High dimensional probability	04/14/2022

Matrix Bernstein inequality

Bowen Zhou *bowen_zz@sjtu.edu.cn*

Theorem 1 (Matrix Bernstein inequality)**.** *Let $X_1, X_2, \cdots, X_N$ be independent mean-zero $n \times n$ symmetric random matrices, such that $\|X_i\| \le K$ almost surely for all i. Then, for every $t \ge 0$, we have*

$$\mathbb{P}\left\{\|\sum_{i=1}^{N} X_i\| \ge t\right\} \le 2n \exp\left(-\frac{t^2/2}{\sigma^2 + Kt/3}\right).$$

Here $\sigma^2 = \|\sum_{i=1}^{N} \mathbb{E} X_i^2\|$ is the norm of the matrix variance of the sum.

A proof of the matrix Bernstein inequality can be based on either the Golden-Thompson inequality or Lieb's inequality:

Lemma 2 (Golden-Thompson inequality)**.** *For any $n \times n$ symmetric matrices A and B, we have*

$$\operatorname{tr}(e^{A+B}) \le \operatorname{tr}(e^A e^B).$$

Lemma 3 (Lieb's inequality)**.** *Let H be an $n \times n$ symmetric matrix. Define the function on matrices*

$$f(X) := \operatorname{tr} \exp(H + \log X).$$

Then f is concave on the space of positive-definite $n \times n$ symmetric matrices.

lemma 2, 3 belong to the rich family of trace inequalities.

Now we prove the matrix Bernstein inequality, using Golden-Thompson inequality.

Proof of Theorem 1 Denote the sum

$$S := \sum_{i=1}^{N} X_i,$$

and note that

$$\|S\| = \max(\lambda_{\max}(S), \lambda_{\max}(-S)),$$

so

$$\mathbb{P}\{\|S\| \ge t\} \le \mathbb{P}\{\lambda_{\max}(S) \ge t\} + \mathbb{P}\{\lambda_{\max}(-S) \ge t\}.$$

We control the R.H.S separately. By the MGF approach, for $\lambda > 0$,

$$\mathbb{P}\{\lambda_{\max}(S) \ge t\} \le e^{-\lambda t}\mathbb{E}e^{\lambda\lambda_{\max}(S)} = e^{-\lambda t}\mathbb{E}\lambda_{\max}(e^{\lambda S}) \le e^{-\lambda t}\mathbb{E}\operatorname{tr}(e^{\lambda S}). \tag{0.1}$$

Then we can apply the Golden-Thompson inequality, indeed,

$$\begin{aligned}
\mathbb{E}\operatorname{tr}(e^{\lambda S}) &= \mathbb{E}\operatorname{tr}\exp\left(\sum_{i=1}^{N-1}\lambda X_i + \lambda X_N\right) \le \mathbb{E}\operatorname{tr}\left[\exp\left(\sum_{i=1}^{N-1}\lambda X_i\right)\exp(\lambda X_N)\right] \\
&= \mathbb{E}_{1\cdots N-1}\operatorname{tr}\left[\exp\left(\sum_{i=1}^{N-1}\lambda X_i\right)\mathbb{E}_N\exp(\lambda X_N)\right] \\
&\le \big\|\mathbb{E}_N\exp(\lambda X_N)\big\|\mathbb{E}_{1\cdots N-1}\operatorname{tr}\exp\left(\sum_{i=1}^{N-1}\lambda X_i\right) \\
&\le \cdots \\
&\le \operatorname{tr} I_n \prod_{i=1}^{N}\big\|\mathbb{E}\exp(\lambda X_i)\big\| = n\prod_{i=1}^{N}\big\|\mathbb{E}\exp(\lambda X_i)\big\|.
\end{aligned}$$

Then we need to control $\left\|\mathbb{E}\exp(\lambda X_i)\right\|$ for each term X_i. This is similar to the scalar case.

Lemma 4 (Moment generating function)**.** *Let X be an $n \times n$ symmetric mean-zero random matrix such that $\|X\| \le K$ almost surely. Then*

$$\mathbb{E}\exp(\lambda X) \preceq \exp\big(g(\lambda)\mathbb{E}X^2\big) \quad \textit{where} \quad g(\lambda) = \frac{\lambda^2/2}{1 - |\lambda|K/3},$$

provided that $|\lambda| \le 3/K$.

Using this lemma, we obtain

$$\begin{aligned}
\mathbb{E}\operatorname{tr}(e^{\lambda S}) &\le n \prod_{i=1}^{N} \left\|\mathbb{E}\exp(\lambda X_i)\right\| \le n \prod_{i=1}^{N} \left\|\exp\big(g(\lambda)\mathbb{E}X_i^2\big)\right\| \\
&= n \prod_{i=1}^{N} \exp\big(g(\lambda)\left\|\mathbb{E}X_i^2\right\|\big) \\
&= n \exp\left(g(\lambda) \sum_{i=1}^{N} \left\|\mathbb{E}X_i^2\right\|\right),
\end{aligned}$$

Let $\hat{\sigma}^2 = \sum_{i=1}^{N} \left\|\mathbb{E}X_i^2\right\|$, substituting this bound into (0.1), we obtain

$$\mathbb{P}\{\lambda_{\max}(S) \ge t\} \le n \exp\big(-\lambda t + g(\lambda)\hat{\sigma}^2\big), \forall\, 0 \le \lambda \le 3/K.$$

Minimize it in λ, it's attained for $\lambda = t/(\hat{\sigma}^2 + Kt/3)$, which gives

$$\mathbb{P}\{\lambda_{\max}(S) \ge t\} \le n \exp\left(-\frac{t^2/2}{\hat{\sigma}^2 + Kt/3}\right).$$

Repeating the argument for $-S$ we complete the proof. □