MECH 942 Project 1: Thermodynamics and material modeling-1D

P R O J E C T R E P O R T

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SYMBOLS

The symbols used for this project are summarized as below (*2*):

$\cdot$	: material time derivative
$\lozenge$	: time derivative for constant plastic strain and hardening parameter
A	: cross-sectional area
A_o	: cross-sectional area in reference configuration
c	: heat capacity
f	: yield function or generic continuous function
$\overset{\lozenge}{f}$	: $\dot{f}$ calculated assuming constant plastic strain and hardening
P	: axial load
P^b	: back load
F_y	: yield load
G	: shear modulus
k	: coefficient of thermal conductivity
L	: velocity gradient tensor
α	: coefficient of thermal expansion
β	: scaling factor for plastic flow
η	: specific entropy (entropy per unit mass)
θ	: temperature
λ	: stretch ratio
λ^e	: elastic stretch ratio
λ^p	: plastic stretch ratio
λ^θ	: thermal stretch ratio
ε	: strain
ε^e	: elastic strain
ε^p	: plastic strain
ε^θ	: thermal strain
ρ	: density in terms of mass per unit current length
ρ_o	: density in reference configuration
Ψ	: specific free energy (free energy per unit mass)
Δt	: time step size for numerical analysis

1 ABSTRACT

Typically, materials are not only subject to mechanical loads but also to temperature loads. The mechanical behavior of materials can be affected by different strain histories and temperature histories. The project developed a 1D thermodynamic plasticity model for elastic rubber materials based on the modification of yield load with temperature by using MATLAB^{TM}. In this model, the elastic strain was substituted for the total strain in the constitutive functions. This project considered four cases of combined thermomechanical and isothermal loadings. The stress and entropy responses for each case were calculated. The developed model generally predicts responses well and can potentially be applied to other loading situations.

2 INTRODUCTION

The responses of material stress and entropy to thermomechanical loading are influenced by the combined effects of temperature and strain. Understanding how the material behaves at different temperatures and strain histories is essential. There is a thermoelastic/plastic rubber elasticity model in the MECH 942 note (*1*), but it is not yet clear how the model will behave when modified to include temperature dependent yield stress under the given strain and temperature histories. This project updated the nonlinear thermos-elastic/plastic model of rubber elasticity material (Figure 1) to include temperature-dependent yield stresses (Chapter 3). Chapter 4 proposes the procedure for programming the integration of variables based on the temperature-dependent plasticity integration method discussed in the book (*2*). A MATLAB™ (*3*) model was developed with the assumption that the temperature dependence of the parameters is linear, as shown in Project description (*4*). This study examined four different strain histories: isothermal combined with constant strain rate, constant strain combined with constant heating/cooling, isothermal combined with cyclic strain, and rapid heating combined with a high strain rate (*4*). Stress and entropy responses under these conditions are plotted and discussed in Chapters 5 and 6. The key findings of this project are summarized in Chapter 7. Overall, this nonlinear thermos-elastic/plastic model effectively predicts stress and entropy responses given temperature and strain histories, and future work is needed to introduce other types of material models.

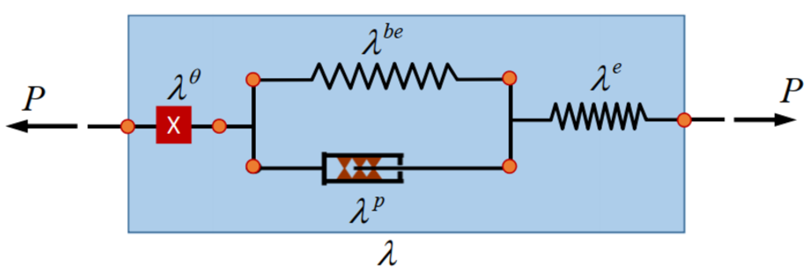

Figure 1. Thermos-elastic/plastic mechanical analog

3 DESCRIPTION OF THE MODEL

The thermodynamics of the plasticity model (*1*) has been modified to include temperature-dependent yield stress as per the Project description (*4*). Here is a brief description of the developed model that provides everything needed to calculate stress and entropy responses:

Kinematics: $\lambda = \lambda^e \lambda^p \lambda^\theta$ (1)

Yield function: $f = \left|P - P^b\right| - P_y$ (2)

Thermodynamic loads: $P^e = \dfrac{AG(\theta)\lambda^e}{\lambda}(\lambda^e - \dfrac{1}{\lambda^{e2}})$; $P^b = \dfrac{AG(\theta)\lambda^p}{\lambda}(\lambda^p - \dfrac{1}{\lambda^{p2}})$; $P^\theta = 0$ (3)

Thermal expansion: $\varepsilon^\theta = \lambda_o{}^\theta e^{\alpha(\theta-\theta_o)} - 1$; $\lambda^\theta = \lambda_o{}^\theta e^{\alpha(\theta-\theta_o)}$ (4)

The trial derivative $\overset{\diamond}{f}$:

$$\overset{\diamond}{f} = \begin{Bmatrix} \overset{\diamond}{P} - \overset{\diamond}{P^b} - \overset{\diamond}{P_y} & P - P^b > 0 \\ -(\overset{\diamond}{P} - \overset{\diamond}{P^b}) - \overset{\diamond}{P_y} & P - P^b \le 0 \end{Bmatrix} \tag{5}$$

$$\overset{\diamond}{P} = \left(k_t{}^e \lambda^e\right)\frac{\dot{\lambda}}{\lambda} + (k_t{}^\theta - \alpha k_t{}^e \lambda^e)\dot{\theta}\,,\ \overset{\diamond}{P^b} = \left(-P^b\right)\frac{\dot{\lambda}}{\lambda} + (k_t{}^{b\theta} + \alpha P^b)\dot{\theta} + (k_t{}^{b\theta}\lambda_p + P^b)\frac{\dot{\lambda}_p}{\lambda_p}\,,\ \overset{\diamond}{P_y} = k_{P_{yo}}\dot{\theta} \tag{6}$$

$$k_t{}^e = \frac{AG(\theta)}{\lambda}(\lambda^e + \frac{2}{\lambda^{e2}})\,,\ k_t{}^\theta = P^e(\frac{1}{G(\theta)}\frac{dG(\theta)}{d\theta} - \alpha) \tag{7}$$

$$k_t{}^{be} = \frac{AG^b(\theta)}{\lambda}(\lambda^p + \frac{2}{\lambda^{p2}})\,,\ k_t{}^{b\theta} = P^b(\frac{1}{G^b(\theta)}\frac{dG^b(\theta)}{d\theta} - \alpha)\,,\ k_{P_y} = \frac{dP_y(\theta)}{d\theta} \tag{8}$$

Flow rule:

$$L^p = \frac{\dot{\lambda}^p}{\lambda^p} = \begin{Bmatrix} \beta_L L + \beta_\theta \dot{\theta} & if\ f \ge 0\ and\ \overset{\diamond}{f} > 0 \\ 0 & All\, other\, cases \end{Bmatrix} \tag{9}$$

$$\beta_L = \frac{\left(P^b + k_t{}^e \lambda^e\right)}{(k_t{}^e\lambda^e + k_t{}^{be}\lambda^p + P^e + P^b)} \quad , \quad \beta_\theta = \frac{\left(k_t{}^\theta - k_t{}^{b\theta} - \alpha k_t{}^e \lambda^e - \alpha P^b - k_{P_y}\right)}{(k_t{}^e\lambda^e + k_t{}^{be}\lambda^p + P^e + P^b)} \tag{10}$$

Calculate the entropy response:

$$\eta_i = -\frac{A}{2\rho_o}\frac{dG_i(\theta_i)}{d\theta}(\lambda_i^{e2} + \frac{2}{\lambda_i^e} - 3) - \frac{A}{2\rho_o}\frac{dG_i^b(\theta_i)}{d\theta}(\lambda_i^{p2} + \frac{2}{\lambda_i^p} - 3) + c_o \ln(\theta_i) + \frac{\alpha P_i}{\rho_i} - D_1 \tag{11}$$

$$\rho\lambda = \rho_o \tag{12}$$

This model can determine the stress and entropy responses based on the strain and temperature histories. APPENDIX 9.1 contains more information regarding the derivations.

