中山大学软件学院 20　级软件工程专业(2010 学年秋季学期)

《数值分析》期末试题(A)答案

1. Fill in the blanket with proper answers (5 marks each, total 40 marks)

1） Suppose $\pi \approx 3.1415926$，then the approximate value 3.141601 has__5__significant digits.

2) The numerical integration formula $\int_{-1}^{1} f(x)dx \approx \frac{1}{3}\left[\mathrm{f}\left(-\frac{1}{2}\right)+4\mathrm{f}(0)+\mathrm{f}\left(\frac{1}{2}\right)\right]$ has __1__ degrees of precision.

3) For the numerical integration formula $\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i f(x_i)$， if $\underline{\sum_{i=1}^{n}|A_i| > (b-a)}$, then we think it is not stable.

4) The error term of Lagrange polynomial approximation for the function **f** at the nodes

$$a \le x_0 < x_1 < \ldots < x_n \le b$$

is

$$\underline{R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x),\ \text{其中}\xi \in (a,b)}\text{。}$$

5） The Chebyshev polynomial of degree 4 is

$$\underline{\mathrm{T}_4(x) = 8x^4 - 8x^2 + 1},$$

and it has __5__ points of alternation。

6） The recursive rule of Newton's methods for solving nonlinear equation f(x)=0 is: $\underline{x_{k+1} = x - \frac{f(x_k)}{f'(x_k)}}$, and its speed of convergence is____2 阶___。

7） The fast algorithm for evaluate the polynomial $y = \sum_{i=0}^{n} a_n x^n$ is called __秦九韶算法或 Horner 算法__, and the recursive rule is：

$$\begin{cases} S_n = a_n \\ S_{k-1} = S_k \cdot x + a_{n-1} \qquad k = n, n-1, \ldots, 1\text{。} \end{cases}$$
$$\underline{y = a_0}$$

8) For the following nonlinear system, write out the Seidel iteration formula

$$\begin{cases} 3x + y - z = 3 \\ 2x - 5y - z = -4 \\ x + 3y - 6z = -2 \end{cases}$$

$$\begin{cases} x_{k+1} = -\frac{1}{3} y_k + \frac{1}{3} z_k + 1 \\ y_{k+1} = \frac{2}{5} x_{k+1} - \frac{1}{5} z_k + \frac{4}{5} \\ z_{k+1} = \frac{1}{6} x_{k+1} + \frac{1}{2} y_{k+1} + \frac{1}{3} \end{cases}$$

或

$$\begin{cases} x_{k+1} = -\frac{1}{3} y_k + \frac{1}{3} z_k + 1 \\ y_{k+1} = -\frac{2}{15} y_k - \frac{1}{15} z_k + \frac{6}{5} \\ z_{k+1} = -\frac{11}{90} y_k + \frac{1}{45} z_k + \frac{33}{30} \end{cases}$$

2. **Write out the** three-moment equation **of the cubic spline that pass through** $(-1, y_1)$ **,** $(0, y_2)$ **and** $(1, y_3)$ **with the clamped condition** $S'(x_0) = y_0'$ **and** $S'(x_2) = y_2'$**. (11 marks)**

解：设m_0、m_1、m_2分别为三次样条函数在$x_0 = -1$、$x_1 = 0$、$x_2 = 1$的二次导数，在端点的一次导数，即边界条件为y_0'、y_2'，则三弯矩方程为

$$\begin{pmatrix} 2 & 1 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$$

其中

$$g_0 = 6 \frac{f[x_0, x_1] - y_0'}{h_0} = 6\left(f[x_0, x_1] - y_0'\right) = 6\left(y_1 - y_0 - y_0'\right)$$

$$g_1 = 6 \frac{f[x_1, x_2] - f[x_0, x_1]}{h_0 + h_1} = 3\left[\left(y_2 - y_1\right) - \left(y_1 - y_0\right)\right] = 3\left(y_2 - 2y_1 + y_0\right)$$

$$g_2 = 6 \frac{y_2' - f[x_1, x_2]}{h_1} = 6\left(y_2' - f[x_1, x_2]\right) = 6\left(y_2' - y_2 + y_1\right)$$

3. Given the function constrain table

x	0	2	3	4
f(x)	5	5	23	69

first construct the divided difference table， and then find the Newton interpolation polynomial.（12 marks）

解：均差表为

X_k	$f(x_k)$	一阶均差	二阶均差	三阶均差
0	5			
2	5	0		
3	23	18	6	
4	69	46	14	2

所以

$$P_3(x) = 5 + 6x(x-2) + 2x(x-2)(x-3) = 2x^3 - 4x^2 + 5$$

4. **Given the** 3×3 **matrix**

$$A=\begin{bmatrix} 0 & -4 & -2 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

1、**Find out the PA=LU factorization for A. The factorization should be correspondent to Gauss elimination.（10 marks）**

2、Using the factorization to solve the linear system.（5 marks）

$$\begin{cases} -4x_2 - 2x_3 = -16 \\ x_1 + x_2 + x_3 = 4 \\ 2x_1 - 2x_2 + x_3 = -6 \end{cases}$$

