One-story Shear Building Displacement Evaluation (Free vibration)

1. Introduction

Single-degree-of-freedom system (SDOF) is very common in the structural engineering such as one-story buildings. These structures are usually subjected to the dynamic loads such as wind loads, live loads, seismic, and impact. It is very important to evaluate the maximum deformation of structures under these dynamic loads to determine if they are still in the safe mode or not. SDOF structures can be idealized as a system with a lumped mass and one massless supporting structure. The viscous damper is widely used in the shear building to decrease the displacement for the structures subjected to external loads. There are two commonly used idealizations to evaluate these structures: the one-story frame (Fig.1a) and mass-spring-damper system (Fig.1b).

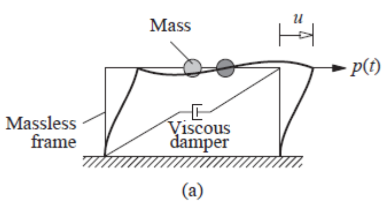

Fig1. (a)

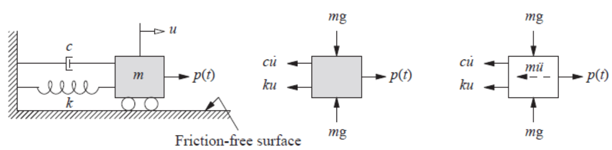

Fig1. (b)

The one-story frame is usually used in the structural engnieering for the shear building. However, the classic SDOF system, shown in Fig.1b, is the mass-spring-damper system connected by the linear springs and viscous dampers. Free vibration is the motion of structures without any sort of external loads or support motion. It is important to evaluate how the structures behave when its free vibration, since it provides the most important dynamic properties of the structure the natural frequencies. This project evaluated the deformation (u) of a normal single degree-of-freedom structure with provided initial conditions and without any external loads p(t) as shown in Fig1.a. The centered differential method, euler forward numerical method and the build-in ODE solver in the MATLAB were conducted to solve ordinary differential equation (equation of motion) considered in this project and the numerical deformation results are compared to analytical deformation results. The results show that both numerical methods and ODE built-in solver are pretty close to the analytical results and all of their true errors are smaller than 10%. For Euler forward method, if the time step reduced from the 0.01s to 0.001s, the true error would decrese from 80% to 6.3%.

2. Scope of this Project and Problem Statement

It is important to evaluate how the structures behave when its free vibration, since it provides the most important dynamic properties of the structure the natural frequencies. For this project, a normal single degrees of freedom system without any external loads p(t) (free vibration) was considered. The considered single degree-of-freedom system is in the linear range and the inelastic range is out of the scope. The hooke's law is applicable to calculate the displacement for the structures in the linear behavior. The initial conditions and the normal SDOF (shown in Fig.2) evaluated for this project are given below:

Stiffness: k =320 kN/cm, c = 4 kN-sec/cm, m = 5 ton;

Initial conditions: u (0) = 1 cm, $\dot{u}$ (0) = 7.6 cm/s. The main goal is to use the different numerical methods to predict the displacement of one-story shear building with the above conditions (Consider t = 0 to 3s) and evaluate their accuracy.

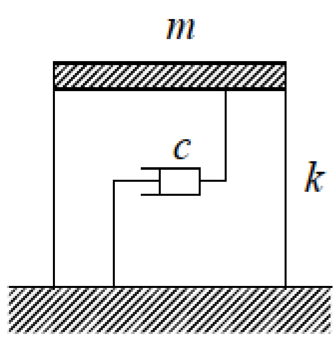

Fig.2

3. Methodology

3.1 Determine the Equation of Motion

The first step to evaluate the structures response is to determine the number of DOFs and in this project the degree of freedom is one. Then, the Newton's second law will be used to establish the equation of motion. In physics, equations of motion are equations that describe the behavior of a physical system in terms of its motion as a function of time. The forces acting on the shear building at a time are shown in Fig. 2. These include the external force $p(t)$, the elastic resisting force f_S , and the damping resisting force f_D The external force $p(t)$ equals to the sum of resisting force f_S and the damping resisting force f_D According to D'Alembert's principle, it means that the mass develops the inertia force f_I proportional to its acceleration in an opposing direction. The f_S is the stiffness k times the displacement and the f_D is proportional to the mass velocity.

