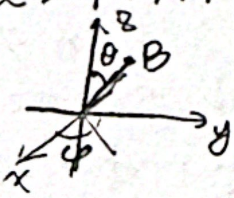
Quantum Computing HW3 房佰文 2024010794

[1] (a) $B_x = B\sin\theta\cos\varphi$



$B_y = B\sin\theta\sin\varphi$

$B_z = B\cos\theta$

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$H = -\mu B\cdot\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\ \cancel{-\sin\theta}\end{pmatrix}$

(b) $\begin{pmatrix}\cos\theta-\lambda & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\ \cancel{-\sin\theta}-\lambda\end{pmatrix}$

$= -\cos\theta\,\cancel{\sin\theta}\cos\theta - \cancel{\lambda(\cos\theta-\sin\theta)} + \lambda^2$

$-\sin^2\theta$

$= \cancel{\lambda^2 - (\cos\theta-\sin\theta)\lambda - \sin\theta(\cos\theta+\sin\theta)}$

$\lambda^2-1$

$\therefore \lambda = \pm 1$

① For $\lambda = 1$

$\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta-1\end{pmatrix}\psi_1 = 0$

$\Rightarrow \psi_1 = \begin{pmatrix}\sin\theta e^{-i\varphi}\\ 1-\cos\theta\end{pmatrix}$,

② For $\lambda = -1$

$\psi_2 = \begin{pmatrix}\sin\theta e^{-i\varphi}\\ -1-\cos\theta\end{pmatrix}$

↓ normalize

$\psi_1 = \begin{pmatrix}\cos\theta/2\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}$ $\psi_2 = \begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\varphi}\cos\theta/2\end{pmatrix}$

(c) $\frac{\partial\psi_1}{\partial\varphi} = \begin{pmatrix}\cancel{\cos\theta/2}\ 0\\ i\sin\theta/2\, e^{i\varphi}\end{pmatrix},\ \frac{\partial\psi_2}{\partial\varphi} = \begin{pmatrix}\cancel{-\sin\theta/2}\ 0\\ i\cos\theta/2\, e^{i\varphi}\end{pmatrix}$

$\langle\psi_1|\dot\psi_1\rangle = \dot\varphi\cdot i\sin\frac{\theta}{2} = i\frac{1-\cos\theta}{2}\dot\varphi$

$\langle\psi_2|\dot\psi_2\rangle = i\frac{1+\cos\theta}{2}\dot\varphi$

We use

$$\begin{cases}\dot a_1(t) = -\langle\psi_1|\dot\psi_1\rangle a_1(t) = -i\dot\varphi\frac{1-\cos\theta}{2}a_1(t)\\ \dot a_2(t) = -\langle\psi_2|\dot\psi_2\rangle a_2(t) = -i\dot\varphi\frac{1+\cos\theta}{2}a_2(t)\end{cases}$$

$\ln a_1 = -i\frac{1-\cos\theta}{2}\Delta\varphi + C$

$$\begin{cases}a_1 = C_1 e^{-i\frac{1-\cos\theta}{2}\varphi}\\ a_2 = C_2 e^{-i\frac{1+\cos\theta}{2}\varphi}\end{cases}$$

& when $\varphi = 0$, $a_1 = 1$ $a_2 = -1$

$\therefore C_1 = 1$ $C_2 = -1$

$$\begin{cases}a_1 = e^{-i\frac{1-\cos\theta}{2}\varphi}\\ a_2 = -e^{-i\frac{1+\cos\theta}{2}\varphi}\end{cases}$$

$\Delta\varphi_1 = -\pi(1-\cos\theta)$ $\Delta\varphi_2 = -\pi(1+\cos\theta)$

$\Delta\varphi = \Delta\varphi_1 - \Delta\varphi = 2\pi\cos\theta$

[2] (a) $f(\lambda) = e^{\lambda A+B} = \sum_n\frac{(\lambda A+B)^n}{n!}$

$\cancel{\frac{\partial f}{\partial\lambda}} = A\sum_n\frac{(\lambda A+B)^{n-1}}{(n-1)!} =$

$\frac{\partial}{\partial\lambda}(\lambda A+B)^n = A(\lambda A+B)^{n-1} + (\lambda A+B)A(\lambda A+B)^{n-2} + \cdots$

$= a_1 + a_2 + \cdots + a_n$

$a_{i+1} - a_i = (\lambda A+B)^{i-1}[(\lambda A+B)A - A(\lambda A+B)](\lambda A+B)^{n-i-1}$

