中 山 大 学 软 件 学 院 **2011** 级 软 件 工 程 专 业 (2012 学年秋季学期)

《SE-205+数 值 计 算 方 法》 期 末 试 题 答 案 (B)

1. Fill in the blankets with proper answers (5 marks each, total 40 marks)

1) Suppose $x \approx 0.0473996$, then the approximate value 0.0474000 has __5__ significant digits.

2) For the numerical integration formula $\int_a^b f(x)dx \approx (b-a)\sum_{i=1}^{n} C_i f(x_i)$, if $\underline{\sum_{i=1}^{n}|C_i| > 1}$, then we think it is not stable.

3) Suppose $g(x) \in C^1[a,b]$, and P is an attractive fixed point, then the fixed point iteration $\{p_n\}$, defined by $p_n = g(p_{n-1})$, satisfies the afterwards (事后) error estimation:

$$|P-p_n| \le \underline{\frac{|g'(P)|}{1-|g'(P)|}|p_n-p_{n-1}|}$$

4) The recursive rule of Newton's methods for solving nonlinear equation $x^2 - A = 0$ is $\underline{x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right)}$, and its speed of convergence is __2阶__.

5) Let $X = (0,-2,5)$ and $A = \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$, then $\|X\|_\infty =$ __5__ , $\|A\|_1 =$ __9__ .

6) Suppose that $\{(x_k, y_k)\}_{k=1}^{N}$ are N points, the normal equation for the least-square line $y = Ax + B$ is

$$\begin{cases} \left(\sum_{k=1}^{N} x_k^2\right)A + \left(\sum_{k=1}^{N} x_k\right)B = \sum_{k=1}^{N} x_k y_k \\ \left(\sum_{k=1}^{N} x_k\right)A + NB = \sum_{k=1}^{N} y_k \end{cases}$$

7) Newton iteration for the following nonlinear system

$$\begin{cases} y = x^2 - \dfrac{1}{2} \\ x^2 + y^2 = 1 \end{cases}$$

is

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where $f_1 = \underline{y_k - x_k^2 + \frac{1}{2}}$, $f_2 = \underline{x_k^2 + y_k^2 - 1}$ and $J = \underline{\begin{pmatrix} -2x_k & 1 \\ 2x_k & 2y_k \end{pmatrix}}$.

8) Let $y(t)$ be the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and $\{(t_k, y_k)\}$ is the sequence of approximations generated by Euler's method, then the local discretization error is $\underline{\varepsilon_{k+1} = y(t_{k+1}) - y_k - hf(t_k, y_k) = O(h^2)}$.

2. (10 Marks)The following tableau illustrates Romberg algorithm for calculating definite integral $\int_a^b f(x)dx$. Suppose the data on the first three lines have been calculated out, please write out the formula to calculate all the data on the last line.

R(0, 0)			
R(1, 0)	**R(1, 1)**		
R(2, 0)	**R(2, 1)**	**R(2, 2)**	
R(3, 0)=?	**R(3, 1)=?**	**R(3, 2)=?**	**R(3, 3)=?**

解：

$$\begin{cases} N = 2^2, h = \dfrac{b-a}{N} \\ R(3,0) = \dfrac{R(2,0)}{2} + \dfrac{h}{2}\left[\sum_{k=1}^{N} f\left(a + \left(k - \dfrac{1}{2}\right) \cdot h\right)\right] \end{cases}$$

$$\begin{cases} R(3,1)=\frac{4}{3}R(3,0)-\frac{1}{3}R(2,0) \\ R(3,2)=\frac{16}{15}R(3,1)-\frac{1}{15}R(2,1) \\ R(3,3)=\frac{64}{63}R(3,2)-\frac{1}{63}R(2,2) \end{cases}$$

3. (10 Marks)Given the function constrain table

x	0	2	5	6
f(x)	5	7	22	11

first construct the divided difference table， and then find the Newton interpolation polynomial.（12 marks）

解：均差表为

X_k	$f(x_k)$	一阶均差	二阶均差	三阶均差
0	5			
2	7	1		
5	22	5	4/5	
6	11	-11	-4	-4/5

所以

$$N_3(x)=5+x+\frac{4}{5}x(x-2)-\frac{4}{5}x(x-2)(x-5)$$
$$=-\frac{4}{5}x^3+\frac{32}{5}x^2-\frac{43}{5}x+5$$