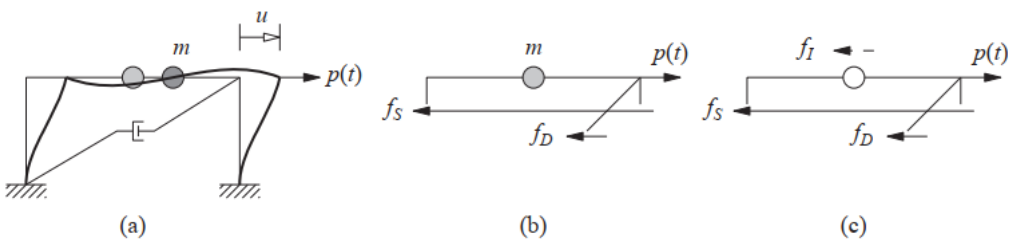

Fig.2 Establish the Equation of Motion for One-story Frame

Using the Newton's second law based on the Fig.2, the equation is given below:

$$m\ddot{u} + c\dot{u} + ku = p(t) \tag{1}$$

where the u is the displacement, m is the mass, c is the damping stiffness and the k is the structure stiffness. Since in this project, the SDOF system is in free vibration situation so that the $p(t)$ equals to zero. Therefore, the equation of motion for SDOF system subjected to free vibration is presented in Eq.2.

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{2}$$

The mass, external force, and damping and structure stiffness are known and then estimate the displacement for the structure. Therefore, the Euler's forward and centered difference numerical methods can be utilized to predict the deformation (u) shown in the above equation.

3.2 Determine Stiffness of Structures

The mass, external force, and damping and structure stiffness can be determined and then estimate the displacement for the structure. The SDOF structures can be idealized as a system with a lumped mass and one massless supporting structure as shown in Fig1.(a). To determine the k stiffness in structural, there are two extreme cases to get the lateral stiffness of the frame. In Fig3. (b), the beam is rigid, and the flexural rigidity is $EI_b = \infty$. In Fig.3 (a), it shows the definitions of width L, height h, elastic modulus E, and moment of inertia about the axis of bending = I_b and I_c for the beam and columns, respectively; The another extreme case is $EI_b = 0$.

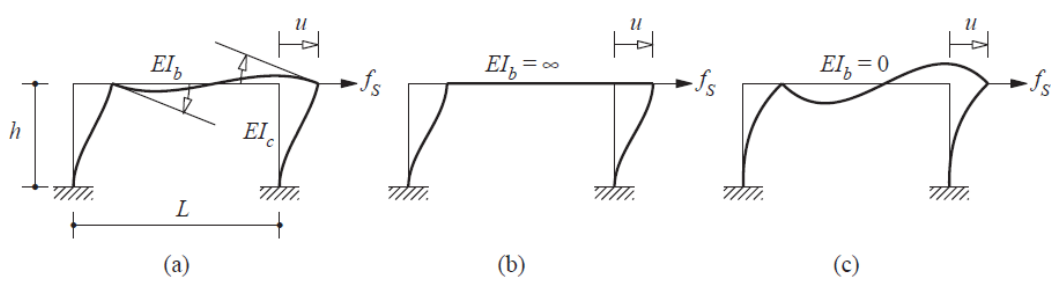

Fig. 3

The stiffness calculations for the beam is rigid and beam has no stiffness are given in Eq. 3 and Eq.4, respectively:

$$k = \sum_{\text{columns}} \frac{12EI_c}{h^3} = 24\frac{EI_c}{h^3} \tag{3}$$

$$k = \sum_{\text{columns}} \frac{3EI_c}{h^3} = 6\frac{EI_c}{h^3} \tag{4}$$

The viscous damper is widely used in the shear building to decrease the displacement for the structures subjected to external loads. The structures without damper will oscillate in free vibration mode as shown in Fig. 4 (a) and the deformation of structures with damper will decrease with the increasing of time as shown in Fig. 4(b). The expected results for this project is kind of looks like Fig4. (b), since the evaluated the SDOF considered the damping effect.

