Lecture Notes on Linear Algebra

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1. Linear Transformation

1.1. Linear Operator.

Definition (Linear Transformation). *Let V and* W *be vector spaces (over* F*). We call a function* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *a linear transformation from* V *to* W *if, for all* x, $y \in \mathrm{V}$ *and* $c \in \mathrm{F}$, *we have*

(*a*) $\mathrm{T}(x+y) = \mathrm{T}(x)+\mathrm{T}(y)$

(*b*) $\mathrm{T}(\mathrm{c}x) = \mathrm{cT}(x)$

Definition. *Let* $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ *be arbitrary functions, where V and W are vector spaces over F, and let* $a \in \mathrm{F}$. *We define* $\mathrm{T}+\mathrm{U} : \mathrm{V} \to \mathrm{W}$ *by* $(\mathrm{T}+\mathrm{U})(x) = \mathrm{T}(x)+\mathrm{U}(x)$ *for all* $x \in \mathrm{V}$, *and* $a\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *by* $(a\mathrm{T})(x) = a\mathrm{T}(x)$ *for all* $x \in \mathrm{V}$.

Remark. *If* $F = Q$ *then* $(a) \Rightarrow (b)$.

Remark. *If* $T : C \to C$ *be function defined by* $T(\delta) = \delta$. T *is additive but not linear.*

Propersition 1.1.

1. *If* T *is linear, then* $T(0) = 0$.
2. T *is linear if and only if* $T(cx+y) = cT(x)+T(y)$ *for all* $x, y \in V$ *and* $c \in F$.
3. *If* T *is linear, then* $T(x-y) = T(x)-T(y)$ *for all* $x, y \in V$

4. *T is linear if and only if, for $x_1, x_2, \cdots, x_n \in V$ and $a_1, a_2, \cdots, a_n \in F$, we have*

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i).$$

5. *For all $a \in F, aT + U$ is linear.*

6. *Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F.*

Definition (Null Space, Range). *Let* V *and* W *be vector spaces and let* T : V $\to$ W *be linear. We define the null space(or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$; that is , $N(T) = \{x \in V : T(x) = 0\}$.*
We define the range(or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V; that is, $R(T) = \{T(x) : x \in V\}$

Remark. *Let V and W be vector spaces, and let $I : V \to V$ and $T_0 : V \to W$ be the identity and zero transformations, respectively. Then $N(I) = \{0\}$, $R(I) = V$, $N(T_0) = V$, and $R(T_0) = \{0\}$.*

Theorem 1.2. *Let V and W be vector spaces and* T : V $\to$ W *be linear. Then $N(T)$ and $R(T)$ are subspaces of* V *and* W, *respectively.*

Remark. *Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$*

Theorem 1.3. *Let V and W be vector spaces, and let* T : V $\to$ W *be linear. If $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for* V *, then*

$$\mathrm{R(T)} = \mathrm{span}(\mathrm{T}(\beta)) = \mathrm{span}(\{\mathrm{T}(v_1), \mathrm{T}(v_2), ..., \mathrm{T}(v_n)\})$$

Example. *Prove Theorem 2.2 for the case that β is infinite, that is, $R(T) = span(\{T(v) : v \in \beta\})$.*

Definition (Nullity and Rank). *Let* V *and* W *be vector spaces and let* T : V $\to$ W *be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted $nullity(T)$, and the rank of T, denoted $rank(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.*

Theorem 1.4 (Dimension Theorem)**.** *Let V and W be vector spaces, and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear.*
If V is finite-dimensional, then

$$nullity(T) + rank(T) = dim(V).$$

Theorem 1.5. *Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.*

Propersition 1.6. *Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.*

(a) Prove that if $\dim(V) < \dim(W)$*, then T cannot be onto.*

(b) Prove that if $\dim(V) > \dim(W)$*, then T cannot be one-to-one.*

Definition (Invertible)**.** *Let V and W be vector spaces, and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. A function* $\mathrm{U} : \mathrm{W} \to \mathrm{V}$ *is said to be an inverse of T if* $\mathrm{TU} = \mathrm{I_W}$ *and* $\mathrm{UT} = \mathrm{I_V}$*. If T has an inverse, then T is said to be invertible. As noted in Appendix B, if T is invertible, then the inverse of T is unique and is denoted by* T^{-1}*.*

Propersition 1.7.

