中 山 大 学 软 件 学 院 **2010** 级 软 件 工 程 专 业 (2011 学年秋季学期)

《SE-205+数值计算方法》期末试题答案(A)

1. Fill in the blankets with proper answers (5 marks each, total 40 marks)

1) Suppose the approximate value 3.141601 has 5 significant digits, then the relative error bound is about $\frac{0.000050}{3.141601}=\frac{\frac{1}{2}\times10^{-4}}{3.141601}$.

2) For the numerical integration formula $\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i f(x_i)$, if $\sum_{i=1}^{n}|A_i| > (b-a)$, then we think it is not stable.

3) The error term of Lagrange polynomial approximation for the function **f** at the nodes

$$a \le x_0 < x_1 < \ldots < x_n \le b$$

is

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)\text{，其中}\xi\in(a,b)\text{。}$$

4) The Chebyshev polynomial of degree 3 is

$$\mathrm{T}_3(x)=4x^3-3x,$$

and it can be represented in trigonometric form as $T_3(x)=\cos(3\arccos(x))$.

5) The recursive rule of Newton's methods for solving nonlinear equation f(x)=0 is $x_{k+1}=x-\frac{f(x_k)}{f'(x_k)}$, and its speed of convergence is **2 阶** for single root.

6) Let $X=(1,2,3)$ and $A=\begin{pmatrix}2 & -2\\ 0 & 3\end{pmatrix}$, then $\|X\|_2=\sqrt{14}$, $\|A\|_\infty=4$.

7) Consider the Euler's methods for solving the initial value problem $y'=f(t,y)$ with $y(t_0)=y_0$, the global discretization error is $O(h)$.

8) For the following linear system,

$$\begin{cases}3x+y=3\\ 2x-3y=-4\end{cases}$$

Seidel iteration formula in matrix form $X_{k+1}=BX_k+f$ is

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The iteration will **converge** (converge or not converge), because $\|B\|_\infty = \frac{1}{3} < 1$.

2. **（12 marks）Consider the quadrature rule**

$$\int_0^h f(x)dx \approx \frac{h}{2}\left[f(0)+f(h)\right]+\lambda h^2\left[f'(0)-f'(h)\right],$$

where λ is some unknown parameter. Find the value of λ so that the quadrature rule has as high degree of precision as possible.

解：

$$f(x)=1: \int_0^h f(x)dx = \frac{h}{2}\left[f(0)+f(h)\right]+\lambda h^2\left[f'(0)-f'(h)\right]=h$$

$$f(x)=x: \int_0^h f(x)dx = \frac{h}{2}\left[f(0)+f(h)\right]+\lambda h^2\left[f'(0)-f'(h)\right]=\frac{h^2}{2}$$

$$f(x)=x^2: \frac{h^3}{3} = \int_0^h f(x)dx = \frac{h}{2}\left[f(0)+f(h)\right]+\lambda h^2\left[f'(0)-f'(h)\right]=\frac{h^3}{2}-2h^3\lambda$$

$$\therefore \lambda = \frac{1}{12}$$

3. **（13 marks）Given 4 end points （1,0），（2,1），（3,0） and（5,0），find the three moment equation（三弯矩方程）for the cubic spline S(x) that satisfies the third, i.e. the periodic boundary condition.**

解：待定的参数有三个 m_1, m_2, m_3，它们是样条函数在后三个节点的二阶导数。g_1, g_2, g_3 可以按如下的方法列表计算，其中给出了 g_3 的两种计算方法。特别注意，g_3 的第一种计算方法是考虑到周期性而虚拟出了一个点！

xk	f(xk)	一阶均差	二阶均差
1	0		
2	1	$f[x_0,x_1]=1$	
3	0	$f[x_1,x_2]=-1$	$\frac{g_1}{6}=f[x_0,x_1,x_2]=-1$
5	0	$f[x_2,x_3]=0$	$\frac{g_2}{6}=f[x_1,x_2,x_3]=\frac{1}{3}$
6	1	1	$\frac{g_3}{6}=\frac{1}{3}$