$= (\lambda A+B)^{i-1}[B,A](\lambda A+B)^{n-i-1}$

$= [B,A](\lambda A+B)^{n-2} = T$

$\therefore \sum_{i=1}^{n} a_i = a_1 + (a_1+T) + (a_1+2T) + \cdots + (a_1+(n-1)T)$

$= na_1 + \frac{n(n-1)}{2}T$

$\therefore \frac{\partial f}{\partial\lambda} = \sum_n\frac{1}{n!}\left[nA(\lambda A+B)^{n-1} + \frac{1}{2}n(n-1)[B,A](\lambda A+B)^{n-2}\right]$

$= Af(\lambda) - \frac{1}{2}[A,B]f(\lambda)$

$\int\cancel{\frac{1}{f}\frac{\partial f}{\partial\lambda}d\lambda}$ $d\ln f(\lambda) = A - \frac{1}{2}[A,B]\,d\lambda$

take integral $\ln f(1) - \ln f(0) = A - \frac{1}{2}[A,B]$

$\Rightarrow e^{A+B} = \cancel{e^B e^A}\, e^A e^{-\frac{1}{2}[A,B]}e^B$

$= e^A e^B e^{-\frac{1}{2}[A,B]}$

(b) $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

$= e^{\alpha a^\dagger}e^{-\alpha^* a}e^{\frac{1}{2}[a,a^\dagger]|\alpha|^2}$

$= e^{\alpha a^\dagger}e^{-\alpha^* a}e^{-|\alpha|^2/2}$

$D(\alpha)\cdot D(\beta) = e^{\alpha a^\dagger - \alpha^* a}e^{\beta a^\dagger - \beta^* a}$

$= D(\alpha+\beta)\,e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)\cancel{[a^\dagger,a]}}$

$\therefore D(iv)D(u) = D(u+iv)\,e^{ivu}$

$D(-iv)D(-u) = D(-u-iv)\,e^{ivu}$

$$U = \underbrace{D(-u-iv)D(u+iv)}e^{2ivu}$$
$$= [D(0)\cdot e^{i0}]e^{2ivu}$$
$$= e^{2ivu}\cdot I$$
$$\therefore \phi = 2vu$$

(c) $|\alpha\rangle = D(\alpha)|0\rangle$
$$= e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}|0\rangle$$
$$e^{-\alpha^* a} = 1 - \alpha^* a + \frac{(\alpha^* a)^2}{2!} + \cdots$$
$$e^{-\alpha^* a}|0\rangle = |0\rangle$$
$$e^{\alpha a^\dagger} = \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} \qquad \frac{(\alpha a^\dagger)^n}{n!}|0\rangle = \frac{\alpha^n\sqrt{n!}}{n!}|n\rangle = \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$
$$\therefore |\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

[3] $[A,[A,B]]=0,\ [B,[A,B]]=0$

(a) $e^A e^B = e^B e^A e^{[A,B]}$ // $e^A e^B e^{-A} = e^B e^{[A,B]}$
$$g(\lambda) = e^{\lambda A} B e^{-\lambda A} \qquad g'(\lambda) = A e^{\lambda A} B e^{-\lambda A} + e^{\lambda A} B(-A) e^{-\lambda A}$$
$$= e^{\lambda A}[A,B]e^{-\lambda A}$$
$$\therefore \forall X \quad f(\lambda) = e^{\lambda A} X e^{-\lambda A}$$
$$f'(\lambda) = e^{\lambda A} A X e^{-\lambda A} + e^{\lambda A}(-XA)e^{-\lambda A}$$
$$= e^{\lambda A}[A,Y]e^{-\lambda A}$$
$$\therefore g^{n}(\lambda) = e^{\lambda A}\underbrace{[A,[\cdots[A,B]]]}_{\text{"n" } A} e^{-\lambda A}$$
$$\therefore g(\lambda) = \cancel{B+}\, g(0) + \frac{1}{1!}g'(0)(\lambda-0) + \frac{1}{2!}g''(0)(\lambda-0)^2 + \cdots$$
$$g(1) = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$$
$$\Rightarrow e^A B e^{-A} = \sum_n \frac{1}{n!}\underbrace{[A,\cdots[A,[A,B]]\cdots]}_{n \text{ times}}$$

