Lecture Notes on Linear Algebra

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1. Vector Space

<u>Definition</u> (Vector Space). *A vector space (or linear space) V over a Field $\mathcal{F}$ consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y, in V there is a unique element $x+y$ in V, and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold.*

§ Subspace .

<u>Definition</u> (Subspace). *A subset W of a vector space V over a field $\mathbb{F}$ is called a subspace of W if W is a vector space over* F *with the operations of addition and scalar multiplication defined on W.*

Remark. *Trivial subspaces of a vector space V, namely V itself and $\{0\}$. Note that empty set ϕ is not a vector space, since it does not contains a zero vector.*

Theorem 1.1. *Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.*
(a) $0 \in W$.
(b) $x + y \in W$ *whenever* $x \in W$ *and* $y \in W$.
(c) $cx \in W$ *whenever* $c \in \mathrm{F}$ *and* $x \in W$.

Corollary 1.1.1. *Let W be a subset of vector space V. W is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and, $x, y \in W$.*

Proof. (⇒) Since W is a subspace $\implies$ $0 \in W$, $ax \in W$ for $a \in \mathcal{F}$ and $ax, y \in W$ $\implies$ $ax + y \in W$. (⇐) $0 \in W$. Since $ax + y \in W$, $a \in \mathcal{F}$, take $a = 1$ $\implies$ $x + y \in W$, and also $\because$ $0 \in W$, take $y = 0$ $\implies$ $ax \in W$, for $a \in \mathcal{F}$. Hence, W isa subspace. ∎

Theorem 1.2. *Any intersection of subspaces of a vector space W is a subspace of V.*

Theorem 1.3. *Let W_1 and W_2 be subspaces of a vector space V, then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

Proof. Suppose $W_1 \cup W_2$ is a subspace of V, we assume that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Let $x \in W_1 \backslash W_2$, $y \in W_2 \backslash W_1$, then $x + y \in W_1$ or W_2, say W_1. But $y = (x + y) - x \in W_1$, which is a contradiction. So we have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Conversely is trivial. ∎

Definition. *If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.*

Theorem 1.4. *Let W_1 and W_2 be subspaces of a vector space V.*

(a) $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof. To prove (a), we first show that $W_1 + W_2$ is a subspace of V. Clearly, $0 = 0 + 0 \in W_1 + W_2$. Let $x = x_1 + x_2$, $y = y_1 + y_2$, $c \in \mathbb{F}$
$cx + y = c(x_1 + x_2) + (y_1 + y_2) = (cx_1 + y_1) + (cx_2 + y_2) \in W_1 + W_2$
By corrolary 1.1.1, $W_1 + W_2$ is a subspace of V.
Then we show that $W_1, W_2 \subseteq W_1 + W_2$. $\forall x \in W_1 y \in W_2$, $x = x + 0 \in W_1 + W_2$, $y = 0 + y \in W_1 + W_2$, we have $W_1, W_2 \subseteq W_1 + W_2$.
To prove (b), Let W be a subspace of V contains both W_1 and W_2, then $\forall x \in W_1, y \in W_2$, $x + y \in W_1 + W_2$. Thus $W_1 + W_2 \subseteq W$. ∎

Definition (Direct Sum). *A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.*

Theorem 1.5. *Let W_1 and W_2 be subspaces of a vector space V. V is the direct sum of W_1 and W_2 i.e.*

$$V = W_1 \oplus W_2$$

if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$

Solution. Suppose that we can write $v = x_1 + x_2 = y_1 + y_2$ with $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Since $x_1 + x_2 = y_1 + y_2$ and W_1, W_2 are subsapces of a vector space V, so $x_1 - y_1 = x_2 - y_2 \in W_1 \cap W_2$. As $V = W_1 \oplus W_2$, $x_1 - y_1 = x_2 - y_2 = \{0\}$, that is, $x_1 = y_1$ and $x_2 = y_2$. Hence it is a unique representation. Conversely, suppose that the condition holds. We claim that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Since $W_1 + W_2$ is the smallest subspace of V, so $W_1 + W_2 \subseteq V$. By assumption, it is clear to get that $V \subseteq W_1 + W_2$. This proves $V = W_1 + W_2$. Since W_1, W_2 are subspaces of V, so $0 \in W_1$ and $0 \in W_2$. Let v be a vector in $W_1 \cap W_2$. From the uniqueness of decomposition and $v = 0 + v = v + 0$, it implies that $v = 0$. This proves $W_1 \cap W_2 = \{0\}$. In conclusion, $V = W_1 \oplus W_2$. ■

§ Linear Combinations and Bases.

Definition (Linearly Dependent)**.** *A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, ..., u_n$ in S and scalars $a_1, a_2, ..., a_n$, not all zero, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Linearly Independent)**.** *A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.*

Remark.

1. *The empty set is linearly independent, for linearly dependent sets must be nonempty.*

2. *A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a. Thus*

$$u = a^{-1}(au) = a^{-1}0 = 0$$

3. *A set is linearly independent if and only if the only representations of* 0 *as linear combinations of its vectors are trivial representations.*

Theorem 1.6. *Let* V *be a vector space, and let* $S_1 \subseteq S_2 \subseteq V$*. If* S_1 *is linearly dependent, then* S_2 *is linearly dependent.*

Proof. Since $S_1 = \{u_1, \cdots, u_m\}$ is linearly dependent i.e. $\exists\ a_1, \cdots, a_m \in \mathcal{F}$ are not all zero s.t.

$$a_1u_1 + \cdots + a_mu_m = 0 \qquad (1)$$

Since $S_1 \subseteq S_2$ say $S_2 = S_1 \cup \{u_{m+1}, \cdots, u_n\}$. Let

$$b_1u_1 + \cdots + b_mu_m + b_{m+1}u_{m+1} + \cdots + b_nu_n = 0\ b_j \in \mathcal{F}, j = 1, 2, 3, \cdots$$

Now, we take $b_{m+1} = \cdots = b_n = 0$ and by (1) there exists $a_1, \cdots, b_n$ are not all zeros. Hence, S_2 is linearly dependent. ∎

Corollary 1.6.1. *Let* S *be a linearly independent subset of a vector space* V*, and let* v *be a vector in* V *that is not in* S*. Then* $S \cup \{v\}$ *is linearly dependent if and only if* $v \in span(S)$*.*

Proof. Suppose $S \cup v$ is linear independent, then $\exists v_1, ..., v_n \in S \cup v$ such that $a_1u_1 + ... + a_nu_n = 0$ for some nonzero scalars $a_1, ..., a_n$. Because S is linear independent, say $u_1 = v$. We have $v = a_1^{-1}(-a_2u_2 - ... - a_nu_n)$, which is a linear combination of $u_2, ..., u_n$. Thus $v \in span(S)$. Conversely, suppose $v \in span(S)$. Then $\exists v_1, ..., v_n \in S$ and scalars $b_1, ..., b_m$ such that $v = b_1v_1 + ... + b_mv_m$. Thus $0 = b_1v_1 + ... + b_mv_m + (-1)v$ which means $S \cup v$ is linear independent. ∎