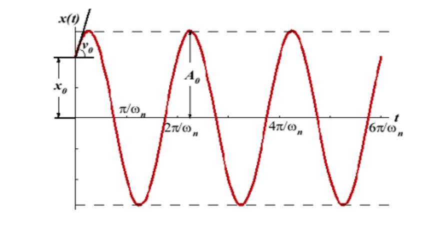

Fig4. (a)

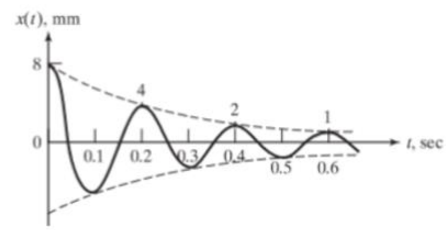

Fig4. (b)

4. Analytical Solution for the SDOF Free Vibration Mode

In order to solve the analytical results for the SDOF equation of motion, there are several terminologies were used in this project to determine that. Natural frequency (shown in Fig4. (a)), w_n, is the frequency at which a system tends to oscillate in the absence of any driving or damping force and it determined in Eq. 5. The k and m are the stiffness for structures determined above.

$$w_n = \sqrt{\frac{k}{m}} \tag{5}$$

The considered damper stiffness belongs to underdamped system, it is because the c is less than the critical damping, $c_c = 2\sqrt{km}$. The damping ratio, ζ, is a dimensionless measure describing how oscillations in a system decay after a disturbance and it is determined as the ratio of applied damping stiffness (c) divided by the critical damping (c_c). There is a term, $w_d = w_n\sqrt{1-\zeta^2}$, the damped circular frequency also used to calculate the analytical results.

From the Eq. (2), the solution of the equation of motion for underdamped SDOF free vibration is conventionally given as follows:

$$u = De^{st}$$

$$\dot{u} = sDe^{st}$$

$$\ddot{u} = s^2De^{st} \tag{6}$$

where D is an arbitrary constant. Substituting these equations into Eq.2 and the value inside the parentheses should be equal to zero and get the below equation:

$$ms^2 + cs + k = 0 \tag{7}$$

Therefore, solve the above Eq. 7 and the root is given below:

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\begin{aligned} s_{1,2} &= -\zeta\omega_{\mathrm{n}} \pm \sqrt{(\zeta\omega_{\mathrm{n}})^2 - \omega_{\mathrm{n}}^2} \\ &= -\zeta\omega_{\mathrm{n}} \pm i\,\omega_{\mathrm{d}} \end{aligned} \tag{8}$$

where the $w_d = w_n\sqrt{1-\zeta^2}$, and substitute the s1 and s2 in below equation, therefore the solution for the free vibration underdamped system is:

$$u(t) = D_1 e^{s_1 t} + D_2 e^{s_2 t}$$

$$u(t) = e^{-\zeta w_n t}(Acosw_d t + Bsinw_d t) \tag{9}$$

Since the stiffness for mass, stiffness, and for damping are known, then use the above equations can solve the analytical results for the evaluated SDOF system under free vibration. The detailed solution of this project problem is given in the APPENDIX.

4. Numerical Methods for the SDOF Free Vibration Mode

Such problems can be tackled by numerical time-stepping methods for integration of differential equations. Since the equation of motion for this project is an ordinary equation, it can be solved by the MATLAB build-in equation and other numerical methods introduced in this course such as centered difference method, Euler's forward (backward) method, and RK-4th method. In this project, the Euler's forward method and centered difference method were utilized to predict the deformation of this structures with given initial conditions.

