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Final Project-Infinitesimal Plasticity Model and Applications -Theory of Plasticity (MECH 942)

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SYMBOLS

The symbols used for this project are summarized as below (*2*):

.	: material time derivative
$\Diamond$	: time derivative for constant plastic strain and hardening parameter
σ	: axial stress
σ_{ave}	: average stress
τ	: shear stress
$\tau_{\max}$	: maximum shear stress
τ_{yo}	: initial yield stress in shear
τ_{y}	: hardening derivative of equivalent shear yield stress
ξ	: hardening parameter
r	: radial distance from the center
$\dfrac{d\phi}{dX}$	: twist (rotation) per unit axial length
A	: cross-sectional area
A_o	: cross-sectional area in reference configuration
f	: yield function or generic continuous function
$\overset{\Diamond}{f}$	: $\dot{f}$ calculated assuming constant plastic strain and hardening
G	: shear modulus
L	: velocity gradient tensor
α	: coefficient of thermal expansion
β	: scaling factor for plastic flow
θ	: temperature
ε	: strain
ε^e	: elastic strain
ε^p	: plastic strain
ε^θ	: thermal strain
Δt	: time step size for numerical analysis

1 ABSTRACT

Plastic flow can occur under a significant portion of the response of some materials, such as aluminum. During small strains, the strain decomposes additively into elastic, plastic, and thermal strains. This infinitesimal strain assumption simplifies the equations of continuum mechanics considerably. This project reviewed three infinitesimal strains solutions for homogenous, inhomogeneous, and nonuniform loading of bars according to the Final Project Description (*1*). For each solution, a simulation procedure was developed. This project evaluated several problems for each solution presented in the book (*2*), which can serve as a learning reference. Additionally, a homogenous model was simulated using MATLAB™ and evaluated by solving problems from the book (*2*).

2 INTRODUCTION

The objective of this project is to investigate the model and applications for plasticity of infinitesimal strain solutions. The strain decomposes additively into elastic, plastic, and thermal strains during small strains. This assumption simplifies the equations of continuum mechanics considerably. It is also known as small displacement theory, small deformation theory, or small displacement gradient theory. Therefore, the resulting models are greatly simplified, and we start with this special case before considering the case for finite deformations. Even though the extensions are quite small, some materials will undergo plastic flow during a significant portion of their response. Three cases of infinitesimal plasticity are considered here, as stated in the Final Project Description (*1*). Homogeneous case: compression in-plane strain (Section 8.3.3 (*2*)); Inhomogeneous case: torsion loading and unloading (Sections 8.5 and 8.6 (*2*)); Nonuniform loading of bars and beams (Section 8.8.3 (*2*)). In these cases, simulation procedures are presented that can be applied to various loading histories. Several problems are addressed for each section (*2*) in this project, which can be used as a learning reference. MATLAB™ was used for the simulations in the project to generate a homogeneous model, and the results were evaluated by solving problems in the book (*2*). It is recommended that the applicability of the models be assessed by further experiments and the development of codes for infinitesimal strain solutions.

3 SECTION 8.3

3.1 A Summary Introduction

The world we live in is three-dimensional (3D), or maybe four-dimensional (4D), if we consider time and space. Nevertheless, in engineering analysis, 2D approximations are often used to reduce modeling and computing resources. An object constrained in the z-direction between two rigid walls will exhibit a state of plane strain. Section 8.3 of the book (*2*) shows a homogeneous model for plasticity at small strains which is discussed in this section. The deformation considered here is plane strain tension or compression, as illustrated in Figure 1. The body is pulled or compressed along the first direction and restrained along the second direction, but is free along the third direction, which is an example of plane strain. According to the book (*2*), the following model assumed only isotropic hardening, and the G^b was set to zero to ignore the kinematic hardening. The background of the model (*2*) and the general simulation procedure are presented here. Several simulations were conducted to validate the simulation model. Generally, this model is applicable to structures that are loaded in a single plane. The plain compression/tension model has been implemented in the software such as COMSOL™ (*3*).

Moreover, this model may be applied to plane-strain compression modeling, which is typically carried out to determine mechanical properties as well as to investigate the development of microstructure during thermomechanical treatment. After modeling, it can generate a stress-strain curve for the specimen (*4*). To evaluate the models in the future, numerical results can be compared to the plane-strain compression test.

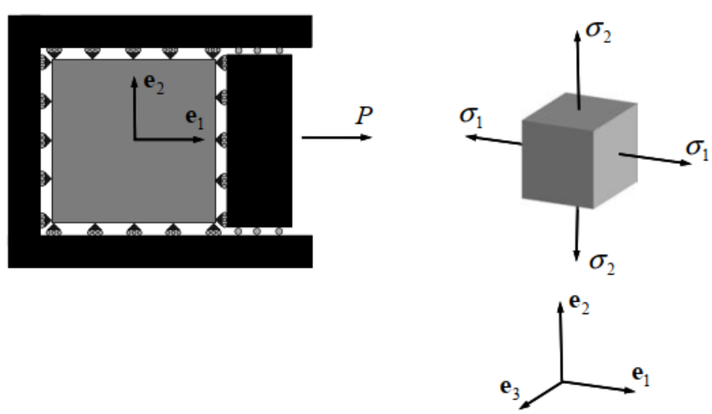


Figure 1. Plane strain tension/compression (*2*)

3.2 A Detailed Description

This background section is mainly based on the book (*2*) which contains all the information needed to conduct simulation analysis. A detailed derivation of this plain strain in compression/tension model can be found in the book (*2*). A brief discussion and description of the model is presented, which will lay the foundation for the simulations.

3.2.1 A Model Description

In the following, consider a mechanical model that is isotropic with the stress given in the book (*2*). No strain is permitted in the second direction, and no stress is applied in the third direction. Therefore, we can calculate the average stress and deviatoric stress. As for the small strains case, the strain decomposes additively into elastic, plastic, and thermal strains. Based on the expression for deviatoric stress for this isotropic case, we can calculate τ. Equivalent shear yield stress model is also provided (*2*). Therefore, we can determine the von Mises yield function. As we know that, the flow rule is depended on two conditions ($f = 0$ and $\overset{\lozenge}{f} > 0$). Then we need to determine the expression of $\overset{\lozenge}{f}$ which assumes the plastic strain rate equals to zero. Both the elastic and plastic flow are provided in the book (*2*). For the elastic flow, the book (*2*) Section 8.3.1 used the "dot" on stress rates/force and mentioned that they represent the "diamond" in the text. The stress rates/force trial derivatives sign ("diamond") will be used directly here. During the plastic flow, it must maintain the consistence ($\dot{f} = 0$). Then, the flow rule can be determined which used to update the plastic strain rate for three directions.

Stress model: $\sigma = (\kappa - \frac{2}{3}G)tr(\varepsilon^e) + 2G\varepsilon^e$ (1)

Average stress: $\sigma_{ave} = \frac{1}{3}(\sigma_1 + \sigma_2 + 0) = \frac{1}{3}(\sigma_1 + \sigma_2)$ (2)

Deviatoric stress: $S = \frac{1}{3}(2\sigma_1 - \sigma_2)e_1 \otimes e_1 + \frac{1}{3}(2\sigma_2 - \sigma_1)e_2 \otimes e_2 - \frac{1}{3}(\sigma_2 + \sigma_1)e_3 \otimes e_3$ (3)

Shear stress: $\tau = \sqrt{\frac{1}{2}S:S} = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)}$ (4)

Equivalent shear yield stress: $\tau_y = \tau_{yo} + \frac{\partial \tau_y}{\partial \theta}\Delta\theta + \frac{\partial \tau_y}{\partial \xi}\xi + \frac{\partial \tau_y}{\partial \sigma_{ave}}\sigma_{ave}$ (5)

Von Mises yield function: $f = \tau - \tau_y = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)} - \tau_y$ (6)

Shear stress derivative: $\dot{\tau} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2]$ (7)

Average normal stress derivative: $\dot{\sigma}_{ave} = \frac{1}{3}(\dot{\sigma}_1 + \dot{\sigma}_2)$ (8)

Elastic flow:

Strain rate at the third direction: $\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$ (9)

Strain trial derivatives for two directions:

$$\overset{\diamond}{\sigma}_1 = \frac{4G(3\kappa + G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \quad (10)$$

$$\overset{\diamond}{\sigma}_2 = \frac{2G(3\kappa - 2G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \quad (11)$$

Trial derivative of f $\overset{\diamond}{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\overset{\diamond}{\sigma}_1 + (2\sigma_2 - \sigma_1)\overset{\diamond}{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\overset{\diamond}{\sigma}_1 + \overset{\diamond}{\sigma}_2)$ (12)

Plastic flow:

If meet the conditions ($f = 0$ and $\overset{\diamond}{f} > 0$), then we can determine:

Strain rate at the third direction: $\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{2G}{3\kappa + 4G}(\sigma_1 + \sigma_2)\beta + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$ (13)

Strain derivatives for two directions:

$$\dot{\sigma}_1 = \overset{\diamond}{\sigma}_1 - \frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta \quad (14)$$

$$\dot{\sigma}_2 = \overset{\diamond}{\sigma}_2 + \frac{2G}{3\kappa + 4G}[2G\sigma_1 - (3\kappa + 2G)\sigma_2]\beta \quad (15)$$

Derivative of the *f*: $\dot{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\dot{\sigma}_1 + \dot{\sigma}_2)$ (16)

Which can be rewritten as:

$$\begin{aligned}\dot{f} &= \overset{\diamond}{f} + [\frac{1}{6\tau}(2\sigma_1 - \sigma_2) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}]\{-\frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta\} \\ &+ [\frac{1}{6\tau}(2\sigma_2 - \sigma_1) - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}]\{\frac{2G}{3\kappa + 4G}[2G\sigma_1 - (3\kappa + 2G)\sigma_2]\beta\} - 2\frac{\partial\tau_y}{\partial\xi}\tau\beta\end{aligned} \quad (17)$$

During plastic flow we need the *consistency condition* to hold: $\dot{f} = 0$

$$\beta = \frac{\tau(3\kappa + 4G)\overset{\diamond}{f}}{2G[(\kappa + G)(\sigma_1^{\ 2} + \sigma_2^{\ 2}) - (\kappa + 2G)\sigma_1\sigma_2 - \kappa\tau\frac{\partial\tau_y}{\partial\sigma_{ave}}(\sigma_1 + \sigma_2)] + 2\tau^2(3\kappa + 4G)\frac{\partial\tau_y}{\partial\xi}} \tag{18}$$

Flow rule for this isotropic case:

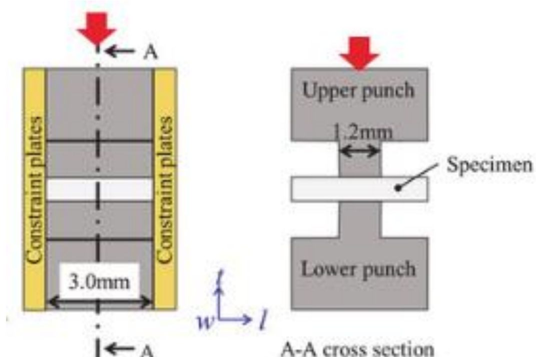
$$\dot{\varepsilon}^p = \begin{Bmatrix} \beta S & if\ f \geq 0\ and\ \overset{\diamond}{f} > 0 \\ 0 & All\, other\, cases \end{Bmatrix} \tag{19}$$

A more detailed review can be found in APPENDIX 8.1.

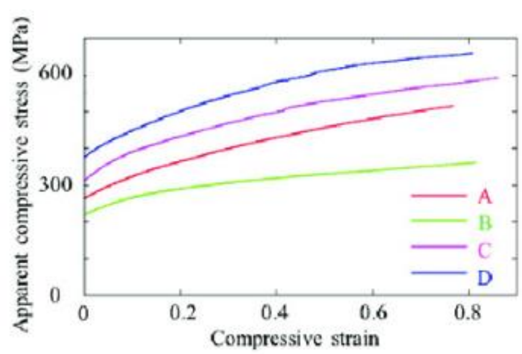
3.2.2 An Application Description

A paper reports results of material tests on single crystals of pure iron (*5*). Four kinds of specimens with various crystal orientations were cut from one crystal rod (*5*). A plane strain compression test was conducted on the specimens (Figure 2a). Based on the load-stroke curves in these material tests, stress-strain relationships were determined (Figure 2b).

