Statistics 620
Final exam, Fall 2011

1. Let $\{N(t)\}$ be a rate λ Poisson process, with arrival times $\{S_n, n = 0, 1, \ldots\}$. Evaluate the expectected sum of squares of the arrival times occurring before t,

$$E(t) = \mathbb{E}\Big[\sum_{n=1}^{N(t)} S_n^2\Big],$$

where we define $\sum_{n=1}^{0} S_n = 0$.

<u>Solution</u>:

$$\mathbb{E}\Big[\sum_{k=1}^{N(t)} S_k^2 \big| N(t) = n\Big] = \mathbb{E}\Big[\sum_{k=1}^{n} U_k^2\Big], \quad \text{where } U_1, U_2, \ldots \text{ are iid } U[0,t].$$

Since $\mathbb{E}[U_k^2] = \int_0^t \frac{1}{t} s^2\, ds = t^3/3$, we find

$$\mathbb{E}\Big[\sum_{k=1}^{N(t)} S_k^2 \big| N(t) = n\Big] = nt^2/3,$$

and thus

$$\mathbb{E}\Big[\sum_{k=1}^{N(t)} S_k^2 \big| N(t)\Big] = N(t)\, t^2/3.$$

Taking expectations then gives

$$\mathbb{E}\Big[\sum_{k=1}^{N(t)} S_k^2\Big] = \mathbb{E}[N(t)]\, t^2/3 = \lambda t^3/3.$$

2. Consider two machines, operating simultaneously and independently, where both machines have an exponentially distributed fime to failure with mean $1/\mu$ (μ is the failure rate). There is a single repair facility, and the repair times are exponentially distributed with rate λ. What is the long run probability that no machine is operating?

<u>Solution</u>: Let $X(t)$ be the number of machines operating at time t. Note that $\{X(t)\}$ is a continuous time Markov chain with states $\{0, 1, 2\}$ and rates $q_{01} = q_{12} = \lambda$, $q_{02} = q_{20} = 0$, $q_{21} = 2\mu$, $q_{10} = \mu$. This can be recognized as a birth/death process, so the limiting probabilities satisfy detailed balance:

$$\begin{aligned} \lambda P_0 &= \mu P_1, \\ \lambda P_1 &= 2\mu P_2. \end{aligned}$$

Since $P_0 + P_1 + P_2 = 1$, we can solve for the required quantity,

$$P_0 = \frac{1}{1 + \lambda/\mu + \lambda^2/2\mu^2}.$$

3. Let $\{U_1, U_2, \dots\}$ be a sequence of independent, identically distributed Uniform random variables taking values in the interval $(0,1)$. Define $X_n = 2^n \prod_{k=1}^{n} U_k$ for $n > 1$, with $X_0 = 1$. Show that $\{X_n\}$ is a martingale, and discuss the limiting behavior of X_n and $\mathbb{E}[X_n]$ as n increases.

Solution:

$$\begin{aligned}
\mathbb{E}[X_n \,|\, X_1, \dots, X_{n-1}] &= \mathbb{E}[2U_n X_{n-1} \,|\, X_1, \dots, X_{n-1}] \\
&= \mathbb{E}[2U_n]\, X_{n-1} \quad \text{by independence} \\
&= X_{n-1}.
\end{aligned}$$

This is the martingale property. Since $\{X_n\}$ is a non-negative martingale, the convergence theorem implies that $\{X_n\}$ has an almost sure finite limit. The only possible such limit is zero. Convergence to any positive value c cannot happen: $\mathbb{P}[|X_{n+1} - c| < \delta \,|\, |X_n - c| < \delta] = \delta/c + o(\delta)$, which does not tend to 1 as $n \to \infty$. Therefore, $\lim_{n\to\infty} X_n = 0$ with probability one. However, by the martingale property, $\mathbb{E}[X_n] = 1$ for all n. This is a situation where the limit of the expectation is not equal to the expectation of the limit.

4. Let $\{B(t)\}$ be a standard Brownian motion, and define $Y(t) = e^{B(t)} \big/ \big(1 + e^{B(t)}\big)$. (a) Explain why $\{Y(t)\}$ is a diffusion process; (b) obtain its infinitesimal mean and variance; (c) explain whether or not sample paths of $\{Y(t)\}$ converge to a limit with probability one.

