ST 790, Homework 1 Solutions
Spring 2015

All of you wrote beautiful programs for problems 2, 4, and 5 and made the expected observations that simulations bear out the analytical results for the linear regression complete case estimator (problem 2), that the GEE available case analysis can be misleading under MAR (problem 4), and that LOCF is a stupid method (problem 5). We give solutions to the two analytical problems (1 and 3) below.

1. Because $\pi(y)$ is an increasing function in y, we have that $Y > \mu$ is equivalent to $\pi(Y) > \pi(\mu)$, where $\mu = E(Y)$. Consequently, $(Y - \mu)$ and $\{\pi(Y) - \pi(\mu)\}$ have the same sign, which implies that
$$E[(Y - \mu)\{\pi(Y) - \pi(\mu)\} > 0,$$
or
$$E\{(Y - \mu)\pi(Y)\} - E\{(Y - \mu)\pi(\mu)\} > 0.$$
But $E\{(Y - \mu)\pi(\mu)\} = \pi(\mu)E\{(Y - \mu)\} = 0$, so this implies that
$$E\{(Y - \mu)\pi(Y)\} > 0,$$
or
$$E\{Y\pi(Y)\} > \mu E\{\pi(Y)\}.$$
Because $\pi(Y)$ is a probability that is strictly > 0, we have that $E\{\pi(Y)\} > 0$, which implies that
$$\frac{E\{Y\pi(Y)\}}{E\{\pi(Y)\}} > \mu,$$
which is the result.

3. (a) With $\lambda_j(Z) = \text{pr}(D = j|D \geq j, Z)$, using Bayes' rule, we have
$$\begin{aligned}
\lambda_j(Z) &= \frac{\text{pr}(D = j, Z)}{\text{pr}(D \geq j, Z)} = \frac{\text{pr}(D = j|Z)p_Z(Z)}{\text{pr}(D \geq j|Z)p_Z(Z)} \\
&= \frac{\text{pr}(D = j|Z)}{\text{pr}(D \geq j|Z)} \\
&= \frac{\text{pr}(D = j|Z)}{1 - \sum_{k=1}^{j-1} \text{pr}(D = k|Z)} \quad j = 2, \dots, T \qquad (1)
\end{aligned}$$
and
$$\lambda_1(Z) = \text{pr}(D = 1|Z), \quad \lambda_{T+1}(Z) = 1.$$
Conversely,
$$\begin{aligned}
\text{pr}(D = j|Z) &= \text{pr}(D > 1, D > 2, \dots, D > j-1, D = j|Z) \\
&= pr(D > 1|Z)\text{pr}(D > 2|D > 1, Z) \times \cdots \times \text{pr}(D > j-1|D > j-2, Z)\text{pr}(D = j|D > j-1, Z) \\
&= \text{pr}(D \geq 1|D \geq 1, Z)\text{pr}(D > 2|D \geq 2, Z) \times \cdots \times \text{pr}(D > j-1|D \geq j-1, Z)\text{pr}(D = j|D \geq j, Z). \qquad (2)
\end{aligned}$$

Noting that

$$\mathrm{pr}(D > k | D \geq k, Z) = 1 - \mathrm{pr}(D = k | D \geq k, Z) = 1 - \lambda_k(Z),$$

(2) is equal to

$$\left[\prod_{k=1}^{j-1} \{1 - \lambda_k(Z)\} \right] \lambda_j(Z) = \mathrm{pr}(D = j | Z), \tag{3}$$

as desired.

(b) Assume

$$\mathrm{pr}(D = j | Z) = \mathrm{pr}(D = j | Z_{(j)}), \quad Z_{(j)} = (Z_1, \dots, Z_{j-1}).$$

Because of (1) above,

$$\lambda_j(Z) = \frac{\mathrm{pr}(D = j | Z_{(j)})}{1 - \sum_{k=1}^{j-1} \mathrm{pr}(D = k | Z_{(k)})}.$$

Because $k < j$ in the denominator, $\mathrm{pr}(D = k | Z_{(k)})$ is a function of $Z_{(j)}$; that is, $Z_{(k)}$ is a subset of $Z_{(j)}$ for $k = 1, \dots, j-1$. Thus, $\lambda_j(Z) = \lambda_j(Z_{(j)})$, a function of $(Z_1, \dots, Z_{j-1})$ only.

Conversely, if $\lambda_j(Z) = \lambda_j(Z_{(j)})$, then by (3),

$$\mathrm{pr}(D = j | Z) = \left[\prod_{k=1}^{j-1} \{1 - \lambda_k(Z_{(k)})\} \right] \lambda_j(Z_(j)),$$

which is a function of $Z_{(j)}$.