Based on the developed model in this section, we can perhaps compare the stress-strain curves developed *numerically* and compared to this type of *test* stress-strain curve to evaluate the model applicability in the future.



a)



b)

Figure 2. a) Plane strain compression test; b) Stress–strain curves obtained from plane strain compression test (*5*)

3.3 A Description of the Simulation

For this model, the inputs are the 1st direction strain and temperature functions. Given sufficiently small steps, any of the proposed integration methods should result in quite accurate simulations, even though they are not optimized to achieve the desired accuracy with a minimum number of steps (*2*). The general simulation procedures are given here. The following parameters are used for considered simulations based on the beginning of Section 8 (*2*):

$$G(\theta) = 300 - 10\theta$$

$$\kappa(\theta) = 800 - 10\theta$$

$$G_b(\theta) = 0$$

$$\tau_y = 30 - \theta + 10\xi - \sigma_{ave}$$

$$\theta_0 = 0 \quad \alpha = 0.01 \quad \Delta t = 0.00001 \quad \varepsilon_{1o} = 0 \quad \varepsilon_{3o} = 0 \quad \xi_o = 0$$

The general programming procedure for this case is proposed as follows (*2*):

1. Input the first direction strain and temperature histories (at t_i) to get $\varepsilon_{1i}, \theta_i$
2. Calculate the current parameters: $G_i, \kappa_i, \tau_{yi}, \varepsilon_i^{\theta}, \varepsilon_i^{p}, \varepsilon_i^{e}, \xi_i$
3. Calculate the current stress σ_1 and σ_2: $\sigma = (\kappa - \frac{2}{3}G)tr(\varepsilon^e) + 2G\varepsilon^e$

$$\sigma_1 = (\kappa + \frac{4}{3}G)(\varepsilon_1 - \varepsilon_1{}^p) + (\kappa - \frac{2}{3}G)(\varepsilon_3 - \varepsilon_3{}^p) - 3\kappa a\theta$$

$$\sigma_2 = (\kappa - \frac{2}{3}G)(\varepsilon_1 - \varepsilon_1{}^p) + (\kappa - \frac{2}{3}G)(\varepsilon_3 - \varepsilon_3{}^p) - 3\kappa a\theta$$

4. Calculate the current average normal stress: $\sigma_{ave} = \frac{1}{3}(\sigma_1 + \sigma_2 + 0) = \frac{1}{3}(\sigma_1 + \sigma_2)$
5. Substitute the parameters to determine the τ_y:

$$\tau_y = \tau_{yo} + \frac{\partial \tau_y}{\partial \theta}\theta + \frac{\partial \tau_y}{\partial \xi}\xi + \frac{\partial \tau_y}{\partial \sigma_{ave}}\sigma_{ave} = 30 - \theta + 10\xi - \sigma_{ave}$$

6. Calculate the current: $\tau = \sqrt{\frac{1}{2}S:S} = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)}$
7. Calculate the yield function: $f = \tau - \tau_y = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)} - \tau_y$

8. Calculate the current strain rate $\dot{\varepsilon}_{1i} = \dot{\varepsilon}_1(t_i)$ and temperature rate $\dot{\theta}_i = \dot{\theta}(t_i)$, which will be used to calculate the strain rate of the third direction $\dot{\varepsilon}_{3i}$

9. If $f \geq 0$ (consider $f > 0$ for numerical simulation), calculate current $\overset{\diamond}{f}_i$ from:

$$\overset{\diamond}{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\overset{\diamond}{\sigma}_1 + (2\sigma_2 - \sigma_1)\overset{\diamond}{\sigma}_2] - \frac{\partial \tau_y}{\partial \theta}\dot{\theta} - \frac{1}{3}\frac{\partial \tau_y}{\partial \sigma_{ave}}(\overset{\diamond}{\sigma}_1 + \overset{\diamond}{\sigma}_2)$$

$$\overset{\diamond}{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\overset{\diamond}{\sigma}_1 + (2\sigma_2 - \sigma_1)\overset{\diamond}{\sigma}_2] + \dot{\theta} + \frac{1}{3}(\overset{\diamond}{\sigma}_1 + \overset{\diamond}{\sigma}_2)$$

$$\overset{\diamond}{\sigma}_1 = \frac{4G(3\kappa + G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$$

$$\overset{\diamond}{\sigma}_2 = \frac{2G(3\kappa - 2G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$$

10. Calculate the current plastic strain rate:

- If $f_i \geq 0$ and $\overset{\diamond}{f}_i > 0$, then

$$\dot{\varepsilon}^p = \begin{Bmatrix} \beta S & \text{if } f \geq 0 \text{ and } \overset{\diamond}{f} > 0 \\ 0 & \text{All other cases} \end{Bmatrix}$$

$$\beta = \frac{\tau(3\kappa + 4G)\overset{\diamond}{f}}{2G[(\kappa + G)(\sigma_1^2 + \sigma_2^2) - (\kappa + 2G)\sigma_1\sigma_2 - \kappa\tau\frac{\partial \tau_y}{\partial \sigma_{ave}}(\sigma_1 + \sigma_2)] + 2\tau^2(3\kappa + 4G)\frac{\partial \tau_y}{\partial \xi}}$$

$$\beta = \frac{\tau(3\kappa + 4G)\overset{\diamond}{f}}{2G[(\kappa + G)(\sigma_1^2 + \sigma_2^2) - (\kappa + 2G)\sigma_1\sigma_2 + \kappa\tau(\sigma_1 + \sigma_2)] + 10 \times 2\tau^2(3\kappa + 4G)}$$

$$S = \frac{1}{3}(2\sigma_1 - \sigma_2)e_1 \otimes e_1 + \frac{1}{3}(2\sigma_2 - \sigma_1)e_2 \otimes e_2 - \frac{1}{3}(\sigma_2 + \sigma_1)e_3 \otimes e_3$$

$$\dot{\varepsilon}_i^p = \begin{bmatrix} \dot{\varepsilon}_1^p = \beta \times \frac{1}{3}(2\sigma_1 - \sigma_2) & & \\ & \dot{\varepsilon}_2^p = \beta \times \frac{1}{3}(2\sigma_2 - \sigma_1) & \\ & & \dot{\varepsilon}_3^p = \beta \times -\frac{1}{3}(\sigma_1 + \sigma_2) \end{bmatrix}$$

$$\dot{\xi}_i = \sqrt{2\dot{\varepsilon}_i^p : \dot{\varepsilon}_i^p} = 2\beta\tau$$

$$\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{2G}{3\kappa + 4G}(\sigma_1 + \sigma_2)\beta + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$$

- Else

$$\dot{\varepsilon}_i{}^p = 0 \quad \dot{\xi}_i = 0$$

$$\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta}$$

11. Store the t_i , $\varepsilon_{1i}, \sigma_{1i}, \sigma_{2i}, \varepsilon_{3i}, \dot{\varepsilon}_{3i}, \theta_i$, for the stress plots
12. Update the variables

$$\varepsilon_{i+1}{}^p = \varepsilon_i{}^p + \dot{\varepsilon}_i{}^p \Delta t$$

$$\xi_{i+1} = \xi_i + \dot{\xi}_i \Delta t$$

$$\varepsilon_{3i+1} = \varepsilon_{3i} + \dot{\varepsilon}_{3i} \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

13. Increment *i* and return to 1st step.

The general simulation is presented here which could be used to simulate the section problems as mentioned in Section 8.3 (*2*).

3.4 A Discussion for Section Problems

Several problems at the sections 8.3(*2*) are selected and discussed here. This information can be used as a reference for better understanding of materials discussed in this section 8.3.

8.3-1: Consider isothermal plane strain compression. Evaluate the slope of the compressive stress-strain plot during a monotonic loading, including both the initial elastic and the following plastic response (i.e., the slopes of the σ_1-ε_1 plot). Calculate the ratio of the transverse to axial strain rates in each range (i.e., the ratio of $\dot{\varepsilon}_3$-$\dot{\varepsilon}_1$).

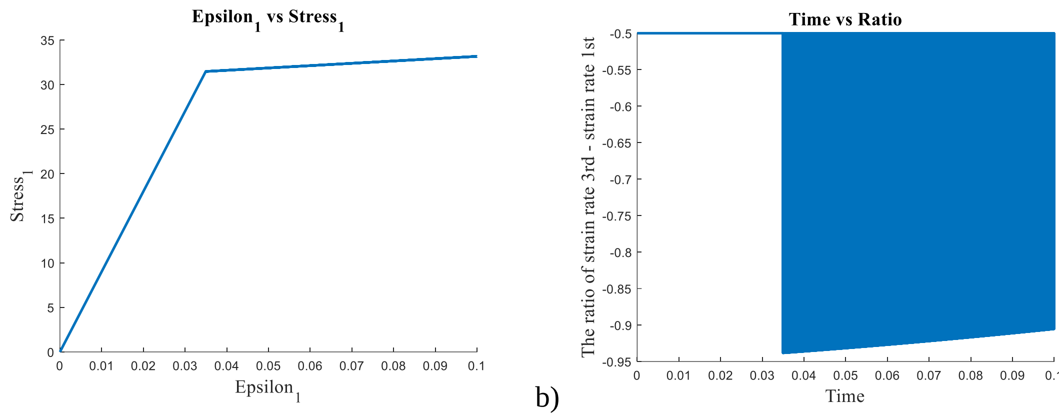
This is the isothermal plan strain case. Therefore, the codes in the APPENDIX 8.4.1.do not consider the temperature effects. The general simulation described before was used to do this simulation. The strain rate at the 1st direction assumed to 1.0 and unchanged temperature is 0. As can be seen in Figure 3a, the slope of the σ_1-ε_1 decreases when the material yields. The transverse to axial strain rates in each range (the ratio of $\dot{\varepsilon}_3$-$\dot{\varepsilon}_1$) is shown in Figure 3b. The ratio is -0.5 and unchanged until the material yields. That makes sense since the strain rate at the 3rd direction is constant for this isothermal case. As the material yields, the transverse increases but in the negative direction.

8.3-2: Calculate the minimum temperature rise that will force the response to reach the yield surface. Calculate this for both the case of ε_1= 0 and also for the case of σ_1 = 0

For this case, we do not have to update the plastic strain rate when the material yields since we are interested in the minimum temperature that causing yielding. Therefore, in the APPENDIX 8.4.2 and 8.4.3, the code used a break function in MATLAB™ to stop the program when the material yields. Both the case of ε_1= 0 and for the case of σ_1 = 0 are considered. The minimum temperature is *63.916* to cause yield for ε_1= 0 case. The minimum temperature is *0.382* to cause yield for σ_1 = 0 case.

