# Lecture Notes on Linear Algebra

CARAPAERO

## 1. Vector Space

<u>**Definition**</u> (Vector Space). *A vector space (or linear space) $W$ over a Field $\mathcal{F}$ consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements $x, y$, in $W$ there is a unique element $x + y$ in $W$, and for each element $a$ in $F$ and each element $x$ in $W$ there is a unique element $ax$ in $W$, such that the following conditions hold.*

### § Subspace .

<u>**Definition**</u> (Subspace). *A subset $W$ of a vector space $W$ over a field $\mathbb{F}$ is called a subspace of $W$ if $W$ is a vector space over $\mathrm{F}$ with the operations of addition and scalar multiplication defined on $W$.*

**Remark.** *Trivial subspaces of a vector space $V$, namely $V$ itself and $\{0\}$. Note that empty set $\phi$ is not a vector space, since it does not contains a zero vector.*

**Theorem 1.1.** *Let $V$ be a vector space and $W$ a subset of $V$. Then $W$ is a subspace of $V$ if and only if the following three conditions hold for the operations defined in $V$.*
*(a)* $0 \in W$.
*(b)* $x + y \in W$ *whenever* $x \in W$ *and* $y \in W$.
*(c)* $cx \in W$ *whenever* $c \in \mathrm{F}$ *and* $x \in W$.

**Corollary 1.1.1.** *Let $W$ be a subset of vector space $V$. $W$ is a subspace of $V$ if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and, $x, y \in W$.*

*Proof.* (⇒) Since W is a subspace $\implies$ $0 \in W$ , $ax \in W$ for $a \in \mathcal{F}$ and $ax, y \in W \implies ax + y \in W$. (⇐) $0 \in W$. Since $ax + y \in W$, $a \in \mathcal{F}$, take $a = 1 \implies x + y \in W$, and also $\because 0 \in W$, take $y = 0 \implies ax \in W$, for $a \in \mathcal{F}$. Hence, W isa subspace. ■

**Theorem 1.2.** *Any intersection of subspaces of a vector space $W$ is a subspace of $V$.*

**Theorem 1.3.** *Let $W_1$ and $W_2$ be subspaces of a vector space $V$, then $W_1 \cup W_2$ is a subspace of $V$ if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

*Proof.* Suppose $W_1 \cup W_2$ is a subspace of $V$, we assume that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Let $x \in W_1 \backslash W_2$ , $y \in W_2 \backslash W_1$, then $x + y \in W_1$ or $W_2$, say $W_1$. But $y = (x + y) - x \in W_1$, which is a contradiction. So we have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Conversely is trivial. ■

**Definition.** *If $S_1$ and $S_2$ are nonempty subsets of a vector space $V$, then the sum of $S_1$ and $S_2$, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.*

**Theorem 1.4.** *Let $W_1$ and $W_2$ be subspaces of a vector space $V$.*

*(a) $W_1 + W_2$ is a subspace of $V$ that contains both $W_1$ and $W_2$.*

*(b) Any subspace of $V$ that contains both $W_1$ and $W_2$ must also contain $W_1 + W_2$.*

*Proof.* To prove $(a)$, we first show that $W_1 + W_2$ is a subspace of $V$. Clearly, $0 = 0 + 0 \in W_1 + W_2$. Let $x = x_1 + x_2$, $y = y_1 + y_2$, $c \in \mathbb{F}$
$cx + y = c(x_1 + x_2) + (y_1 + y_2) = (cx_1 + y_1) + (cx_2 + y_2) \in W_1 + W_2$
By corrolary 1.1.1, $W_1 + W_2$ is a subspace of $V$.
Then we show that $W_1, W_2 \subseteq W_1 + W_2$. $\forall x \in W_1 y \in W_2$, $x = x + 0 \in W_1 + W_2$, $y = 0 + y \in W_1 + W_2$, we have $W_1, W_2 \subseteq W_1 + W_2$.
To prove $(b)$, Let $W$ be a subspace of $V$ contains both $W_1$ and $W_2$, then $\forall x \in W_1, y \in W_2$, $x + y \in W_1 + W_2$. Thus $W_1 + W_2 \subseteq W$. ■

**Definition** (Direct Sum). *A vector space $V$ is called the direct sum of $W_1$ and $W_2$ if $W_1$ and $W_2$ are subspaces of $V$ such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that $V$ is the direct sum of $W_1$ and $W_2$ by writing $V = W_1 \oplus W_2$.*