4.1 Centered Difference Method

This method is based on a finite difference approximation of the time derivations of displacement. The velocity and acceleration of the structures with a constant time step Δt can be approximate by using this method and shown in Eq. 10.

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \qquad \ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} \tag{10}$$

Substituting these approximate expressions for velocity and acceleration into Eq.1. The reason is that for free vibration is a special case that $p(t)$ equals to zero and it can be used for structures under external load.

$$m\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} + c\frac{u_{i+1} - u_{i-1}}{2\Delta t} + ku_i = p_i \tag{11}$$

In this equation u_i and u_{i-1} are assumed known, since initial conditions are given, however, for u_{-1} there should be another approximation based on the initial conditions and given in Eq.12.

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t} \qquad \ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$$

$$u_{-1} = u_0 - \Delta t(\dot{u}_0) + \frac{(\Delta t)^2}{2}\ddot{u}_0 \tag{12}$$

Rearrange the Eq.11 to let the left side of the equation is u_{i+1} that needs to be predicted.

$$\left[\frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}\right]u_{i+1} = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right]u_i \tag{13}$$

Then use the following symbols to simplify the Eq. 13:

$$\hat{p}_i = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right]u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2}\right]u_i$$

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}$$

$$a = \frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}$$

$$b = k - \frac{2m}{(\Delta t)^2} \tag{14}$$

The initial conditions for u_{-1} is in Eq. 12 and $\ddot{u}_0$ are calculated below:

$$\ddot{u}_0 = \frac{p_0 - c\dot{u}_0 - ku_0}{m} \tag{15}$$

Calculations for time step I, and the u_{i+1} are expressed as Eq. 16.

$$\hat{p}_i = p_i - au_{i-1} - bu_i$$

$$u_{i+1} = \frac{\hat{p}_i}{\hat{k}} \tag{16}$$

The process and steps for centered difference method utilized in this project are given above, and the MATLAB codes are created for this method and attached in APPENDIX.

4.2 Euler's Forward Method

Euler's forward method is a numerical method to solve first order first degree differential equation with a given initial value. The Euler's forward method for solving ODE is used the following equation:

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{17}$$

The Fig. 5 shows how this method looks like. Note that the method increments a solution through an interval h while using derivative information from only the beginning of the interval.

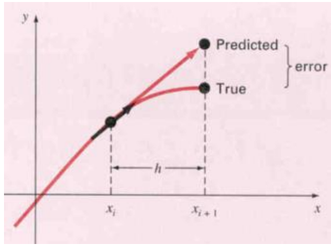

Fig. 5

The equation of motion of the SDOF system under free vibration mode is a second order ODE. Therefore, to solve this second order ODE, a new symbol should be used to represent the first derivative of the displacement (velocity). Solving a system of first order ODE instead of a second order ODE strategy was used for this project and it shows in the MATLAB codes. The process and steps for Euler's forward method utilized in this project are given above, and the MATLAB codes are created for this method and attached in APPENDIX.

5. Analytical and Numerical Results

5.1 Analytical Results

The process to solve the analytical solution for this underdamped SDOF free vibration case with the given initial conditions has already been discussed in the Section 4 above. There is no point to repeat how to calculate it for this structure problem. The detailed of the developed process of the analytical solution for this project is attached in the APPENDIX. The analytical solution for this project ODE is given in Eq. 18.

$$u(t) = e^{-0.4t}(cos8t + sin8t) \tag{18}$$

The considered time is from 0 to 3s for this project, and the total time can be longer if needed. The time step is considered as 0.01s for this plot. The MATLAB code is developed, and the deformation of this structure vs time is plotted in Fig.6. It is a reasonable plot because the deformation decreases with time increases (damping effect) and the maximum deformation of the structure is 1.36 cm.