4 PROGRAMMED PROCESSES FOR THE INTEGRATION OF VARIABLES

For this thermos-elastic/plastic model, the inputs are strain and temperature functions. It is assumed that we can calculate the associated rates for these functions (*2*). The general programming procedure for this case is proposed as follows (*2*):

Assumed unchanged parameters for this project: $A_o, k_o, \alpha_o, c_o, \rho_o, D_1, \lambda_o{}^{\theta}, \Delta t$

The initial conditions are: $t = 0$, $\varepsilon_o{}^p = 0$, $\varepsilon_o = 0$, $\theta = \theta_o$

Parameters assumed for different evaluated cases: $\dot{\varepsilon}_o, \dot{\theta}_o$

The algorithm used to integrate the response is given by:

1. Input the strain and temperature histories (at t_i) to get ε_i, θ_i
2. Calculate the current parameters: $G_i, G_i{}^b, P_{yi}, \varepsilon_i{}^{\theta}, \varepsilon_i{}^p,\ \lambda_i{}^e,\ \lambda_i{}^p, \lambda_i{}^{\theta}$

   (Note: $\lambda_i{}^e = \dfrac{\lambda_i}{\lambda_i{}^p \lambda_i{}^{\theta}}$, $\lambda_i{}^p = (\varepsilon_i{}^p + 1)$, $\lambda_i{}^{\theta} = \lambda_o{}^{\theta} e^{\alpha(\theta_i - \theta_o)}$, and assumed $\lambda_o{}^{\theta} = 1.0$)
3. Calculate current derivatives to temperature: $\dfrac{dG_i(\theta_i)}{d\theta}, \dfrac{dG_i{}^b(\theta_i)}{d\theta}, k_{P_{y_i}} = \dfrac{dP_{y_i}(\theta_i)}{d\theta}$
4. Calculate the current loads: $P_i = P_i^e$, P_i^b, P_i^{θ}
5. Calculate the yield function: $f_i = \left| P_i^e - P_i^b \right| - P_{yi}$
6. Calculate the current strain rate $\dot{\varepsilon}_i = \dot{\varepsilon}(t_i) = \dot{\lambda}_i$ and temperature rate $\dot{\theta}_i = \dot{\theta}(t_i)$
7. If $f \geq 0$, calculate current $\overset{\diamond}{f}_i$ from:

$$\overset{\diamond}{f}_i = \begin{cases} \overset{\diamond}{P}_i - \overset{\diamond}{P}{}_i^b - \overset{\diamond}{P}_{y_i} & P_i - P_i^b > 0 \\ -(\overset{\diamond}{P}_i - \overset{\diamond}{P}{}_i^b) - \overset{\diamond}{P}_{y_i} & P_i - P_i^b \leq 0 \end{cases}$$

8. Calculate the current plastic strain rate:

- If $f_i \geq 0$ and $\overset{\Diamond}{f_i} > 0$, then

$$L_i^p = \frac{\dot{\lambda}_i^p}{\lambda_i^p} = \begin{cases} \beta_L L + \beta_\theta \dot{\theta} & if\ f \geq 0\ and\ \overset{\Diamond}{f} > 0 \\ 0 & All\ other\ cases \end{cases}, \quad \dot{\varepsilon}_i^p = (1+\varepsilon_i^p)(\beta_L \frac{\dot{\varepsilon}_i}{\varepsilon_i + 1} + \beta_\theta \dot{\theta})$$

- Else

$$\dot{\varepsilon}_i^p = 0$$

9. Calculate the current entropy response:

$$\rho_i \lambda_i = \rho_o$$

$$\eta_i = -\frac{A}{2\rho_o}\frac{dG_i(\theta_i)}{d\theta}(\lambda_i^{e2} + \frac{2}{\lambda_i^e} - 3) - \frac{A}{2\rho_o}\frac{dG_i^b(\theta_i)}{d\theta}(\lambda_i^{p2} + \frac{2}{\lambda_i^p} - 3) + c_o \ln(\theta_i) + \frac{\alpha P_i}{\rho_i} - D_1$$

10. Store the t_i , ε_i , θ_i , P_i^e , P_{y_i} , η_i for the stress and entropy plots

11. Update the variables

$$\varepsilon_{i+1}^p = \varepsilon_i^p + \dot{\varepsilon}_i^p \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

12. Increment *i* and return to *1st* step.

APPENDIX 9.2 provides additional information regarding the models developed in MATLAB™.

5 RESULTS

This section shows stress and response plots for each loading case. The stress shown below corresponds to plot a, whereas entropy is referred to as plot b. The red dashed line in the stress figures represents yield stress. The following parameters are used for considered simulations (4):

$$G(\theta) = 500 \times (1 - 0.002 \times \theta) = 500 - 1 \times \theta$$

$$G^b(\theta) = 100 \times (1 - 0.001 \times \theta) = 100 - 0.1 \times \theta$$

$$P_y(\theta) = 10 \times (1 - 0.002 \times \theta) = 10 - 0.02 \times \theta$$

$$A = 1 \quad \alpha = 1.2 \times 10^{-4} \quad c = 1 \quad \rho_o = 1 \quad D_1 = 0 \quad \lambda_o^{\theta} = 1 \quad \Delta t = 0.0001$$

In each case, the initial conditions are the same as follows:

$$t = 0 \qquad \varepsilon_o{}^p = 0 \qquad \varepsilon_o = 0 \qquad \theta_o = 20$$

Evaluate the response under the following loading cases (4):

a. Isothermal response at θ_o due to constant strain rate: $\varepsilon = \dot{\varepsilon}_o t$. The $\dot{\varepsilon}_o$ is not given (4), therefore, assumed the $\dot{\varepsilon}_o = 1$.

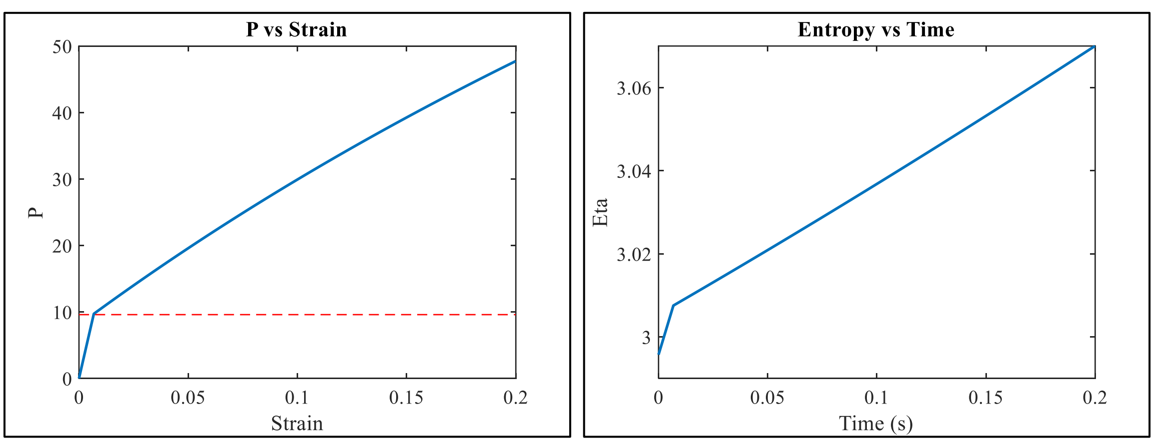

Figure 2. Elastic-plastic stress and entropy responses for isothermal and constant strain rate

b. Constant $\varepsilon = 0$ and constant heating/cooling $\theta = \dot{\theta}_o t$. The $\dot{\theta}_o$ is not given (*4*), therefore, assumed the $\dot{\theta}_o = +30$ for the heating case and $\dot{\theta}_o = -30$ the cooling case.

