# 1. Vector Space

## §1-3 Subspace.

10 Prove that $W_1 = \{\,(a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \,|\, a_1x_1 + \cdots + a_nx_n = 0\,\}$ is a subspace of $\mathrm{F}^n$, but $W_2 = \{\,(a_1, a_2, \cdots, a_n) \in \mathrm{F}^n \,|\, a_1 + \cdots + a_n = 1\,\}$ is not.

*Solution.* Let $x, y \in W_1$ , $c \in F$ , $x = (x_1, x_2, \cdots, x_n)$ , $y = (y_1, y_2, \cdots, y_n)$
**Claim :** $W_1$ is a subspace of $\mathrm{F}^n$

(a)

$$
\begin{aligned}
x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\
\because x_1 + y_1 + \cdots + x_n + y_n &= x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_n \\
&= 0 + 0 \\
&= 0 \\
&\therefore x + y \in W_1
\end{aligned}
$$

(b)

$$
\begin{aligned}
&cx = (cx_1 + cx_2 + \cdots + cx_n) \qquad c \in F \\
&\because cx_1 + cx_2 + \cdots + cx_n \\
&= c(x_1 + x_2 + \cdots + x_n) \\
&= c * 0 \\
&= 0 \\
&\therefore cx \in W_1
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\because 0 + 0 + \cdots + 0 = 0 \\
&\therefore (0, 0, \cdots, 0) \in W_1
\end{aligned}
$$

$\therefore W_1$ is a subspace of F ■

13 Let $S$ be a nonempty set and $F$ a field. Prove that for any $s_0 \in S$,$\{\, f \in F(S, F) \,|\, f(s_0) = 0\,\}$ , is a subspace of $F(S, F)$.

*Solution.* **Claim.** $\{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$ is a subspace of $F(S,F)$

(a) let $f_a, f_b \in \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$
$\because (f_a + f_b)(s_0) = f_a(s_0) + f_b(s_0) = 0$
$\therefore (f_a + f_b)(s_0) \in \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$

(b) let $f_a \in \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}, c \in$ F
$\because cf_a(s_0) = c * 0 = 0$
$\therefore cf_a(s_0) \in \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$

(c) $f(s) = 0 \in \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$

$\therefore \{\, f \in F(S,F) \,|\, f(S_0) = 0 \,\}$ is a subspace of $F(S,F)$. ■

14 Let $S$ be a nonempty set and $F$ a field. Let $C(S,F)$ denote the set of all functions $f \in F(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of $S$. Prove that $C(S,F)$ is a subspace of $F(S,F)$

*Solution.* **Claim.** $C(S,F)$ is a subspace of $F(S,F)$

(a) let $f, g \in C(S,F)$
$f(s) \neq 0$ when $s \in \{\, s_1, s_2, \cdots, s_n \,\}$ $g(s) \neq 0$ when $s \in \{\, s_1, s_2, \cdots, s_m \,\}$
$(f+g)(s)$
$= f(s) + g(s)$
$f(s) + f(s) \neq 0$ only if $s \in (\{\, s_1, s_2, \cdots, s_n \,\} \cup \{\, s'_1, s'_2, \cdots, s'_n \,\})$
$\because \#(\{\, s_1, s_2, \cdots, s_n \,\} \cup \{\, s'_1, s'_2, \cdots, s'_n \,\}) \leq n + m$ is finite
$\therefore (f+g)(s) \in C(S,F)$

(b) let $c \in$ F
$cf(s) \neq 0$ only if $s \in \{\, s_1, s_2, \cdots, s_n \,\}$
$\because \#(\{\, s_1, s_2, \cdots, s_n \,\}) = n$ is finite
$\therefore cf(s) \in C(S,F)$

(c) $f' \in F(S,F)$, $f'(s) = 0$ $s \in S$, $f' \in C(S,F)$

■

20 Max

23 Let $W_1$ and $W_2$ be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both $W_1$ and $W_2$.

(b) Prove that any subspace of V that contains both $W_1$ and $W_2$ must also contain $W_1 + W_2$.

*Solution.*
(a)**Claim** $W_1 + W_2$ is a subspace of V

(1) let $x_1, x_2 \in W_1$ , $y_1, y_2 \in W_2$
$(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = (x_1 + x_2) + (y_1 + y_2)$
$\because W_1, W_2$ is a subspace of V
$\therefore (x_1 + x_2) \in W_1, (y_1 + y_2) \in W_2 \implies (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$

(2) let $x_1 \in W_1, y_1 \in W_2$ , $c \in$ F
$c(x_1 + y_1) = cx_1 + cy_1$
$\because W_1, W_2$ is a subspace of V, $cx_1 \in W_1, cy_1 \in W_2$
$\therefore cx_1 + cy_1 \in W_1 + W_2$

