ST 790, Homework 1 Solutions
Spring 2017

1. Because $\pi(y)$ is an increasing function in y, we have that $Y > \mu$ is equivalent to $\pi(Y) > \pi(\mu)$, where $\mu = E(Y)$. Consequently, $(Y - \mu)$ and $\{\pi(Y) - \pi(\mu)\}$ have the same sign, which implies that
$$E[(Y - \mu)\{\pi(Y) - \pi(\mu)\} > 0,$$
or
$$E\{(Y - \mu)\pi(Y)\} - E\{(Y - \mu)\pi(\mu)\} > 0.$$
But $E\{(Y - \mu)\pi(\mu)\} = \pi(\mu)E\{(Y - \mu)\} = 0$, so this implies that
$$E\{(Y - \mu)\pi(Y)\} > 0,$$
or
$$E\{Y\pi(Y)\} > \mu E\{\pi(Y)\}.$$
Because $\pi(Y)$ is a probability that is strictly > 0, we have that $E\{\pi(Y)\} > 0$, which implies that
$$\frac{E\{Y\pi(Y)\}}{E\{\pi(Y)\}} > \mu,$$
which is the result.

2. With $X = (X_1, X_2)$, case (i) corresponds to missingness depending only on X, case (ii) corresponds to missingness depending only on Y, and case (iii) is missingness that does not depend on X or$Y$. Under case (i), as in the notes, we expect the complete case OLS estimator to be consistent. Under (ii), we expect it to be possibly inconsistent, and under (iii) we expect it to be consistent. See attached program. The simulations appear to bear out the analytical results.

3. (a) With $\lambda_j(Z) = \mathrm{pr}(D = j|D \geq j, Z)$, using Bayes' rule, we have
$$\begin{aligned}\lambda_j(Z) &= \frac{\mathrm{pr}(D = j, Z)}{\mathrm{pr}(D \geq j, Z)} = \frac{\mathrm{pr}(D = j|Z)p_Z(Z)}{\mathrm{pr}(D \geq j|Z)p_Z(Z)} \\ &= \frac{\mathrm{pr}(D = j|Z)}{\mathrm{pr}(D \geq j|Z)} \\ &= \frac{\mathrm{pr}(D = j|Z)}{1 - \sum_{k=1}^{j-1} \mathrm{pr}(D = k|Z)} \quad j = 2, \dots, T \qquad (1)\end{aligned}$$
and
$$\lambda_1(Z) = \mathrm{pr}(D = 1|Z), \quad \lambda_{T+1}(Z) = 1.$$
Conversely,
$$\begin{aligned}\mathrm{pr}(D = j|Z) &= \mathrm{pr}(D > 1, D > 2, \dots, D > j - 1, D = j|Z) \\ &= pr(D > 1|Z)\mathrm{pr}(D > 2|D > 1, Z) \times \cdots \times \mathrm{pr}(D > j - 1|D > j - 2, Z)\mathrm{pr}(D = j|D > j - 1, Z) \\ &= \mathrm{pr}(D \geq 1|D \geq 1, Z)\mathrm{pr}(D > 2|D \geq 2, Z) \times \cdots \times \mathrm{pr}(D > j - 1|D \geq j - 1, Z)\mathrm{pr}(D = j|D \geq j, Z). \qquad (2)\end{aligned}$$

Noting that

$$\mathrm{pr}(D > k|D \geq k, Z) = 1 - \mathrm{pr}(D = k|D \geq k, Z) = 1 - \lambda_k(Z),$$

(2) is equal to

$$\left[\prod_{k=1}^{j-1}\{1 - \lambda_k(Z)\}\right] \lambda_j(Z) = \mathrm{pr}(D = j|Z), \tag{3}$$

as desired.

(b) Assume

$$\mathrm{pr}(D = j|Z) = \mathrm{pr}(D = j|Z_{(j)}), \quad Z_{(j)} = (Z_1, \ldots, Z_{j-1}).$$

Because of (1) above,

$$\lambda_j(Z) = \frac{\mathrm{pr}(D = j|Z_{(j)})}{1 - \sum_{k=1}^{j-1} \mathrm{pr}(D = k|Z_{(k)})}.$$

Because $k < j$ in the denominator, $\mathrm{pr}(D = k|Z_{(k)})$ is a function of $Z_{(j)}$; that is, $Z_{(k)}$ is a subset of $Z_{(j)}$ for $k = 1, \ldots, j-1$. Thus, $\lambda_j(Z) = \lambda_j(Z_{(j)})$, a function of $(Z_1, \ldots, Z_{j-1})$ only.

Conversely, if $\lambda_j(Z) = \lambda_j(Z_{(j)})$, then by (3),

$$\mathrm{pr}(D = j|Z) = \left[\prod_{k=1}^{j-1}\{1 - \lambda_k(Z_{(k)})\}\right] \lambda_j(Z_(j)),$$

which is a function of $Z_{(j)}$.

4. (a) You should have found the straight line model with possibly different intercepts and slopes to be reasonable. See attached R program for (b), (c), and (d), which shows that the estimates of β do not agree, which seems consistent with concern over inconsistency of the estimator under MAR. In particular, the parameter of interest β_4 seems attenuated relative to that from the full data. For (e), note that based on the full data the null hypothesis that $\beta_4 = 0$ is strongly rejected, but with the data with induced MAR missingness, there is not enough evidence to reject at level 0.05. This is consistent with the contention that the naive analysis under MAR can result in compromised inferences.

5. See attached program for partial results. The t-test for mean difference at age 14 strongly rejects the null hypothesis that the means are the same for each gender based on the full data, but the evidence favoring a difference is borderline based on the LOCF data. As in the previous problem, this is consistent with the contention that using LOCF to "correct" for missingness is a naive, potentially flawed (i.e., stupid) approach.