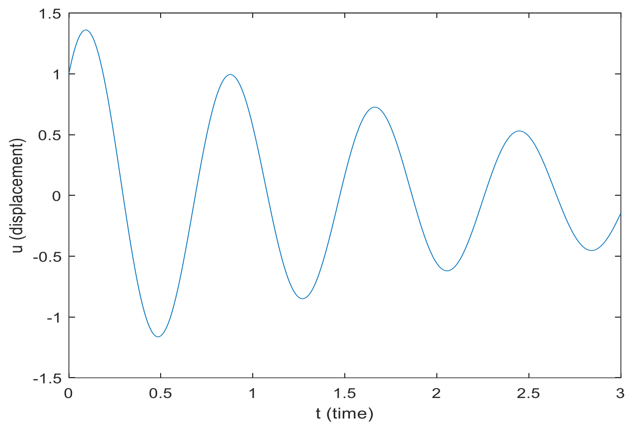

Fig. 6

5.2 Centered Difference Method Results

The process to utilize the centered difference method for this underdamped SDOF free vibration case with the given initial conditions has already been discussed in the Section 4.1 above. There is no point to repeat how to calculate it for this structure problem. The considered time period for this method is also from 0 to 3s for this project, and the results for this method will be compared to the analytical results. The time step is also considered as 0.01s for this method. The MATLAB code for this method is provided in APPENDIX, and the deformation of this structure vs time is plotted in Fig.7. It is a reasonable plot because the deformation decreases with time increases (damping effect) and the maximum deformation of the structure is also smaller than 1.5cm.

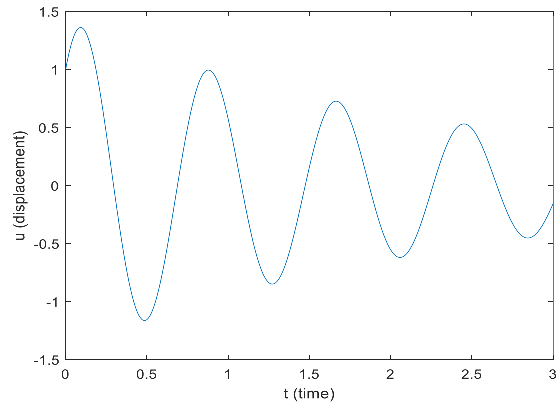

Fig. 7

5.3 Euler's Forward Method Results

There is a tricky way to solve the second ODE for this project is that considering the second ODE as a system of first order ODE and then solve that by this method. The MATLAB code for this method is also provided in APPENDIX, and the deformation of this structure versus time is plotted in Fig.8. The time step is also considered as 0.001s in Fig.8. However, the first trial the time step was set up to 0.01s and the approximation of the displacement was quite different from the analytical results. The refined time step gives more reasonable results compared to analytical results and it will be discussed in the next section.

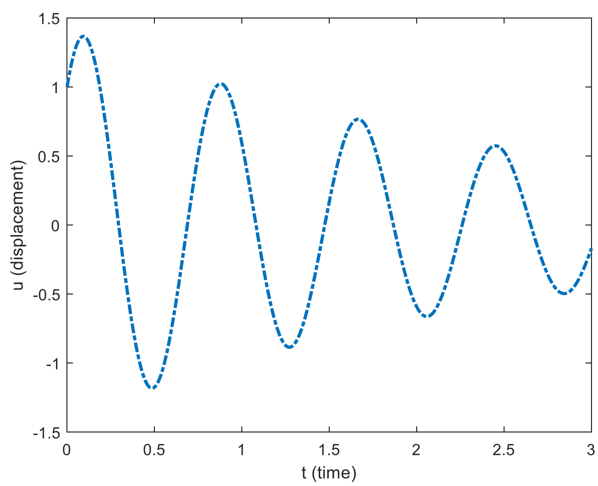

Fig. 8

5.4 ODE45 built-in Equation in MATLAB

A numerical ODE solver is used as the main tool to solve the ODE's. The MATLAB function ode45 will be used for this project. The ode45 can only solve a first order ODE in the program. Therefore, to solve the second order ODE for this project, the ODE must be first converted to a set of first order ODE as mentioned before. The ode45 is used for the non-stiff problem and it should be the first try in MATLAB. The displacement versus time plot shown in Fig.9 and the time range is from 0 to 3s.