**Theorem 1.5.** *Let $W_1$ and $W_2$ be subspaces of a vector space $V$. $V$ is the direct sum of $W_1$ and $W_2$ i.e.*

$$V = W_1 \oplus W_2$$

*if and only if each vector in $V$ can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$*

*Solution.* Suppose that we can write $v = x_1 + x_2 = y_1 + y_2$ with $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Since $x_1 + x_2 = y_1 + y_2$ and $W_1, W_2$ are subsapces of a vector space $V$, so $x_1 - y_1 = x_2 - y_2 \in W_1 \cap W_2$. As $V = W_1 \oplus W_2$, $x_1 - y_1 = x_2 - y_2 = \{0\}$, that is, $x_1 = y_1$ and $x_2 = y_2$. Hence it is a unique representation. Conversely, suppose that the condition holds. We claim that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Since $W_1 + W_2$ is the smallest subspace of $V$, so $W_1 + W_2 \subseteq V$. By assumption, it is clear to get that $V \subseteq W_1 + W_2$. This proves $V = W_1 + W_2$. Since $W_1, W_2$ are subspaces of $V$, so $0 \in W_1$ and $0 \in W_2$. Let $v$ be a vector in $W_1 \cap W_2$. From the uniqueness of decomposition and $v = 0 + v = v + 0$, it implies that $v = 0$. This proves $W_1 \cap W_2 = \{0\}$. In conclusion, $V = W_1 \oplus W_2$. ■

### § Linear Combinations and Bases.

**<u>Definition</u>** (Linearly Dependent). *A subset $S$ of a vector space $W$ is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, ..., u_n$ in $S$ and scalars $a_1, a_2, ..., a_n$, not all zero, such that*

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

*In this case we also say that the vectors of $S$ are linearly dependent.*

**<u>Definition</u>** (Linearly Independent). *A subset $S$ of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of $S$ are linearly independent.*

**Remark.**

1. *The empty set is linearly independent, for linearly dependent sets must be nonempty.*

2. *A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar $a$. Thus*

$$u = a^{-1}(au) = a^{-1}0 = 0$$

*3. A set is linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations.*

**Theorem 1.6.** *Let $V$ be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If $S_1$ is linearly dependent, then $S_2$ is linearly dependent.*

*Proof.* Since $S_1 = \{u_1, \cdots, u_m\}$ is linearly dependent i.e. $\exists\ a_1, \cdots, a_m \in \mathcal{F}$ are not all zero s.t.

$$a_1u_1 + \cdots + a_mu_m = 0 \qquad (1)$$

Since $S_1 \subseteq S_2$ say $S_2 = S_1 \cup \{u_{m+1}, \cdots, u_n\}$. Let

$$b_1u_1 + \cdots + b_mu_m + b_{m+1}u_{m+1} + \cdots + b_nu_n = 0\ b_j \in \mathcal{F}, j = 1, 2, 3, \cdots$$

Now, we take $b_{m+1} = \cdots = b_n = 0$ and by (1) there exists $a_1, \cdots, b_n$ are not all zeros. Hence, $S_2$ is linearly dependent. ∎

**Corollary 1.6.1.** *Let $S$ be a linearly independent subset of a vector space $V$, and let $v$ be a vector in $V$ that is not in $S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.*

*Proof.* Suppose $S \cup v$ is linear independent, then $\exists v_1, ..., v_n \in S \cup v$ such that $a_1u_1 + ... + a_nu_n = 0$ for some nonzero scalars $a_1, ..., a_n$. Because $S$ is linear independent, say $u_1 = v$. We have $v = a_1^{-1}(-a_2u_2 - ... - a_nu_n)$, which is a linear combination of $u_2, ..., u_n$. Thus $v \in span(S)$. Conversely, suppose $v \in span(S)$. Then $\exists v_1, ..., v_n \in S$ and scalars $b_1, ..., b_m$ such that $v = b_1v_1 + ... + b_mv_m$. Thus $0 = b_1v_1 + ... + b_mv_m + (-1)v$ which means $S \cup v$ is linear independent. ∎

**Theorem 1.7.** *Let $S$ be a linearly independent subset of a vector space $V$, and let $v$ be a vector in $V$ that is not in $S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in$ span($S$).*