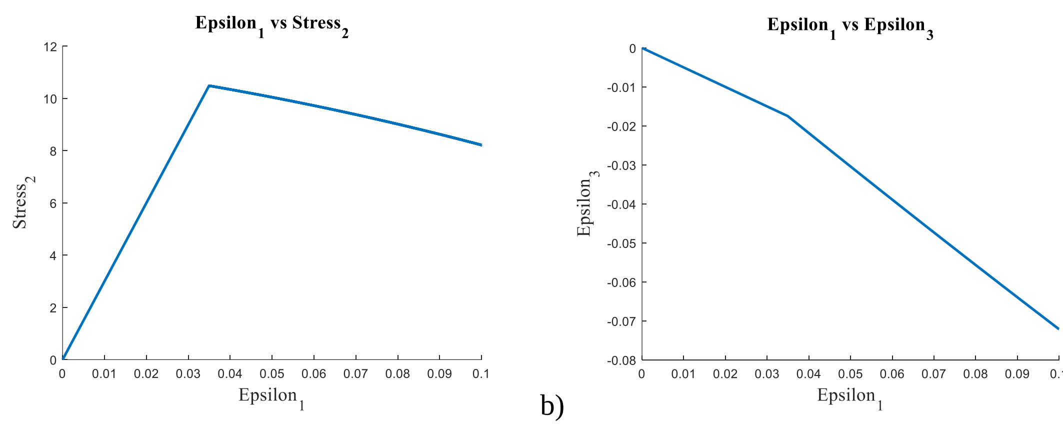
8.3-3: Simulate the response under plane strain compression in monotonic compression and cyclic compression. Plot as a function of ε_1 the stresses σ_1 and σ_2, and the strain ε_3

This is assumed as the isothermal plan strain case. The same parameters as problem 8.3.1 were used for this case. Note only monotonic compression is presented here. For the cyclic loads, the general procedure would be the same but the strain at the 1st direction should be cyclic. As can be seen in Figure 4a, the slope of the ε_1- σ_2 decreases when the material yields. The axial to transverse strains in each range is shown in Figure 4b. Figure 4b shows that the transverse strain direction is negative as axial strain increases in positive direction.



a) b)

Figure 3. a) The slopes of the σ1-ε1 plot ; b) the ratio of $\dot{\varepsilon}_3$-$\dot{\varepsilon}_1$



a) b)

Figure 4. a) The plot of ε_1- σ_1 ; b) The plot of ε_1- ε_3

4 SECTION 8.5 and 8.6

4.1 A Summary Introduction

In the above analysis, the homogenous case has been evaluated. This section introduces an inhomogeneous problem defined by deformation and simplified constitutive assumptions. When a circular bar is twisted, it tends to twist; the stresses developed in the bar are called torsional stresses. As a result of torque or twisting moment, the bar is considered to be in torsion, or under torsional load. Torque is measured in length and force. An example is the shape of the shafts used in cars and power plants for transmission (as shown in Figure 5). Torsion in noncircular sections is also an option. Torsion in hollow sections, such as those found on airplane wings or wind turbines, is an obvious example. As shown in Figure 6a, the torsion strain increases linearly from zero at the center of the cross-section to its maximum at the outer edge. Elastic torsion and elastic-plastic torsion for circular bars are discussed here. The background of the model (*2*), and the general simulation procedure are presented here. Over the cross-sectional area, the axial strain is assumed to be constant. The basic kinematic assumptions are, therefore, given as:

$$\varepsilon = \text{Constant} \tag{20}$$

$$\gamma_{r\theta} = r\frac{d\phi}{dX} \tag{21}$$

Later of this section derives a derivation of a pure torsion for a hollow cylindrical bar. It was found from that the hollow shafts are much better to take torsional loads compared to solid shafts due to lower weight and cost. As shown in the Figure 6a, shear stress varies linearly from zero at the center to the maximum at the boundary. Inside a solid shaft, most of the material experiences a shear stress whose value is much below the maximum shear stress. But at the same they are adding to the weight, without contributing much to the capability of the shaft to carry torsional load. This section developed a simulation procedure that can be used for modeling.

To see the results of the unloading, a pure torsion solution is presented in this section (Figure 6b). If the entire bar remains elastic, the unloading will also be elastic. The more interesting case is that in which we applied a large enough twist to create an elastic core section of radius r_e and an elastic-plastic outer section. The point here is that a structure loaded beyond its elastic limit, but below its plastic limit, will show self-equilibrating residual forces after the load has been removed (Figure 6b). Due to this, subsequent variable repeated load applications exceeding the elastic limit load can be accommodated without further yielding (*10*). In plastic design, this is known as the ***shakedown*** limit state, which is typically beneficial for steel bridges.

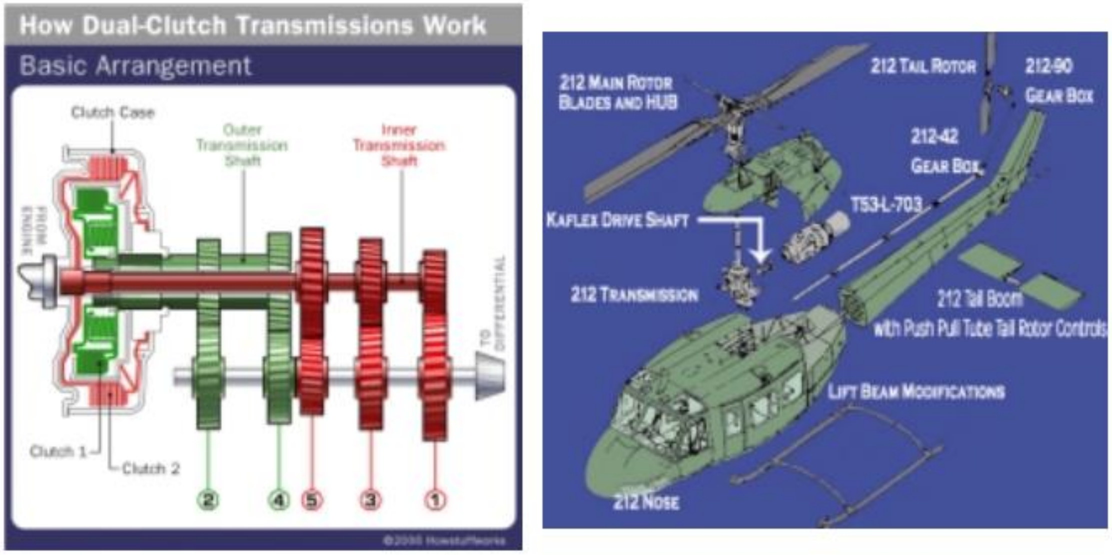
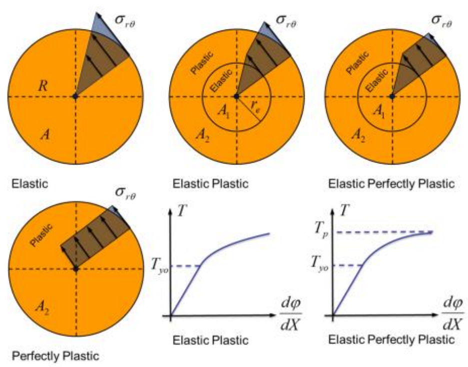


Figure 5. An important use of shafts is to transmit power between parallel planes, as in cars, aircraft engines or helicopters (Google image)

a) 

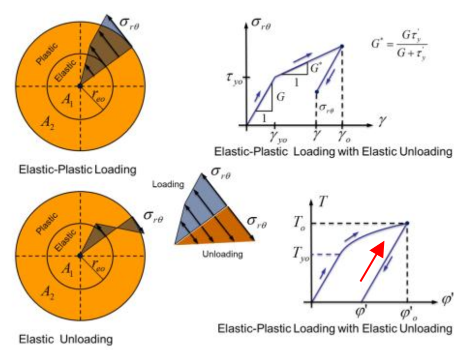
b) 

Figure 6. a)The elastic-plastic response of a circular bar subjected to an axial load and a torsional moment; b) The elastic-plastic loading of a circular member in pure torsion followed by anelastic unloading of the member to zero torsional moment (*2*)

4.2 A Detailed Description

4.2.1 A Model Description

For circular cross sections, we normally assume the response is a combination of simple shear and uniaxial extension. A detailed derivation of this elastic/elastic-plastic torsion and unloading of circular member can be found in the book (*2*). The brief discussion and description of the model are presented here, and a detailed review can be found in APPENDIX 8.2.

Elastic Torsion

For a uniform axial load P, the axial stress can be calculated by:

$$\sigma = E\varepsilon = \frac{P}{A} = \frac{EA\varepsilon}{A} \tag{22}$$

Based on the torsion equation, the shear stress for the elastic member is given by:

$$T = GI_p \frac{d\phi}{dX} \tag{23}$$

$$\sigma_{r\theta} = Gr\frac{d\phi}{dX} = \frac{Tr}{I_p} \tag{24}$$

From the expression of deviatoric stress for this case, we can calculate τ as:

$$\tau = \sqrt{\frac{1}{2}S:S} = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} \tag{25}$$

The *yield function* for the fully elastic bar is taken to have the form of the von Mises yield function as:

$$f = \tau - \tau_y = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} \tag{26}$$

$$f = \sqrt{\frac{1}{3}(\frac{P}{A})^2 + \left(\frac{T^2 r^2}{I_p{}^2}\right)} - \tau_{yo} \tag{27}$$

This condition can be rewritten as by squaring the above yield function (f = 0):

$$\tau_{yo}{}^2 = \frac{1}{3}(\frac{P}{A})^2 + \left(\frac{T^2 R^2}{I_p{}^2}\right) \tag{28}$$

If there is no axial load (pure torsion), the maximum torsional moment that the bar can take as:

$$T = \frac{\tau_{yo} I_p}{R} \tag{29}$$

Elastic-plastic Torsion

Note the pure torsion case (no axial loads applied) will be considered for the rest of this section. Figure 6b shows the loading resulting for a bar that undergoes plastic flow. Both the axial load and the torsional moment play a role in yielding so the radius r_e of initial yield can be obtained from the equation $f = 0$ given by:

$$f = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} = \sqrt{\frac{1}{3}(E\varepsilon)^2 + (Gr_e\frac{d\phi}{dX})^2} - \tau_{yo} = 0 \tag{30}$$

That is, the point of initial yield is given in terms of the twist by:

$$r_e = \frac{\tau_{yo}}{G\frac{d\phi}{dX}} \tag{31}$$

From pure shear, we know that the expression for stress in the region of plastic flow is given by:

$$\sigma_{r\theta} = G\gamma_{yo} + \frac{G\tau_y'}{G + G\tau_y'}(\gamma_{r\theta} - \gamma_{yo}) = \frac{G}{G+\tau_y'}(\tau_{yo} - \tau_y'\gamma_{r\theta}) \tag{32}$$

The torque now can be found from the equation:

$$T = G\frac{d\phi}{dX}I_{p1} + \frac{G}{G+\tau_y'}(\tau_{yo}Q_{p2} + \tau_y'\frac{d\phi}{dX}I_{p2}) \tag{33}$$

Where:

$$Q_{p2} = \frac{2\pi}{3}(R^3 - r_e^3)\,;\, I_{p1} = \frac{\pi}{2}(r_e^4)\,;\, I_{p2} = \frac{\pi}{2}(R^4 - r_e^4) \tag{34}$$

Unloading Torsion

The unloading is shown through pure torsion because it is a simple solution. This section consider that the applied torque is T_o and the corresponding twist is ϕ_o' for the initial monotonic loading. Based on the torsion equations, we can determine:

$$\gamma_o = r\phi_o' \tag{35}$$

$$r_{eo} = \frac{\tau_{yo}}{G\phi_o'} \tag{36}$$

$$T_o = G\phi_o' I_{p1o} + \frac{G}{G+\tau_y'}(\tau_{yo}Q_{p2o} + \tau_y'\phi_o' I_{p2o}) \tag{37}$$

where I_{p1o} and Q_{p2o} are the corresponding values of I_{p1} and Q_{p2} for $r_e = r_{eo}$.

In the elastic range, we have the expression for the stress given by:

$$\sigma_{r\theta} = G\gamma = G\phi' r \tag{38}$$

In the plastic zone, as shown in Figure 6b, the stress is given by:

$$\sigma_{r\theta} = \tau_{yo} + \frac{G\tau_y'}{G+\tau_y'}(\gamma_o - \gamma_{yo}) - G(\gamma_o - \gamma) = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}) - G(\gamma_o - \gamma) \tag{39}$$

Considering that $\tau_{yo} = G\gamma_{yo}$; $\gamma_o = r\phi_o'$; $\gamma = r\phi'$, we will get:

$$\sigma_{r\theta} = \tau_{yo} + G^*(\phi_o' r - \frac{\tau_{yo}}{G}) - G(\phi_o' r - \phi' r) \tag{40}$$

The torsional moment can be evaluated by putting the stresses into equation for the moment to obtain:

$$T = GI_{p1o}\phi' + \tau_{yo}Q_{p2o} + G^*\phi_o' I_{p2o} - \frac{G^*\tau_{yo}}{G}Q_{p2o} - GI_{p2o}\phi_o' + GI_{p2o}\phi' \tag{41}$$

As can be seen in Figure 6b, ultimately the unloading is elastic and represents the subtraction of a linearly increasing stress with the radius. This can be written as

$$T = T_o - GI_p(\phi_o' - \phi') \tag{42}$$

Thus, for elastic unloading, the relationship between the torsional moment and the angle of twist is linear. From this equation, one can also calculate the twist as a function of the applied torsional moment:

$$\phi' = \phi_o' + \frac{T - T_o}{GI_p} \tag{43}$$

For total unloading, we will have T = 0 to get

$$\phi' = \phi_o' - \frac{T_o}{GI_p} \tag{44}$$

4.2.2 An Application Description

A study presents an experimental and numerical analysis of the mechanical behavior of a fixed-end cylinder specimen made of SAE 1045 steel during a torsion test (*6*). Experimental measurements and numerical predictions are quite similar, as shown in Figure 7a. Most of the difference between these curves appears at the beginning of the process when the material begins to harden. This effect is more pronounced in the experiments than in the simulation (*6*). An investigation was conducted on the distribution of shear stress on the cross-section of a plastic metal solid circular shaft under pure torsion yielding, and on the applicability of the complete plastic model assumption and the shear stress formula (*7*). Figure 7b shows a typical torque-torsion angle curve for a plastic metal shaft under pure torsion. There were several tests performed on circular cylindrical bars to obtain torque-twist curves (*the torsional shear stress versus torsional shear strain plots*) to show the importance of plasticity in structural design and metal forming to engineering undergraduate students(*8*). As a result, an experiment is *recommended* to evaluate the applicability of the model described in this section in the future.