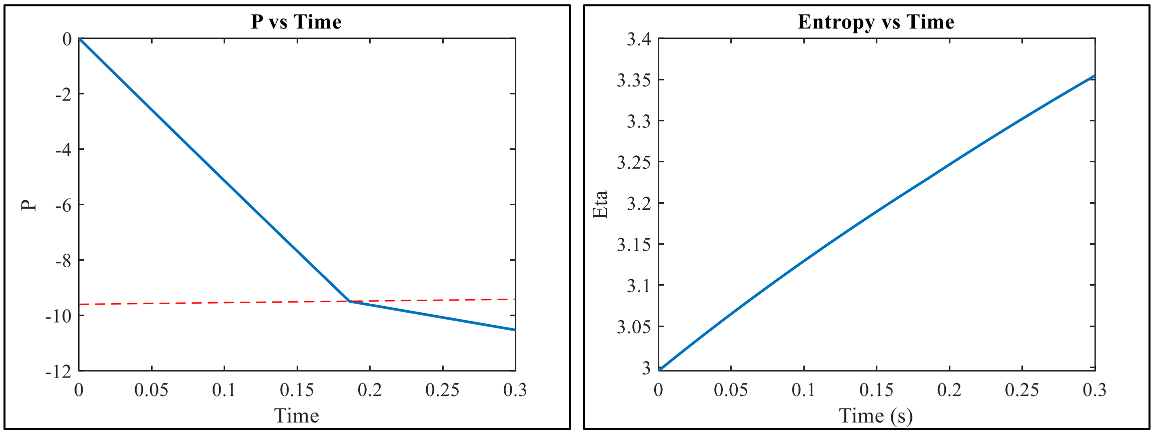

Figure 3. Elastic-plastic stress and entropy responses for $\varepsilon = 0$ and constant heating

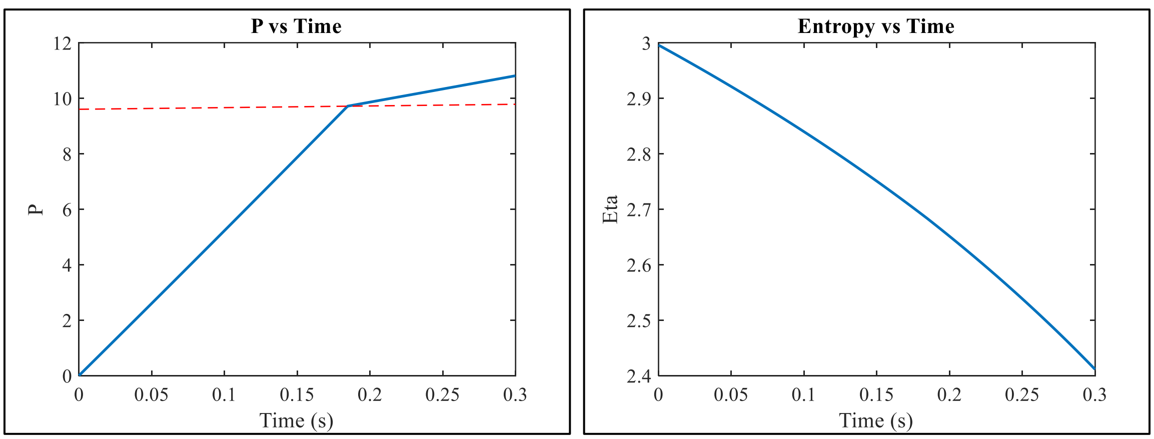

Figure 4. Elastic-plastic stress and entropy responses for $\varepsilon = 0$ and constant cooling

c. Isothermal response at θ_o on cyclic straining $\varepsilon = at\,\text{Sin}(wt)$. The parameters were selected as $a = 0.03$ and $w = 8\pi$ to match the requirement as a response below yield for the first cycle and go beyond yield in the second cycle (*4*).

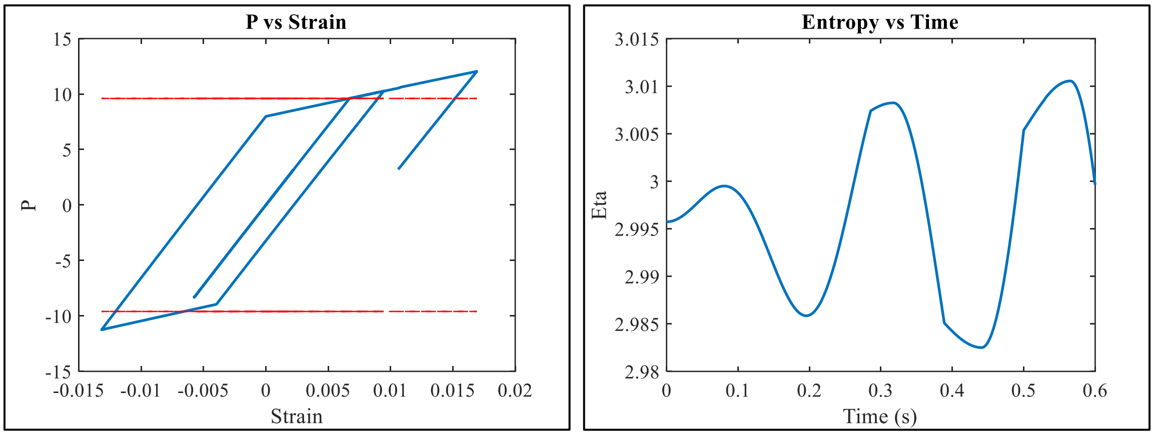

Figure 5. Elastic-plastic stress and entropy responses for isothermal and cyclic straining

d. Combined thermo-mechanical loading as: $\varepsilon = \dot{\varepsilon}_o t$ and $\theta = \dot{\theta}_o t$. Use $\dot{\varepsilon}_o = 30$ and $\dot{\theta}_o = 50$ to simulate the rapid change of strain and temperature case.

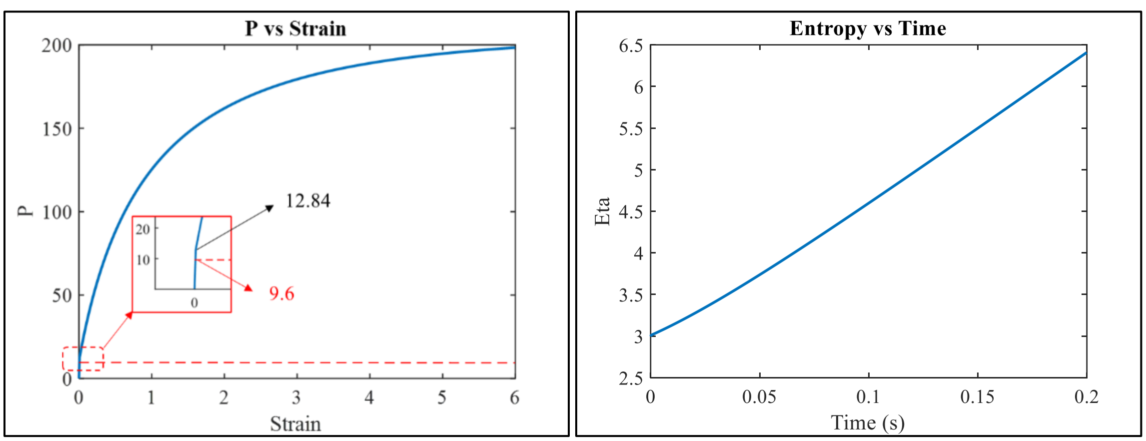

Figure 6. Elastic-plastic stress and entropy responses for the combined thermo-mechanical

6 DISCUSSION

As shown in Figure 2a, the yield load overshoot error (*2*) is very small, indicating that the time step Δt is small enough for $\dot{\varepsilon}_o = 1$. Figure 2a suggests that the temerpature dependent yield stress does not change much before the material yields. Figure 2b displays that the entropy increases with time. In this isothermal case, the heat should be transferred to the system to ensure a constant temperature is maintained, which would result in an increase in entropy. It is also possible to explain this behavior by examining how the terms in Eqn.11 change with time. Since the temperature is constant for this case, it would not affect the change in entropy in Figure 2b. Figure 7d shows that the term that is related P in Eqn.11 increases with time and it has a greater effect than other terms. Figure 7d shows the detailed effect of terms on entropy in Eqn.11.