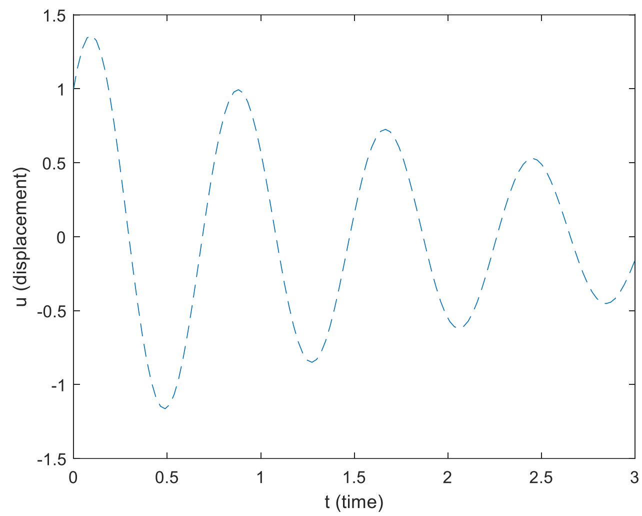

Fig.9

5.5 True Error Analysis

The true error is the difference between the true value and the approximation. In this project, the true values are the results from analytical solution and the approximations are the results for numerical methods and MATLAB build-in function. As shown in Fig.10(a), ode45 build-in function and centered difference method predict enough accurate results compared to analytical results. However, the Euler's method does not work well since the time step was set up to 0.01s for Fig.10(a). The Euler's method gets more accurate results as shown in Fig.10 (b) with smaller time step equals to 0.001s.

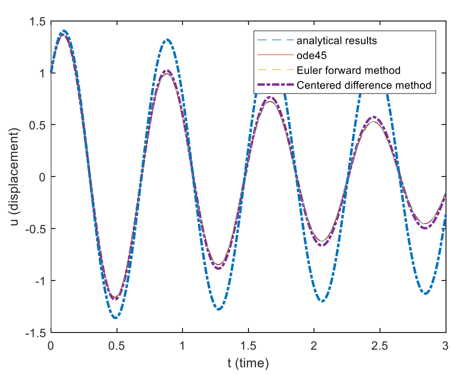

Fig.10 (a)

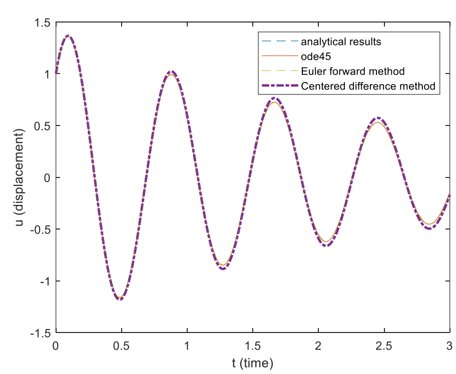

Fig.10 (b)

The time range considered from 0 to 3s and the true error analysis used each 0.5s to compare the results for analytical and numerical results. The Table 1 presents the error percentage for each method. The mean true error of Euler's forward method (0.01s time step) is about 80% that is because the time step is not small enough and later the time step reduced to 0.001s as discussed before. The mean true errors of Euler's forward method (0.001s time step), ode45 function and centered difference method are all smaller than 8%. The bolted number for ode45 error at 1.5s is about 30.543%. The reason makes the error is kind of huge is that the ode45 varies time step automatically and the ode45 value in the Table 1 was used the value at 1.49s instead of 1.5s. The 1.49s value for ode45 then was compared to the true value at 1.49s and it is very close to each other and then mean error decrease to 3.5%. Therefore, all the considered numerical methods with small enough time step and MATLAB built-in function used can predict accurate displacements for the SDOF system in free vibration mode with given initial conditions.