(b) $D(\alpha) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}$

① $D^\dagger(\alpha) = e^{-\alpha a^\dagger} e^{\alpha^* a} e^{-|\alpha|^2/2} = e^{-(\alpha a^\dagger - \alpha^* a)}$

let $A = -\alpha a^\dagger + \alpha^* a$
$$D^\dagger(\alpha)\, a\, D(\alpha) = e^{A} a e^{-A} = \sum_n \frac{1}{n!}\underbrace{[A,\cdots[A,[A,a]]\cdots]}_{n \text{ times}}$$
$$[A,a] = -\alpha a^\dagger a - \alpha^* a^2 - (-\alpha a a^\dagger - \alpha^* a^2)$$
$$= -\alpha[a^\dagger, a] = \alpha$$
$$[A,[A,a]] = 0 \quad \cdots \quad \underbrace{[A,\cdots[A,[A,B]]\cdots]}_{n \text{ times}} = 0 \quad n \geq 2$$
$$\therefore D^\dagger(\alpha)\, a\, D(\alpha) = a + \alpha$$
$$D^\dagger(\alpha)\, a^\dagger D(\alpha) = a^\dagger + \alpha^*$$

② $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$
$$D^\dagger(\alpha)\, x\, D(\alpha) = x + \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*) = x + \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha)$$
$$D^\dagger(\alpha)\, p\, D(\alpha) = p + i\sqrt{\frac{\hbar m\omega}{2}}(\alpha^*-\alpha) = p + \sqrt{2\hbar m\omega}\,\mathrm{Im}(\alpha)$$

[4] $H = 2\pi\sqrt{3}\, \vec{n}\cdot\vec{\sigma} \qquad \vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$
$$U(t) = e^{-iHt} = e^{i\theta(\vec{n}\cdot\vec{\sigma})} = \cos\theta I - i\sin\theta(\vec{n}\cdot\vec{\sigma})$$
$$\theta = 2\sqrt{3}\pi t$$

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$
$$|\psi(t)\rangle = (\cos\theta I - i\sin\theta(\vec{n}\cdot\vec{\sigma}))|0\rangle$$
$$= \cos\theta|0\rangle - i\sin\theta\frac{1}{\sqrt{3}}(|1\rangle + i|1\rangle + |0\rangle)$$
$$= \left(\cos\theta - \frac{i\sin\theta}{\sqrt{3}}\right)|0\rangle - \frac{i\sin\theta}{\sqrt{3}}(1+i)|1\rangle$$
$$P_0(t) = \left|\cos\theta - \frac{i\sin\theta}{\sqrt{3}}\right|^2$$
$$= \cos^2\theta + \frac{1}{3}\sin^2\theta = \frac{2\cos^2(2\sqrt{3}\pi t)+1}{3}$$

(b) See the code uploaded on Github

(c) Also see the code

[5]

(a) Suppose $N = n_5 n_4 n_3 n_2 n_1$

$p = p_2 p_1 1 \qquad q = q_2 q_1 1$

~~$n_1 = p_1 + q_1$~~
$$\begin{cases} n_1 + 2c_1 = p_1 + q_1 \\ n_2 + 2c_2 = p_2 + q_2 + p_1 q_1 + c_1 \\ n_3 + 2c_3 = p_1 q_2 + p_2 q_1 + c_2 \\ n_4 + 2c_4 = p_2 q_2 + c_3 \\ n_5 = c_4 \end{cases}$$

there are 8 bits needed

Let $s_1 = 2p_1 - 1$ ; $s_2 = 2p_2 - 1$

$s_3 = 2q_1 - 1$ ; $s_4 = 2q_2 - 1$

$s_5 = 2c_1 - 1$ ; $s_6 = 2c_2 - 1$

$s_7 = 2c_3 - 1$ ; $s_8 = 2c_4 - 1$
$$[n_1 + 2c_1 - (p_1+q_1)]^2$$
$$= \left[n_1 + s_5 + 1 - \left(\frac{s_1+1}{2} + \frac{s_2+1}{2}\right)\right]^2$$
$$= \left(n_1 + s_5 - \frac{1}{2}s_1 - \frac{1}{2}s_2\right)^2$$
$$[n_2 + 2c_2 - p_2 - q_2 - p_1 q_1 - c_1]^2$$
$$= \left(n_2 + s_6 + 1 - \frac{s_2}{2} - \frac{s_4}{2} - 1 - \frac{s_1+1}{2}\cdot\frac{s_3+1}{2} - \frac{s_5+1}{2}\right)^2$$
$$\cancel{= n_2 + s_6 - \frac{1}{2}s_3 - \frac{1}{2}s_4}$$

I think it's not very possible to calculate this 13×13 matrix by hand. I'll upload to code later on the github to calculater this.