Solution: Write $f(x) = e^x/(1+e^x)$.

(a) $f(x)$ is continuous and increasing and is therefore invertible as a map from $\mathbb{R}$ to $(0,1)$. $\{Y(t)\}$ acquires the Markov property from $\{B(t)\}$ via the invertibility of f, and continuous sample paths via the continuity of f. Possession of these two properties makes $\{Y(t)\}$ satisfy the definition of a diffusion.

(b) Compute $f'(x) = \frac{e^x}{1+e^x} - \frac{e^{2x}}{(1+e^x)^2} = f(x) - f(x)^2$. Therefore,

$$f'' = f' - 2f\,f' = f - f^2 - 2f(f - f^2) = f(f-1)(2f-1).$$

Applying the transformation formula, with $\mu_X(x) = 0$, $\sigma_X(x) = 1$ and $x = x(y) = f^{-1}(y)$ gives

$$\begin{aligned}
\mu_Y(y) &= \mu_X(x) f'(x) + \frac{1}{2}\sigma_X^2 f''(x) = y(y-1)(2y-1)/2, \\
\sigma_Y^2(y) &= \sigma_X^2(x)[f'(x)]^2 = [y(1-y)]^2.
\end{aligned}$$

(c) Brownian motion in one dimension is null recurrent (e.g., think of it as the limit of a random walk) and so

$$\mathbb{P}[B(t) = 0 \text{ infinitely often}] = 1.$$

Thus, $\mathbb{P}[Y(t) = 1/2 \text{ infinitely often}] = 1$ so sample paths of $\{Y(t)\}$ do not converge to either 0 or 1 (the only two possible almost sure limits) with probability one. This is despite the observation that $\lim_{t\to\infty} \mathbb{P}[Y(t) \in (\epsilon, 1-\epsilon)] = 0$ for all $\epsilon > 0$.

5. Let $\{B(t)\}$ be standard Brownian motion. The event that $\{B(t)\}$ has a zero crossing between s and t is $A(s,t) = \{B(u) = 0 \text{ for some } u \text{ with } s < u < t\}$. We wish to find $\mathbb{P}\{A(s,t)\}$. As a preliminary step, in part (a), we consider the hitting time $\tau_x = \min\{u \geq 0 : B(u) = x\}$ for $x > 0$. The calculation continues in part (b), on the following page.

(a) Find $\mathbb{P}\{\tau_x \leq t\}$ in terms of the standard normal distribution function, $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\{\frac{-u^2}{2}\}\, du$.

Solution: By symmetry and independent increments, $\mathbb{P}[B(t) > x] = (1/2)\mathbb{P}[\tau_x \leq t]$. Therefore,

$$\mathbb{P}\{\tau_x \leq t\} = 2\big(1 - \Phi(x/\sqrt{t})\big).$$

(b) By conditioning on $B(s)$, find an expression for $\mathbb{P}\{A(s,t)\}$. Evaluate this expression using the identity

$$\int_0^\infty e^{-v^2/2s}\Big\{\int_v^\infty e^{-u^2/2(t-s)}\,du\Big\}\,dv = \sqrt{s(t-s)}\ \arccos\sqrt{s/t}, \tag{1}$$

where arccos is the inverse of the cosine function. You are not asked to prove (1) and so no specific knowledge about the arccos function is required for this question.

Solution:

$$\begin{aligned}
\mathbb{P}\{A(s,t)\} &= \mathbb{E}\left[\mathbb{P}\{A(s,t)\mid B(s)\}\right] \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi s}}e^{-v^2/2s}\,2\big(1-\Phi(v/\sqrt{t-s})\big)\,dv \\
&= 4\int_0^\infty \frac{1}{\sqrt{2\pi s}}e^{-v^2/2s}\Big\{\int_v^\infty \frac{1}{\sqrt{2\pi(t-s)}}e^{-u^2/2(t-s)}\,du\Big\}dv.
\end{aligned}$$

Then, using (1), we get

$$P_A(s,t) = \frac{2}{\pi}\arccos\sqrt{\frac{s}{t}}.$$

(Incidentally, note that $\lim_{s\to 0} P_A(s,t) = 1$.)