Definition (Basis)**.** *A basis* β *for a vector space* V *is a linearly independent subset of* V *that generates* V*. If* β *is a basis for* V*, we also say that the vectors of* β *form a basis for* V*.*

Theorem 1.7. *Let* V *be a vector space and* $\beta = \{u_1, u_2, ..., u_n\}$ *be a subset of* V*. Then* β *is a basis for* V *if and only if each* $v \in V$ *can be uniquely expressed as a linear combination of vectors of* β*, that is, can be expressed in the form*

$$V = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \cdots, a_n$*.*

Proof. If $S = \emptyset$ or $S = 0$, then $V = \{0\}$ and $\emptyset$ is a subset of S which is a basis of V. Otherwise S contains a nonzero vector $\{u_1\}$. We continue choosing vectors $u_2, ..., u_k \in S$ such that $\beta = \{u_1, ..., u_k\}$ is linear independent and $\{u_1, ..., u_k, v\}$ is linear dependent for some vector $v \in V \backslash S$. To show that β is a basis, it suffices toshow that β generates V. Let $v \in S$. If $v \in \beta$, then clearly $\beta \in span(S)$. If $v \in S \backslash \beta$, then we have $\beta \cup v$ is linear dependent. So $v \in span(\beta)$. Thus $S \subseteq span(\beta)$. ■

Theorem 1.8. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Proof. We prove it by mathematical induction on m. For $m = 0$, $L = \emptyset$, so take $H = G$. Assume the theorem holds for some integer $m \geq 0$. For $m+1$, let $L = \{v_1, ..., v_{m+1}\}$ be a linear independent subset of V containing $m + 1$ vectors. Because $\{v_1, ..., v_m\}$ is linear independent, so by the induction hypothesis, we have $m \geq n$ and $\exists$ a subset $\{u_1, ..., u_{n-m}\}$ of G such that $\{v_1, ..., v_m\} \cup \{u_1, ..., u_{n-m}\}$ generates v. Thus $\exists$ scalars $a_1, ..., a_m, b_1, ..., bn - m$ such that $a_1v_1 + ... + a_mv_m + b_1u_1 + ... + b_{n-m}u_{n-m} = v_{m+1}$. We have $n - m > 0$ and v_{m+1} is a linear combination of $\{v_1, ..., v_m\}$ which contradicts the assumption that L is linear independent. Hence $n > m$ that is $n \geq m + 1$. Moreover, say b_1 is nonzero, otherwise we obtain the same contradiction. We have $u_1 = (-b_1^{-1}a_1)v_1 + ... + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1} + (-b_1^{-1}b_2)u_2 + ... + (-b_1^{-1}b_{n-m})u_n - m$. Let $H = \{u_2, ..., u_{n-m}\}$. Then $u_1 \in span(L \cup H)$ and $v_1, ..., v_m, u_2, ..., u_{n-m} \in span(L \cup H)$, so $\{v_1, ..., v_m, u_1, ..., u_{n-m}\} \subseteq span(L \cup H)$. We have $span(L \cup H) = V$. Since H is a subset of G contains $(n - m) - 1 = n - (m + 1)$ vectors. So by mathematical induction, we are done. ■

Theorem 1.9 (Replacement Theorem)**.** *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

Proof. Let $dim(V) = n$. If $W = \{0\}$, then W is finite-dimensional and $dim(W) = 0 \leq n$. Otherwise, W contains a nonzero vector x_1, so $\{_1\}$ is a linear independent set. We continue choosing vectors $x_1, ..., x_k \in W$ such that $x_1, ..., x_k$ is linear independent. This process must stop at $k \leq n$ and $\{x_1, ..., x_k\}$ is linear independent and other vector in W produces a linear dependent set. Hence it is a basis of W. Thus $dim(W) = k \leq n$. If $dim(W) = n$, a basis of W is a linear independent subset of V contains n vectors, so a basis of W is also a basis of V. Thus $W = V$.

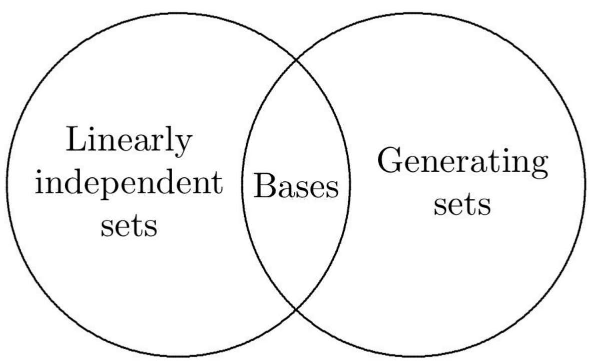
■

Corollary 1.9.1. *Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.*

Definition (Finite-Dimensional)**.** *A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.*

Corollary 1.9.2. *Let V be a vector space with dimension n.*

1. *Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.*
2. *Any linearly independent subset of V that contains exactly n vectors is a basis for V.*
3. *Every linearly independent subset of V can be extended to a basis for V.*



Theorem 1.10. *Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and* $\dim(W) \leq \dim(V)$*. Moreover, if* $\dim(W) = \dim(V)$*, then $V = W$.*

Proof. Since V is a finite-dimensional vector space, say $\dim(V) = n$. If $W = \{0\}$, then W is a finite-dimensional and $\dim(W) = 0 \leq n$. Otherwise, W contains a nonzero vector x_1 ; so $\{x_1\}$ is linearly independent set. Continue choosing vectors, $x_1, \cdots, x_k$ in W such that $S = \{x_1, \cdots, x_k\}$ is linearly independent. Since no linearly independent subset can contain more than n vectors in V, therefore $S \cup \{v\}$ is linearly dependent for $\{v\} \in span(S)$ then by Thm 1.7 S generates W. S is a basis of $W \implies \dim(W) = k \leq n$. If $\dim(W) = n$, then a basis for W is a linearly independent subset of V containing n vectors. By replacement theorem any linearly independent subset of V contains exactly n vectors is also a basis of V. Hence $V = W$. ■

Propersition 1.11. *Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. $W_1 \subseteq W_2$ if and only if* $\dim(W_1 \cap W_2) = \dim(W_1)$

Theorem 1.12. *Let $v_1, v_2, \cdots, v_k, v$ be vectors in a vector space V, and define $W_1 = span(\{v_1, v_2, \cdots, v_k\})$, and $W_2 = span(\{v_1, v_2, \cdots, v_k, v\})$.Then* $v \in \text{span}(W_1)$ *if and only if* $\dim(W_1) = \dim(W_2)$.

Remark. *We may give an example for satisfying the conditions on above but* $\dim(W_1) \neq \dim(W_2)$.

