Problem 1: $X_L = L\pi$, $L = X_1 X_2^T - X_2 X_1^T$ $X_1 = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ T_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ T_2 \end{bmatrix}$

$$\therefore L = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ T_1 \end{bmatrix} [X_2\ Y_2\ Z_2\ T_2] - \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ T_2 \end{bmatrix} [X_1\ Y_1\ Z_1\ T_1]$$

$$= \begin{bmatrix} X_1X_2 & X_1Y_2 & X_1Z_2 & X_1T_2 \\ Y_1X_2 & Y_1Y_2 & Y_1Z_2 & Y_1T_2 \\ Z_1X_2 & Z_1Y_2 & Z_1Z_2 & Z_1T_2 \\ T_1X_2 & T_1Y_2 & T_1Z_2 & T_1T_2 \end{bmatrix} - \begin{bmatrix} X_2X_1 & X_2Y_1 & X_2Z_1 & X_2T_1 \\ Y_2X_1 & Y_2Y_1 & Y_2Z_1 & Y_2T_1 \\ Z_2X_1 & Z_2Y_1 & Z_2Z_1 & Z_2T_1 \\ T_2X_1 & T_2Y_1 & T_2Z_1 & T_2T_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & X_1Y_2 - X_2Y_1 & X_1Z_2 - X_2Z_1 & X_1T_2 - X_2T_1 \\ Y_1X_2 - Y_2X_1 & 0 & Y_1Z_2 - Y_2Z_1 & Y_1T_2 - Y_2T_1 \\ Z_1X_2 - Z_2X_1 & Z_1Y_2 - Z_2Y_1 & 0 & Z_1T_2 - Z_2T_1 \\ T_1X_2 - T_2X_1 & T_1Y_2 - T_2Y_1 & T_1Z_2 - T_2Z_1 & 0 \end{bmatrix}$$

$X_L = L\pi$

$$\therefore X_L = L \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b(X_1Y_2 - X_2Y_1) + c(X_1Z_2 - X_2Z_1) + d(X_1T_2 - X_2T_1) \\ a(Y_1X_2 - X_1Y_2) + c(Y_1Z_2 - Y_2Z_1) + d(Y_1T_2 - Y_2T_1) \\ a(Z_1X_2 - Z_2X_1) + b(Z_1Y_2 - Z_2Y_1) + d(Z_1T_2 - Z_2T_1) \\ a(T_1X_2 - T_2X_1) + b(T_1Y_2 - T_2Y_1) + c(T_1Z_2 - T_2Z_1) \end{bmatrix}$$

Pencil pts: $X(\lambda) = \lambda X_1 + (1-\lambda) X_2$.

$$X(\lambda) = \lambda \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ T_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} \lambda(X_1 - X_2) + X_2 \\ \lambda(Y_1 - Y_2) + Y_2 \\ \lambda(Z_1 - Z_2) + Z_2 \\ \lambda(T_1 - T_2) + T_2 \end{bmatrix}$$

$X(\lambda_\pi)^T \pi = 0$ solve for λ_π

$$\therefore \lambda = \frac{(-aX_2 - bY_2 - cZ_2 - dT_2)}{a(X_1 - X_2) + b(Y_1 - Y_2) + c(Z_1 - Z_2) + d(T_1 - T_2)}$$

Take λ back to $X(\lambda_\pi)$.

$$X(\lambda\pi) = \lambda\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ T_1 \end{bmatrix} + (1-\lambda)\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ T_2 \end{bmatrix}$$

$$\alpha = \frac{-1}{a(X_1-X_2)+b(Y_1-Y_2)+c(Z_1-Z_2)+d(T_1-T_2)}$$

$$= \frac{-1}{a(X_1-X_2)+b(Y_1-Y_2)+c(Z_1-Z_2)+d(T_1-T_2)}\left[(aX_2+bY_2+cZ_2+dT_2)\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ T_1 \end{bmatrix} - (aX_1+bY_1+cZ_1+dT_1)\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ T_2 \end{bmatrix}\right]$$

$$= \alpha\begin{bmatrix} b(Y_2X_1-Y_1X_2)+c(Z_2X_1-Z_1X_2)+d(T_2X_1-T_1X_2) \\ a(Y_1X_2-X_1Y_2)+c(Y_1Z_2-Y_2Z_1)+d(Y_1T_2-Y_2T_1) \\ a(Z_1X_2-Z_2X_1)+b(Z_1Y_2-Z_2Y_1)+d(Z_1T_2-Z_2T_1) \\ a(T_1X_2-T_2X_1)+b(T_1Y_2-T_2Y_1)+c(T_1Z_2-T_2Z_1) \end{bmatrix} = \alpha X_L$$

$\therefore X_L$ is equal to $X(\lambda\pi)$ up to scale.

Problem 2.

$x(\lambda)$ intersects a quadric Q. $\quad X = \lambda X_1 + (1-\lambda) X_2$.

$\therefore X^T Q X = 0$

$= (\lambda X_1 + (1-\lambda) X_2)^T Q (\lambda X_1 + (1-\lambda) X_2)$

$= \lambda^2 X_1^T Q X_1 + (\lambda X_1^T Q (1-\lambda) X_2) + (1-\lambda) X_2^T Q \lambda X_1 + (1-\lambda) X_2^T Q (1-\lambda) X_2$.

$= (X_1^T Q X_1 + X_2^T Q X_2 - X_1^T Q X_2 - X_2^T Q X_1)\lambda^2 + (X_1^T Q X_2 + X_2^T Q X_1 - 2 X_2^T Q X_2)\lambda$
$\quad + X_2^T Q X_2$.

X_1: $4*1$ matrix $\quad Q$: $4*4$ (symmetric matrix) $\quad X_2$: $4*1$.

Let $X_1^T Q X_2 = a$ where a is a scale. $\xrightarrow{\text{transpose.}} X_2^T Q^T X_1 = a$.

$X_2^T Q X_1 = b$

$\because Q$ is symm $\therefore Q = Q^T$

$\therefore X_2^T Q X_1 = a = b$

$\therefore X_1^T Q X_2 = X_2^T Q X_1$

$\therefore X^T Q X = (X_1^T Q X_1 - 2X_1^T Q X_2 + X_2^T Q X_2)\lambda^2 + 2(X_1^T Q X_2 - X_2^T Q X_2)\lambda$
$\quad + X_2^T Q X_2 = 0$

$\therefore C_2 = X_1^T Q X_1 - 2X_1^T Q X_2 + X_2^T Q X_2$

$C_1 = 2(X_1^T Q X_2 - X_2^T Q X_2)$

$C_0 = X_2^T Q X_2$.