Homework #3
Due by 11/12/2025 before class

1. Adiabatic theorem and geometric phase. Consider a spin-1/2 particle in a magnetic field $\vec{B}$. The Hamiltonian is given by

$$H = -\mu\vec{\sigma}\cdot\vec{B}, \tag{1}$$

where μ is the magnetic moment of the particle and $\vec{\sigma}$ represents the three Pauli matrices. Suppose the magnetic field has a fixed magnitude B and a time-dependent orientation in space described by the polar angle θ and the azimuthal angle φ.

(a) [5 pt] Express the x, y and z components of the magnetic field in terms of θ and φ. Further give an expression for the Hamiltonian in the matrix form.

(b) [5 pt] Solve the instantaneous eigenstates of this Hamiltonian at the polar angle θ and the azimuthal angle φ.

(c) [10 pt] Suppose we keep θ constant, and increase φ from 0 to 2π adiabatically. In the class, we know the coefficient on an instantaneous eigenstate $|n\rangle$ evolves as

$$\dot{a}_n(t) = -\langle n(t)|\dot{n}(t)\rangle a_n(t). \tag{2}$$

Now express the coefficient a_n in terms of φ for both instantaneous eigenstates, and derive an expression for the accumulated phase between these two states when φ increases from 0 to 2π. Note that when $\varphi = 2\pi$, we return to the original basis at $\varphi = 0$. Therefore we simply get an additional phase on the original basis. It shall only depend on the trajectory of the adiabatic evolution, and is thus called a geometric phase.

2. (a) [10 pt] When two operators A and B satisfy $[A,[A,B]] = 0$ and $[B,[A,B]] = 0$, we have a special case to the Zassenhaus formula: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$. To prove this formula, we can construct a function $f(\lambda) \equiv e^{\lambda A+B} \equiv \sum_n \frac{(\lambda A+B)^n}{n!}$. Prove that

$$\frac{df}{d\lambda} = Af(\lambda) - \frac{1}{2}[A,B]f(\lambda). \tag{3}$$

Further integrate it from $\lambda = 0$ to $\lambda = 1$ to get the desired formula.

(b) [5 pt] As an application, consider the displacement operator for a simple harmonic oscillator $D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a}$, where $\alpha \in \mathbb{C}$ is the direction of the displacement on the complex plane. Suppose we apply 4 displacement operations sequentially as the unitary time evolution of the system: $D(u)$, $D(iv)$, $D(-u)$ and $D(-iv)$, where $u, v \in \mathbb{R}$. Prove that the overall unitary $U = D(-iv)D(-u)D(iv)D(u) = e^{i\phi}I$ is a global phase, and that the phase ϕ is related to the area enclosed by this trajectory on the complex plane.

(c) [5 pt] A coherent state is an eigenstate of the annihilation operator: $a|\alpha\rangle = \alpha|\alpha\rangle$ ($\alpha \in \mathbb{C}$). We can obtain a coherent state by applying a displacement to the ground state (vacuum) of the simple harmonic oscillator: $|\alpha\rangle = D(\alpha)|0\rangle$. Give an expression for the coherent state $|\alpha\rangle$ in the particle number basis ($\{|n\rangle\}$, also known as the Fock basis).

3. (a) [10 pt] From the above formula, we have $e^A e^B = e^B e^A e^{[A,B]}$, or $e^A e^B e^{-A} = e^B e^{[A,B]}$ when $[A,[A,B]] = 0$ and $[B,[A,B]] = 0$. Substitute $B \leftarrow \log B$, we have $e^A B e^{-A} = B + [A,B]$ when $[A,[A,B]] = 0$ and $[B,[A,B]] = 0$. In general when $[A,B]$ does not commute with A or B, we have

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots = \sum_n \frac{1}{n!} \underbrace{[A, \cdots [A, [A}_{n \text{ times}}, B]] \cdots]. \tag{4}$$

To prove this formula, we can construct $g(\lambda) \equiv e^{\lambda A} B e^{-\lambda A}$. Show that this formula is given by the Taylor series of $g(\lambda)$ around $\lambda = 0$.

(b) [5 pt] Again, consider the displacement operator $D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a}$. Compute the unitary transformation of the annihilation and creation operators under $D(\alpha)$: $D^\dagger(\alpha) a D(\alpha)$ and $D^\dagger(\alpha) a^\dagger D(\alpha)$. Further show that the position x and the momentum p operators will be shifted by constants proportional to $\mathrm{Re}[\alpha]$ and $\mathrm{Im}[\alpha]$, respectively, under such a unitary transformation. These are the reasons why $D(\alpha)$ is called a displacement operator.

4. Numerically simulate the single-qubit dynamics by discretizing into single-qubit gates. Specifically, consider the Hamiltonian

$$H = B_x \sigma_x + B_y \sigma_y + B_z \sigma_z \tag{5}$$

with $B_x = B_y = B_z = 2\pi$. Start from $|0\rangle$ and evolve the system for time $T = 2$. Decompose the unitary evolution into a sequence of rotations around x, y and z axes.

(a) [5 pt] This problem can be solved analytically as a rotation around the $[1,1,1]$ axis. Compute how the population in the state $|0\rangle$ evolves with time.

(b) [10 pt] Consider the first-order discretization and Trotter decomposition. Plot the population in the state $|0\rangle$ versus time t (at the discretized time points) for different number of steps $M = 100, 200, 500, 1000$. Observe how the curves converge to the exact solution as M increases.

(c) [5 pt] Repeat the calculation for the second-order discretization and Trotter decomposition. Compare your results with the first-order method.

5. Given a 6-bit odd integer N. Factor it into two 3-bit integers $N = p \times q$ by solving a combinatorial optimization problem.

(a) [5 pt] For a general 6-bit odd integer N, following the procedure in the class to express the factoring problem as to find the ground state of an Ising model Hamiltonian

$$H = -\sum_i h_i s_i - \sum_{i<j} J_{ij} s_i s_j + C. \tag{6}$$

Express the matrices h and J, and the constant C, in terms of the bits in the binary representation of the integer $N = n_5 n_4 n_3 n_2 n_1 1$.

(b) [5 pt] Below we take $N = 35$. Classically, solve this optimization problem by the brutal-force method to enumerate all the possible spin states. Verify that the ground state has an energy of zero, and that it corresponds to the correct factoring $35 = 5 \times 7$ or $35 = 7 \times 5$.

(c) [10 pt] Write a Qiskit code to solve this optimization problem by quantum annealing with first order Trotter decomposition. Run a classical simulation for an evolution time $T = 10$ and a step number $M = 100$. You can use an initial transverse field $B = 1$, and vary your Hamiltonian linearly with time. Repeat the simulation for 1000 trials. For each of the measured spin configuration, compute the corresponding energy, and plot a histogram of the energy for 1000 trials.

(d) [5 pt] Run your quantum annealing code for $N = 27$. What is the result you get? Note that for $N = 27$, no factoring into two 3-bit integers exists.