Theorem 1.13. *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V.*

(a) Then the subspace $W_1 + W_2$ is finite-dimensional, and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(b) Let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if

$$\dim(V) = \dim(W_1) + \dim(W_2)$$

Proof.

(a). Let $\beta = \{\, u_1, u_2, \cdots, u_k \,\}$ is a basis of $W_1 \cap W_2$ with $\dim(W_1) = k+m$, $\dim(W_2) = k+n$ and $\dim(W_1 \cap W_2) = k$ for $k, m, n \in \mathbb{N}$. Since $\beta \in W_1$ and $\beta \in W_2$ by Replacement Theorem, every linearly independent subset of V can be extended to a basis for V.

$\exists\ \beta_1 = \{\, u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m \,\}$ is a basis of W_1 and $\beta_2 = \{\, u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_n \,\}$ is a basis of W_2. Let $x \in W_1 + W_2$.

Claim. $Span(\{\, u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \,\}) = W_1 + W_2$

$x = (a_1u_1 + a_2u_2 + \cdots + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m) + (b_1u_1 + b_2u_2 + \cdots + b_ku_k + b_{k+1}w_1 + b_{k+2}w_2 + \cdots + b_{k+n}w_n), a_i, b_j \in \mathcal{F}$ for $i, j = 1, 2, \cdots$

$= c_1u_1 + c_2u_2 + \cdots + c_ku_k + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m + b_{k+1}w_1 + b_{k+2} + \cdots + b_{k+n}w_n, c_1, c_2, \cdots, c_k \in \mathcal{F} \implies \forall\ x \in W_1 + W_2$

$\therefore W_1 + W_2 \subseteq span(\{\, u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \,\})$

$\because W_1 + W_2$ is a subspace, any linear combination of $W_1 + W_2$'s subset are in $W_1 + W_2$

$\therefore span(\{\, u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \,\}) \in W_1 + W_2$

$\therefore span(\{\, u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \,\}) = W_1 + W_2$

Claim. $\{\, u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \,\}$ is linearly independent

$$\sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i + \sum_{i=1}^{n} c_iw_i = 0 \implies \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i = -\sum_{i=1}^{n} c_iw_i$$

$$\because \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i \in W_1\ ,\quad -\sum_{i=1}^{n} c_iw_i \in W_2$$

$$\therefore \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i\ ,\quad -\sum_{i=1}^{n} c_iw_i \in W_1 \cap W_2$$

$$\implies \exists\ d_i \in \mathcal{F}, \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i = -\sum_{i=1}^{n} c_iw_i = \sum_{i=1}^{k} d_iu_i$$

$\because \beta_1, \beta_2$ is linearly independent $\therefore -\sum_{i=1}^{n} c_iw_i = \sum_{i=1}^{k} d_iu_i$ only for scalars are all zeros. $\sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i = 0$ only for scalars are all zero

$\therefore \{u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n\}$ is linearly independent

$\therefore \{\, u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \,\}$ is a basis of $W_1 + W_2$

$\therefore \dim(W_1 + W_2) = k + m + n$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2)$

(b). $W_1 \cap W_2 = \{ 0 \}$

by Exercise 1.16.29(a), if W_1 and W_2 are finite-dimensional subspace of a vector space V, dim(V) = dim(W_1) + dim(W_2) - dim($W_1 \cap W_2$) = dim(W_1) + dim(W_2) - dim($\{ 0 \}$) = dim(W_1) + dim(W_2)

■

Theorem 1.14. *Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$ if and only if there exist base β_1 , β_2 of W_1 , W_2, respectively such that $\beta_1 \cup \beta_2$ is a basis for V.*

Proof. Let $\beta_1 = \{ v_1, v_2, \cdots , v_n \} , v_1, v_2, \cdots , v_n \in W_1$,
$\beta_2 = \{ u_1, u_2, \cdots , u_m \} , u_1, u_2, \cdots , u_m \in W_2$

$$W_1 + W_2 = \left\{ \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j u_j \mid a_1, \cdots , a_n, b_1, \cdots , b_m \in \mathcal{F} \right\}$$

Claim : Span($\beta_1 \cup \beta_2$) $\subseteq W_1 + W_2$
let $x \in$ Span($\beta_1 \cup \beta_2$) , $x = a_1v_1 + a_2v_2 + \cdots + b_1u_1 + b_2u_2 + \cdots + b_mu_m$

$\because W_1, W_2$ is a subspace of V $\therefore \sum_{i=1}^{n} a_i v_i \in W_1$, $\sum_{i=1}^{m} b_i u_i \in W_2$

$\therefore x \in W_1 + W_2, \text{Span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$

Claim. $W_1 + W_2 \subseteq$ Span($\beta_1 \cup \beta_2$)
Let $x \in W_1 + W_2, x = (a_1v_1 + \cdots + a_nv_n) + (b_1u_1 + \cdots + b_mu_m)$
$\therefore x \in \text{Span}(\beta_1 \cup \beta_2) \therefore W_1+W_2 \subseteq \text{Span}(\beta_1 \cup \beta_2) \therefore \text{Span}(\beta_1 \cup \beta_2) = W_1+W_2$

Claim $\beta_1 \cup \beta_2$ is linearly independent

$$\sum_{i=1}^{n} a_i v_1 + \sum_{j=1}^{m} b_j u_j = 0 \implies \sum_{i_1}^{n} a_i v_i = -\sum_{j=1}^{m} b_j u_j$$

$\because \sum_{i=1}^{n} a_i v_i \in W_1$, $-\sum_{j=1}^{m} b_j u_j \in W_2$ and $\sum_{i=1}^{n} a_i v_i$, $-\sum_{i=1}^{m} b_i u_i \in W_1 \cap W_2$

$\because W_1 \cap W_2 \therefore \sum_{i=1}^{n} a_i v_i = -\sum_{j=1}^{m} b_j u_j = 0 \quad \because \beta_1, \beta_2$ is linearly independent

$\therefore a_1 = \cdots = a_n = b_1 = \cdots = b_m = 0 \therefore \beta_1 \cup \beta_2$ is linearly independent
$\therefore \beta_1 \cup \beta_2$ is a basis of V.

■

Theorem 1.15.