Table 1. The True Error Percentage for Different Approximate Methods

time (s)	*u(cm) true*	*u(cm) ode45*	*u(cm) euler(0.01)*	*u(cm) euler(0.001)*	*u(cm) centered*	*error ode45(%)*	*error euler 0.01(%)*	*error euler 0.001(%)*	*error centered(%)*
0.5	-1.155	-1.163	-1.356	-1.1744	-1.156	0.750	17.405	1.699	0.128
1	0.566	0.565	0.775	0.5903	0.573	0.041	36.990	4.356	1.273
1.5	0.169	0.117	0.281	0.1682	0.159	**30.543**	66.617	0.276	5.630
2	-0.560	-0.516	-1.067	-0.5921	-0.555	7.844	90.679	5.799	0.815
2.5	0.486	0.489	1.077	0.5302	0.490	0.590	121.645	9.091	0.859
3	-0.145	-0.156	-0.364	-0.1691	-0.155	7.779	151.013	16.632	6.832
					Mean	7.925	80.725	6.309	2.589

Note： if change the 1.49s instead of 1.5s for ode45 evaluation,the mean error for ode45 will decrese to 3.5%.

6. Conclusions

There are the conclusions for this project:

- The displacements obtained from each method (include analytical results) are reasonable and reliable in structural engineering view; The plot for each method is reasonable and the amplitude of one-story building decreases with the time due to damping effect;
- The Euler forward method at time step (0.01s) performed not well in this project and the true error of it is about 80% which is very high. The error for this method reduces to 6.3% when the time step equals to 0.001s;
- The Centered difference method has the smallest true error compared to other considered numerical methods;
- All the considered numerical methods with small enough time step and MATLAB built-in function used can predict accurately displacements for the SDOF system in free vibration mode with given initial conditions.

7. Future Research

There are the recommendations for future project:

- SDOF system displacement prediction under the effect of Blasting;
- SDOF system displacement prediction under the effect of periodic loads (example: cos(t));
- SDOF system displacement prediction under the effect of Earthquakes;

- Multiple Degree of freedom systems displacement prediction subjected to dynamic loads;
- Nonlinear structures displacement and velocity evaluations subjected to dynamic loads.

5. References

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Appendix

Analytical solution

The equation of motion:

$$5\ddot{u} + 4\dot{u} + 320u = 0$$

u: displacement

$\dot{u}$: velocity

$\ddot{u}$: acceleration

The characteristic equation is:

$$5r^2 + 4r + 320 = 0$$

Then use $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-4 \pm \sqrt{(-4)^2 - 4 \times 5 \times 320}}{2 \times 5}$$

$$= \frac{-4 \pm \sqrt{-6384}}{10}$$

$$= -0.4 \pm 7.99i \approx -0.4 \pm 8i$$

Therefore:

$$u(t) = e^{-0.4t}(A\cos 8t + B\sin 8t)$$

The first derivative of $u(t)$ equation:

$$\dot{u}(t) = -0.4e^{-0.4t}(A\cos 8t + B\sin 8t) + e^{-0.4t}(-8A\sin 8t + 8B\cos 8t)$$

substitute $u(0)$ and $\dot{u}(0)$ initial conditions $\Rightarrow A=1,\ B=1$

$\Rightarrow$ exact solution: $\boxed{u(t) = e^{-0.4t}(\cos 8t + \sin 8t)}$

Centered Difference Method MATLAB Codes

```
%% Centered difference method
clear
clc
format longg
% stiffness and initial conditions
M = 5;
C = 4;
K = 320;
dt= 0.01;
x0= 1;
v0= 7.6;
time_start = 0;
time_end = 3;
h = (time_end - time_start)/dt;
F = zeros(1,h); % for this case we assume all F (t) =
0,(no external force)
y = zeros(size(F));
% initial acceleration
a0 = (F(1)-C.*v0-K.*x0)./M;
% initialisation of y (first 2 values).
y0 = x0-dt.*v0+dt.^2/2.*a0;  %% U-1 value represents
here as y0
y(:,1) = x0;
K_prime = (M./dt.^2+C./(2.*dt));
a = (M./dt.^2-C./(2.*dt));
b = K - 2*M./dt.^2;
F_prime = (F(:,1)-b.*y(:,1)-a.*y0);
y(:,2) = F_prime/K_prime;
N = size(F,2);
for i = 2:N ;
    K_prime = (M./dt.^2+C./(2.*dt));
    F_prime = (F(:,i)-b.*y(:,i)-a.*y(:,i-1));
    y(:,i+1) = F_prime/K_prime;
end
x = 0:0.01:3;
plot(x,y,'-')
```