Figure 3a and Figure 4a indicate that the material yields in both heating and cooling based on *assumed temperature strain rate*, but in compression for heating and in tension for cooling. Figure 3a shows that as temperature increases, the P^e is increasing in the compression side. Recall that $\lambda^e = \lambda / (\lambda^p \lambda^\theta)$, the $\lambda^\theta > 1$ according to Eqn. 4, and $\lambda = 1$ in this case. In the elastic range, the $\lambda^p = 1$ as discussed previously. Therefore, the $\lambda^e < 1$ indicates that the P^e is in the compression according to Eqn. 3. Figure 3b and Figure 8b illustrate that entropy increases as temperature increases. According to Figure 8d, the change in entropy for this case is dominated by the temperature term in Eqn.11. On the other hand, Figure 4a indicates that P^e is increasing on the tension side as temperature decreases, which can be explained by $\lambda^e > 1$ in Eqn. 3. Figure 4b and Figure 9b present that entropy decreases as temperature decreases.

Figure 5a and Figure 10c demonstrate that the material does not yield at the first cycle and yields at the second cycle based on $a = 0.03$ and $w = 8\pi$. This is because strain amplitude increases as time passes as shown in the $\varepsilon = 0.03t\,\mathrm{Sin}(8\pi t)$. According to Figure 5b, the entropy varies according to the cyclic strain, and the variation increases as the strain increases. To maintain a constant temperature in this isothermal and cyclic strain case, heat should either be transferred into or out of the system, which will result in a varied entropy. The entropy change is determined by terms related to λ_i^e, λ_i^p, and P in Eqn.11, as depicted in Figure 10d.

For this project, the combined thermomechanical case was made to simulate the rapid change of strain and temperature scenario. Figure 6a illustrates that, due to the high strain rate, the load overshoot the yield surface by *34%*, indicating a sizeable propagating error in the form of a vertical shift of the plastic loading portion of the response (*2*). Figure 11a shows the solution to this problem by using a smaller $\Delta t = 0.00001$. However, the small increment size costs considerable time, making this algorithm less than ideal for practical use. There are two other solutions: *the elastic return* algorithm and the *plastic return* algorithm (*2*). Figure 11b illustrates that the entropy increases with time. Heat is increasing in this case, resulting in an increase in entropy, as shown Figure 11c and Figure 11d.

7 CONCLUSION

This rubber elasticity thermodynamics of the plasticity model showed ***different*** stresses and entropies when subjected to ***different*** strain and temperature histories. Overall, the stress and entropy response trends are reasonable. This model can potentially be used to solve other load scenarios. Here are some key findings based on ***assumed and given*** parameters:

- According to the stress data presented, the material yield occurs at very small strains for all cases;
- When materials begin to yield, the yield stress does not change much in comparison to the initial yield stress for all cases;
- The stress response of the isothermal and cyclic straining cases exhibits the Bauschinger effect as kinematic hardening (*2*);
- When the material is subjected to $\varepsilon = 0$ and constant heating/cooling with the same rate, the stress and entropy change with the same trend but inversely.
- Comparing Figure 2a with Figure 3a and Figure 4a, for $\varepsilon = 0$ and constant heating/cooling, the material takes a little longer time to yield than for isothermal and constant strain rate case;
- If the strain rate is very high near the yield point, the time step Δt should be very small (close to zero) to reduce yield overshoot, as shown in Figure 11a. The *elastic or plastic return* methods should be used for updating this model in the future (*2*).

8 REFERENCES

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3. MATLAB 2022a, The MathWorks, Inc., Natick, Massachusetts, United States.

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9 APPENDICES

9.1 DERIVATIONS OF THE MODEL

The model should consider temperature-dependent yield stress (*4*). There will be changes to the yield function and flow rule for the model presented in the note (*1*), but the procedure will remain the same. The kinematics and constitutive assumptions for the nonlinear thermos-elastic/plastic model are presented in the note (*1*) as follows:

Kinematics: $\lambda = \lambda^e \lambda^p \lambda^\theta$ (13)

Constitutive assumptions: $\Psi = \frac{AG(\theta)}{2\rho_o}(\lambda^{e2} + \frac{2}{\lambda^e} - 3) + \frac{AG^b(\theta)}{2\rho_o}(\lambda^p + \frac{2}{\lambda^p} - 3) + \Psi^\theta(\theta)$

$$c|_{P=0} = c_o \qquad \alpha = \alpha_o \qquad q = -k\frac{\partial\theta(x,t)}{\partial x} = -kg \tag{14}$$

Thermal expansion: $\frac{\dot{\lambda}^\theta}{\lambda^\theta} = \alpha_o\dot{\theta} \qquad \ln(\frac{\dot{\lambda}^\theta}{\lambda^\theta}) = \alpha_o(\theta - \theta_o) \qquad \lambda^\theta = \lambda_o{}^\theta e^{\alpha(\theta-\theta_o)} \qquad \varepsilon^\theta = \lambda_o{}^\theta e^{\alpha(\theta-\theta_o)} - 1$ (15)

Load: $P = \rho\frac{\partial\Psi}{\partial\varepsilon^e}\lambda^e = \rho\frac{\partial\Psi}{\partial\lambda^e}\frac{\partial\lambda^e}{\partial\varepsilon^e}\lambda^e = (\frac{\rho_o}{\lambda})\left[\frac{AG(\theta)}{2\rho_o}(2\lambda^e - \frac{1}{\lambda^{e2}})\right](1)\lambda^e$ (16)

$$P^b = \rho\frac{\partial\Psi}{\partial\varepsilon^p}\lambda^p = \rho\frac{\partial\Psi}{\partial\lambda^p}\frac{\partial\lambda^p}{\partial\varepsilon^p}\lambda^p = (\frac{\rho_o}{\lambda})\left[\frac{AG(\theta)}{2\rho_o}(2\lambda^p - \frac{1}{\lambda^{p2}})\right](1)\lambda^p \tag{17}$$

$$P^b = \rho\frac{\partial\Psi}{\partial\varepsilon^\theta}\lambda^\theta \tag{18}$$

Therefore, we will get load expressions:

$$P^e = \frac{AG(\theta)\lambda^e}{\lambda}(\lambda^e - \frac{1}{\lambda^{e2}}) \tag{19}$$

$$P^b = \frac{AG(\theta)\lambda^p}{\lambda}(\lambda^p - \frac{1}{\lambda^{p2}}) \tag{20}$$

$$P^\theta = 0 \tag{21}$$

For a given time and temperature and strain histories, we can determine the yield function value as follows:

Yield load function: $f = \left|P - P^b\right| - P_y$ (22)