If W_1 is any subspace of vector space of V, then there exists a subspace W_2 of V such that

$$V = W_1 \oplus W_2$$

Proof. let $\beta_1 = \{\, v_1, v_2, \cdots, v_m \,\}$ and $\dim(V) = n$. By Corollary of Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V then $\exists\ \beta = \{\, v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_{n-m} \,\}$ is a basis of V. Let $W_2 = \operatorname{span}(\{u_1, u_2, \cdots, u_{n-m}\})$ $\because u_1, u_2, \cdots, u_{n-m} \in V, V$ is a vector space by Thm 1.5, the span of any subset S of a vector space V is a subspace.
$\therefore W_2$ is a subspace of V. Now we claim that $W_1 \cap W_2 = \{0\}$
$\because\ W_1, W_2$ is a subspace of V $\therefore\ 0 \in W_1, W_2$. Assume $\exists\ r \in V$ and $r \in W_1\ ,\ r \in W_2\ ,\ r \neq 0$.

$$\begin{aligned} r &= a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \\ &= b_1 u_1 + b_2 u_2 + \cdots + b_{n-m} u_{n-m} \\ &= c_1 v_1 + \cdots + c_m v_m + d_1 u_1 + \cdots + d_{n-m} u_{n-m} \end{aligned}$$

$$\implies \begin{cases} (c_1 - a_1)v_1 + \cdots + (c_m - a_m)v_m + d_1 u_1 + \cdots + d_{n-m}u_{n-m} = 0 \\ c_1 v_1 + \cdots + c_m v_m + (d_1 - b_1)u_1 + \cdots + (d_{n-m} - b_{n-m}) \end{cases}$$
$\implies\ c_1 = a_1, c_2 = a_2, \cdots, c_m = a_m, d_1 = b_1, \cdots, d_{n-m} = b_{n-m}$
$\Rightarrow r = r + r \Rightarrow r = 0 \rightarrow\leftarrow \therefore W_1 \cap W_2 = \{0\}$. ■

2. Linear Transformation

§ Linear Operator

Definition (Linear Transformation)**.** *Let V and* W *be vector spaces (over* F*). We call a function* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *a linear transformation from* V *to* W *if, for all* $x,\ y \in \mathrm{V}$ *and* $c \in \mathrm{F}$*, we have*

(*a*) $\mathrm{T}(x + y) = \mathrm{T}(x) + \mathrm{T}(y)$

(*b*) $\mathrm{T}(\mathrm{c}x) = \mathrm{cT}(x)$

Definition. *Let* $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ *be arbitrary functions, where V and W are vector spaces over F, and let* $a \in \mathrm{F}$*. We define* $\mathrm{T} + \mathrm{U} : \mathrm{V} \to \mathrm{W}$ *by* $(\mathrm{T} + \mathrm{U})(x) = \mathrm{T}(x) + \mathrm{U}(x)$ *for all* $x \in \mathrm{V}$*, and* $a\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *by* $(a\mathrm{T})(x) = a\mathrm{T}(x)$ *for all* $x \in \mathrm{V}$*.*

Remark. *If $F = Q$ then $(a) \Rightarrow (b)$.*

Remark. *If $T : C \to C$ be function defined by $T(\delta) = \delta$. T is additive but not linear.*

Propersition 2.1.

1. *If T is linear, then $T(0) = 0$.*
2. *T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.*
3. *If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$*
4. *T is linear if and only if, for $x_1, x_2, \cdots, x_n \in V$ and $a_1, a_2, \cdots, a_n \in F$, we have*
$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i).$$
5. *For all $a \in F, aT + U$ is linear.*
6. *Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F.*

Definition (Null Space, Range)**.** *Let* V *and* W *be vector spaces and let* T : V $\to$ W *be linear. We define the null space(or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$; that is , $N(T) = \{x \in V : T(x) = 0\}$.*
We define the range(or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V; that is, $R(T) = \{T(x) : x \in V\}$

Remark. *Let V and W be vector spaces, and let $I : V \to V$ and $T_0 : V \to W$ be the identity and zero transformations, respectively. Then $N(I) = \{0\}$, $R(I) = V$, $N(T_0) = V$, and $R(T_0) = \{0\}$.*

Theorem 2.2. *Let V and W be vector spaces and* T : V $\to$ W *be linear. Then $N(T)$ and $R(T)$ are subspaces of* V *and* W*, respectively.*

Proof.
Let $0_V \in V, 0_W \in W$ be a zero vectors
$\because T(0_V) = 0_W \therefore 0_V \in N(T)$
Let $x, y \in N(T), c \in F$,
$\because T(x+y) = T(x) + T(y) = 0_W + 0_W = 0_W$
$\therefore x + y \in N(T)$
$\because T(cx) = cT(x) = c0_W = 0_W$
$\therefore cx \in N(T)$
$\implies N(T)$ is a subspace of V
$\because T(0_V) = 0_W \therefore 0_W \in R(T)$
Let $x, y \in R(T)$ and $c \in F$. $\exists v, w \in V, T(v) = x \in V$ and $T(w) = y$
$\because x + y = T(v) + T(w) = T(v+w)$
$\therefore x + y \in R(T)$
$\because cx = cT(v) = T(cv)$
$\therefore cx \in R(T) \implies R(T)$ is a subspace of W

■

Remark. *Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$*

Theorem 2.3. *Let V and W be vector spaces, and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. If $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for* V *, then*

$$\mathrm{R(T)} = \mathrm{span}(\mathrm{T}(\beta)) = \mathrm{span}(\{\mathrm{T}(v_1), \mathrm{T}(v_2), ..., \mathrm{T}(v_n)\})$$

Proof.
Clearly, $T(v_i) \in R(T)$
$\because R(T)$ is a subspace $\therefore$ By thm 1.5 : $span(T(\beta)) \subseteq R(T)$
Let $w \in R(T)$
$\because w = T(v)$ for some $v \in V \therefore \beta$ is a basis of V
$\therefore v = \sum_{i=1}^{n} a_i v_i$ for some $a_i \in F, \quad 1 \leq i \leq n$
$\because T$ is linear $\therefore w = T(v) = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i) \in span(T(\beta))$
$\therefore R(T) \subseteq span(T(\beta)) \implies R(T) = span(T(\beta)) = span(\{T(v_1), \cdots, T(v_n)\})$
■

Example. *Prove Theorem 2.2 for the case that β is infinite, that is, $R(T) = span(\{T(v) : v \in \beta\})$.*

Definition (Nullity and Rank)**.** *Let* V *and* W *be vector spaces and let* T : V $\to$ W *be linear. If* $N(T)$ *and* $R(T)$ *are finite-dimensional, then we define the nullity of* T*, denoted* $nullity(T)$*, and the rank of* T*, denoted* $rank(T)$*, to be the dimensions of* $N(T)$ *and* $R(T)$*, respectively.*

Theorem 2.4. *Let* V *and* W *be vector spaces, and let* T : V $\to$ W *be linear. If* V *is finite-dimensional, then*

$$nullity(T) + rank(T) = dim(V).$$

Proof.
Suppose $\dim(V) = n$, $\dim(N(T)) = k$ and $\{v_i = \forall_i = 1, \cdots, k\}$ is a basis of $N(T)$
$\because$ Cor of thm 1.11 $\therefore$ We can extend $\{v_i : \text{for } i = 1, \cdots, k\}$ to a basis $\beta = \{v_i : \text{for } i = 1, \cdots, k\}$ of V
Claim : $S = \{T(v_i) : \text{for } i = k+1, \cdots, n\}$ is a basis of $R(T)$