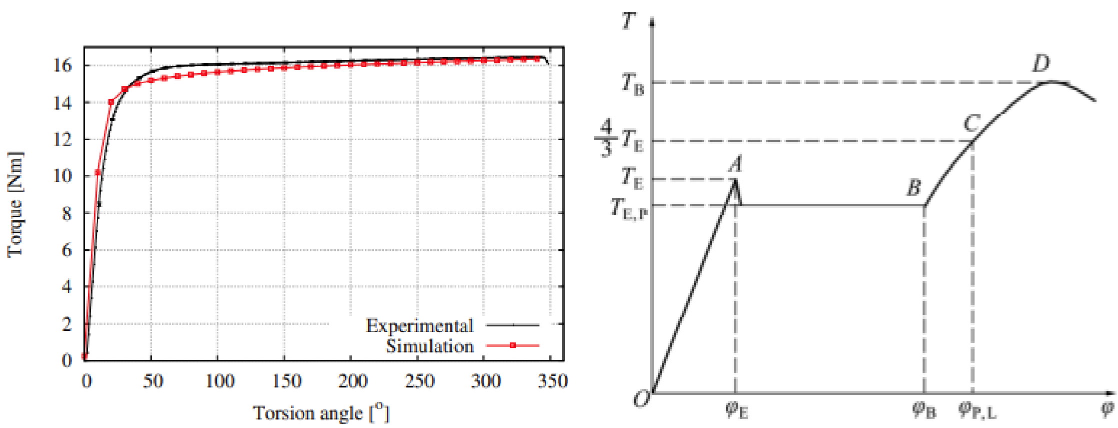


Figure 7. a) Average experimental and numerical results for the torque as a function of the torsion angle (*6*); b) Torque-torsion curve of plastic metal shaft under pure torsion (*7*)

After an initial phase of limited plastic deformation, a structure made of elastic-plastic material can exhibit elastic behavior without increasing residual deformations within the load scheme. Such behavior is referred to as shakedown (Figure 8). For bridge engineering, the shakedown is usually used in the continuous span bridge design which will be discussed further in the next section.

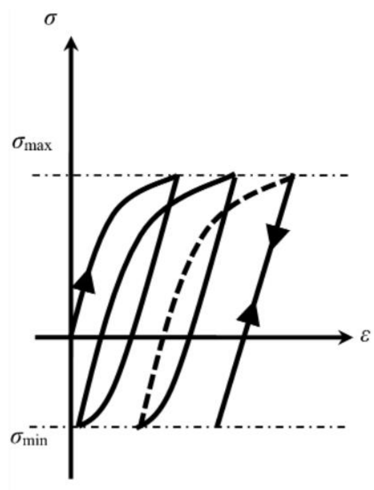


Figure 8. Elastic shakedown behavior (*9*)

4.3 A Description of the Simulation

This simulation procedure is developed for the *pure torsion and isotropic*. This can be simplified as a *pure shear* case. The detailed for pure shear model can be found in the section 8.1.2 (*2*). Since this is assumed to be an isothermal case, the temperature strain will not be considered in the simulation. The following parameters are based on the beginning of Section 8(*2*):

$$G(\theta) = 300$$

$$G^b(\theta) = 30$$

$$\tau_y = \tau_{yo} + \tau_y{}'\xi = 30 + 10\xi$$

$$\Delta t = 0.0001 \quad \gamma_{r\theta o} = 0 \quad \xi_o = 0$$

The general programming procedure for this case is proposed as follows (*2*):

1. Input the shear strain $\gamma_{r\theta i}$ (at t_i) histories
2. Calculate the current parameters: $G_i, G^b{}_i$, $\gamma_{r\theta i}{}^e, \gamma_{r\theta i}{}^p$
3. Calculate the current stresses: $\sigma_{r\theta} = \dfrac{G}{G + \tau_y{}'}(\tau_{yo} - \tau_y{}'\gamma_{r\theta}) = \dfrac{G}{G+10}(30 - 10\gamma_{r\theta})$
4. Substitute the parameters to determine the τ_y: $\tau_y = 30 + 10\xi$
5. Calculate the deviatoric stress (pure shear): $S = \sigma_{r\theta}(e_r \otimes e_\theta + e_\theta \otimes e_r)$
6. Calculate the current shear stress: $\tau = \sqrt{\dfrac{1}{2} S : S} = \sqrt{\dfrac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} = \sqrt{\sigma_{r\theta}{}^2} = \sigma_{r\theta}$

7. Calculate the yield function: $f = \tau - \tau_y = \sqrt{\sigma_{r\theta}^{\ 2}} - \tau_y = \sigma_{r\theta} - \tau_y$

8. Calculate the current stress rate $\dot{\sigma}_{r\theta_i}$

9. If $f \geq 0$, calculate current $\overset{\diamond}{f_i}$ ignore the plastic strain rate from: $\overset{\diamond}{f_i} = \dot{\sigma}_{r\theta}$

10. If $f_i \geq 0$ and $\overset{\diamond}{f_i} > 0$, the flow rule for pure shear is got from section 8.1.2 (*2*):

$$\dot{\gamma}_{r\theta}^{\ p} = \begin{cases} \begin{bmatrix} 0 & \dfrac{\dot{\sigma}_{r\theta}}{2(G^b + \tau_y')} & 0 \\ \dfrac{\dot{\sigma}_{\theta r}}{2(G^b + \tau_y')} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & if\ f \geq 0\ and\ \overset{\diamond}{f} > 0 \\ 0 & All\ other\ cases \end{cases}$$

$$\dot{\xi}_i = \sqrt{2\dot{\varepsilon}_i^{\ p} : \dot{\varepsilon}_i^{\ p}}$$

- Else

$$\gamma_{r\theta}^{\ p} = 0 \quad \dot{\xi}_i = 0$$

11. Store the t_i, $\gamma_{r\theta}, \sigma_{r\theta}$ for the shear strain and stress plots

12. Update the variables

$$\gamma_{r\theta i+1}^{\ p} = \gamma_{r\theta i}^{\ p} + \dot{\gamma}_{r\theta i}^{\ p} \Delta t$$

$$\xi_{i+1} = \xi_i + \dot{\xi}_i \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

13. Increment *i* and return to 1st step.

4.4 A Discussion for Section Problems

Several problems at the sections 8.5 (*2*) are selected and discussed here. This information can be used as a reference for better understanding of materials discussed in this section 8.5.

8.5-2: Calculate the pure torsion moment T_p for the case of perfect plasticity

Figure 6a shows the case of perfect plasticity case for a circular bar. We can consider one area A1, representing the region of perfect plasticity flow. The torque now can be found from the equation:

$$T_p = \int_{A1} \sigma_{r\theta} r dA = \int_{A1} G\gamma_{r\theta} r dA = \int_{A1} G\frac{d\phi}{dX} r^2 dA = G\frac{d\phi}{dX}\int_{A1} r^2 dA = G\frac{d\phi}{dX} I_{p1} \quad (45)$$

Where:

$$I_{p1} = \frac{\pi}{2}(R^4) \quad (46)$$

8.5-3: Develop the case of pure torsion for a hollow cylindrical bar

It can be expected the shear stress distribution in a hollow cylindrical bar is different from solid bar discussed before. Figure 9 shows shear stress distribution across a hollow cylindrical bar and noted that there is no shear stress in the hollow circle (R_S region). Hollow shafts weigh less for a given length and diameter than solid shafts. Most of the material inside a solid shaft experiences / carries a shear stress whose value is well below the maximum shear stress. In the meantime, they add to the shaft's weight without adding much to its ability to carry torsional loads. The derivation of the pure torsion for a hollow cylindrical bar is presented here.

Elastic torsion

We consider pure isotropic hardening. Therefore, the yield function can be written as:

$$f = \tau - \tau_y = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} = \sqrt{\frac{1}{3}(\frac{P}{A})^2 + \left(\frac{T^2 r^2}{I_p{}^2}\right)} - \tau_{yo} \quad (47)$$

Because we are dealing with the pure torsion case, the axial stress term was removed as:

$$f = \sqrt{\left(\frac{T^2 r^2}{I_p{}^2}\right)} - \tau_{yo} \quad (48)$$

Where I_p is referred to the Figure 9:

$$I_p = \frac{\pi}{2}(R_L{}^4 - R_S{}^4) \tag{49}$$

Elastic-plastic torsion

The yield function can be written as:

$$f = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} = \sqrt{\frac{1}{3}(E\varepsilon)^2 + (Gr\frac{d\phi}{dX})^2} - \tau_{yo} = \sqrt{(Gr\frac{d\phi}{dX})^2} - \tau_{yo} \tag{50}$$

That is, the point of initial yield is given in terms of the twist by:

$$r_e = \frac{\tau_{yo}}{G\dfrac{d\phi}{dX}} \tag{51}$$

This procedure for a hollow bar is similar to that for a solid bar, however the Polar Moment of Inertia differs. There are two areas A1 and A2 represent elastic flow and elastic-plastic flow, respectively. From pure shear, we know that the expression for stress in the region of plastic flow is given by:

$$\sigma_{re} = G\gamma_{yo} + \frac{G\tau_y{}'}{G + G\tau_y{}'}(\gamma_{r\theta} - \gamma_{yo}) = \frac{G}{G + \tau_y{}'}(\tau_{yo} - \tau_y{}'\gamma_{r\theta}) \tag{52}$$

The torque now can be found from the equation:

$$T = G\frac{d\phi}{dX}I_{p1} + \frac{G}{G + \tau_y{}'}(\tau_{yo}Q_{p2} + \tau_y{}'\frac{d\phi}{dX}I_{p2}) \tag{53}$$

Where:

$$Q_{p2} = \frac{2\pi}{3}(R_L{}^3 - r_e{}^3)\,;\, I_{p1} = \frac{\pi}{2}(r_e{}^4 - R_s{}^4)\,;\, I_{p2} = \frac{\pi}{2}(R_L{}^4 - r_e{}^4) \tag{54}$$

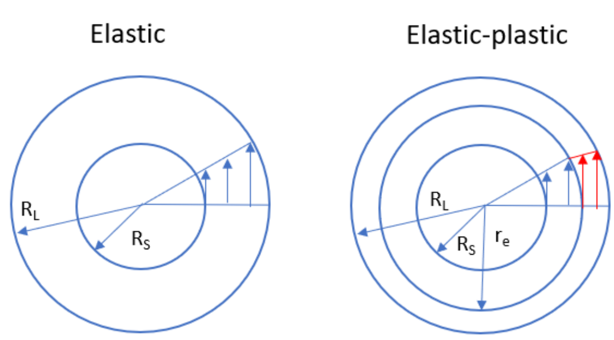


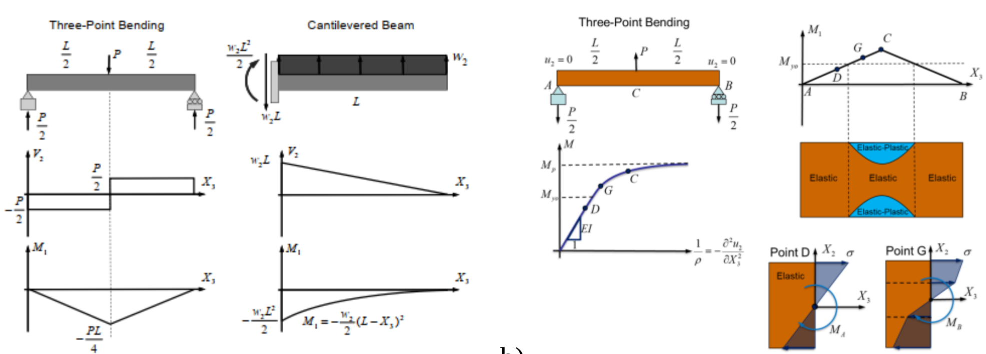
Figure 9. Stress-distribution in a hollow cylindrical bar

5 SECTION 8.8.3

5.1 A Summary Introduction

The deformation caused by multiple loading is considered in this section, which can vary along the length of the bar. These problems are important in that they result in one-dimensional equations with applications in civil and industrial structures. This is typically used in the plastic collapse analysis to determine the collapse load for beams. The book considered the simple span beam/ cantilever beam in the section 8.8.3 (*2*) which are presented here. Figure 10a shows two statically determinate problems and their associated shear and moment diagrams. The background of the model (*2*) and the general simulation procedure are presented here. Several problems are evaluated for the cantilever beam model in this section (*2*).