*Proof.* Let $\beta$ be a basis of $v$. Suppose $v = a_1u_1 + ... + a_nu_n = b_1u_1 + ... + b_nu_n$. We have $0 = (a_1 - b_1)u_1 + ... + (a_n - b_n)u_n$. Since $\beta$ is linear independent, we have $(a_1 - b_1) = ... = (a_n - b_n)$, hence $a_1 = b_1, ..., a_n = b_n$, which means $v$ is uniquely expressed as a linear combination of $\beta$. Conversely, suppose $v$ can be uniquely expressed as a linear combination of $\beta$. Clearly, $\beta$ generates $V$ and the representation of 0 is unique. So if $0 = a_1u_1 + ... + a_nu_n$, we have $a_1 = ... = a_n$. Thus $\beta$ is linear independent and is a basis of $V$. ∎

**Definition** (Basis). *A basis $\beta$ for a vector space $V$ is a linearly independent subset of $V$ that generates $V$. If $\beta$ is a basis for $V$, we also say that the vectors of $\beta$ form a basis for $V$.*

**Theorem 1.8.** *Let $V$ be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of $V$. Then $\beta$ is a basis for $V$ if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$, that is, can be expressed in the form*

$$V = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

*for unique scalars $a_1, a_2, \cdots, a_n$.*

*Proof.* If $S = \emptyset$ or $S = 0$, then $V = \{0\}$ and $\emptyset$ is a subset of $S$ which is a basis of $V$. Otherwise $S$ contains a nonzero vector $\{u_1\}$. We continue choosing vectors $u_2, ..., u_k \in S$ such that $\beta = \{u_1, ..., u_k\}$ is linear independent and $\{u_1, ..., u_k, v\}$ is linear dependent for some vector $v \in V \backslash S$. To show that $\beta$ is a basis, it suffices toshow that $\beta$ generates $V$. Let $v \in S$. If $v \in \beta$, then clearly $\beta \in span(S)$. If $v \in S \backslash \beta$, then we have $\beta \cup v$ is linear dependent. So $v \in span(\beta)$. Thus $S \subseteq span(\beta)$. ■

**Theorem 1.9.** *If a vector space $V$ is generated by a finite set $S$, then some subset of $S$ is a basis for $V$. Hence $V$ has a finite basis.*

*Proof.* We prove it by mathematical induction on $m$. For $m = 0$, $L = \emptyset$, so take $H = G$. Assume the theorem holds for some integer $m \geq 0$. For $m+1$, let $L = \{v_1, ..., v_{m+1}\}$ be a linear independent subset of $V$ containing $m+1$ vectors. Because $\{v_1, ..., v_m\}$ is linear independent, so by the induction hypothesis, we have $m \geq n$ and $\exists$ a subset $\{u_1, ..., u_{n-m}\}$ of $G$ such that $\{v_1, ..., v_m\} \cup \{u_1, ..., u_{n-m}\}$ generates $v$. Thus $\exists$ scalars $a_1, ..., a_m, b_1, ..., bn - m$ such that $a_1v_1 + ... + a_mv_m + b_1u_1 + ... + b_{n-m}u_{n-m} = v_{m+1}$. We have $n - m > 0$ and $v_{m+1}$ is a linear combination of $\{v_1, ..., v_m\}$ which contradicts the assumption that $L$ is linear independent. Hence $n > m$ that is $n \geq m + 1$. Moreover, say $b_1$ is nonzero, otherwise we obtain the same contradiction. We have $u_1 = (-b_1^{-1}a_1)v_1 + ... + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1} + (-b_1^{-1}b_2)u_2 + ... + (-b_1^{-1}b_{n-m})u_n - m$. Let $H = \{u_2, ..., u_{n-m}\}$. Then $u_1 \in span(L \cup H)$ and $v_1, ..., v_m, u_2, ..., u_{n-m} \in span(L \cup H)$, so $\{v_1, ..., v_m, u_1, ..., u_{n-m}\} \subseteq span(L \cup H)$. We have $span(L \cup H) = V$. Since $H$ is a subset of $G$ contains $(n - m) - 1 = n - (m + 1)$ vectors. So by mathematical induction, we are done. ■

**Theorem 1.10** (Replacement Theorem)**.** *Let $V$ be a vector space that is generated by a set $G$ containing exactly $n$ vectors, and let $L$ be a linearly independent subset of $V$ containing exactly $m$ vectors. Then $m \leq n$ and there exists a subset $H$ of $G$ containing exactly $n-m$ vectors such that $L \cup H$ generates $V$.*