1. Claim $span(S) = R(T)$
   $\because$ Thm 2.2 $\therefore$ $R(T) = span(T(\beta)) = span(\{T(v_1), \cdots, T(v_n)\})$
   $\because$ $T(v_i) = 0$ for $i = 1, \cdots, k$
   $\therefore$ $R(T) = span(\{T(v_{k+1}, \cdots, T(v_n)\}) = span(S)$

2. Claim : S is linear independent
   Suppose $\displaystyle\sum_{i=k+1}^{n} b_i T(v_i) = 0 \quad \forall b_{k+1}, \cdots b_n \in F$
   $\because$ T is linear $\therefore$ $T(\displaystyle\sum_{i=k+1}^{n} b_i v_i) = 0 \implies \sum_{i=k+1}^{n} b_i v_i \in N(T)$
   Hence, $\exists c_i \in F$ for $i = 1, \cdots, k$
   $$\implies \sum_{i=k+1}^{n} b_i v_i = \sum_{i=k+1}^{n} c_i v_i \implies \sum_{i=k+1}^{n} b_i v_i + \sum_{i=k+1}^{n} (-c_i) v_i = 0$$
   $\because$ β is a basis of V $\therefore$ $b_i, \forall_i = k+1, \cdots, n = 0$
   $\therefore$ S^n is linear independent
   $S = \{T(v_i) : \text{for } i = k+1, \cdots, n\} \implies rank(T) = n - k$

$\blacksquare$

Theorem 2.5. *Let* V *and* W *be vector spaces, and let* $T : V \to W$ *be linear. Then* T *is one-to-one if and only if* $N(T) = \{0\}$*.*

Proof. T is one-to-one $\iff N(T) = \{0\}$
$\Rightarrow$
Suppose T is one-to-one and $x \in N(T)$
$T(x) = 0 = T(0)$
$\because T$ is one-to-one $\therefore x = 0$
$\Rightarrow N(T) = \{0\}$
$\Leftarrow$
Suppose $N(T) = \{0\}$ and $T(x) = T(y)$
$0 = T(x) - T(y) = T(x - y)$
$\because N(T) = \{0\} \therefore x - y = 0 \therefore x = y$
$\Rightarrow T$ is one-to-one
$\Rightarrow T$ is one-to-one $\iff N(T) = \{0\}$

■

Propersition 2.6. *Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.*

(a) *Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.*

(b) *Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.*

Definition (Invertible)**.** *Let V and W be vector spaces, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ be linear. A function $\mathrm{U} : \mathrm{W} \to \mathrm{V}$ is said to be an inverse of T if $\mathrm{TU} = \mathrm{I_W}$ and $\mathrm{UT} = \mathrm{I_V}$. If T has an inverse, then T is said to be invertible. As noted in Appendix B, if T is invertible, then the inverse of T is unique and is denoted by T^{-1}.*

Propersition 2.7.

1. $(TU)^{-1} = U^{-1}T^{-1}$.
2. $(T^{-1})^{-1} = T$*; in particular, T^{-1} is invertible.*

Definition. *Let A be an $n \times n$ matrix. Then A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I$.*
If A is invertible, then the matrix B such that $AB = BA = I$ is unique. (If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$.) The matrix B is called the inverse of A and is denoted by A^{-1}.

Definition (Isomorphism)**.** *Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ that is invertible. Such a linear transformation is called an isomorphism from V onto W.*

Corollary 2.7.1. *Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.*

Theorem 2.8. *Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if* $\dim(\mathrm{V}) = \dim(\mathrm{W})$.

Corollary 2.8.1. *Let V be a vector space over F. Then V is isomorphic to F^n if and only if* $\dim(V) = n$.

Remark. *The Linearity and Finite-dimensional is essential arguement.*

Example. *Recall the definition of $P(R)$ on page 10. Define*

$$T : P(R) \to P(R) \quad by \quad T(f(x)) = \int_0^x f(t)dt.$$

Prove that T linear and one-to-one, but not onto.

Example. *Let $T : P(R) \to P(R)$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.*

Example. *Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U : V \to V$ by*

$$T(a_1, a_2, \cdots) = (a_2, a_3, \cdots) \quad and \quad U(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

T and U are called the left shift and right shift operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Theorem 2.9. *Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \cdots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$ then $U = T$.*

Proof. Suppose $U(v_i) = T(v_i) = w_i$ for $i = 1, \cdots, n$

Let $x \in U, V$ and $x = \sum_{i=1}^{n} a_i v_i$ where $a_i \in F$

$U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i T(v_i) = T(x)$

$\implies U = T$ ∎

Propersition 2.10. *Let V and W be vector spaces and $T : V \to W$ be linear.*

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for W.

§ Matrix Representation

Definition. *Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*

Definition. *Let $\beta = \{u_1, u_2, \cdots, u_n\}$ be an ordered basis for a finite- dimensional vector space V. For $x \in V$, let $a_1, a_2, \cdots, a_n$, be the unique scalars such that*

$$x = \sum_{i=1}^{\mathrm{n}} a_i u_i$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Remark. *Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.*

Definition. *Using the notation above, we call the $\mathrm{m} \times \mathrm{n}$ matrix A defined by $\mathrm{A}_{ij} = \mathrm{a}_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathrm{A} = [\mathrm{T}]_\beta^\gamma$. If $\mathrm{V} = \mathrm{W}$ and $\beta = \gamma$, then we write $\mathrm{A} = [\mathrm{T}]_\beta$.*
Notice that the jth column of A is simply $[\mathrm{T}(v_j)]_\gamma$. Also observe that if $\mathrm{U} : \mathrm{V} \to \mathrm{W}$ is a linear transformation such that $[\mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma$, then $\mathrm{U} = \mathrm{T}$ by the corollary to Theorem 2.6 (p. 73).

Theorem 2.11. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let* $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ *be linear transformations. Then*

(a) $[\mathrm{T}+\mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma + [\mathrm{U}]_\beta^\gamma$ *and*

(b) $[a\mathrm{T}]_\beta^\gamma = a[\mathrm{T}]_\beta^\gamma$ *for all scalars a.*

Proof. Let $\beta = \{v_j \mid 1 \le j \le n\}$ and $\gamma = \{w_i \mid 1 \le i \le m\} \implies \exists! a_{ij}, b_{ij} \in F$ for $1 \le j \le n$, $1 \le i \le m$ s.t.

