cdots, v_m \}$,dim(V) $= n$
by Corollary of Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V
$\Rightarrow \exists \beta = \{ v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_{n-m} \}$ is a basis of V
Claim. $\text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}) \bigoplus W_1 = V$ let $v' \in W_1, u' \in \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \})$
$\because \beta$ is a linearly independent set $\therefore v' \notin \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}), u' \notin W_1$
$\Rightarrow W_1 \cap W_2 = \{ 0 \} \therefore \text{span}(\{ u_1, u_2, \cdots, u_{n-m} \}) \bigoplus W_1 = V$ ■