*Proof.* Let $dim(V) = n$. If $W = \{0\}$, then $W$ is finite-dimensional and $dim(W) = 0 \leq n$. Otherwise, $W$ contains a nonzero vector $x_1$, so $\{_1\}$ is a linear independent set. We continue choosing vectors $x_1, ..., x_k \in W$ such that $x_1, ..., x_k$ is linear independent. This process must stop at $k \leq n$ and $\{x_1, ..., x_k\}$ is linear independent and other vector in $W$ produces a linear dependent set. Hence it is a basis of $W$. Thus $dim(W) = k \leq n$. If $dim(W) = n$, a basis of $W$ is a linear independent subset of $V$ contains $n$ vectors, so a basis of $W$ is also a basis of $V$. Thus $W = V$.

■

**Corollary 1.10.1.** *Let $V$ be a vector space having a finite basis. Then every basis for $V$ contains the same number of vectors.*

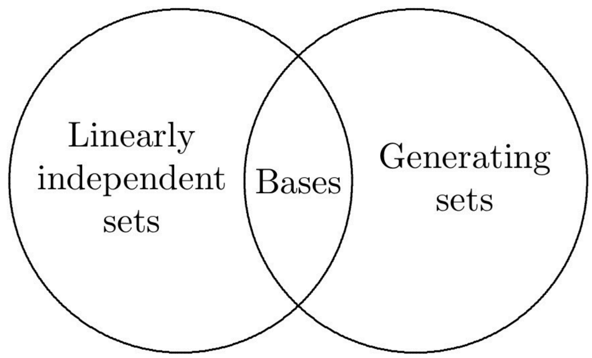
**Definition** (Finite-Dimensional)**.** *A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for $V$ is called the dimension of $V$ and is denoted by $dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.*

**Corollary 1.10.2.** *Let $V$ be a vector space with dimension $n$.*

1. *Any finite generating set for $V$ contains at least $n$ vectors, and a generating set for $V$ that contains exactly $n$ vectors is a basis for $V$.*

2. *Any linearly independent subset of $V$ that contains exactly $n$ vectors is a basis for $V$.*

3. *Every linearly independent subset of $V$ can be extended to a basis for $V$.*




**Theorem 1.11.** *Let $W$ be a subspace of a finite-dimensional vector space $V$. Then $W$ is finite-dimensional and* $\dim(W) \leq \dim(V)$*. Moreover, if* $\dim(W) = \dim(V)$*, then* $V = W$.

*Proof.* Since $V$ is a finite-dimensional vector space, say $\dim(V) = n$. If $W = \{0\}$, then $W$ is a finite-dimensional and $\dim(W) = 0 \leq n$. Otherwise, $W$ contains a nonzero vector $x_1$ ; so $\{x_1\}$ is linearly independent set. Continue choosing vectors, $x_1, \cdots, x_k$ in $W$ such that $S = \{x_1, \cdots, x_k\}$ is linearly independent. Since no linearly independent subset can contain more than $n$ vectors in $V$, therefore $S \cup \{v\}$ is linearly dependent for $\{v\} \in span(S)$ then by Thm 1.7 $S$ generates $W$. $S$ is a basis of $W \implies \dim(W) = k \leq n$. If $\dim(W) = n$, then a basis for $W$ is a linearly independent subset of $V$ containing $n$ vectors. By replacement theorem any linearly independent subset of $V$ contains exactly $n$ vectors is also a basis of $V$. Hence $V = W$. ∎

**Propersition 1.12.** *Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $V$. $W_1 \subseteq W_2$ if and only if* $\dim(W_1 \cap W_2) = \dim(W_1)$

**Theorem 1.13.** *Let $v_1, v_2, \cdots, v_k, v$ be vectors in a vector space $V$, and define $W_1 = span(\{v_1, v_2, \cdots, v_k\})$, and $W_2 = span(\{v_1, v_2, \cdots, v_k, v\})$. Then* $v \in \mathrm{span}(W_1)$ *if and only if* $\dim(W_1) = \dim(W_2)$.