1. $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ and $U(v_j) = \sum_{i=1}^{m} b_{ij} w_i$, hence

$$(T+U)(v_j) = \sum_{i=1}^{m} (a_{ij} + b_{ij}) wi$$

Thus $\left([T+U]_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = \left([T]_\beta^\gamma\right)_{ij} + \left([U]_\beta^\gamma\right)_{ij} = \left([T]_\beta^\gamma + [U]_\beta^\gamma\right)_{ij} \implies [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

2. $T(V_j) = \sum_{i=1}^{m} a_{ij} w_i$ for $j = 1, \cdots n$ and $a \in F$ then $aT(v_j) = a \sum_{i=1}^{m} a_{ij} w_i = \sum_{i=1}^{m} a a_{ij} w_i = T(a v_j)$ thus $\left([aT]_\beta^\gamma\right)_{jj} = a a_{ij} = \left(a[T]_\beta^\gamma\right)_{ij} \implies [aT]_\beta^\gamma = a[T]_\beta^\gamma$

■

Theorem 2.12. *Let V, W, and Z be vector spaces over the same field F, and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *and* $\mathrm{U} : \mathrm{W} \to \mathrm{Z}$ *be linear. Then* $\mathrm{UT} : \mathrm{V} \to \mathrm{Z}$ *is linear.*

Proof. Let $x, y \in V$, $a \in F$ $UT(ax+y) = U(T(ax+y)) = U(aT(x)+T(y)) = aUT(x) + UT(y) \Rightarrow UT$ is linear. ■

Theorem 2.13. *Let V be a vector space. Let T,* U_1, $\mathrm{U}_2 \in L(V)$*. Then*

(a) $\mathrm{T}(\mathrm{U}_1 + \mathrm{U}_2) = \mathrm{TU}_1 + \mathrm{TU}_2$ *and* $(\mathrm{U}_1 + \mathrm{U}_2)\mathrm{T} = \mathrm{U}_1\mathrm{T} + \mathrm{U}_2\mathrm{T}$

(b) $\mathrm{T}(\mathrm{U}_1\mathrm{U}_2) = (\mathrm{TU}_1)\mathrm{U}_2$

(c) $TI = IT = T$

(d) $a(\mathrm{U}_1\mathrm{U}_2) = (a\mathrm{U}_1)\mathrm{U}_2 = \mathrm{U}_1(a\mathrm{U}_2)$ *for all scalars* a.

Proof. Let $x \in V$

1. $T\left(U_1 + U_2\right)(x) = T\left(U_1(x) + U_2(x)\right) = TU_2(x) + TU_2(x) \Rightarrow T\left(U_1 + U_2\right) = TU_1 + TU_2$ $\left(U_1 + U_2\right)T(x) = U_1T(x) + U_2T(x) \Rightarrow \left(U_1 + U_2\right)T = U_1T + U_2T$
2. $T\left(U_1U_2\right)(x) = T\left(U_1\left(U_2(x)\right)\right) = \left(TU_1\right)\left(U_2(x)\right) \Rightarrow T\left(U_1U_2\right) = \left(TU_1\right)U_2$
3. $TI(x) = T(x) = IT(x) \Rightarrow TI = T = IT$
4. $a\left(U_1U_2\right)(x) = a\left(U_1\left(U_2(x)\right) = \left(aU_3\right)\left(U_2(x)\right) = U_1\left(aU_2(x)\right) \Rightarrow a\left(U_1U_2\right) = \left(aU_1\right)U_2 = U_1\left(aU_2\right)$

■

Definition (The Product Of Two Matrices)**.** *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times\ p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{\mathrm{n}} A_{ik}B_{kj} \quad for \quad 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

Theorem 2.14. *Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let* $\mathrm{T} : \mathrm{V} \to \mathrm{w}$ *and* $\mathrm{U} : \mathrm{W} \to \mathrm{Z}$ *be linear transformations. Then*

$$[\mathrm{UT}]_\alpha^\gamma = [\mathrm{U}]_\beta^\gamma[\mathrm{T}]_\alpha^\beta$$

.

Proof. Let $\alpha = \{v_i \mid i = 1 \cdots n\}, \beta = \{w_j \mid j = 1 \cdots m\}, \gamma = \{z_k \mid k = 1 \cdots p\}$ $\Longrightarrow\ \exists! a_{ij}, b_{ki} \in F$ for $j = 1, \cdots, m \quad i = 1, \cdots n \quad$ and $k = 1, \cdots, p$ such that

$$T(v_i) = \sum_{i=1}^{m} a_{ij}wj \quad U(w_j) = \sum_{k=1}^{p} b_{kj}z_k$$

$$UT(v_i) = U(T(v_i)) = U\left(\sum_{j=1}^{m} a_{ij} w_j\right) = \sum_{j=1}^{m} a_{ij} U(w_j) = \sum_{j=1}^{m} a_{ji}\left(\sum_{k=1}^{p} b_{kj} z_k\right)$$

$$= \sum_{k=1}^{p}\sum_{j=1}^{m} b_{kj} a_{ji} z_k \quad \text{for} \quad i = 1, \cdots n$$

$$\implies ([UT]_\alpha^\alpha)_{ij} = \sum_{k=1}^{p}\sum_{j=1}^{m} b_{kj} a_{ji} = \left([U]_\beta^\alpha [T]_\alpha^\beta\right)_{ij} \implies [UT]_\alpha^\alpha = [U]_\beta^\alpha [T]_\alpha^\beta$$

■

Corollary 2.14.1. *Let V be a finite-dimensional vector space with an ordered basis β. Let $\mathrm{T}, \mathrm{U} \in L(V)$. Then $[\mathrm{UT}]_\beta = [\mathrm{U}]_\beta [\mathrm{T}]_\beta$.*

Proof. **not yet** ■

Definition. *We define the Kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix In is defined by $(I_n)_{ij} = \delta_{ij}$. Thus, for example,*

$$I_1 = \begin{bmatrix} 1 \end{bmatrix} \; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textit{ and } \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 2.15. *Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices, and D and E be $q \times m$ matrices. Then*

(a) $A(B+C) = AB + AC$ and $(D+E)A = DA + EA$.

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a.

(c) $\mathrm{I}_m A = A = A\mathrm{I}_n$.

(d) If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.

not input now

Corollary 2.15.1. *Let A be an $m \times n$ matrix, $B_1, B_2, \cdots, B_k$ be $n \times p$ matrices, $C_1, C_2, \cdots, C_k$ be $q \times m$ matrices, and $a_1, a_2, \cdots, a_k$ be scalars. Then*

$$A(\sum_{i=1}^{k} a_i B_i) = \sum_{i=1}^{k} a_i A B_i$$

and

$$(\sum_{i=1}^{k} a_i C_i) A = \sum_{i=1}^{k} a_i C_i A.$$

not input now

With this notation, we see that if

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then $A^2 = O$ (the zero matrix) even though $A \neq O$. Thus the cancellation property for multiplication in fields is not valid for matrices. To see why, assume that the cancellation law is valid. Then, from $A \cdot A = A^2 = O = A \cdot O$, we would conclude that $A = O$, which is false.

Theorem 2.16. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each $j (1 \leq j \leq p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then*

(a) $u_j = Av_j$

(b) $v_j = Be_j$, *where e_j is the jth standard vector of* F^p.

