Integer Points in Dynamical Systems

PM reading 1 - Dynamics and Arithmetic of Post-critically Finite Polynomials

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Dynamical system

Definition:
Let $\phi_1, \ldots, \phi_r : \mathbb{P}^1 \to \mathbb{P}^1$ be a collections of separable rational maps ($\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$).
The dynamical system generated by $\phi_1, \cdots, \phi_r$ denoted by

$$\Phi = \langle \phi_1, \ldots, \phi_r \rangle$$

is defined as all possible compositions of ϕ_i where $1 \leq i \leq r$, more formally:

$$\Phi = \{\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k} : k \leq 0, 1 \leq i_j \leq r\}$$

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The orbit of a point P is the collection of all points such that P can be mapped into by some $\phi \in \Phi$, denoted as

$$O^+_\Phi(P) = \{\phi(P) : \phi \in \Phi\}$$

Problem

Given a dynamical system, does the orbit of a point contain finitely many integer?

Examples of dynamical system

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Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\phi : z \to z + 1$. Then $\Phi_1 = \langle \phi \rangle = \{z + 1, z + 2, z + 3, \dots\}$.

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The question on the number of integer points is easy in this case: if the initial input is an integer, then Φ can generate infinitely many integer points.

Note that this is true for all Φ with at least one polynomial with integer coefficient among its generators.

Examples

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\phi : z \to \frac{1}{z+1}$. Then

$$\Phi_2 = \langle \phi \rangle = \{\frac{1}{z+1}, \frac{1}{\frac{1}{z+1}+1} = \frac{z+1}{z+2}, \frac{1}{\frac{1}{\frac{1}{z+1}+1}+1} = \frac{z+2}{z+3}, \frac{1}{\frac{1}{\frac{1}{\frac{1}{z+1}+1}+1}+1} = \frac{z+3}{z+5} \dots \}$$

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However, if we change the coordinate with this coordinate function $y(z) = \frac{1}{z}$. Then the generator of Φ_2 becomes $y \circ \phi = z + 1$.
Therefore, Φ_2 is just Φ_1 under a different coordinate system.

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ϕ is totally ramified at a point P if $e_P(\phi) = deg(\phi)$.

Example of ramification index

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Let $\phi(z) = (z-1)(z-2)^2$, then $e_1(\phi) = 1$ and $e_2(\phi) = 2$

Polynomial

Definition:
Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, be a separable rational map. ϕ is a polynomial if any of the following equivalent condition are true:

1. There exist a point $P \in \mathbb{P}^1$ such that $\phi(P) = P$ and ϕ is totally ramified at P
2. There exists a coordinate function y (i.e. y is a rational function of degree 1) on $\mathbb{P}^1$ such that $y \circ \phi \in k[y]$

If ϕ satisfies these conditions, then we say ϕ is polynomial with respect to P or that ϕ is polynomial with respect to y.

Decomposition property of dynamical system

Let Φ be a dynamical system, $P_0 \in \mathbb{P}^1$ be a point. We say Φ has the decomposition property if there exist finite subsets $\Lambda_1, \Lambda_2 \subset \Phi$ satisfying the following conditions:

1. $\Phi = \Lambda_1 \cup \bigcup_{\lambda \in \Lambda_2} \lambda \circ \Phi$
2. every $\lambda \in \Lambda_2$ satisfies $\#\lambda^{-1}(P_0) \geq 3$

Polynomial generator and decomposition property

Theorem 1

Let $\Phi = \langle \phi_1, \cdots, \phi_r \rangle$ be a dynamical system of degree at least two. Then

1. Φ has the decomposition property for every $P \in \mathbb{P}^1 \iff \Phi$ contains no nontrivial polynomial maps.
2. Suppose Φ is generated by a single element, say $\Phi = \langle \phi \rangle$. Fix a point $P_0 \in \mathbb{P}^1$. Then Φ has the decomposition property $\iff \Phi$ does not contain a nontrivial map which is polynomial with respect to P_0.

Recall: Theorem 1.1 (forward direction)

Let $\Phi = \langle \phi_1, \cdots, \phi_r \rangle$ be a dynamical system of degree at least two. Then:

Φ has the decomposition property for every $P \in \mathbb{P}^1 \implies \Phi$ contains no nontrivial polynomial maps.