**Remark.** *We may give an example for satisfying the conditions on above but* $\dim(W_1) \neq \dim(W_2)$.

**Theorem 1.14.** *Let $W_1$ and $W_2$ be finite-dimensional subspaces of a vector space $V$.*

*(a) Then the subspace $W_1 + W_2$ is finite-dimensional, and*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(b) *Let $V = W_1 + W_2$. Deduce that $V$ is the direct sum of $W_1$ and $W_2$ if and only if*

$$\dim(V) = \dim(W_1) + \dim(W_2)$$

*Proof.*

(a). Let $\beta = \{ u_1, u_2, \cdots, u_k \}$ is a basis of $W_1 \cap W_2$ with $\dim(W_1) = k + m$, $\dim(W_2) = k + n$ and $\dim(W_1 \cap W_2) = k$ for $k, m, n \in \mathbb{N}$. Since $\beta \in W_1$ and $\beta \in W_2$ by Replacement Theorem, every linearly independent subset of $V$ can be extended to a basis for $V$.

$\exists\ \beta_1 = \{ u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m \}$ is a basis of $W_1$ and $\beta_2 = \{ u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_n \}$ is a basis of $W_2$. Let $x \in W_1 + W_2$.

**Claim.** $Span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) = W_1 + W_2$

$x = (a_1u_1 + a_2u_2 + \cdots + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m) + (b_1u_1 + b_2u_2 + \cdots + b_ku_k + b_{k+1}w_1 + b_{k+2}w_2 + \cdots + b_{k+n}w_n), a_i, b_j \in \mathcal{F}$ for $i, j = 1, 2, \cdots$

$= c_1u_1 + c_2u_2 + \cdots + c_ku_k + a_{k+1}v_1 + a_{k+2}v_2 + \cdots + a_{k+m}v_m + b_{k+1}w_1 + b_{k+2} + \cdots + b_{k+n}w_n, c_1, c_2, \cdots, c_k \in \mathcal{F} \implies \forall\ x \in W_1 + W_2$

$\therefore W_1 + W_2 \subseteq span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \})$

$\because W_1 + W_2$ is a subspace, any linear combination of $W_1 + W_2$'s subset are in $W_1 + W_2$

$\therefore span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) \in W_1 + W_2$

$\therefore span(\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}) = W_1 + W_2$

**Claim.** $\{ u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n \}$ is linearly independent

$$\sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i + \sum_{i=1}^{n} c_iw_i = 0 \implies \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i = -\sum_{i=1}^{n} c_iw_i$$

$$\because \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i \in W_1\ ,\quad -\sum_{i=1}^{n} c_iw_i \in W_2$$

$$\therefore \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i\ ,\quad -\sum_{i=1}^{n} c_iw_i \in W_1 \cap W_2$$

$$\implies \exists\ d_i \in \mathcal{F}, \sum_{i=1}^{k} a_iu_i + \sum_{i=1}^{m} b_iv_i = -\sum_{i=1}^{n} c_iw_i = \sum_{i=1}^{k} d_iu_i$$

$\because \beta_1, \beta_2$ is linearly independent $\therefore -\sum_{i=1}^{n} c_i w_i = \sum_{i=1}^{k} d_i u_i$ only for scalars are all zeros. $\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i = 0$ only for scalars are all zero

$\therefore \{u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_n\}$ is linearly independent

$\therefore \{\, u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_n \,\}$ is a basis of $W_1 + W_2$

$\therefore \dim(W_1 + W_2) = k + m + n$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) + \dim(W_1 \cap W_2)$

(b). $W_1 \cap W_2 = \{\, 0 \,\}$

by Exercise 1.16.29(a), if $W_1$ and $W_2$ are finite-dimensional subspace of a vector space V, dim(V) = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(W_1 \cap W_2)$ = $\dim(W_1)$ + $\dim(W_2)$ - $\dim(\{\, 0 \,\})$ = $\dim(W_1)$ + $\dim(W_2)$

■

**Theorem 1.15.** *Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that $V = W_1 \oplus W_2$ if and only if there exist base $\beta_1$ , $\beta_2$ of $W_1$ , $W_2$, respectively such that $\beta_1 \cup \beta_2$ is a basis for $V$.*

*Proof.* Let $\beta_1 = \{\, v_1, v_2, \cdots, v_n \,\}, v_1, v_2, \cdots, v_n \in W_1$,
$\beta_2 = \{\, u_1, u_2, \cdots, u_m \,\}, u_1, u_2, \cdots, u_m \in W_2$

$$W_1 + W_2 = \left\{ \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j u_j \mid a_1, \cdots, a_n, b_1, \cdots, b_m \in \mathcal{F} \right\}$$