Euler's Forward Method MATLAB Codes

```
function [x,y]=euler_explicit(f,xinit,yinit,xfinal,h)
  n = (xfinal-xinit)/h;
  % Initialization of x and y as column vectors
  x = [xinit;zeros(n,1)];
  y = [yinit;zeros(n,2)];
  % Calculation of x and y
  for i = 1:n
    x(i+1) = x(i) + h;
    y(i+1,:) = y(i,:) + h*f(x(i),y(i,:));
  end
end
```

```
  clear
  clc
  xinit = 0;
  xfinal = 3;
  yinit = [1 7.6];
  h = 0.001;
  [x,y] =
euler_explicit(@project_equ,xinit,yinit,xfinal,h);
  plot(x,y(:,1))
  xlim([0 3]);
  ylim([-3 3]);
```

ODE45 Build-in MATLAB Codes

```
function dy = fun_ode45(t,u);
dy = zeros(2,1);
dy(1) = u(2);
dy(2) = -320/5*u(1)-4/5*u(2);
end
```

```
clear
clc
timerange = [0 3]; %% seconds
initialvalues = [1 7.6];
[t u] = ode45(@fun_ode45,timerange,initialvalues)
plot(t,u(:,1))
```

Plots MATLAB Codes

```
%% project plots
% %% The equation of motion for SDOF without external
force is:
% %  M*du^2/dt^2+C*du/dt+K*u = 0
%
 %% analytical results
clear
clc
t = 0:0.01:3;
u = sqrt(2).*exp(-0.4.*t).*cos(8.*t-pi.*0.25);
plot(t,u,'-')
hold on
 %% use ode45 to solve the problem
clear
clc
timerange = [0 3]; %% seconds
initialvalues = [1 7.6];
[t u] = ode45(@fun_ode45,timerange,initialvalues);
plot(t,u(:,1),'--')
hold on
%% Euler forward method
clear
clc
xinit = 0;
xfinal = 3;
yinit = [1 7.6];
h = 0.001;
[x,y] =
euler_explicit(@project_equ,xinit,yinit,xfinal,h);
plot(x,y(:,1),'-.','LineWidth',2)
hold on
%% Centered difference method
clear
clc
format longg
M = 5;
C = 4;
K = 320;
dt= 0.01;
x0= 1;
v0= 7.6;
time_start = 0;
```

```
time_end = 3;
h = (time_end - time_start)/dt;
F = zeros(1,h); % for this case we assume all F (t) =
0,(no external force)
y = zeros(size(F));
% initial acceleration
a0 = (F(1)-C.*v0-K.*x0)./M;
% initialisation of y (first 2 values).
y0 = x0-dt.*v0+dt.^2/2.*a0;  %% U-1 value represents
here as y0
y(:,1) = x0;
K_prime = (M./dt.^2+C./(2.*dt));
a = (M./dt.^2-C./(2.*dt));
b = K - 2*M./dt.^2;
F_prime = (F(:,1)-b.*y(:,1)-a.*y0);
y(:,2) = F_prime/K_prime;
N = size(F,2);
for i = 2:N ;
    K_prime = (M./dt.^2+C./(2.*dt));
    F_prime = (F(:,i)-b.*y(:,i)-a.*y(:,i-1));
    y(:,i+1) = F_prime/K_prime;
end
x = 0:0.01:3;
plot(x,y,':')
legend('analytical results','ode45','Euler forward
method','Centered difference method ')
xlabel('t (time)');
ylabel('u (displacement)');
```