1. $(TU)^{-1} = U^{-1}T^{-1}$.
2. $(T^{-1})^{-1} = T$*; in particular, T^{-1} is invertible.*

Definition. *Let A be an $n \times n$ matrix. Then A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I$.*
If A is invertible, then the matrix B such that $AB = BA = I$ is unique. (If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$.) The matrix B is called the inverse of A and is denoted by A^{-1}*.*

Definition (Isomorphism)**.** *Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *that is invertible. Such a linear transformation is called an isomorphism from V onto W.*

Corollary 1.7.1. *Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.*

Theorem 1.8. *Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if* $\dim(\mathrm{V}) = \dim(\mathrm{W})$.

Corollary 1.8.1. *Let V be a vector space over F. Then V is isomorphic to F^n if and only if* $\dim(V) = n$.

Remark. *The Linearity and Finite-dimensional is essential arguement.*

Example. *Recall the definition of $P(R)$ on page 10. Define*

$$T : P(R) \to P(R) \;\; by \;\; T(f(x)) = \int_0^x f(t)dt.$$

Prove that T linear and one-to-one, but not onto.

Example. *Let $T : P(R) \to P(R)$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.*

Example. *Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U : V \to V$ by*

$$T(a_1, a_2, \cdots) = (a_2, a_3, \cdots) \;\; and \;\; U(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

T and U are called the left shift and right shift operators on V, respectively.

(a) *Prove that T and U are linear.*

(b) *Prove that T is onto, but not one-to-one.*

(c) *Prove that U is one-to-one, but not onto.*

Theorem 1.9. *Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \cdots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$ then $U = T$.*

Propersition 1.10. *Let V and W be vector spaces and $T : V \to W$ be linear.*

(a) *Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.*

(b) *Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.*

(c) *Suppose $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for W.*

§ Matrix Representation

<u>Definition</u>. *Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*

<u>Definition</u>. *Let $\beta = \{u_1, u_2, \cdots, u_n\}$ be an ordered basis for a finite- dimensional vector space V. For $x \in V$, let $a_1, a_2, \cdots, a_n$, be the unique scalars such that*

$$x = \sum_{i=1}^{\mathrm{n}} a_i u_i$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Remark. *Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.*

<u>Definition</u>. *Using the notation above, we call the $\mathrm{m} \times \mathrm{n}$ matrix A defined by $\mathrm{A}_{ij} = \mathrm{a}_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathrm{A} = [\mathrm{T}]_\beta^\gamma$. If $\mathrm{V} = \mathrm{W}$ and $\beta = \gamma$, then we write $\mathrm{A} = [\mathrm{T}]_\beta$.*
Notice that the jth column of A is simply $[\mathrm{T}(v_j)]_\gamma$. Also observe that if $\mathrm{U} : \mathrm{V} \to \mathrm{W}$ is a linear transformation such that $[\mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma$, then $\mathrm{U} = \mathrm{T}$ by the corollary to Theorem 2.6 (p. 73).

Theorem 1.11. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ be linear transformations. Then*

(a) $[\mathrm{T} + \mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma + [\mathrm{U}]_\beta^\gamma$ *and*

(b) $[a\mathrm{T}]_\beta^\gamma = a[\mathrm{T}]_\beta^\gamma$ *for all scalars a.*

Proof. Proof. Let $\beta = \{v_j \mid 1 \le j \le n\}$ and $\gamma = \{w_i \mid 1 \le i \le m\} \implies \exists! a_{ij}, b_{ij} \in F$ for $1 \le j \le n$, $1 \le i \le m$ s.t.

1. $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ and $U(v_j) = \sum_{i=1}^{m} b_{ij} w_i$, hence

$$(T+U)(v_j) = \sum_{i=1}^{m} (a_{ij} + b_{ij}) wi$$

   Thus $\left([T+U]_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = \left([T]_\beta^\gamma\right)_{ij} + \left([U]_\beta^\gamma\right)_{ij} = \left([T]_\beta^\gamma + [U]_\beta^\gamma\right)_{ij}$ $\implies [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

2. $T(V_j) = \sum_{i=1}^{m} a_{ij} w_i$ for $j = 1, \cdots n$ and $a \in F$ then $aT(v_j) = a \sum_{i=1}^{m} a_{ij} w_i = \sum_{i=1}^{m} a a_{ij} w_i = T(av_j)$ thus $\left([aT]_\beta^\gamma\right)_{jj} = a a_{ij} = \left(a[T]_\beta^\gamma\right)_{ij} \implies [aT]_\beta^\gamma = a[T]_\beta^\gamma$