When $f \geq 0$, it violates the yield function smaller than zero theory. Them, we need to calculate the rate of change of the yield function:

$$\dot{f} = \begin{cases} \dot{P} - \dot{P}^b - \dot{P}_{y_o} & P - P^b > 0 \\ -(\dot{P} - \dot{P}^b) - \dot{P}_{y_o} & P - P^b \leq 0 \end{cases} \tag{23}$$

If we take the derivatives of loads as functions of temperature and time, we obtain:

$$\dot{P} = \left(k_t^{\,e}\lambda^e\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,\theta} - \alpha k_t^{\,e}\lambda^e)\dot{\theta} + (-P^e - k_t^{\,e}\lambda^e)\frac{\dot{\lambda}_p}{\lambda_p}$$

$$\dot{P}^b = \left(-P^b\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,b\theta} + \alpha P^b)\dot{\theta} + (k_t^{\,b\theta}\lambda_p + P^b)\frac{\dot{\lambda}_p}{\lambda_p}$$

$$\dot{P}_{y_0} = k_{P_{yo}}\dot{\theta} \qquad k_{P_{yo}} = \frac{dP_{y_o}(\theta)}{d\theta} \tag{24}$$

During plastic flow we need the consistency condition to hold: $\dot{f} = 0$

Therefore, we will get: $\dot{P} = \dot{P}^b + \dot{P}_{y_o}$

By substituting the derivates for elastic strain rate in the above equation, we will obtain:

$$\left(k_t^{\,e}\lambda^e\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,\theta} - \alpha k_t^{\,e}\lambda^e)\dot{\theta} + (-P^e - k_t^{\,e}\lambda^e)\frac{\dot{\lambda}_p}{\lambda_p} = \left(-P^b\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,b\theta} + \alpha P^b)\dot{\theta} + (k_t^{\,b\theta}\lambda_p + P^b)\frac{\dot{\lambda}_p}{\lambda_p} + k_{P_{yo}}\dot{\theta} \tag{25}$$

Now, we need to determine the trial derivative $\overset{\diamond}{f}$, and the strain rate of plasticity is assumed to be zero for determining trial derivative $\overset{\diamond}{f}$.

The trial derivative $\overset{\diamond}{f}$:

$$\overset{\diamond}{f} = \begin{cases} \overset{\diamond}{P} - \overset{\diamond}{P^b} - \overset{\diamond}{P_y} & P - P^b > 0 \\ -(\overset{\diamond}{P} - \overset{\diamond}{P^b}) - \overset{\diamond}{P_y} & P - P^b \leq 0 \end{cases} \tag{26}$$

$$\overset{\diamond}{P} = \left(k_t^{\,e}\lambda^e\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,\theta} - \alpha k_t^{\,e}\lambda^e)\dot{\theta}\,,\ \overset{\diamond}{P^b} = \left(-P^b\right)\frac{\dot{\lambda}}{\lambda} + (k_t^{\,b\theta} + \alpha P^b)\dot{\theta} + (k_t^{\,b\theta}\lambda_p + P^b)\frac{\dot{\lambda}_p}{\lambda_p}\,,\ \overset{\diamond}{P_y} = k_{P_{yo}}\dot{\theta} \tag{27}$$

$$k_t^{\,e} = \frac{AG(\theta)}{\lambda}(\lambda^e + \frac{2}{\lambda^{e2}})\,;\ k_t^{\,\theta} = P^e(\frac{1}{G(\theta)}\frac{dG(\theta)}{d\theta} - \alpha) \tag{28}$$

$$k_t^{\,be} = \frac{AG^b(\theta)}{\lambda}(\lambda^p + \frac{2}{\lambda^{p2}})\,;\ k_t^{\,b\theta} = P^b(\frac{1}{G^b(\theta)}\frac{dG^b(\theta)}{d\theta} - \alpha)\,;\ k_{P_y} = \frac{dP_y(\theta)}{d\theta} \tag{29}$$

Flow rule:

$$L^p = \frac{\dot{\lambda}^p}{\lambda^p} = \begin{cases} \beta_L L + \beta_\theta\dot{\theta} & if\ f \geq 0\ and\ \overset{\diamond}{f} > 0 \\ 0 & All\ other\ cases \end{cases} \tag{30}$$

Reform Eqn. 25 by removing terms related to the strain rate of plasticity, we will get betas as below:

$$\beta_L = \frac{\left(P^b + k_t^{\,e}\lambda^e\right)}{(k_t^{\,e}\lambda^e + k_t^{\,be}\lambda^p + P^e + P^b)} \qquad \beta_\theta = \frac{\left(k_t^{\,\theta} - k_t^{\,b\theta} - \alpha k_t^{\,e}\lambda^e - \alpha P^b - k_{P_y}\right)}{(k_t^{\,e}\lambda^e + k_t^{\,be}\lambda^p + P^e + P^b)} \tag{31}$$

Specify entropy:

$$\eta = -\frac{A}{2\rho_o}\frac{dG(\theta)}{d\theta}(\lambda^{e2} + \frac{2}{\lambda^e} - 3) - \frac{A}{2\rho_o}\frac{dG^b(\theta)}{d\theta}(\lambda^{p2} + \frac{2}{\lambda^p} - 3) - \frac{d\Psi^\theta(\theta)}{d\theta} + \frac{\alpha P}{\rho} \tag{32}$$

$$\frac{d\Psi^\theta(\theta)}{d\theta} = -c_o \ln(\theta) + D_1 \tag{33}$$

Substitute Eqn.33 into Eqn.32, we will get the entropy equation:

$$\eta_i = -\frac{A}{2\rho_o}\frac{dG_i(\theta_i)}{d\theta}(\lambda_i^{\,e2} + \frac{2}{\lambda_i^{\,e}} - 3) - \frac{A}{2\rho_o}\frac{dG_i^{\,b}(\theta_i)}{d\theta}(\lambda_i^{\,p2} + \frac{2}{\lambda_i^{\,p}} - 3) + c_o \ln(\theta_i) + \frac{\alpha P_i}{\rho_i} - D_1 \tag{34}$$