或

$$g_3 = 6\frac{f[x_0, x_1] - f[x_2, x_3]}{h_0 + h_2} = 6 \cdot \frac{1-0}{1+2} = 2$$

列方程组

$$\begin{bmatrix} 2 & \lambda_1 & \mu_1 \\ \mu_2 & 2 & \lambda_2 \\ \lambda_3 & \mu_3 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1/2 & 1/2 \\ 1/3 & 2 & 2/3 \\ 1/3 & 2/3 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$$

4.（10 marks）Given the function constrain table

x	−1	1	3	4
f(x)	−4	−2	−8	1

first construct the divided difference table，and then find the Newton interpolation polynomial.

解：均差表为

X_k	$f(x_k)$	一阶均差	二阶均差	三阶均差
−1	−4			
1	−2	1		
3	−8	−3	−1	
4	1	9	4	1

所以

$$P_3(x) = -4 + (x+1) - (x+1)(x-1) + (x+1)(x-1)(x-3) = x^3 - 4x^2 + 1$$

5. **Given the** 3×3 **matrix**

$$A=\begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

1、（10 marks）Find out the PA=LU factorization for A. The factorization should be correspondent to Gauss elimination.

2、（5 marks）Using the factorization to solve the linear system.

$$\begin{cases} -3x_2 - x_3 = 3 \\ x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \end{cases}$$

解 1、三阶矩阵做高斯消去法需要两次消去的过程，具体如下：

$$A=\begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, P=(1,2,3)$$

$$A=\begin{bmatrix} 0 & -3 & -1 \\ 1/2 & 3/2 & 1/2 \\ 2 & 1 & 1 \end{bmatrix}, P=(3,2,1)$$

$$A=\begin{bmatrix} 0 & -3 & -1 \\ 1/2 & -1/2 & 0 \\ 2 & 1 & 1 \end{bmatrix}, P=(3,1,2)$$

于是可以得到 PA=LU 分解中的各个矩阵：

$$P=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}, U=\begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

2、在已知 PA=LU 分解的基础上，求解方程组的基本思想是：

$$Ax=b$$
$$PAx=Pb$$
$$LUx=Pb$$
$$\begin{cases} Ly=Pb \\ Ux=y \end{cases}$$

也就是我们要解两个方程组

$$\begin{cases} y_1=5 \\ y_2=3 \\ \frac{1}{2}y_1-\frac{1}{2}y_2+y_3=1 \end{cases} \qquad \begin{cases} 2x_1+x_2+x_3=y_1 \\ -3x_2-x_3=y_2 \\ 0\cdot x_3=y_3 \end{cases}$$

解方程得

$$y_1=5, y_2=3, y_3=0$$

$$x_3\text{取任意值}, x_2=-1-\frac{x_3}{3}, x_1=3-\frac{x_3}{3}$$

6. （10 marks）Given the differential equation,

$$\begin{cases} \frac{dy}{dt}=f(t) \\ y(a)=0 \end{cases}$$

and the step size is chosen to be $h=\frac{b-a}{n}$**,where n is some positive number.**

1、 **Find the Euler recursive rule for numerically solving the equation;**

2、 **Find out the approximate solution at** $t=b$**.**

解：1、微分方程离散化

$$\frac{y_{k+1}-y_k}{h}=f(t_k)=f(a+kh)$$

即可得到 Euler 递推法则

$$y_{k+1}=y_k+hf(a+kh) \qquad k=0,1,2,...,n-1$$

2、$\therefore$

$$\sum_{k=0}^{n-1} y_{k+1}=\sum_{k=0}^{n-1} y_k+h\sum_{k=0}^{n-1} f(a+kh)$$

$$y_1+y_2+...+y_n=y_0+y_1+...+y_{n-1}+h\sum_{k=0}^{n-1} f(a+kh)$$

$$y_n=y_0+h\sum_{k=0}^{n-1} f(a+kh)$$

$$y(b)\approx y_n=y(a)+h\sum_{k=0}^{n-1} f(a+kh)=h\sum_{k=0}^{n-1} f(a+kh)$$