However, in bridge engineering, this type of loading is of greater interest for continuous spans. Structures with ductile members and connections are recognized to possess a reserve of strength beyond the first plastic hinge that can be utilized in the design. At the limit, a "plastic mechanism" forms, and the structure is said to have attained its "ultimate strength"(*10*). However, it does not mean that a continuous 2-span bridge cannot support more loads once the first "*plastic hinge*" forms at the interior support. As a result, the continuous bridge has become two simply-supported beams, and it will fail until a second "*plastic hinge*" appears. This again shows that continuous bridges have more capacity than engineers thought, and this type of behavior has been adopted as the <u>*moment redistribution method*</u> in the AASHTO LRFD BDS (*11*).



a) b)

Figure 10. a) Diagrams to calculate balance laws and jump conditions for beams under distributed and point loads and moments; b)Example of calculating the curvature and stress distribution for a beam under three-point bending that has a part that is elastic and one that is elastic-plastic (*2*)

5.2 A Detailed Description

5.2.1 A Statically Determined Beam Description

A detailed derivation of these statically determinate models can be found in the book (*2*). There are two problems consider in the Section 8.8.3 (*2*). A brief discussion and description of the model is presented, and a detailed review can be found in APPENDIX 8.3. The first problem is that of a simply supported beam with a central point load. The other is a cantilevered beam under a constant distributed load. Note that a beam subjected to pure bending is bent into an arc of a circle and that the moment-curvature relationship can be expressed mentioned below. If it is below the maximum elastic moment then the beam is under elastic loading and its deflection can be studied with the standard relation, in this case, of

$$M_1 = \frac{EI_1}{\rho_1} = -EI_1 \frac{d^2 u_2}{dX_3{}^2} \tag{55}$$

For example, for the cantilevered beam, we have

$$\frac{d^2 u_2}{dX_3{}^2} = \frac{w_2}{2EI_1}(L - X_3)^2 \tag{56}$$

so we can do the integration to get the slope and deflection:

$$\frac{du_2}{dX_3} = \frac{w_2}{6EI_1}(L - X_3)^3 + D_1 \tag{57}$$

$$u_2 = \frac{w_2}{24EI_1}(L - X_3)^4 + D_1 X_3 + D_2 \tag{58}$$

where D1 and D2 are constants of integration that need to be fit to the *boundary conditions* given at the root of the beam as:

$$u_2\Big|_{X_3=0} = \frac{du_2}{dX_3}\Big|_{X_3=0} = 0 \tag{59}$$

We can also approximate the response as elastic perfectly plastic and identify for each moment the range of the section that has experienced plastic deformation. If we set this equal to the limit moment M_p for perfectly plastic response, we can calculate the conditions under which we get a section that no longer can carry higher moments. For instance, in the three-point bending problem, the plastic hinge occurs when

$$M_p = \frac{PL}{4} \tag{60}$$

which means the maximum load we can apply on the beam before failure is

$$P_{\max} = \frac{4M_p}{L} \tag{61}$$

When observing simultaneous loadings in elastic response, we know that the system is linear, so the solutions are additive and can be considered separately. Figure 10b shows a typical situation for a beam under three-point bending. As can be seen, the central portion has undergone both elastic and plastic deformation, whereas the ends have only undergone elastic deformation.

5.2.2 An Indeterminate Beam Description

Plastic analysis is primarily beneficial in that it takes advantage of the ductility of the material (e.g., steel) in order to maximize the performance of an indeterminate structure. As shown in Figure 11, apply the load *P* at the point D of a two-span beam (*12,13*). The first plastic hinge will happen at D and when:

$$M_p = 0.203PL \tag{62}$$

$$P = \frac{4.926M_p}{L} \tag{63}$$

If we add ΔP at D, we will have additional moment as shown in Figure 11b. Maintain the load in *AC* member, and apply a $(P + \Delta P)$ load at the point E. The plastic hinge will form at the interior support point C as shown in Figure 11c. Based on Figure 11abc, we can solve the ΔP:

$$-0.463M_p - 0.5\Delta PL - 0.094(P + \Delta P)L = -M_p \tag{64}$$

$$\Delta P = 0.125\frac{M_p}{L} \tag{65}$$

If we unload the $(P + \Delta P)$ at the point E, the moment diagram will be changed to Figure 11d. Note that unloading process is always elastic. Keeping $(P + \Delta P)$ in the *AC* member, and loading and unloading at the point E meets the following requirements, the beam will be in the shakedown limit condition and behave like Figure 8.

$$(P + \Delta P)_{sh} = (4.926 + 0.125)\frac{M_p}{L} = 5.051\frac{M_p}{L} \tag{66}$$

This $(P + \Delta P)_{sh}$ is also known as a ***shakedown*** load limit.

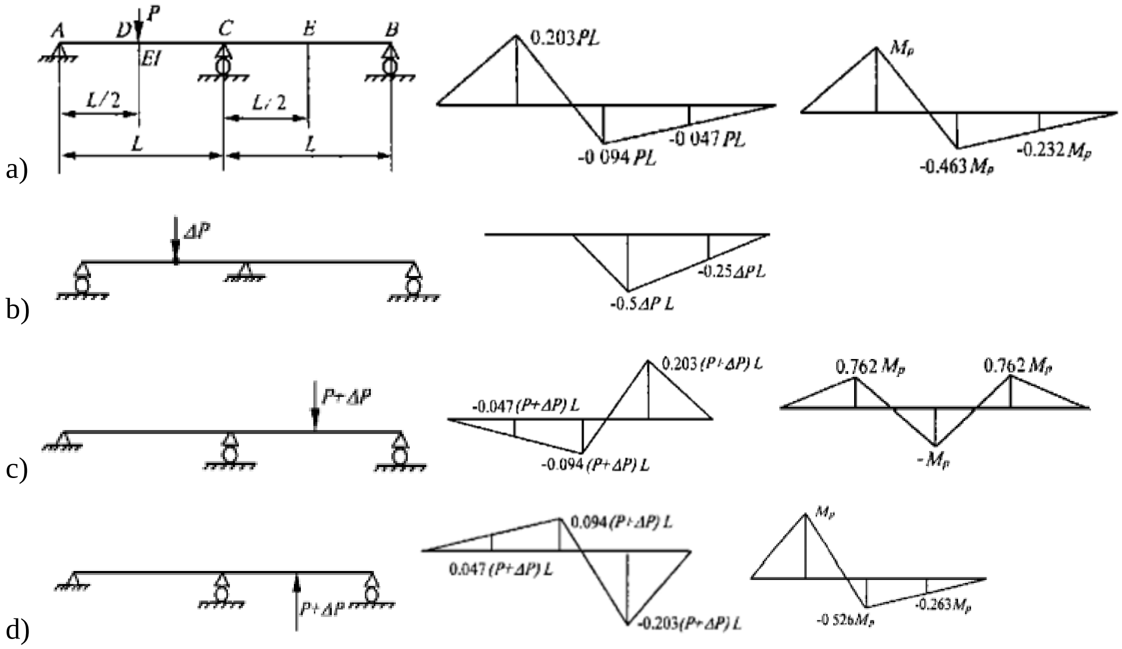


Figure 11. a) Load P at point D; b)Load *ΔP at point D; c) Load* $P + \Delta P$ at point E; d) Unload $P + \Delta P$ at point E (*12*,*13*)

5.3 A Description of the Simulation

This simulation procedure is developed for the *pure bending and isotropic*. This can be simplified as a *uniaxial extension* case. The detailed for uniaxial model can be found in the section 8.1.3 (*2*). Since this is assumed to be an isothermal case, the temperature strain will not be considered in the simulation. The following parameters are used for controlled axial stress history case based on the beginning of Section 8 (*2*):

$$G(\theta) = 300$$

$$G^b(\theta) = 30$$

$$\tau_y = \tau_{yo} + \tau_y{}'\xi = 30 + 10\xi$$

$$\Delta t = 0.0001 \quad \gamma_{r\theta o} = 0 \quad \xi_o = 0 \quad E = 29000$$

The general programming procedure for this case is proposed as follows (*2*):

1. Input the strain ε_{1i} (at t_i) histories
2. Calculate the current stress: $\sigma_{1i} = E\varepsilon = E(\varepsilon_{1i} - \varepsilon_{1i}{}^p)$
3. Get deviatoric stress: $S = \frac{1}{3}(2\sigma_1)e_1 \otimes e_1 - \frac{1}{3}(\sigma_1)e_2 \otimes e_2 - \frac{1}{3}(\sigma_1)e_3 \otimes e_3$

4. Substitute the parameters to determine the τ_y: $\tau_y = 30 + 10\xi$

5. Calculate the yield function: $f = \tau - \tau_y = \sqrt{\frac{1}{3}\sigma_1^{\ 2}} - \tau_y = \sqrt{\frac{1}{3}}\sigma_1 - \tau_y$

6. Calculate the current τ and τ_y rate: $\dot{\tau} = \frac{1}{3}\frac{\sigma_1}{\tau}\dot{\sigma}_1$; $\dot{\tau}_y = \frac{1}{3}\frac{\sigma_1}{\tau}\dot{\sigma}_1 + \frac{\partial \tau_y}{\partial \xi}\dot{\xi} = \frac{1}{3}\frac{\sigma_1}{\tau}\dot{\sigma}_1 + 10\dot{\xi}$

7. Calculate the current τ rate: $\dot{\tau} = \frac{1}{3}\frac{\sigma_1}{\tau}\dot{\sigma}_1$

8. If $f \geq 0$, calculate current $\overset{\lozenge}{f}_i$ ignore the plastic strain rate from: $\overset{\lozenge}{f}_i = \frac{1}{3}\frac{\sigma_1}{\tau}\dot{\sigma}_1$

9. If $f_i \geq 0$ and $\overset{\lozenge}{f}_i > 0$, the flow rule is got from section 8.1.3 (*2*):

$$\dot{\varepsilon}_i^{\ p} = \begin{cases} \begin{bmatrix} \dot{\varepsilon}_1^{\ p} = \beta \times \frac{1}{3}(2\sigma_1) & 0 & 0 \\ 0 & \dot{\varepsilon}_2^{\ p} = -\beta \times \frac{1}{3}(\sigma_1) & 0 \\ 0 & 0 & \dot{\varepsilon}_3^{\ p} = -\beta \times \frac{1}{3}(\sigma_1) \end{bmatrix} & if\ f \geq 0\ and\ \overset{\lozenge}{f} > 0 \\ 0 & All\ other\ cases \end{cases}$$

$$\dot{\xi}_i = \sqrt{2\dot{\varepsilon}_i^{\ p} : \dot{\varepsilon}_i^{\ p}} = 2\beta\tau$$

$$\beta = \frac{\overset{\lozenge}{f}_i}{2\tau \frac{\partial \tau_y}{\partial \xi}} = \frac{\overset{\lozenge}{f}_i}{20\tau}$$

➢ Else

$$\dot{\varepsilon}_i^{\ p} = 0 \qquad \dot{\xi}_i = 0$$

10. Store the t_i , $\dot{\varepsilon}_i^{\ p}$, σ_1 for the strain stress plots

11. Update the variables

$$\varepsilon_{i+1}^{\ p} = \varepsilon_i^{\ p} + \dot{\varepsilon}_i^{\ p}\Delta t \quad \xi_{i+1} = \xi_i + \dot{\xi}_i \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

12. Increment *i* and return to 1st step.

5.4 A Discussion for Section Problems

Several problems at the sections 8.8.3 (*2*) are selected and discussed here. This information can be used as a reference for better understanding of materials discussed in this section 8.8.3.