(3) $\because W_1, W_2$ is a subspace of V,$\therefore 0 \in W_1, 0 \in W_2$,
$0 + 0 = 0 \in W_1 + W_2 \therefore W_1 + W_2$ is a subspace of V

$W_1 = \{x + 0 \,|\, x \in W_1\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$W_2 = \{0 + y \,|\, y \in W_2\} \subseteq \{\, x + y \,|\, x \in W_1, y \in W_2 \,\}$
$\therefore W_1 + W_2$ contains both $W_1$ and $W_2$
(b)let $W_3$ is a subspace of $V$, $W_3 \subseteq W_1$,$W_3 \subseteq W_2$

let $x \in W_1, y \in W_2$,$\because W_3$ is a subspace.$\therefore x + y \in W_3 \implies W_1 + W_2 \in W_3$ ■

30 Let $W_1$ and $W_2$ be subspaces of a vector space V Prove that V is the direct sum of $W_1$ and $W_2$ if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

*Solution.*

($\Rightarrow$) $W_1 \cap W_2 = \{0\}, W_1 + W_2 = \text{V}$
**Claim.** each vector in V can not be only one written as $x + y$
where $x \in W_1, y \in W_2$
let $u \in \text{V}, u = x_1 + y_1 = x_2 + y_2$,
$x_1, x_2 \in W_1, y_1, y_2 \in W_2, x_1 \neq x_2, y_1 \neq y_2$
$x_1 + y_1 = x_2 + y_2 \implies x_1 - x_2 = y_2 - y_1$
$\because$ $W_1$ is a subspace, $(x_1 - x_2) \in W_1$, $W_2$ is a subspace, $(y_2 - y_1) \in W_2$
$W_1 \cap W_2 = \{0\} \therefore (x_1 - x_2) = (y_2 - y_1) = 0 \implies x_1 = x_2, y_1 = y_2 \rightarrow\leftarrow$
$\therefore$ each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$and $x_2 \in W_2$

($\Leftarrow$) $\text{V} = \{x + y \,|\, x \in W_1, y \in W_2\} = W$
**Claim.** $W_1 \cap W_2$ not only 0
$\exists\, u \in W_1 \cap W_2, u = 0 + u = u + 0 \rightarrow\leftarrow$
$\therefore W_1 \oplus W_2 = V$

■

**§1-4 Linear Combination.**

13 Show that if $S_1$ and $S_2$ are subsets of a vector space V such that $S_1 \subseteq S_2$,then span($S_1$) $\subseteq$ span($S_2$). In particular, if $S_1 \subseteq S_2$ and span($S_1$) = V, deduce that span($S_2$) = V

*Solution.* **Claim** span($S_1$) $\subseteq$ span($S_2$)
let $S_1 = \{v_1, v_2, \cdots, v_n\}, S_2 = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_m\}, x \in \text{span}(S_1)$
$x = a_1v_1 + \cdots + a_nv_n, a_1, a_2, \cdots, a_n \in \text{F}$
$= a_1v_1 + \cdots + a_nv_n + 0u_1 + 0u_2 + \cdots + 0u * n \in \text{Span}(S_2)$
$\therefore \text{span}(S_1) \subseteq \text{span}(S_2)$

■

14 Show that if $S_1$ and $S_2$ are arbitrary subsets of a vector space V, then span($S_1 \cup S_2$ )= span($S_1$) + span($S_2$).

*Solution.* Let $S_1 \cap S_2 = \{v_1, v_2, \cdots, v_n\}$,
$S_1 = \{u_1, u_2, \cdots, u_m, v_1, \cdots, v_n\}, S_2 = \{r_1, \cdots, r_k, v_1, \cdots, v_n\}$
**Claim. I can do it better (not now)** span($S_1$)+span($S_2$) $\subseteq$ span($S_1 \cup S_2$)
let $x = (a_1v_1 + a_2v_2 + \cdots + a_nv_n) + (b_1u_1 + b_2u_2 + \cdots + b_nv_n)$
$= a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_1 + b_2u_2 + \cdots + b_nv_n \in \text{span}S_1 \cup S_2$
$\therefore \text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$
**Claim.** $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$

■

## §1-5 Linear Independent.

13 Let V be a vector space over a field of characteristic not equal to two.

(a) Let $u$ and $v$ be distinct vectors in V. Prove that $\{ u, v \}$ is linearly independent if and only if $\{ u + v, u - v \}$ is linearly independent.

(b) Let $u, v$ and $w$ be distinct vectors in V. Prove that $\{ u, v, w \}$ is linearly independent if and only if $\{ u + v, u + w, v_w \}$ is linearly independent.

*Solution.* ■

16 Prove that a set $S$ of vectors is linearly independent if and only if each finite subset of $S$ is linearly independent.

*Solution.* ■

20 Let $f, g \in F(R, R)$ be the functions defined by $f(t) = e^{et}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that $f$ and $g$ are linearly independent in $F(R, R)$.

*Solution.* ■