解 1、三阶矩阵做高斯消去法需要两次消去的过程，具体如下：

$$A=\begin{bmatrix} 0 & -4 & -2 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}, P=(1,2,3)$$

$$A=\begin{bmatrix} 0 & -4 & -2 \\ 1/2 & 2 & 1/2 \\ 2 & -2 & 1 \end{bmatrix}, P=(3,2,1)$$

$$A=\begin{bmatrix} 0 & -4 & -2 \\ 1/2 & -1/2 & -1/2 \\ 2 & -2 & 1 \end{bmatrix}, P=(3,1,2)$$

于是可以得到 PA=LU 分解中的各个矩阵：

$$P=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}, U=\begin{pmatrix} 2 & -2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1/2 \end{pmatrix}$$

2、在已知 PA=LU 分解的基础上，求解方程组的基本思想是：

$$Ax=b$$
$$PAx=Pb$$
$$LUx=Pb$$
$$\begin{cases} Ly=Pb \\ Ux=y \end{cases}$$

也就是我们要解两个方程组

$$\begin{cases} y_1=-6 \\ y_2=-16 \\ \frac{1}{2}y_1-\frac{1}{2}y_2+y_3=4 \end{cases} \qquad \begin{cases} 2x_1-2x_2+x_3=y_1 \\ -4x_2-2x_3=y_2 \\ -\frac{1}{2}x_3=y_3 \end{cases}$$

解方程得

$$y_1=-6, y_2=-16, y_3=-1$$
$$x_3=2, x_2=3, x_1=-1$$

5. Using the Romberg algorithm to find the numerical integration for $\int_0^1 x^2dx$ **.**

It's only to calculate up to that [0,1] is divided into 4 subintervals.（**12 marks**）

解： $h=\frac{1}{4}=0.25$

T(f, 4h)=0.5		
T(f, 2h)=0.375	**S(f, 2h)=0.375**	

T(f, h)=0.34375	**S(f, h)=0.33333**	**B(f, h)=0.33056**

6. Given the differential equation,

$$\begin{cases} \dfrac{dy}{dx} = 0.5y \\ y(0) = 2 \end{cases}$$

and the step size is chosen to be $h = 0.2$,

1、 **Write the Euler recursive formula.（6 marks）**

2、 **Find out the approximate solution at x=1.（4 marks）**

解 1、

$$y_0 = 2$$

$$\frac{y_{k+1} - y_k}{h} = 0.5y_k$$

$$y_{k+1} - y_k = 0.1y_k$$

$$y_{k+1} = 1.1y_k$$

$$y_k = 1.1^k \cdot y_0 = 2\times 1.1^k$$

2、在 x=1 处的近似值为

$$y_5 = 2\times 1.1^5$$

(EXTRA WORK) Given three points on the plane: $(-1, y_1), (0, y_2), (1, y_3)$,

and three base functions

$$f_1(x) = 1$$

$$f_2(x) = x$$

$$f_3(x) = a + bx + x^2$$

Where a and b are some fixed constants.

1. **Derive the normal equation for the linear least square problem for the linear combination: (6 marks)**

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$$

2. **Do you think what are the best choice for a and b, so that the normal equation is easy to solve?（4 marks）**

解 1、令

$$E_2 = \left(c_1 - c_2 + c_3\left(a - b + 1\right) - y_1\right)^2 + \left(c_1 + ac_3 - y_2\right)^2 + \left(c_1 + c_2 + c_3\left(a + b + 1\right) - y_3\right)^2$$

两边求导数，并令导数等于 0，得

$$\begin{cases}\left(c_1-c_2+c_3\left(a-b+1\right)-y_1\right)+\left(c_1+ac_3-y_2\right)+\left(c_1+c_2+c_3\left(a+b+1\right)-y_3\right)=0\\-\left(c_1-c_2+c_3\left(a-b+1\right)-y_1\right)+\left(c_1+c_2+c_3\left(a+b+1\right)-y_3\right)=0\\\left(c_1-c_2+c_3\left(a-b+1\right)-y_1\right)\left(a-b+1\right)+\left(c_1+ac_3-y_2\right)\cdot a+\left(c_1+c_2+c_3\left(a+b+1\right)-y_3\right)\left(a+b+1\right)=0\end{cases}$$

$$\begin{pmatrix}3 & 0 & 3a+2\\0 & 2 & 2b\\3a+2 & 2b & \left(a-b+1\right)^2+a^2+\left(a+b+1\right)^2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}y_1+y_2+y_3\\-y_1+y_3\\\left(a-b+1\right)y_1+ay_2+\left(a+b+1\right)y_3\end{pmatrix}$$

2、为了使方程组易解，只需要 a=-2/3，b=0。此时方程组变形为

$$\begin{pmatrix}3 & 0 & 0\\0 & 2 & 0\\0 & 0 & 4\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}y_0+y_1+y_2\\-y_0+y_2\\3y_0-6y_1+3y_2\end{pmatrix}$$