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Theorem 1.12. *Let V, W, and Z be vector spaces over the same field F, and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *and* $\mathrm{U} : \mathrm{W} \to \mathrm{Z}$ *be linear. Then* $\mathrm{UT} : \mathrm{V} \to \mathrm{Z}$ *is linear.*

Proof. Proof. Let $x, y \in V$, $a \in F$ $UT(ax+y) = U(T(ax+y)) = U(aT(x) + T(y)) = aUT(x) + UT(y) \Rightarrow UT$ is linear. ■

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Theorem 1.13. *Let V be a vector space. Let T, U_1, $\mathrm{U}_2 \in L(V)$. Then*

(a) $\mathrm{T}(\mathrm{U}_1 + \mathrm{U}_2) = \mathrm{TU}_1 + \mathrm{TU}_2$ *and* $(\mathrm{U}_1 + \mathrm{U}_2)\mathrm{T} = \mathrm{U}_1\mathrm{T} + \mathrm{U}_2\mathrm{T}$

(b) $\mathrm{T}(\mathrm{U}_1\mathrm{U}_2) = (\mathrm{TU}_1)\mathrm{U}_2$

(c) $TI = IT = T$

(d) $a(\mathrm{U}_1\mathrm{U}_2) = (a\mathrm{U}_1)\mathrm{U}_2 = \mathrm{U}_1(a\mathrm{U}_2)$ *for all scalars a.*

Proof. Proof. Let $x \in V$

1. $T(U_1 + U_2)(x) = T(U_1(x) + U_2(x)) = TU_2(x) + TU_2(x) \Rightarrow T(U_1 + U_2) = TU_1 + TU_2$ $(U_1 + U_2)T(x) = U_1T(x) + U_2T(x) \Rightarrow (U_1 + U_2)T = U_1T + U_2T$

2. $T(U_1U_2)(x) = T(U_1(U_2(x))) = (TU_1)(U_2(x)) \Rightarrow T(U_1U_2) = (TU_1)U_2$

3. $TI(x) = T(x) = IT(x) \Rightarrow TI = T = IT$

4. $a(U_1U_2)(x) = a(U_1(U_2(x)) = (aU_3)(U_2(x)) = U_1(aU_2(x)) \Rightarrow a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$

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Definition (The Product Of Two Matrices)**.** *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{\mathrm{n}} A_{ik}B_{kj} \quad for \quad 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

Theorem 1.14. *Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let* $\mathrm{T} : \mathrm{V} \to \mathrm{w}$ *and* $\mathrm{U} : \mathrm{W} \to \mathrm{Z}$ *be linear transformations. Then*

$$[\mathrm{UT}]_\alpha^\gamma = [\mathrm{U}]_\beta^\gamma [\mathrm{T}]_\alpha^\beta$$

.

Proof. Proof. Let $\alpha = \{v_i \mid i = 1 \cdots n\}, \beta = \{w_j \mid j = 1 \cdots m\}, \gamma = \{z_k \mid k = 1 \cdots p\}$ $\implies \exists! a_{ij}, b_{ki} \in F$ for $j = 1, \cdots, m \quad i = 1, \cdots n \quad$ and $k = 1, \cdots, p$ such that

$$T(v_i) = \sum_{i=1}^{m} a_{ij} wj \quad U(w_j) = \sum_{k=1}^{p} b_{kj} z_k$$

$$UT(v_i) = U(T(v_i)) = U\left(\sum_{j=1}^{m} a_{ij} w_j\right) = \sum_{j=1}^{m} a_{ij} U(w_j) = \sum_{j=1}^{m} a_{ji}\left(\sum_{k=1}^{p} b_{kj} z_k\right)$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{m} b_{kj} a_{ji} z_k \quad \text{for} \quad i = 1, \cdots n$$

$$\implies ([UT]_\alpha^\alpha)_{ij} = \sum_{k=1}^{p} \sum_{j=1}^{m} b_{kj} a_{ji} = \left([U]_\beta^\alpha [T]_\alpha^\beta\right)_{ij} \implies [UT]_\alpha^\alpha = [U]_\beta^\alpha [T]_\alpha^\beta$$