9.2 MATLAB CODES

```
% GOAl: Solve the stress and entropy response numerically with strain and
% temperature history

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% Apr 2022

% Parameters definitions
% A : cross-sectional area (normally same as Ao)
% alpha : coefficient of thermal expansion
% c : heat capacity
% Epsilon: strain
% Theta: temperature
% Eta: specific entropy (entropy per unit mass)
% Lambda: stretch ratio
% Xi: hardening parameter (not used in the codes)
% Py: yield load
% dt : time step take as 0.0001s
% Var_dot : take the derivative of variable with respect to temperature
% Var_dot_time : take the derivative of variable with respect to time
% Eta_1st_term : means the 1st term in specify entropy equation

%------------- BEGIN CODE ----------------------------------------------

% Clear and clc command and workspace
clear
clc

% Assumed unchanged parameters
A = 1;
alpha = 1.2*10e-4;
c = 1;
Lamada_theta_o = 1.0;
D1 = 0;
rho_o = 1;
dt = 0.0001;

% Initialization of assumed paramteres
theta_o = 20 ;% theta_o = 20 % can redefine it if desired
Epsilon_o_dot = 1 ;% assume a positive value
theta_o_dot = -30;  % +- 30 postive for healing negative for cooling
Epsilon_o = 0;
```

```
% Input the function of G,Gb and Py
syms theta
G_theta = 500 - 1 * theta; % delta_Go = 1.0
Gb_theta = 100 - 0.1 * theta; % delta_Go = 0.1
Py_theta = 10 - 0.02 * theta; % delta_Pyo = 0.02

% Ask which case you would like to run? eg., Case a = Case 1;
Case = input("Which case you would like to run? Case 1, 2, 3, or 4? ");

% Switch function for selected case
switch Case
   case 1
      syms t
      theta_t = 0 * t + theta_o; % theta function
      Epsilon_t = Epsilon_o_dot * t + Epsilon_o; % Epsilon function
   case 2
      syms t
      theta_t = theta_o_dot * t + theta_o; % theta function
      Epsilon_t = 0 * t + Epsilon_o; % Epsilon function
   case 3
      a = 0.03;
      w = 8 * pi;  % a = 1, w = 4, why 0.966s huge jump?
      syms t
      theta_t = 0 * t + theta_o; % theta function
      Epsilon_t = a * t * sin(w * t) + Epsilon_o; % Epsilon function
   case 4
      theta_o_dot = 50;
      Epsilon_o_dot = 30;
      syms t
      theta_t =  theta_o_dot * t + theta_o; % theta function
      Epsilon_t = Epsilon_o_dot * t + Epsilon_o; % Epsilon function
end

% Initialization of time and strain_p
t = 0; % time start from 0
Epsilon_p = 0; % initial value


% set up different the number of steps for the plot
if Case == 3 % cyclic load case needs more steps than other cases
   steps = 6000;
else
   steps = 3000;
end
```

```
% preallocating for speed
P = zeros(length(steps));
Strain = zeros(length(steps));
Strain_p = zeros(length(steps));
Yield = zeros(length(steps));
Eta = zeros(length(steps));
temperature = zeros(length(steps));
Eta_1st_term = zeros(length(steps));
Eta_2nd_term = zeros(length(steps));
Eta_3rd_term = zeros(length(steps));
Eta_4th_term = zeros(length(steps));


%% for loop
for i = 1 : steps
   % Input the strain and temperature histories
   Epsilon = double(subs(Epsilon_t));
   theta = double(subs(theta_t));

   G = double(subs(G_theta));
   G_dot= double(vpa(subs(diff(G_theta),theta)));

   Gb = double(subs(Gb_theta));
   Gb_dot = double(vpa(subs(diff(Gb_theta),theta)));

   Py = double(subs(Py_theta));
   Py_dot = double(vpa(subs(diff(Py_theta),theta)));


   Epsilon_theta = Lamada_theta_o * exp(alpha*(theta-theta_o)) - 1 ;

   % Lambda terms
   Lambda =  Epsilon + 1;
   Lambda_theta = Epsilon_theta + 1;
   Lambda_p = Epsilon_p + 1;
   Lambda_e = Lambda/(Lambda_theta*Lambda_p);


   % loads
   Pe = A * G * Lambda_e / Lambda * (Lambda_e - 1/(Lambda_e^2));
   Pb = A * Gb * Lambda_p / Lambda * (Lambda_p - 1/(Lambda_p^2));
   P_theta = 0; % no use for this project

   % calculate yield function

   f = abs(Pe - Pb) - Py;
```

```
% calculate strain rate and temperature rate
Epsilon_dot = double(vpa(subs(diff(Epsilon_t),t)));
theta_dot = double(vpa(subs(diff(theta_t),t)));
Lambda_dot = Epsilon_dot; % this is true for equation


% if the f >= 0, the calculate f_diamond
% define the variables for calculating the f_diamond

kt_e =  A * G / Lambda * (Lambda_e + 2/(Lambda_e^2));
kt_theta =  Pe * (1/G * G_dot - alpha);
kt_be = A * Gb / Lambda * (Lambda_p + 2/(Lambda_p^2));
kt_b_theta =  Pb * (1/Gb * Gb_dot - alpha);
kpy = Py_dot; % define a new define variable dpy(theta)/d(theta)

% f trial derivative terms

P_diamond  = (kt_e * Lambda_e) * (Lambda_dot/Lambda) + ...
   (kt_theta - alpha * kt_e * Lambda_e) * theta_dot;
Pb_diamond  = (-Pb) * (Lambda_dot/Lambda) ...
   + (kt_b_theta + alpha * Pb) * theta_dot;
Py_diamond = kpy * theta_dot;

% beta for L and theta

w_denom = kt_e * Lambda_e + kt_be * Lambda_p + Pe + Pb; % denominator
beta_L = (Pb + kt_e*Lambda_e)/w_denom;
beta_theta = (kt_theta - kt_b_theta - alpha * kt_e * Lambda_e ...
   - alpha * Pb - kpy)/w_denom;

% switch for updating the strain rate of plasticity

if f > 0 || f == 0
   if Pe - Pb > 0
      f_diamond = P_diamond  - Pb_diamond - Py_diamond ;
   elseif Pe - Pb <= 0
      f_diamond = -(P_diamond  - Pb_diamond) - Py_diamond ;
   end
      if f_diamond > 0 % update the Epsilon_p_dot
         Epsilon_p_dot = (1 + Epsilon_p) * ((beta_L * Epsilon_dot) /(Epsilon + 1) ...
            + beta_theta * theta_dot);
      else
         Epsilon_p_dot = 0;
      end
else
   Epsilon_p_dot = 0;
end
```

```
    % determine the entrop
    if i == 1 % initial entropy
        Eta(i) = c * log(theta_o) - D1;  % assumed D1 = 0
    else
        Eta(i) = -A/(2*rho_o)*G_dot*(Lambda_e^2 + 2/(Lambda_e)-3) ...
            -A/(2*rho_o)*Gb_dot*(Lambda_p^2 + 2/(Lambda_p)-3)...
             + c * log(theta) + alpha * Pe/(rho_o/Lambda);
    end

    % store Pe, Py, theta, Eta, Eta terms in Eqn.11 and strain for iterations
    P(i) = Pe;
    Strain(i) = Epsilon;
    Yield(i) = Py;
    temperature (i) = theta;
    Eta_1st_term (i) = -A/(2*rho_o)*G_dot*(Lambda_e^2 + 2/(Lambda_e)-3);
    Eta_2nd_term (i) =-A/(2*rho_o)*Gb_dot*(Lambda_p^2 + 2/(Lambda_p)-3);
    Eta_3rd_term (i) = c * log(theta);
    Eta_4th_term (i) = alpha * Pe/(rho_o/Lambda);

    % update the time and Epsilon_p
    t = t + dt;
    Epsilon_p = Epsilon_p + Epsilon_p_dot * dt ;

end
%% plots
if Case == 3
    createfigure_Fy_both(Strain,P,Yield)
else
    createfigure_Fy_pos(Strain,P,Yield)
end
createfigure_entropy(temperature,Eta)
% plot with repsect to time
time = 0 : dt :(steps-1)*dt;
run("Plot_Epsilon_time.m")
run("Plot_P_time.m")
run("Plot_Eta_time.m")
run ("Plot_Eta_4terms_time.m")

%------------- END OF CODE -------------------------------------
%Please send suggestions for improvement of the above template header
%to Bowen Yang at this email address: byang11@huskers.unl.edu
% Note:
% Other m-files required for plots:
% Plot_Epsilon_time.m;Plot_P_time.m;
% Plot_Eta_4terms_time.m;
% functions: createfigure_Fy_both;createfigure_Fy_pos;createfigure_entropy
% To save some spaces, the plot functions are not provided here in the APPENDIX.
```