8.8-4 For the cantilevered beam subjected to a uniform distributed load, calculate the maximum distributed load w_2 that can be applied before formation of a plastic hinge at its root.

Firstly, we need to determine which place of this cantilevered beam has the maximum moment loads. The moment equation for this case is reformed from the curvature equation as below:

$$\frac{d^2u_2}{dX_3{}^2} = \frac{w_2}{2EI_1}(L - X_3)^2 \tag{67}$$

$$M(X_3) = -\frac{w_2}{2}(L - X_3)^2 \tag{68}$$

The maximum moment place is at $X_3 = 0$, and the maximum moment at this place is :

$$M(X_3 = 0) = -\frac{w_2}{2}(L)^2 \tag{69}$$

Assume that the plastic hinge happens at the maximum moment location if we consider ***prismatic*** cantilevered beam:

$$M_p = -\frac{w_2}{2}(L)^2 \tag{70}$$

Then you can find the maximum w_2 can be applied before formation of a plastic hinge at its root:

$$w_2 = -\frac{M_p}{2}(L)^2 \tag{71}$$

Note the negative sign is for the w_2 direction not for the magnitude.

8.8-5: For the cantilevered beam subjected to a point load P at its free end, calculate the maximum load P that can be applied before formation of a plastic hinge at its root.

Firstly, we need to determine which place of this cantilevered beam has the maximum moment loads. The moment equation for this case is reformed from the curvature equation as below:

$$M(X_3) = -P(L - X_3) \tag{72}$$

The maximum moment place is at $X_3 = 0$, and the maximum moment at this place is :

$$M(X_3 = 0) = -PL \tag{73}$$

Assume that the plastic hinge happens at the maximum moment location if we consider *prismatic* cantilevered beam:

$$M_p = -PL \tag{74}$$

Then you can find the maximum P can be applied before formation of a plastic hinge at its root:

$$P = -\frac{M_p}{L} \tag{75}$$

Note the negative sign is for the P direction not for the magnitude.

6 SUMMARY

This project investigated three infinitesimal strain solutions for homogenous, inhomogeneous, and nonuniform loading of bars and discussed about their applications in the research areas. Several problems in the book (*2*) for each section were solved and can be used for others. A model simulation procedure for each solution was developed in this project. Generally, the simulation results in MATLAB for homogenous problems look reasonable. Engineering undergraduate students are recommended to design relevant experiments or develop codes for validating the proposed simulation procedures to understand the importance of plasticity in structural design and metal forming in the future.

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8 APPENDICES

8.1 Section 8.3 Details Review

In the following, consider a mechanical model that is isotropic with the stress given by:

$$\sigma = (\kappa - \frac{2}{3}G)tr(\varepsilon^e) + 2G\varepsilon^e \tag{76}$$

No strain is permitted in the second direction, and no stress is applied in the third direction. In this way, we can calculate the average stress and deviatoric stress as follows:

$$\sigma_{ave} = \frac{1}{3}(\sigma_1 + \sigma_2 + 0) = \frac{1}{3}(\sigma_1 + \sigma_2) \tag{77}$$

$$S = \frac{1}{3}(2\sigma_1 - \sigma_2)e_1 \otimes e_1 + \frac{1}{3}(2\sigma_2 - \sigma_1)e_2 \otimes e_2 - \frac{1}{3}(\sigma_2 + \sigma_1)e_3 \otimes e_3 \tag{78}$$

As for the small strains case, the strain decomposes additively into elastic, plastic, and thermal strains.

$$\varepsilon = \varepsilon^e + \varepsilon^p + \varepsilon^\theta \tag{79}$$

The plastic strain rate, elastic strain rate and thermal strain rate can be written as:

$$\dot{\varepsilon}^p = \beta S = \dot{\varepsilon}_1{}^p e_1 \otimes e_1 + \dot{\varepsilon}_2{}^p e_2 \otimes e_2 + \dot{\varepsilon}_3{}^p e_3 \otimes e_3 \tag{80}$$

$$\dot{\varepsilon}^e = \dot{\varepsilon}_1{}^e e_1 \otimes e_1 + \dot{\varepsilon}_2{}^e e_2 \otimes e_2 + \dot{\varepsilon}_3{}^e e_3 \otimes e_3 \tag{81}$$

$$\dot{\varepsilon}^\theta = \alpha\dot{\theta}I \tag{82}$$

As a result of the expression for deviatoric stress (calculated from hydrostatic stress – average stress at the diagonal) for this isotropic case, we can calculate τ as follows:

$$S = \frac{1}{3}(2\sigma_1 - \sigma_2)e_1 \otimes e_1 + \frac{1}{3}(2\sigma_2 - \sigma_1)e_2 \otimes e_2 - \frac{1}{3}(\sigma_2 + \sigma_1)e_3 \otimes e_3 \tag{83}$$

$$\tau = \sqrt{\frac{1}{2}S:S} = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)} \tag{84}$$

The yield function in general is taken to have the form of the von Mises yield function, and is given for isotropic hardening by:

$$f = \tau - \tau_y = \sqrt{\frac{1}{3}(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2)} - \tau_y \tag{85}$$

The hardening derivative of equivalent shear yield stress is given as:

$$\tau_y = \tau_{yo} + \frac{\partial\tau_y}{\partial\theta}\Delta\theta + \frac{\partial\tau_y}{\partial\xi}\xi + \frac{\partial\tau_y}{\partial\sigma_{ave}}\sigma_{ave} \tag{86}$$

Take the derivatives with respect to time and stresses, the $\dot{\tau}$ can be represented as:

$$\dot{\tau} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] \quad (87)$$

The derivative of average normal stress is:

$$\dot{\sigma}_{ave} = \frac{1}{3}(\dot{\sigma}_1 + \dot{\sigma}_2) \quad (88)$$

This results in the expression for the rate of change of the yield function given by:

$$\dot{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\dot{\sigma}_1 + (2\sigma_2 - \sigma_1)\dot{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{\partial\tau_y}{\partial\xi}\dot{\xi} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\dot{\sigma}_1 + \dot{\sigma}_2) \quad (89)$$

To determine trial derivatives, we assume that the strain rate of plasticity is zero. In the book (*2*) Section 8.3.1, the elastic flow is considered. The book (*2*) Section 8.3.1 used the "dot" on stress rates/force and mentioned that they represent the "diamond" in the text. The stress rates/force trial derivatives sign ("diamond") will be used directly here. When we substitute the elastic strain rate, we get the following stress rate equation:

$$\dot{\sigma} = (\kappa - \frac{2}{3}G)tr(\dot{\varepsilon}^e) + 2G\dot{\varepsilon}^e = (\kappa - \frac{2}{3}G)tr(\dot{\varepsilon} - \dot{\varepsilon}^\theta) + 2G(\dot{\varepsilon} - \dot{\varepsilon}^\theta) \quad (90)$$

We can get the strain rate at the third direction by:

$$\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \quad (91)$$

We will finally get the strain trial derivatives for two directions:

$$\overset{\diamond}{\sigma}_1 = \frac{4G(3\kappa + G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \quad (92)$$

$$\overset{\diamond}{\sigma}_2 = \frac{2G(3\kappa - 2G)}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{18\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \quad (93)$$

We can easily get the trial derivative of f, by taking out the term related to the strain hardening parameter in the derivative of f:

$$\overset{\diamond}{f} = \frac{1}{6\tau}[(2\sigma_1 - \sigma_2)\overset{\diamond}{\sigma}_1 + (2\sigma_2 - \sigma_1)\overset{\diamond}{\sigma}_2] - \frac{\partial\tau_y}{\partial\theta}\dot{\theta} - \frac{1}{3}\frac{\partial\tau_y}{\partial\sigma_{ave}}(\overset{\diamond}{\sigma}_1 + \overset{\diamond}{\sigma}_2) \quad (94)$$

The book (*2*) Section 8.3.2 considered the plastic flow. By substituting the strain rate of elastic, we will get the stress rate equation for each direction:

$$\dot{\varepsilon}^e = \dot{\varepsilon} - \dot{\varepsilon}^p - \dot{\varepsilon}^\theta = \dot{\varepsilon} - \beta S - \alpha\dot{\theta}I \quad (95)$$

$$\dot{\sigma} = (\kappa - \frac{2}{3}G)tr(\dot{\varepsilon}^e) + 2G\dot{\varepsilon}^e = (\kappa - \frac{2}{3}G)tr(\dot{\varepsilon}) + 2G(\dot{\varepsilon}) - 2G\beta S - 3\alpha\kappa\dot{\theta}I \tag{96}$$

We can get the strain rate at the third direction by:

$$\dot{\varepsilon}_3 = -\frac{3\kappa - 2G}{3\kappa + 4G}\dot{\varepsilon}_1 - \frac{2G}{3\kappa + 4G}(\sigma_1 + \sigma_2)\beta + \frac{9\alpha\kappa G}{3\kappa + 4G}\dot{\theta} \tag{97}$$

Follow the same procedures in the elastic flow above, we will get stress rates as:

$$\dot{\sigma}_1 = \overset{\diamond}{\sigma}_1 - \frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta \tag{98}$$

$$\dot{\sigma}_2 = \overset{\diamond}{\sigma}_2 + \frac{2G}{3\kappa + 4G}[2G\sigma_1 - (3\kappa + 2G)\sigma_2]\beta \tag{99}$$

The derivative of yield function during plastic flow can be written as:

$$\begin{aligned} \dot{f} = \overset{\diamond}{f} &+ [\frac{1}{6\tau}(2\sigma_1 - \sigma_2) - \frac{1}{3}\frac{\partial \tau_y}{\partial \sigma_{ave}}]\{-\frac{2G}{3\kappa + 4G}[(3\kappa + 2G)\sigma_1 - 2G\sigma_2]\beta\} \\ &+ [\frac{1}{6\tau}(2\sigma_2 - \sigma_1) - \frac{1}{3}\frac{\partial \tau_y}{\partial \sigma_{ave}}]\{\frac{2G}{3\kappa + 4G}[2G\sigma_1 - (3\kappa + 2G)\sigma_2]\beta\} - 2\frac{\partial \tau_y}{\partial \xi}\tau\beta \end{aligned} \tag{100}$$

During plastic flow we need the <u>*consistency condition*</u> to hold: $\dot{f} = 0$

$$\beta = \frac{\tau(3\kappa + 4G)\overset{\diamond}{f}}{2G[(\kappa + G)(\sigma_1^2 + \sigma_2^2) - (\kappa + 2G)\sigma_1\sigma_2 - \kappa\tau\frac{\partial \tau_y}{\partial \sigma_{ave}}(\sigma_1 + \sigma_2)] + 2\tau^2(3\kappa + 4G)\frac{\partial \tau_y}{\partial \xi}} \tag{101}$$

Flow rule for this isotropic case:

$$\dot{\varepsilon}^p = \begin{Bmatrix} \beta S & \quad if\ f \ge 0\ and\ \overset{\diamond}{f} > 0 \\ 0 & \quad All\, other\, cases \end{Bmatrix} \tag{102}$$