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Corollary 1.14.1. *Let V be a finite-dimensional vector space with an ordered basis β. Let $\mathrm{T}, \mathrm{U} \in L(V)$. Then $[\mathrm{UT}]_\beta = [\mathrm{U}]_\beta[\mathrm{T}]_\beta$.*

Proof. **not yet** ∎

<u>Definition</u>. *We define the Kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix In is defined by $(I_n)_{ij} = \delta_{ij}$. Thus, for example,*

$$I_1 = \begin{bmatrix}1\end{bmatrix} \; I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \; \text{and} \;\; I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Theorem 1.15. *Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices, and D and E be $q \times m$ matrices. Then*

(a) $A(B+C) = AB + AC$ and $(D+E)A = DA + EA$.

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a.

(c) $\mathrm{I}_m A = A = A\mathrm{I}_n$.

(d) If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.

not input now

Corollary 1.15.1. *Let A be an $m \times n$ matrix, $B_1, B_2, \cdots, B_k$ be $n \times p$ matrices, $C_1, C_2, \cdots, C_k$ be $q \times m$ matrices, and $a_1, a_2, \cdots, a_k$ be scalars. Then*

$$A(\sum_{i=1}^{k} a_i B_i) = \sum_{i=1}^{k} a_i A B_i$$

and

$$(\sum_{i=1}^{k} a_i C_i)A = \sum_{i=1}^{k} a_i C_i A.$$

not input now

With this notation, we see that if

$$A = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

then $A^2 = O$ (the zero matrix) even though $A \neq O$. Thus the cancellation property for multiplication in fields is not valid for matrices. To see why, assume that the cancellation law is valid. Then, from $A \cdot A = A^2 = O = A \cdot O$, we would conclude that $A = O$, which is false.

Theorem 1.16. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each $j (1 \leq j \leq p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then*

(a) $u_j = Av_j$

(b) $v_j = Be_j$*, where e_j is the jth standard vector of* F^p*.*

Theorem 1.17. *Assume the notation in Theorem 2.13.*

(a) Suppose that z is a (column) vector in F^p. Use Theorem 2.13(b) to prove that B_z is a linear combination of the columns of B. In particular, if $z = (a_1, a_2, \cdots, a_p)^t$, then show that

$$Bz = \sum_{j=1}^{p} a_j v_j.$$

(b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column j of B.

(c) For any row vector $w \in F^m$, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w. Hint: Use properties of the transpose operation applied to (a).

(d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.

Theorem 1.18. *Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ be linear. Then, for each $u \in \mathrm{V}$, we have*

$$[\mathrm{T}(u)]_\gamma = [\mathrm{T}]_\beta^\gamma [u]_\beta$$

.

<u>Definition</u>. *Let A be an $m \times n$ matrix with entries from a field F . We denote by L_A the mapping $L_A : \mathrm{F}^n \to \mathrm{F}^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in \mathrm{F}^n$. We call L_A a left-multiplication transformation.*

Theorem 1.19. *The characteristics of Left-Multiplication Transformation Let A be an $m \times n$ matrix with entries from F . Then the left-multiplication transformation $\mathrm{L}_A : \mathrm{F}^n \to \mathrm{F}^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties.*

(a) $[\mathrm{L}_A]_\beta^\gamma = A$.

(b) $\mathrm{L}_A = \mathrm{L}_B$ *if and only if* $A = B$.

(c) $\mathrm{L}_{A+B} = \mathrm{L}_A + \mathrm{L}_B$ *and* $\mathrm{L}_{aA} = a\mathrm{L}_A$ *for all* $a \in F$.

(d) *If* $\mathrm{T} : F^n \to F^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $\mathrm{T} = \mathrm{L}_C$*. In fact,* $C = [\mathrm{T}]_\beta^\gamma$.

(e) *If* E *is an* $n \times p$ *matrix, then* $\mathrm{L}_{AE} = \mathrm{L}_A\mathrm{L}_E$.

(f) *If* $m = nL$*, then* $\mathrm{L}_{I_n} = \mathrm{I}_{F^n}$.

Corollary 1.19.1. *Let V be a finite-dimensional vector space with an ordered basis β, and let $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ be linear. Then T is invertible if and only if $[\mathrm{T}]_\beta$ is invertible. Furthermore, $[\mathrm{T}^{-1}]_\beta = ([\mathrm{T}]_\beta)^{-1}$*

Corollary 1.19.2. *Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(\mathrm{L_A})^{-1} = \mathrm{L}_{A^{-1}}$*

Theorem 1.20. *Let A,B, and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, matrix multiplication is associative.*

Lemma. *Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case,* $\dim(v) = \dim(W)$.