9.3 ADDITIONAL PLOTS

9.3.1 Problem 2a

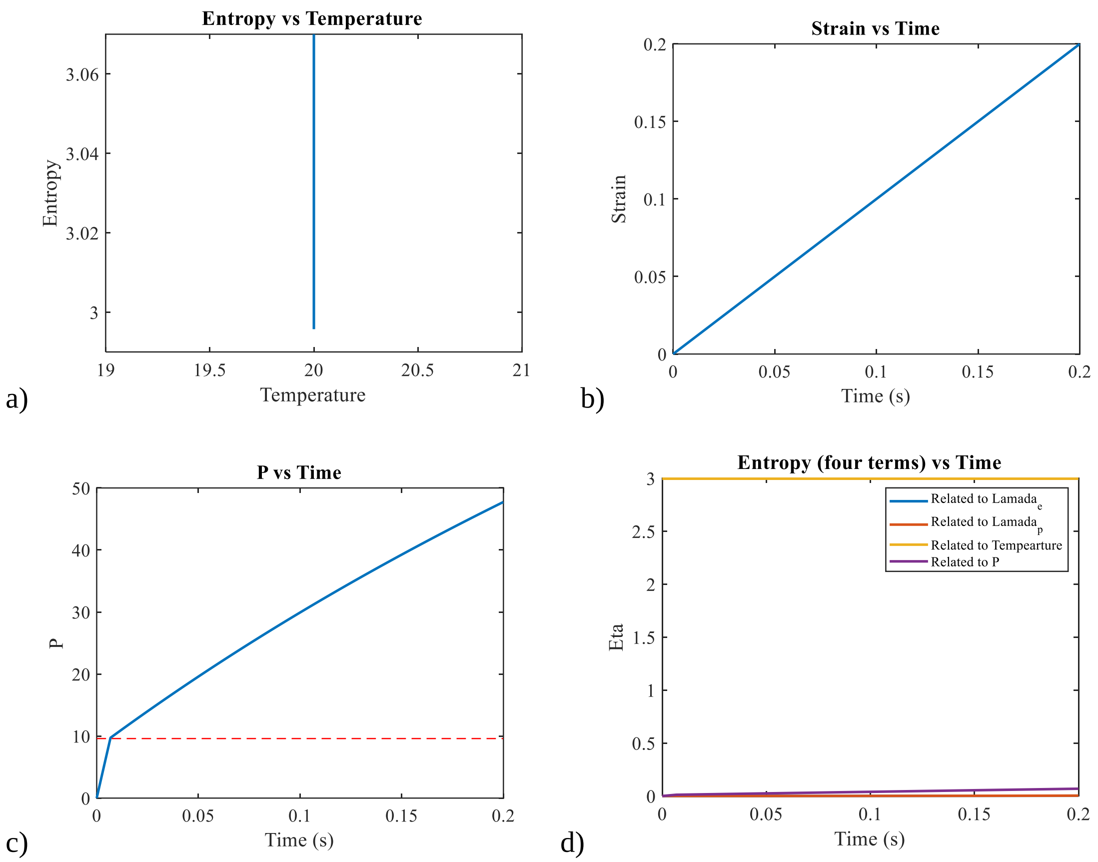

Figure 7. Plots for isothermal response at θ_o and constant strain rate $\varepsilon = \dot{\varepsilon}_o t$