Proof. Let $A \in M_{m \times n}(F)$, $b \in M_{n \times p}(F)$ for $j = 1, \cdots, p$.
Let u_j, v_j be the jth columns of AB and B respectively, then

1. $u_j = v_j$

2. $v_j = Be_j$ where e_j is the jth standard. vector of F^n

1. $$u_j = \begin{bmatrix} (AB)_{1j} \\ \vdots \\ (AB)_{mj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} A_{1k} B_{kj} \\ \vdots \\ \sum_{k=1}^{n} A_{mk} B_{kj} \end{bmatrix} = A \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix} = Av_j$$

2. $v_j = \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix} = \begin{bmatrix} (BI)_{1j} \\ \vdots \\ (BI)_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} B_{1k} I_{kj} \\ \vdots \\ \sum_{k=1}^{p} B_{nk} I_{kj} \end{bmatrix} = B \begin{bmatrix} I_{1j} \\ \vdots \\ I_{pj} \end{bmatrix} = Bej$

■

Theorem 2.17. *Assume the notation in Theorem 2.13.*

(a) Suppose that z is a (column) vector in F^p. Use Theorem 2.13(b) to prove that B_z is a linear combination of the columns of B. In particular, if $z = (a_1, a_2, \cdots, a_p)^t$, then show that

$$Bz = \sum_{j=1}^{p} a_j v_j.$$

(b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column j of B.

(c) For any row vector $w \in F^m$, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w. Hint: Use properties of the transpose operation applied to (a).

(d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.

not yet

Theorem 2.18. *Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ be linear. Then, for each $u \in \mathrm{V}$, we have*

$$[\mathrm{T}(u)]_\gamma = [\mathrm{T}]_\beta^\gamma [u]_\beta$$

.

Proof. Fix $u \in V$ define the linear transformation $T : F \longrightarrow V$ by $f(a) = au$ $g : F \longrightarrow W$ by $g(a) = aT(u)$ $\forall a \in F$ Let $\alpha = 1$ be the standard ordered basis of F, note that $g(a) = aT(u) = T(au) = T(f(a)) \implies g = Tf$, then

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma [1]_\gamma^\alpha = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$$

■

<u>Definition</u>. *Let A be an $m \times n$ matrix with entries from a field F . We denote by L_A the mapping $L_A : \mathrm{F}^n \to \mathrm{F}^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in \mathrm{F}^n$. We call L_A a left-multiplication transformation.*

Theorem 2.19. *The characteristics of Left-Multiplication Transformation Let A be an $m \times n$ matrix with entries from F . Then the left-multiplication transformation $\mathrm{L}_A : \mathrm{F}^n \to \mathrm{F}^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties.*

(a) $[\mathrm{L}_A]_\beta^\gamma = A$.

(b) $\mathrm{L}_A = \mathrm{L}_B$ *if and only if* $A = B$.

(c) $\mathrm{L}_{A+B} = \mathrm{L}_A + \mathrm{L}_B$ *and* $\mathrm{L}_{aA} = a\mathrm{L}_A$ *for all* $a \in F$.

(d) *If* $\mathrm{T} : F^n \to F^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $\mathrm{T} = \mathrm{L}_C$. *In fact,* $C = [\mathrm{T}]_\beta^\gamma$.

(e) *If* E *is an* $n \times p$ *matrix, then* $\mathrm{L}_{AE} = \mathrm{L}_A\mathrm{L}_E$.

(f) *If* $m = nL$, *then* $\mathrm{L}_{I_n} = \mathrm{I}_{F^n}$.

Proof.

1. The jth column of $[L_A]_\beta^\gamma = L_A(e_j) \because$ Thm 2.13 $\therefore L_A(e_j) = Ae_j$ is the jth column of $A \implies [L_A]_\beta^\gamma = A$
2. $(\Rightarrow)$ If $L_A = L_B \implies A = [L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$

   $(\Leftarrow)$ If $A = B \implies L_A(x) = Ax = Bx = L_B(x) \implies L_A = L_B$
3. $L_{A+B}(x) = (A + B)(x) = Ax + Bx = L_A(x) + L_B(x)$

   $\implies L_{A+B} = L_A + L_B$

   $L_{aA}(x) = (aA)x = aAx = aL_A(x) \implies L_{aA} = aL_A \ \forall a \in F$

4. Let $C = [T]_\beta^\gamma$ by Thm 2.14 $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta \Leftrightarrow T(x) = Cx = L_C(x)\ \forall x \in F^n \implies T = L_C$ by (2) C is unique.

5. For $j = 1, \cdots, p$ by Thm 2.13 $(AE)e_j$ is the jth column of AE, $(AE)e_j = A(Ee_j) \therefore L_{AE}(e_j) = (AE)e_j = A(Ee_j) = L_A(Ee_j) = L_A(L_E(e_j)) = L_A L_E e_j$ By cor of Thm 2.6 $\therefore L_{AE} = L_A L_E$

6. $L_{I_n}(x) = I_n x = x\ \forall x \in F^n$, $I_{F^n}(x) = x \implies L_{I_n} = I_{F^n}$

■

Corollary 2.19.1. *Let V be a finite-dimensional vector space with an ordered basis β, and let $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ be linear. Then T is invertible if and only if $[\mathrm{T}]_\beta$ is invertible. Furthermore, $[\mathrm{T}^{-1}]_\beta = ([\mathrm{T}]_\beta)^{-1}$*

Proof. **not yet** ■

Corollary 2.19.2. *Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(\mathrm{L_A})^{-1} = \mathrm{L}_{A^{-1}}$*

Proof. **not now** ■

Theorem 2.20. *Let A,B, and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, matrix multiplication is associative.*

Proof. Let A, B and C be metrics $\ni A(BC)$ is defined $\implies (AB)C$ is also defined and $A(BC) = (AB)C$
Let $B \in M_{m\times n}$ $C \in M_{n\times p} \implies (BC) \in M_{m\times p} \implies A \in M_{q\times m}$
$\because (AB) \in M_{q\times n}$ $C \in M_{n\times p} \implies (AB)C$ is defined
$\because$ By Thm 2.15.5 $\therefore L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB}L_C = L_{(AB)C}$
$\because$ By Thm 2.15.2 $\therefore A(BC) = (AB)C$ ■

Lemma. *Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case,* $\dim(v) = \dim(W)$.

Proof. Let linear transform $T : V \to W$ is invertible $\implies V$ is finite dimensional $\Leftrightarrow W$ is finite dimensional
In this case, $\dim(V) = \dim(W)$
$(\Rightarrow)V$ is finite dimension

Let $\beta = \{x_i \mid \text{for } i = 1, \cdots, n\}$ be a basis of V
$\because$ Thm 2.2 $\therefore R(T) = $span$(T(\beta))$, $\because T$ is invertible $\therefore R(T) = W$, $\because$ Thm 1.9 $\therefore W$ is finite dimensional
$(\Leftarrow) W$ is finite dimensional. Similarly, V is finite dimensional
Suppose V, W are finite dimensional $\because T$ is invertible $\Leftrightarrow T$ is one-to-one and onto, $\therefore$ nullity$(T) = 0$
$\because$By Thm 2.3 nullity$(T)+$ rank(T) = dim(V) $\therefore$ dim(V) = rank (T) = dim$(R(T))$ = dim(W) ■

Theorem 2.21. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. Then T is invertible if and only if* $[\mathrm{T}]_\beta^\gamma$ *is invertible. Furthermore,* $[\mathrm{T}^{-1}]_\gamma^\beta = ([\mathrm{T}_\beta^\gamma])^{-1}$.