8.2 Section 8.5 and 8.6 Details Review

8.2.1 Elastic Torsion

For a uniform axial load *P*, the axial stress can be calculated by:

$$\sigma = E\varepsilon = \frac{P}{A} = \frac{EA\varepsilon}{A} \tag{103}$$

Based on the torsion equation, the shear stress for the elastic member is given by:

$$T = GI_p \frac{d\phi}{dX} \tag{104}$$

$$\sigma_{r\theta} = Gr\frac{d\phi}{dX} = \frac{Tr}{I_p} \tag{105}$$

The corresponding expression for stress and deviatoric stress can be expressed as:

$$\sigma = \sigma e_3 \otimes e_3 + \sigma_{r\theta}(e_r \otimes e_\theta + e_\theta \otimes e_r) \tag{106}$$

$$S = \frac{1}{3}\sigma(-e_r \otimes e_r - e_\theta \otimes e_\theta + 2e_3 \otimes e_3) + \sigma_{r\theta}(e_r \otimes e_\theta + e_\theta \otimes e_r) \tag{107}$$

From the expression of deviatoric stress for this case, we can calculate τ as:

$$\tau = \sqrt{\frac{1}{2}S:S} = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} \tag{108}$$

The <u>*yield function*</u> for the fully elastic bar is taken to have the form of the von Mises yield function as:

$$f = \tau - \tau_y = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} \tag{109}$$

$$f = \sqrt{\frac{1}{3}(\frac{P}{A})^2 + \left(\frac{T^2r^2}{I_p{}^2}\right)} - \tau_{yo} \tag{110}$$

The maximum elastic load the bar can take then is given by reaching yield at the largest radius *R*. This condition can be rewritten as by squaring the above yield function ($f = 0$):

$$\tau_{yo}{}^2 = \frac{1}{3}(\frac{P}{A})^2 + \left(\frac{T^2R^2}{I_p{}^2}\right) \tag{111}$$

As can be seen, the axial and torsion applied <u>*would affect each other*</u>. If there is no axial load (pure torsion), the maximum torsional moment that the bar can take as:

$$T = \frac{\tau_{yo}I_p}{R} \tag{112}$$

8.2.2 Elastic-plastic Torsion

Figure 6b shows the loading resulting for a bar that undergoes plastic flow. The center is undeformed by the torsion, as a result of the kinematic assumption, an elastic central section reaching the initial yield and plastic flow as the radius increases. As was shown by the yield function, both the axial load and the torsional moment play a role in yielding so the radius r_e of initial yield can be obtained from the equation $f = 0$ given by:

$$f = \sqrt{\frac{1}{3}\sigma^2 + \sigma_{r\theta}{}^2} - \tau_{yo} = \sqrt{\frac{1}{3}(E\varepsilon)^2 + (Gr_e\frac{d\phi}{dX})^2} - \tau_{yo} = 0 \tag{113}$$

Note the pure torsion case (no axial loads applied) will be considered for the rest of this part. That is, the point of initial yield is given in terms of the twist by:

$$r_e = \frac{\tau_{yo}}{G\frac{d\phi}{dX}} \tag{114}$$

We use this to identify the two areas A1 and A2, representing, respectively, the region of elastic flow and elastic-plastic flow. From pure shear, we know that the expression for stress in the region of plastic flow is given by:

$$\sigma_{r\theta} = G\gamma_{yo} + \frac{G\tau_y{}'}{G + G\tau_y{}'}(\gamma_{r\theta} - \gamma_{yo}) = \frac{G}{G + \tau_y{}'}(\tau_{yo} - \tau_y{}'\gamma_{r\theta}) \tag{115}$$

The torque now can be found from the equation:

$$T = G\frac{d\phi}{dX}I_{p1} + \frac{G}{G + \tau_y{}'}(\tau_{yo}Q_{p2} + \tau_y{}'\frac{d\phi}{dX}I_{p2}) \tag{116}$$

Where:

$$Q_{p2} = \frac{2\pi}{3}(R^3 - r_e^3)\,;\, I_{p1} = \frac{\pi}{2}(r_e^4)\,;\, I_{p2} = \frac{\pi}{2}(R^4 - r_e^4) \tag{117}$$

8.2.3 Unloading

The unloading will be shown through pure torsion because it is a simple solution. If the entire bar remains elastic then the unloading is elastic and results in a zero stress distribution. It is more interesting to look at that case in which we have applied a large enough twist that results in an elastic core section of radius re and an elastic-plastic outer section. As a result, we consider that the applied torque is T_o and the corresponding twist is ϕ_o' for the initial monotonic loading. Based on the torsion equations, we can determine:

$$\gamma_o = r\phi_o{}' \tag{118}$$

$$r_{eo} = \frac{\tau_{yo}}{G\phi_o'} \tag{119}$$

$$T_o = G\phi_o' I_{p1o} + \frac{G}{G+\tau_y'}(\tau_{yo} Q_{p2o} + \tau_y' \phi_o' I_{p2o}) \tag{120}$$

where I_{p1o} and Q_{p2o} are the corresponding values of I_{p1} and Q_{p2} for $r_e = r_{eo}$.

When unloaded, the torsional moment must be zero, even though the twist will not be zero. To simplify the notation, we will take the twist after unloading to be ϕ'. In the elastic range, we have the expression for the stress given by:

$$\sigma_{r\theta} = G\gamma = G\phi' r \tag{121}$$

In the plastic zone, as shown in Figure 6b, the stress is given by:

$$\sigma_{r\theta} = \tau_{yo} + \frac{G\tau_y'}{G+\tau_y'}(\gamma_o - \gamma_{yo}) - G(\gamma_o - \gamma) = \tau_{yo} + G^*(\gamma_o - \gamma_{yo}) - G(\gamma_o - \gamma) \tag{122}$$

Considering that $\tau_{yo} = G\gamma_{yo}$; $\gamma_o = r\phi_o'$; $\gamma = r\phi'$, we will get:

$$\sigma_{r\theta} = \tau_{yo} + G^*(\phi_o' r - \frac{\tau_{yo}}{G}) - G(\phi_o' r - \phi' r) \tag{123}$$

The torsional moment can be evaluated by putting the stresses into equation for the moment to obtain:

$$T = GI_{p1o}\phi' + \tau_{yo} Q_{p2o} + G^* \phi_o' I_{p2o} - \frac{G^* \tau_{yo}}{G} Q_{p2o} - GI_{p2o}\phi_o' + GI_{p2o}\phi' \tag{124}$$

As can be seen in Figure 6b, ultimately the unloading is elastic and represents the subtraction of a linearly increasing stress with the radius. This can be written as

$$T = T_o - GI_p(\phi_o' - \phi') \tag{125}$$

Thus, for elastic unloading, the relationship between the torsional moment and the angle of twist is linear. From this equation, one can also calculate the twist as a function of the applied torsional moment:

$$\phi' = \phi_o' + \frac{T - T_o}{GI_p} \tag{126}$$

For total unloading, we will have T = 0 to get

$$\phi' = \phi_o' - \frac{T_o}{GI_p} \tag{127}$$

8.3 Section 8.8.3 Details Review

Note that a beam subjected to pure bending is bent into an arc of a circle and that the moment-curvature relationship can be expressed mentioned below. If it is below the maximum elastic moment then the beam is under elastic loading and its deflection can be studied with the standard relation, in this case, of

$$M_1 = \frac{EI_1}{\rho_1} = -EI_1 \frac{d^2 u_2}{dX_3{}^2} \tag{128}$$

For example, for the cantilevered beam, we have

$$\frac{d^2 u_2}{dX_3{}^2} = \frac{w_2}{2EI_1}(L - X_3)^2 \tag{129}$$

so we can do the integration to get the slope and deflection:

$$\frac{du_2}{dX_3} = \frac{w_2}{6EI_1}(L - X_3)^3 + D_1 \tag{130}$$

$$u_2 = \frac{w_2}{24EI_1}(L - X_3)^4 + D_1 X_3 + D_2 \tag{131}$$

where D1 and D2 are constants of integration that need to be fit to the *boundary conditions* given at the root of the beam as:

$$u_2\Big|_{X_3=0} = \frac{du_2}{dX_3}\Big|_{X_3=0} = 0 \tag{132}$$

In the case of statically determinate problems, another important fact is that, as long as we have the moment-curvature diagram of the beam under bending, we can determine the curvature of each point of the beam numerically and then integrate this to determine the displacement. We can also approximate the response as elastic perfectly plastic and identify for each moment the range of the section that has experienced plastic deformation. It is also important that we know the maximum moment. If we set this equal to the limit moment M_p for perfectly plastic response, we can calculate the conditions under which we get a section that no longer can carry higher moments. For instance, in the three-point bending problem, the plastic hinge occurs when

$$M_p = \frac{PL}{4} \tag{133}$$

which means the maximum load we can apply on the beam before failure is

$$P_{\max} = \frac{4M_p}{L} \tag{134}$$

When observing simultaneous loadings in elastic response, we know that the system is linear, so the solutions are additive and can be considered separately. The load V and the moment M can still be calculated as before if the problem is statically determined, but the constitutive equations are not linear, so the solutions for the displacements no longer add up. To calculate the point of yield, we need to consider all loadings simultaneously. Figure 10b shows a typical situation for a beam under three-point bending. As can be seen, the central portion has undergone both elastic and plastic deformation, whereas the ends have only undergone elastic deformation. With the moment-curvature response, we can calculate the curvature for each point in the beam. Using this equation and the two boundary conditions of u2 = 0 at both ends, we can obtain the curve for u2. It is also possible to calculate the stress distribution for each moment, so we have all the information about both the deflection and the stress distribution in the beam.