9.3.2 Problem 2b (heating)

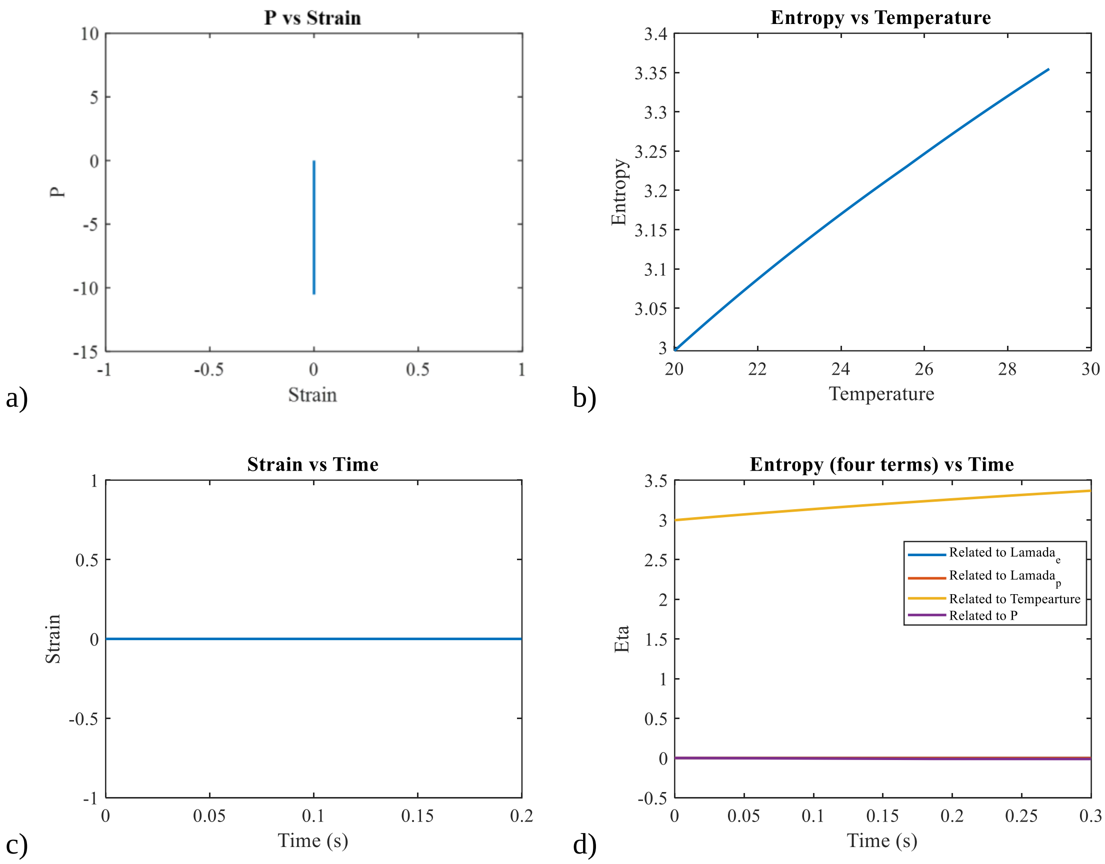

Figure 8. Plots for $\varepsilon = 0$ and constant heating

9.3.3 Problem 2b (cooling)

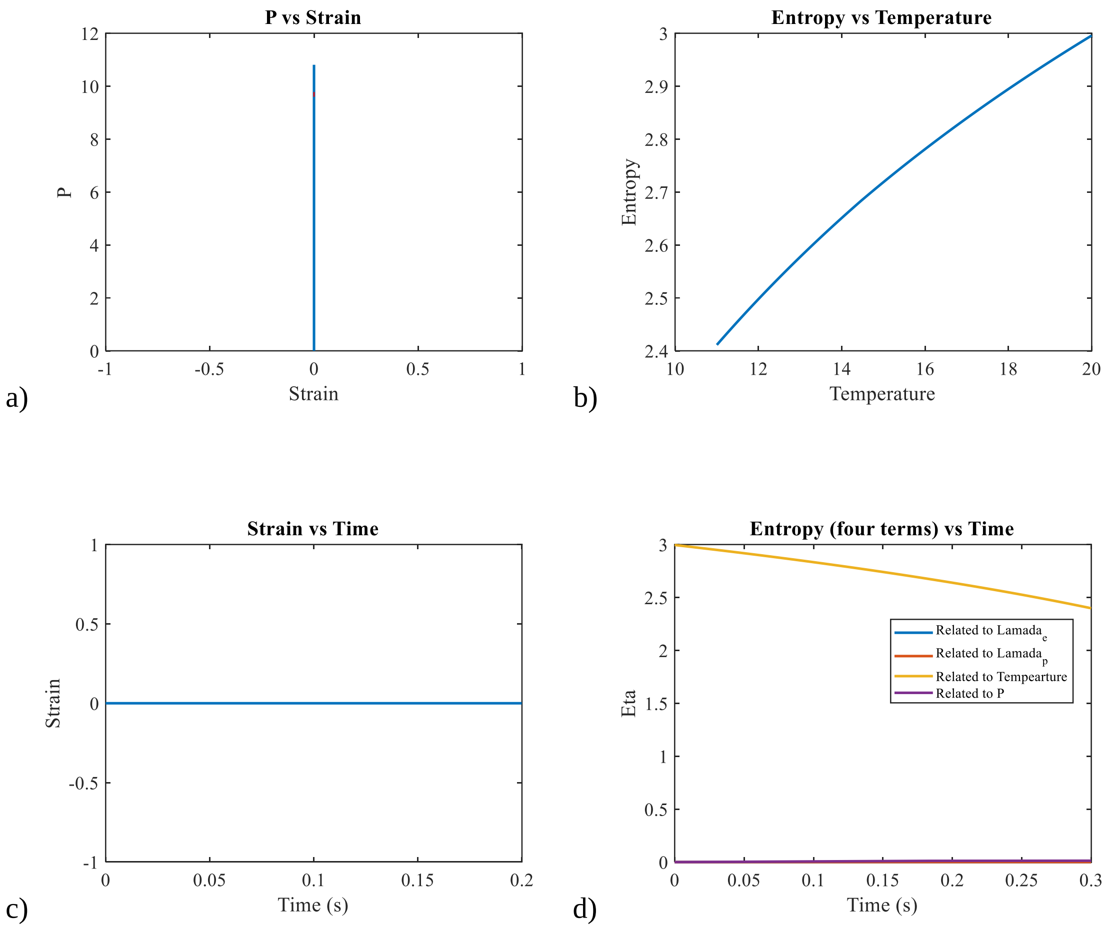

Figure 9. Plots for $\varepsilon = 0$ and constant cooling

9.3.4 Problem 2c

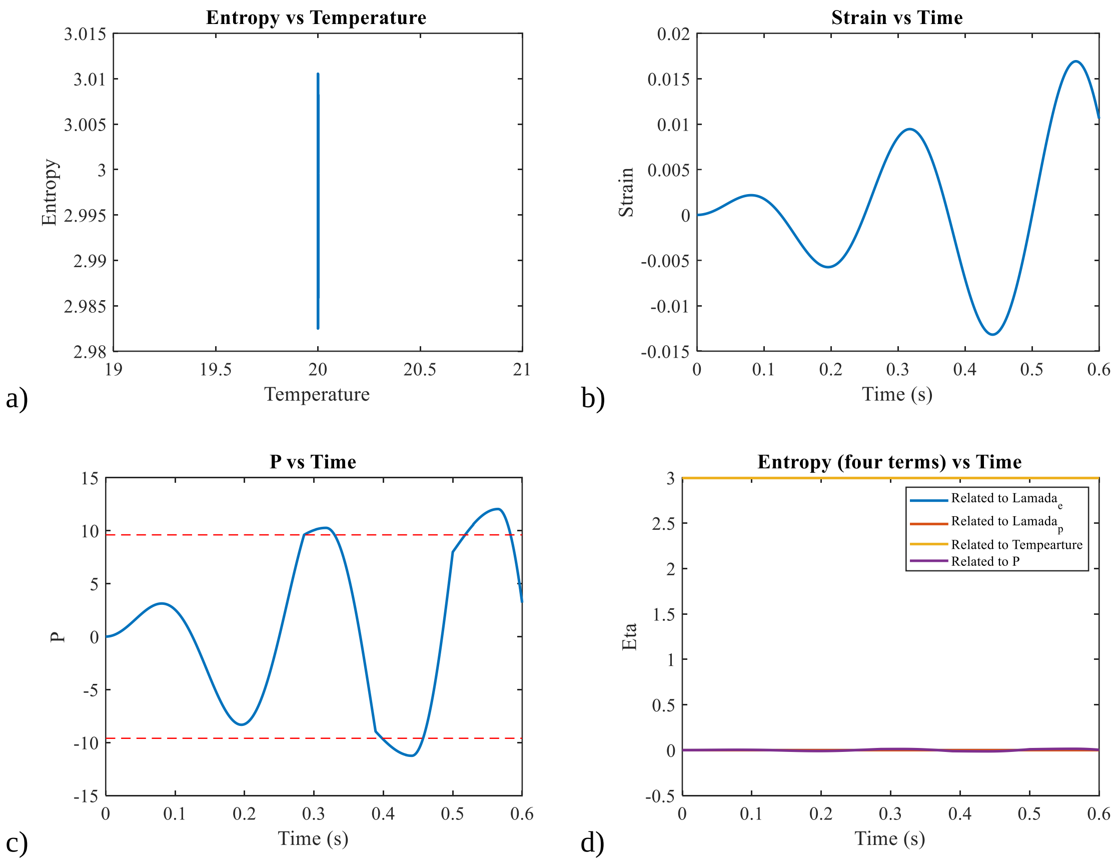

Figure 10. Plots for isothermal and cyclic straining

9.3.5 Problem 2d

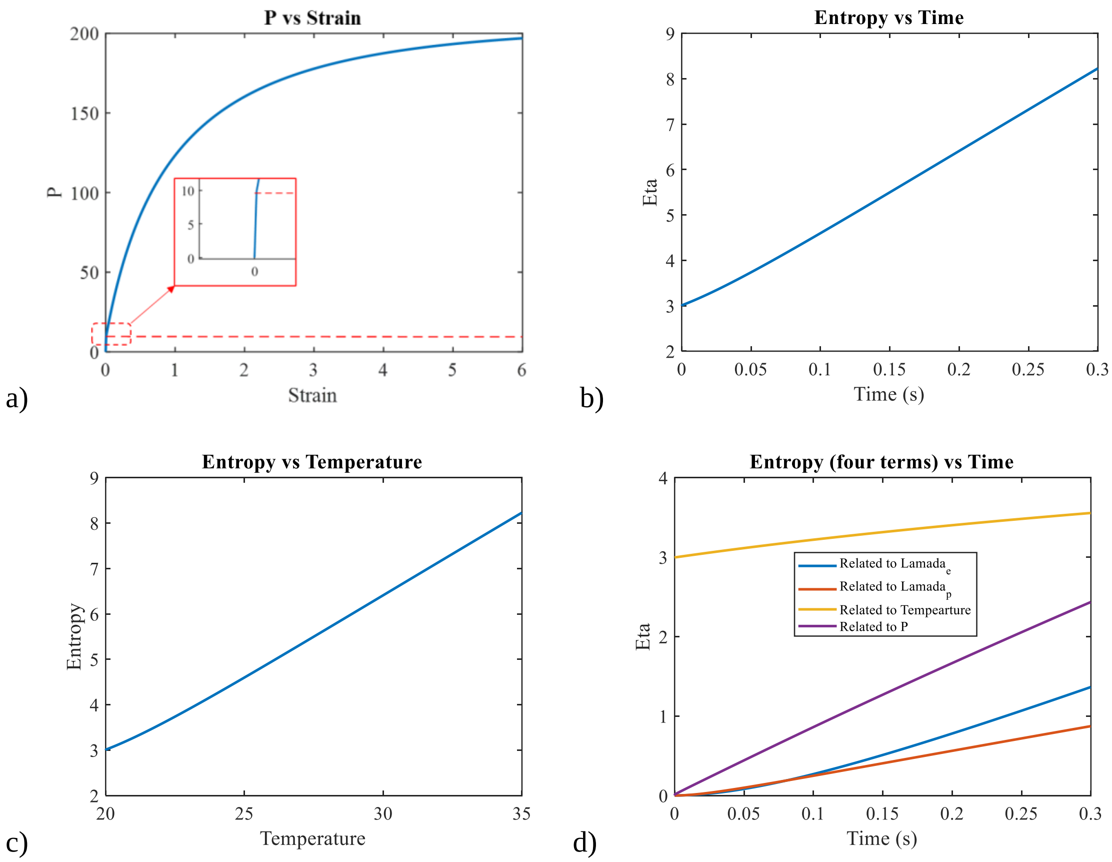

Figure 11. Plots for the combined thermo-mechanical ($\Delta t = 0.00001$)