Proof. Let V, W is finite dimensional vector space, β, γ be a basis of V, W, $T : V \to W$ is a linear transform $\implies T$ is invertible $\Leftrightarrow [T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$
$(\Rightarrow)$ Suppose T is invertible $\because$ lemma of Thm 2.17, we have dim(V) = dim(W). Let $n =$dim(V) $\therefore [T]_\beta^\gamma \in M_{n\times n}$, $T^{-1} : W \to V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$ $\because I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$
$I_n = [I_W]_\gamma = [TT^{-1}]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$
$\therefore [T]_\beta^\gamma [T^{-1}]_{\gamma=I_n}^\beta$ and $[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = I_n \implies [T]_\beta^\gamma$ is invertible and $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$
$(\Leftarrow)$ Suppose $A = [T]_\beta^\gamma$ is invertible $\implies \exists B \in M_{n\times n} \ni AB = BA = I_n$, $\because$ By Thm 2.6 $\therefore \exists U \in \mathscr{L}(W, V) \ni U(w_j) = \sum_{i=1}^{n} B_{ij} v_i$ for $j = 1, \cdots, n$ where $\gamma = \{w_i \mid \text{for } i = 1, \cdots, n\}$ and $\beta = \{v_i \mid \text{for } i = 1, \cdots, n\}$ $\therefore B = [U]_\gamma^\beta$
Claim $U = T^{-1}$, $\because [UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$
$\therefore UT = I_V$ and $TU = I_W$ similarly $\implies U = T^{-1}$ ■

Theorem 2.22. *Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = \left[I_V\right]_{\beta'}^{\beta}$. Then*

(a) Q is invertible.

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Proof. Let β, β' be order basis of finite dimensional vector space V. Let $Q = [I_V]_{\beta'}^\beta$

1. Q is invertible, $\because$ By Thm 2.18 I_V is invertible $\implies$ $[I_V]_{\beta'}^{\beta}$ is invertible $\therefore Q$ is invertible

2. $\forall v \in V,\ [V]_\beta = Q[V]_{\beta'} \because \forall v \in V$

$$[V]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$$

■

Theorem 2.23. *Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then*

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Proof. Let T be linear operator on V which is a finite dimensional vector space, β, β' be order basis of V, Q be a change coordinate matrix from β' to $\beta \implies [T]_{\beta'} = Q^{-1}[T]_\beta Q$
$\because$ By Thm 2.11 $\therefore Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'} = [IT]_{\beta'}^{\beta} = [T]_{\beta'}[I]_{\beta'}^{\beta} = [T]_\beta Q \implies [T]_{\beta'} = Q^{-1}[T]_\beta Q$ ■

Corollary 2.23.1. *Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A] = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.*

Proof. Let $A \in M_{n\times n}(\mathrm{F})$ and γ be an order basis of $\mathrm{F}^n \implies [L_A]_\gamma = Q^{-1}AQ$ where Q is the $n \times n$ matrix whose jth column is the jth vector of γ
Let β be an order basis of $\mathrm{F}^n, \because$ By Thm 2.15 $\therefore [L_A]_\beta = A \because$ By Thm 2.23 $\therefore [L_A]_\gamma = Q^{-1}[L_A]_\beta Q = Q^{-1}AQ$ where $Q = [I]_\gamma^\beta$

■

<u>Definition</u>. *Let A and B be matrices in $M_{n\times n}(\mathrm{F})$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.*

Theorem 2.24. *"is similar to" is an equivalence relation on $M_{n\times n}(\mathrm{F})$*

not yet

Propersition 2.25. *If A and B are similar $n \times n$ matrices, then tr(A) = tr(B).*

not yet

Definition. *Let V and W be vector space over F. We denote the vector space of all linear transformations from V into W by $\mathscr{L}(V,W)$. In the case that $V = W$, we write $\mathscr{L}(V)$ instead of $\mathscr{L}(V,W)$*

Theorem 2.26. *Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then function $\Phi : \mathscr{L}(V,W) \to M_{m\times n}(\mathrm{F})$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathscr{L}(V,W)$, is an isomorphism.*

Corollary 2.26.1. *Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then $\mathscr{L}(V,W)$ is finite-dimensional of dimension mn.*

Lemma 2.27. *Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Then T is an isomorphism if and only if $T(\beta)$ is a basis for W.*

not yet

Theorem 2.28. *Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \cdots, v_n\}$ and $\gamma = \{w_1, w_2, \cdots, w_m\}$, respectively. By Thm 2.6, there exist linear transformations $T_{ij} : V \to W$ such that*

$$T_{ij}(v_k) = \begin{cases} w_i \ \mathit{if} k = j \\ 0 \ \mathit{if} k \neq j \end{cases}$$

First prove that $\{T_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $\mathscr{L}(V,W)$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[T_{ij}]_\beta^\gamma = M^{ij}$. Again by Thm 2.6, there exists a linear transformation $\Phi : \mathscr{L}(V,W) \to M_{m\times n}(\mathrm{F})$ such that $\Phi(T_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

not yet

Definition. *Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to \mathrm{F}^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$.*

Theorem 2.29. *Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspace of V and W, respectively.*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] \arrow[dr, phantom, "(1)\qquad(2)"] & W \arrow[d, "\phi_\gamma"] \\ \mathsf{F}^n \arrow[r, "L_A"'] & \mathsf{F}^m \end{tikzcd}$$

Let V and W be vector spaces of dimension n and m, respectively, and let $T : V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We are now able to use ϕ_β and $\phi\gamma$ to study the relationship between the linear transformations T and $L_A : \mathrm{F}^n \to \mathrm{F}^m$. Let us first consider figure above. Notice that there are two composites of linear transformations that map V into F^m:

1. Map V into F^n with ϕ_β and follow this transformation with L_A; this yields the composite $L_A\phi_\beta$.

2. Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma T$.

These two composites are depicted by the dashed arrows in the diagram. By a simple reformulation of Theorem 2.14 (p. 91), we may conclude that

$$L_A\phi_\beta = \phi_\gamma T$$

that is, the diagram "commutes." Heuristically, this relationship indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ, respectively, we may "identify" T with L_A. This diagram allows us to transfer operations on abstract vector spaces to ones on F^n and F^m.

Theorem 2.30. *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that rank(T) = rank(L_A) and that nullity(T) = nullity(L_A), where $A = [T]_\beta^\gamma$.*

not yet