**Claim :** $\mathrm{Span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$
let $x \in \mathrm{Span}(\beta_1 \cup \beta_2)$ , $x = a_1 v_1 + a_2 v_2 + \cdots + b_1 u_1 + b_2 u_2 + \cdots + b_m u_m$
$\because W_1, W_2$ is a subspace of V $\therefore \sum_{i=1}^{n} a_i v_i \in W_1$ , $\sum_{i=1}^{m} b_i u_i \in W_2$
$\therefore x \in W_1 + W_2, \mathrm{Span}(\beta_1 \cup \beta_2) \subseteq W_1 + W_2$

**Claim.** $W_1 + W_2 \subseteq \mathrm{Span}(\beta_1 \cup \beta_2)$
Let $x \in W_1 + W_2, x = (a_1 v_1 + \cdots + a_n v_n) + (b_1 u_1 + \cdots + b_m u_m)$
$\therefore x \in \mathrm{Span}(\beta_1 \cup \beta_2) \therefore W_1 + W_2 \subseteq \mathrm{Span}(\beta_1 \cup \beta_2) \therefore \mathrm{Span}(\beta_1 \cup \beta_2) = W_1 + W_2$

**Claim** $\beta_1 \cup \beta_2$ is linearly independent

$$\sum_{i=1}^{n} a_i v_1 + \sum_{j=1}^{m} b_j u_j = 0 \implies \sum_{i_1}^{n} a_i v_i = -\sum_{j=1}^{m} b_j u_j$$

$$\because \sum_{i=1}^{n} a_i v_i \in W_1 \ , \ \ -\sum_{j=1}^{m} b_j u_j \in W_2 \text{ and } \sum_{i=1}^{n} a_i v_i \ , \ \ -\sum_{i=1}^{m} b_i u_i \in W_1 \cap W_2$$

$$\because W_1 \cap W_2 \therefore \sum_{i=1}^{n} a_i v_i = -\sum_{j=1}^{m} b_j u_j = 0 \quad \because \beta_1, \beta_2 \text{ is linearly independent}$$

$\therefore a_1 = \cdots = a_n = b_1 = \cdots = b_m = 0 \therefore \beta_1 \cup \beta_2$ is linearly independent
$\therefore \beta_1 \cup \beta_2$ is a basis of V.

■

**Theorem 1.16.**

*If $W_1$ is any subspace of vector space of $V$, then there exists a subspace $W_2$ of $V$ such that*

$$V = W_1 \oplus W_2$$

*Proof.* let $\beta_1 = \{ v_1, v_2, \cdots, v_m \}$ and $\dim(V) = n$. By Corollary of Replacement Theorem, Every linearly independent subset of V can be extended to a basis for V then $\exists\ \beta = \{ v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_{n-m} \}$ is a basis of $V$. Let $W_2 = \operatorname{span}(\{u_1, u_2, \cdots, u_{n-m}\}) \because u_1, u_2, \cdots, u_{n-m} \in V, V$ is a vector space by Thm 1.5, the span of any subset S of a vector space V is a subspace.
$\therefore W_2$ is a subspace of V. Now we claim that $W_1 \cap W_2 = \{0\}$
$\because W_1, W_2$ is a subspace of V $\therefore 0 \in W_1, W_2$. Assume $\exists\ r \in V$ and $r \in W_1$ , $r \in W_2$ , $r \neq 0$.

$$\begin{aligned} r &= a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \\ &= b_1 u_1 + b_2 u_2 + \cdots + b_{n-m} u_{n-m} \\ &= c_1 v_1 + \cdots + c_m v_m + d_1 u_1 + \cdots + d_{n-m} u_{n-m} \end{aligned}$$

$$\implies \begin{cases} (c_1 - a_1)v_1 + \cdots + (c_m - a_m)v_m + d_1 u_1 + \cdots + d_{n-m} u_{n-m} = 0 \\ c_1 v_1 + \cdots + c_m v_m + (d_1 - b_1)u_1 + \cdots + (d_{n-m} - b_{n-m}) \end{cases}$$

$\implies c_1 = a_1, c_2 = a_2, \cdots, c_m = a_m, d_1 = b_1, \cdots, d_{n-m} = b_{n-m}$
$\Rightarrow r = r + r \Rightarrow r = 0 \rightarrow\leftarrow \therefore W_1 \cap W_2 = \{0\}$.

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