Proof part 1

Suppose that Φ has the decomposition property and it has degree at least 2. For the sake of contradiction, assume that Φ contains a nontrivial polynomial map $\phi \in \Phi$, say ϕ is polynomial with respect to the point $P_0 \in \mathbb{P}^1$.

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As for any iteration of $(\phi^n)^{-1}(P_0) = \{P_0\}$, thus $\phi^n \neq \lambda\psi$ for any $\lambda \in \Lambda_2$.
(Note that ϕ^n denotes the n^{th} iteration of ϕ, not the n^{th} power of ϕ.)

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(Note that ϕ^n denotes the n^{th} iteration of ϕ, not the n^{th} power of ϕ.)

Hence, the iterations of $\phi^n \in \Lambda_1$. Since Λ_1 is finite, there are some $n > m \geq 1$ such that $\phi^n = \phi^m$. Therefore, $deg(\phi) = 1$ which contradict the assumption that Φ has degree at least 2.

Recall: Theorem 1.1 (backward direction)

Let $\Phi = \langle \phi_1, \cdots, \phi_r \rangle$ be a dynamical system of degree at least two. Then:

Φ contains no nontrivial polynomial maps $\implies$ Φ has the decomposition property for every $P \in \mathbb{P}^1$

Now suppose that $\Phi = \langle \phi_1, \ldots, \phi_r \rangle$ does not satisfy the decomposition property for some $P_0 \in \mathbb{P}^1$, so there is no way to choose finite sets Λ_1, Λ_2 satisfying the decomposition property.

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For each integer m, let $\Phi_m = \{\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_m} : 1 \le i_1, \ldots, i_m \le r\} \subset \Phi$.

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Clearly each Φ_m is finite set, and for each $M \geq 1$,

$$\Phi = \bigcup_{m=0}^{M-1} \Phi_m \cup \bigcup_{\lambda \in \Phi_M} \lambda \circ \Phi$$

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Since $\psi_m \in \Phi_m$, ψ_m can be written as

$$\psi_m = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_m}$$

and

$$\psi_m^{-1}(P_0) = \phi_{i_m}^{-1} \circ \phi_{i_{m-1}}^{-1} \circ \cdots \circ \phi_{i_2}^{-1} \circ \phi_{i_1}^{-1}(P_0)$$

Since $\#\psi_m^{-1}(P_0) \leq 2$, there is at most one point P_t such that $\#\phi_{i_t}^{-1}(P_t) = 2$ and for all $i \neq t$, $\#\phi_i^{-1}(P_i) = 1$.

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There are only r distinct functions, by pigeon hole principle, there must be a function that appears at least 6 times in the expression for ψ_m. Therefore, either ϕ appears 3 times before ϕ_t, or ϕ appears 3 times after ϕ_t.

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Call this function ϕ and let P_u, P_v, P_w be the inputs of ϕ.

$$\cdots P_{u-1} \overset{\phi}{\leftarrow} P_u \cdots P_{v-1} \overset{\phi}{\leftarrow} P_v \cdots P_{w-1} \overset{\phi}{\leftarrow} P_w \cdots$$

By Hurwitz's formula, we know that

$$2deg(\phi) - 2 \geq \sum_{P \in \mathbb{P}^1} (e_P(\phi) - 1)$$

This means that any map can only have at most 2 distinct totally ramified points.

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Thus, P_u, P_v, P_w are not all distinct, say $P_u = P_w$.

Then take the composition of map between ϕ at P_u and ϕ at P_w. We get a map that has a fixed and totally ramified point which is equivalent as a polynomial.

$$P_0 \xleftarrow{\phi_{i_0}} P_1 \xleftarrow{\phi_{i_1}} \dots \xleftarrow{\phi_{i_{u-1}}} P_u \underbrace{\xleftarrow{\phi_{i_u}=\phi} P_{u+1} \xleftarrow{\phi_{i_{u+1}}} \dots \xleftarrow{\phi_{i_{w-1}}}}_{\text{This is a polynomial with respect to } P_{i_u}} P_w = P_u \xleftarrow{\phi_w = \phi} \dots$$

Recall: Theorem 1.2

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Suppose Φ is generated by a single element, say $\Phi = \langle \phi \rangle$. Fix a point $P_0 \in \mathbb{P}^1$. Then Φ has the decomposition property $\iff$ Φ does not contain a nontrivial map which is polynomial with respect to P_0.