8.4 Section 8.3 MATLAB Codes

8.4.1 Problem 8.3-1 and 8.3-3 (Monotonic Loading) in Book

```
% Author: Bowen Yang, EIT
% Work organization: UNL
% email: byang11@huskers.unl.edu
% Apr 2022
% Parameters definitions
% A : cross-sectional area (normally same as Ao)
% alpha : coefficient of thermal expansion
% Epsilon: strain
% Theta: temperature
% Lambda: stretch ratio
% Xi: hardening parameter, assume no hardering parameter
% Py: yield load
% dt : time step take as 0.001s


%------------ BEGIN CODE -------------------------------------------
---

% Clear and clc command and workspace
clear
clc

% Assumed unchanged parameters
alpha = 0.01;
dt = 0.00001;

% Input the function of G,kappa
G = 300 ; %
kappa = 800 ; %

% Initialization of time and strain_p
Xi = 0;
t = 0; % time start from 0
epsilon_1 = 0;
epsilon_3 = 0;
epsilon_1p = 0; % initial value
epsilon_3p = 0; % initial value
epsilon_1p_dot = 0;
epsilon_3p_dot = 0;

steps = 10000;
```

```
for i = 1 : steps

    % Calculate the current strain rate 1st direction (no tempeature
rate)
    epsilon_1_dot = 1; % assumed value

    epsilon_1 = epsilon_1_dot * t;

    % calculate stress 1 and 2
    sigma_1 = (kappa + 4/3*G)*(epsilon_1-epsilon_1p) + (kappa -
2/3*G)*(epsilon_3-epsilon_3p);
    sigma_2 = (kappa - 2/3*G)*(epsilon_1-epsilon_1p) + (kappa -
2/3*G)*(epsilon_3-epsilon_3p);
    sigma_3 = 0;

    % calculate the average stress
    sigma_avg = 1/3*(sigma_1+sigma_2+sigma_3);

    % Determine the Tau_y and Tau
    tau_y = 30  + 10*Xi - sigma_avg;
    tau = sqrt(1/3*(sigma_1^2-sigma_1*sigma_2+sigma_2^2));

    % Calculate the yield function f
    f = tau - tau_y;


    % if the f >= 0, the calculate f_diamond
    % define the variables for calculating the f_diamond

    sigma_1_diamond = 4*G*(3*kappa+G)/(3*kappa+4*G)*epsilon_1_dot;
    sigma_2_diamond = 2*G*(3*kappa-2*G)/(3*kappa+4*G)*epsilon_1_dot;

    % calculate f_diamond
    f_diamond = 1/(6*tau)*((2*sigma_1-sigma_2)*sigma_1_diamond
+(2*sigma_2-sigma_1)*sigma_2_diamond)...
        +1/3*(sigma_1_diamond+sigma_2_diamond);

    % beta
    beta =
tau*(3*kappa+4*G)*f_diamond/(2*G*((kappa+G)*(sigma_1^2+sigma_2^2)-...
        (kappa+2*G)*sigma_1*sigma_2+
kappa*tau*(sigma_1+sigma_2))+20*tau^2*(3*kappa+4*G));

    % flow rule
    if (f > 0 || f == 0) && f_diamond > 0
        % update the plastic strain rate for the 1st,3rd directions
        epsilon_1p_dot = beta * 1/3*(2*sigma_1-sigma_2);
```

```
        epsilon_3p_dot = beta * -1/3*(sigma_1+sigma_2);

        % update the total strain rate for the third direction
        epsilon_3_dot = -(3*kappa-2*G)/(3*kappa+4*G)-
2*G/(3*kappa+4*G)*(sigma_1+sigma_2)*beta;

        % update the Xi rate
        Xi_dot = 2*beta*tau;

    else

        epsilon_1p_dot = 0;
        epsilon_3p_dot = 0;

        % update the strain rate for the third direction
        epsilon_3_dot = -(3*kappa-2*G)/(3*kappa+4*G);
        Xi_dot = 0;

    end

    % strain 1st direction vs sigma 1st direction
    stress1(i) = sigma_1;
    stress2(i) = sigma_2;
    strain1(i) = epsilon_1;
    strain3(i) = epsilon_3;
    shear_y (i) = tau_y;
    stress_avg (i) = sigma_avg;
    % strain rate 3rd direction vs strain rate 1st direction
    strain1_dot(i) = epsilon_1_dot;
    strain3_dot (i) = epsilon_3_dot;
    rate(i) = strain3_dot (i)/strain1_dot(i);

    % update the variables
    t = t + dt;
    epsilon_1p = epsilon_1p + epsilon_1p_dot * dt ;
    epsilon_3p = epsilon_3p + epsilon_3p_dot * dt ;
    epsilon_3 = epsilon_3 + epsilon_3_dot * dt ;
    Xi = Xi + Xi_dot * dt ;


end
```

```
%% Create figure
figure1 = figure('InvertHardcopy','off','Color',[1 1 1],...
    'Renderer','painters');

% Create axes
axes1 = axes('Parent',figure1);
hold(axes1,'on');

% Create plot
plot(strain1,stress1,'LineWidth',2);

% Create ylabel
ylabel('Stress_1','LineWidth',1,'FontSize',15,'FontName','times new
roman');

% Create xlabel
xlabel('Epsilon_1','LineWidth',1,'FontSize',15,'FontName','times new
roman');

% Create title
title('Epsilon_1 vs Stress_1','LineWidth',1,'FontSize',15,...
    'FontName','times new roman');


% plot with repsect to time
time = 0:dt:(steps-1)*dt;

%% Create figure

figure1 = figure('InvertHardcopy','off','Color',[1 1 1],...
    'Renderer','painters');

axes1 = axes('Parent',figure1);
hold(axes1,'on');

% Create plot
plot(time,rate,'LineWidth',2);

% Create ylabel
ylabel('The ratio of strain rate 3rd - strain rate
1st','LineWidth',1,'FontSize',15,'FontName','times new roman');

% Create xlabel
xlabel('Time','LineWidth',1,'FontSize',15,'FontName','times new
roman');
```

```
% Create title
title('Time vs Ratio','LineWidth',1,'FontSize',15,...
    'FontName','times new roman');



%% Create figure
figure1 = figure('InvertHardcopy','off','Color',[1 1 1],...
    'Renderer','painters');

% Create axes
axes1 = axes('Parent',figure1);
hold(axes1,'on');

% Create plot
plot(strain1,stress2,'LineWidth',2);


% Create ylabel
ylabel('Stress_2','LineWidth',1,'FontSize',15,'FontName','times new
roman');

% Create xlabel
xlabel('Epsilon_1','LineWidth',1,'FontSize',15,'FontName','times new
roman');

% Create title
title('Epsilon_1 vs Stress_2','LineWidth',1,'FontSize',15,...
    'FontName','times new roman');


%% Create figure
figure1 = figure('InvertHardcopy','off','Color',[1 1 1],...
    'Renderer','painters');

% Create axes
axes1 = axes('Parent',figure1);
hold(axes1,'on');

% Create plot
plot(strain1,strain3,'LineWidth',2);


% Create ylabel
ylabel('Epsilon_3','LineWidth',1,'FontSize',15,'FontName','times new
roman');
```

```
% Create xlabel
xlabel('Epsilon_1','LineWidth',1,'FontSize',15,'FontName','times new
roman');

% Create title
title('Epsilon_1 vs Epsilon_3','LineWidth',1,'FontSize',15,...
    'FontName','times new roman');
```

8.4.2 Problem 8.3-2 in Book (ε_1=0)

```
% Author: Bowen Yang, EIT
% Work organization: UNL
% email: byang11@huskers.unl.edu
% Apr 2022
% Parameters definitions
% A : cross-sectional area (normally same as Ao)
% alpha : coefficient of thermal expansion
% Epsilon: strain
% Theta: temperature
% Lambda: stretch ratio
% Xi: hardening parameter, assume no hardering parameter
% Py: yield load
% dt : time step take as 0.001s

%------------- BEGIN CODE --------------------------------------------
---

% Clear and clc command and workspace
clear
clc


% Assumed unchanged parameters
alpha = 0.01;
dt = 0.001;

% Initialization of assumed paramteres
Xi = 0;
epsilon_3 = 0;
epsilon_1p = 0; % initial value
epsilon_3p = 0; % initial value
epsilon_1p_dot = 0;
epsilon_3p_dot = 0;

t = 0; % time start from 0

% strain at the 1st direction equals to zero case
epsilon_1 = 0; % Epsilon 1st function
epsilon_1_dot = 0 ; % assumed value


theta_dot = 0.001;% theta dot (with repect to time)
```

```
for theta = 0:theta_dot:100

    G = 300 - 1 * theta; % delta_Go = 1.0
    kappa = 800 - 10 * theta ; %

    % calculate stress 1 and 2

    sigma_1 = (kappa + 4/3*G)*(epsilon_1-epsilon_1p) + (kappa -
2/3*G)*(epsilon_3-epsilon_3p)-3*alpha*kappa*theta;
    sigma_2 = (kappa - 2/3*G)*(epsilon_1-epsilon_1p) + (kappa -
2/3*G)*(epsilon_3-epsilon_3p)-3*alpha*kappa*theta;
    sigma_3 = 0;

    % calculate the average stress
    sigma_avg = 1/3*(sigma_1+sigma_2+sigma_3);

    % Determine the Tau_y and Tau
    tau_y = 30  - theta + 10*Xi - sigma_avg;
    tau = sqrt(1/3*(sigma_1^2-sigma_1*sigma_2+sigma_2^2));

    % Calculate the yield function f
    f = tau - tau_y;


    % if the f >= 0, the calculate f_diamond
    % define the variables for calculating the f_diamond
    sigma_1_diamond = 4*G*(3*kappa+G)/(3*kappa+4*G)*epsilon_1_dot -
18*alpha*kappa*G/(3*kappa+4*G)*theta_dot;
    sigma_2_diamond = 2*G*(3*kappa-2*G)/(3*kappa+4*G)*epsilon_1_dot -
18*alpha*kappa*G/(3*kappa+4*G)*theta_dot;


    f_diamond = 1/(6*tau)*((2*sigma_1-sigma_2)*sigma_1_diamond
+(2*sigma_2-sigma_1)*sigma_2_diamond)...
    + theta_dot +1/3*(sigma_1_diamond+sigma_2_diamond);


    % beta
    beta =
tau*(3*kappa+4*G)*f_diamond/(2*G*((kappa+G)*(sigma_1^2+sigma_2^2)-...
    (kappa+2*G)*sigma_1*sigma_2+
kappa*tau*(sigma_1+sigma_2))+20*tau^2*(3*kappa+4*G));

    % flow rule
    if (f > 0 || f == 0) && f_diamond > 0
        fprintf("The temperature is :"+string(theta))
        break
```

```
    else
        epsilon_1p_dot = 0;
        epsilon_3p_dot = 0;
    % update the strain rate for the third direction
        epsilon_3_dot = -(3*kappa-2*G)/(3*kappa+4*G) +
9*alpha*kappa*G/(3*kappa+4*G)*theta_dot;
        Xi_dot = 0;
    end

    % update the variables
    epsilon_1p = epsilon_1p + epsilon_1p_dot * dt ;
    epsilon_3p = epsilon_3p + epsilon_3p_dot * dt ;
    epsilon_3 = epsilon_3 + epsilon_3_dot * dt ;
    Xi = Xi + Xi_dot * dt ;

End
```

8.4.3 Problem 8.3-2 in Book (σ_1=0)

```
% Author: Bowen Yang, EIT
% Work organization: UNL
% email: byang11@huskers.unl.edu
% Apr 2022
% Parameters definitions
% A : cross-sectional area (normally same as Ao)
% alpha : coefficient of thermal expansion
% Epsilon: strain
% Theta: temperature
% Lambda: stretch ratio
% Xi: hardening parameter, assume no hardering parameter
% Py: yield load
% dt : time step take as 0.001s

%------------- BEGIN CODE --------------------------------------------
---

% Clear and clc command and workspace
clear
clc


% Assumed unchanged parameters
alpha = 0.01;
dt = 0.001;

% Initialization of assumed paramteres
Xi = 0;
epsilon_3 = 0;
epsilon_1p = 0; % initial value
epsilon_3p = 0; % initial value
epsilon_1p_dot = 0;
epsilon_3p_dot = 0;

t = 0; % time start from 0

% assume montonic loading for strain at the 1st direction
epsilon_1_dot = 1 ; % assumed value
epsilon_1 = epsilon_1_dot * t;

theta_dot = 0.001;% theta dot (with repect to time)
```

```
for theta = 0:theta_dot:100

    G = 300 - 1 * theta; % delta_Go = 1.0
    kappa = 800 - 10 * theta ; %

    % calculate stress 1 and 2
    sigma_1 = 0; % stress at 1st direction = 0
    sigma_2 = (kappa - 2/3*G)*(epsilon_1-epsilon_1p) + (kappa -
2/3*G)*(epsilon_3-epsilon_3p)-3*alpha*kappa*theta;
    sigma_3 = 0;

    % calculate the average stress
    sigma_avg = 1/3*(sigma_1+sigma_2+sigma_3);

    % Determine the Tau_y and Tau
    tau_y = 30  - theta + 10*Xi - sigma_avg;
    tau = sqrt(1/3*(sigma_1^2-sigma_1*sigma_2+sigma_2^2));

    % Calculate the yield function f
    f = tau - tau_y;

    % if the f >= 0, the calculate f_diamond
    % define the variables for calculating the f_diamond

    sigma_1_diamond = 4*G*(3*kappa+G)/(3*kappa+4*G)*epsilon_1_dot -
18*alpha*kappa*G/(3*kappa+4*G)*theta_dot;
    sigma_2_diamond = 2*G*(3*kappa-2*G)/(3*kappa+4*G)*epsilon_1_dot -
18*alpha*kappa*G/(3*kappa+4*G)*theta_dot;


    f_diamond = 1/(6*tau)*((2*sigma_1-sigma_2)*sigma_1_diamond
+(2*sigma_2-sigma_1)*sigma_2_diamond)...
    + theta_dot +1/3*(sigma_1_diamond+sigma_2_diamond);

    % beta

    beta =
tau*(3*kappa+4*G)*f_diamond/(2*G*((kappa+G)*(sigma_1^2+sigma_2^2)-...
    (kappa+2*G)*sigma_1*sigma_2+
kappa*tau*(sigma_1+sigma_2))+20*tau^2*(3*kappa+4*G));

    % flow rule
    if (f > 0 || f == 0) && f_diamond > 0
        fprintf("The temperature is :"+string(theta))
        break
    else
        epsilon_1p_dot = 0;
```

```
        epsilon_3p_dot = 0;
    % update the strain rate for the third direction
        epsilon_3_dot = -(3*kappa-2*G)/(3*kappa+4*G) + 
9*alpha*kappa*G/(3*kappa+4*G)*theta_dot;
        Xi_dot = 0;

    end

    % update the variables
    epsilon_1p = epsilon_1p + epsilon_1p_dot * dt ;
    epsilon_3p = epsilon_3p + epsilon_3p_dot * dt ;
    epsilon_3 = epsilon_3 + epsilon_3_dot * dt ;
    Xi = Xi + Xi_dot * dt ;

end
```