Theorem 1.21. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ be linear. Then T is invertible if and only if $[\mathrm{T}]_\beta^\gamma$ is invertible. Furthermore, $[\mathrm{T}^{-1}]_\gamma^\beta = ([\mathrm{T}_\beta^\gamma])^{-1}$.*

Theorem 1.22. *Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = \left[I_V\right]_{\beta'}^{\beta}$. Then*

(a) Q is invertible.

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Theorem 1.23. *Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then*

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Corollary 1.23.1. *Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A] = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.*

Definition. *Let A and B be matrices in $M_{n\times n}(\mathrm{F})$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.*

Theorem 1.24. *"is similar to" is an equivalence relation on $M_{n\times n}(\mathrm{F})$*

Propersition 1.25. *If A and B are similar $n \times n$ matrices, then tr(A) = tr(B).*

Definition. *Let V and W be vector space over F. We denote the vector space of all linear transformations from V into W by $\mathscr{L}(V,W)$. In the case that $V = W$, we write $\mathscr{L}(V)$ instead of $\mathscr{L}(V,W)$*

Theorem 1.26. *Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then function $\Phi : \mathscr{L}(V,W) \to M_{m\times n}(\mathrm{F})$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathscr{L}(V,W)$, is an isomorphism.*

Corollary 1.26.1. *Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then $\mathscr{L}(V,W)$ is finite-dimensional of dimension mn.*

Lemma 1.27. *Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Then T is an isomorphism if and only if $T(\beta)$ is a basis for W.*

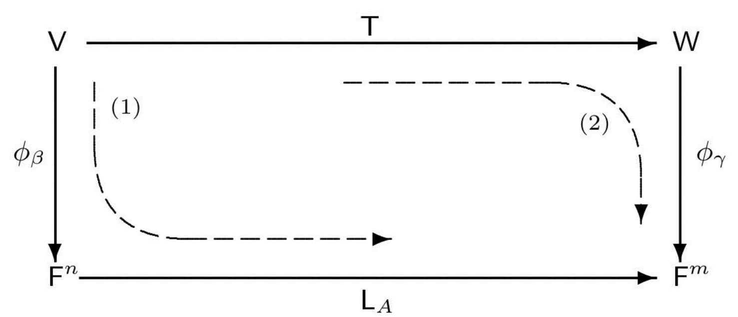
Theorem 1.28. *Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \cdots, v_n\}$ and $\gamma = \{w_1, w_2, \cdots, w_m\}$, respectively. By Thm 2.6, there exist linear transformations $T_{ij} : V \to W$ such that*

$$T_{ij}(v_k) = \begin{cases} w_i \ \mathit{if} k = j \\ 0 \ \mathit{if} k \neq j \end{cases}$$

First prove that $\{T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $\mathscr{L}(V, W)$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[T_{ij}]_\beta^\gamma = M^{ij}$. Again by Thm 2.6, there exists a linear transformation $\Phi : \mathscr{L}(V, W) \to M_{m\times n}(\mathrm{F})$ such that $\Phi(T_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

<u>Definition</u>. *Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to \mathrm{F}^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$.*

Theorem 1.29. *Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspace of V and W, respectively.*



Let V and W be vector spaces of dimension n and m, respectively, and let $T : V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We are now able to use ϕ_β and $\phi\gamma$ to study the relationship between the linear transformations T and $L_A : \mathrm{F}^n \to \mathrm{F}^m$. Let us first consider figure above. Notice that there are two composites of linear transformations that map V into F^m:

1. Map V into F^n with ϕ_β and follow this transformation with L_A; this yields the composite $L_A\phi_\beta$.

2. Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma T$.

These two composites are depicted by the dashed arrows in the diagram. By a simple reformulation of Theorem 2.14 (p. 91), we may conclude that

$$L_A \phi_\beta = \phi_\gamma T$$

that is, the diagram "commutes." Heuristically, this relationship indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ, respectively, we may "identify" T with L_A. This diagram allows us to transfer operations on abstract vector spaces to ones on F^n and F^m.

Theorem 1.30. *Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that rank$(T) =$ rank(L_A) and that nullity$(T) =$ nullity(L_A), where $A = [T]_\beta^\gamma$.*