Proof part 2

Define μ, v as:

$$P_0 \underbrace{\xleftarrow{\phi_{i_0}} P_1 \xleftarrow{\phi_{i_1}} \dots \xleftarrow{\phi_{i_{u-1}}}}_{v} P_u \underbrace{\xleftarrow{\phi_{i_u}} P_{u+1} \xleftarrow{\phi_{i_{u+1}}} \dots \xleftarrow{\phi_{i_{w-1}}}}_{\mu} P_w = P_u \xleftarrow{\phi_w} \dots P_m$$

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Since Φ only has 1 generator ϕ, $\mu = \phi^n$ and $v = \phi^m$ for some n, m.
Note that $v(P_u) = P_0$.

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Choose some integer k such that $nk > m$, then consider:

$$P_u \xrightarrow{\phi^m} P_0 \xrightarrow{\phi^{nk-m}} P_u \xrightarrow{\phi^m} P_0 \xrightarrow{\phi^{nk-m}} P_u$$

It follows that ϕ^{nk} fixes P_0 and ϕ^{nk} is totally ramified at P_0. Since $\phi^{nk} \in \Phi$, Φ contains a polynomial with respect to P_0. $\square$

Theorem 2

Suppose that Φ contains no polynomial maps and fix a point $P \in \mathbb{P}^1(\mathbb{Q})$. Let z be a coordinate function. Then

$$\{Q : Q \in O_\Phi^+(P) \text{ and } z(Q) \in \mathbb{Z}\}$$

is a finite set.

Proof

Since Φ does not contain any polynomial map, there are finite sets $\Lambda_1, \Lambda_2 \subset \Phi$ such that

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Therefore, the orbit of any point P can be written as

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Thus, it is equivalent to show the set $\{\lambda \circ \phi(P) : \phi \in \Phi \text{ and } z \circ \lambda \circ \phi(P) \in \mathbb{Z}\}$ is finite for each $\lambda \in \Lambda_2$.

For any homogeneous coordinate $[X, Y]$ on $\mathbb{P}^1$, let:

- z be the coordinate function with $z = \frac{X}{Y}$
- for any $\phi \in \Phi$, write $\phi P = [u_\phi, v_\phi]$
- $\lambda = [F_\lambda, G_\lambda]$ where $F_\lambda, G_\lambda \in \mathbb{Z}[X, Y]$ are homogeneous polynomial of degree d

Since F_λ, G_λ have no common factors. Thus, the resultant of F_λ and G_λ, $Res(F_\lambda, G_\lambda) \neq 0$.

We observe that

$$z \circ \lambda \circ \phi(P) \in \mathbb{Z} \iff \frac{F_\lambda(u_\phi, v_\phi)}{G_\lambda(u_\phi, v_\phi)} \in \mathbb{Z}$$

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Thus, for any $\lambda \in \Lambda_2$, the set $\{\lambda\phi P : \phi \in \Phi \text{ and } z(\lambda\phi P) \in \mathbb{Z}\}$ is finite which is equivalent as $\{Q : Q \in O_{\Phi}^{+}(P) \text{ and } z(Q) \in \mathbb{Z}\}$ is a finite set. $\square$

Integer points in dynamical system generated by a single function

Theorem 3

Let $\phi(Z) \in K(Z)$ be a rational function of degree at least two and let $t \in K \cup \{\infty\} = \mathbb{P}^1(K)$. If $\phi^2(Z) \notin \bar{K}[Z]$, then the sequence

$$t, \phi(t), \phi^2(t), \phi^3(t), \ldots$$

contains only finitely many elements of R_S.

Thank you!

Reference

J. H. Silverman, Integer points, Diophantine approximation, and iteration of rational maps, Duke Math. J. **71** (1993), no. 3, 793–829