4. (15 Marks)Find the PA=LU factorization for the following matrix. The factorization should correspond to Gauss elimination that selects the element, which has the largest absolute value in column, as the pivotal element.

$$A=\begin{bmatrix} 0 & 2 & 1 & 4 \\ 2 & 0 & 6 & 3 \\ 4 & 2 & 2 & 2 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$

解：

$$\underset{p=(1,2,3,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ 2 & 0 & 6 & 3 \\ 4 & 2 & 2 & 2 \\ -1 & 1 & 3 & 2 \end{bmatrix}} \to \underset{p=(3,2,1,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ 2 & 0 & 6 & 3 \\ 4 & 2 & 2 & 2 \\ -1 & 1 & 3 & 2 \end{bmatrix}} \to \underset{p=(3,2,1,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -1 & 5 & 2 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{2} & \frac{7}{2} & \frac{5}{2} \end{bmatrix}}$$

$$\to \underset{p=(3,2,1,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -1 & 5 & 2 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{2} & \frac{7}{2} & \frac{5}{2} \end{bmatrix}} \to \underset{p=(3,1,2,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -1 & 5 & 2 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{2} & \frac{7}{2} & \frac{5}{2} \end{bmatrix}} \to \underset{p=(3,1,2,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -\frac{1}{2} & \frac{11}{2} & 4 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{4} & \frac{11}{4} & -\frac{1}{2} \end{bmatrix}}$$

$$\to \underset{p=(3,1,2,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -\frac{1}{2} & \frac{11}{2} & 4 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{4} & \frac{11}{4} & -\frac{1}{2} \end{bmatrix}} \to \underset{p=(3,1,2,4)}{\begin{bmatrix} 0 & 2 & 1 & 4 \\ \frac{1}{2} & -\frac{1}{2} & \frac{11}{2} & 4 \\ 4 & 2 & 2 & 2 \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}}$$

所以

$$P=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{2} & 1 \end{pmatrix}, U=\begin{pmatrix} 4 & 2 & 2 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & \frac{11}{2} & 4 \\ 0 & 0 & 0 & -\frac{5}{2} \end{pmatrix}$$

5. **(15 Marks)Show that when Heun's method is used to solve the following ordinary differential equation over** $[a,b]$**:**

$$\begin{cases} \dfrac{dy}{dt} = f(t) \\ y(a) = 0 \end{cases}$$

The result is

$$y(b) = \frac{h}{2}\sum_{k=0}^{N-1}\left(f(t_k) + f(t_{k+1})\right)$$

Where $h = \dfrac{b-a}{N}$.

证明：Heun's 递推公式为

$$y_0 = 0$$

$$\begin{cases} p_{k+1} = y_k + hf\left(t_k, y_k\right) = y_k + hf\left(t_k\right) & \text{这个式子下面没有用到} \\ y_{k+1} = y_k + \dfrac{h}{2}\left(f\left(t_k, y_k\right) + f\left(t_{k+1}, p_{k+1}\right)\right) = y_k + \dfrac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right) \end{cases}$$

所以

$$\sum_{k=0}^{N-1} y_{k+1} = \sum_{k=0}^{N-1} y_k + \sum_{k=0}^{N-1}\frac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right)$$

$$\begin{aligned} y_N &= y_0 + \sum_{k=0}^{N-1}\frac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right) \\ &= \frac{h}{2}\sum_{k=0}^{N-1}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right) \end{aligned}$$

（第 6、7 两题任选一题。两题都做仅给第六题的得分）

6. (10 Marks)Let $g(x)=\sqrt{x+6}$,

1) Can fixed-point iteration $\{x_n\}$ **defined by** $x_n=g(x_{n-1})$ **be used to find the solution(s) to the equation** $x=g(x)$.

2) Prove that for any initial value $x_0\ge -6$**, the fixed point iteration** $\{x_n\}$ **always converges?**

解： $x=g(x)$ 等价于

$$\begin{aligned}&x=\sqrt{x+6}\\&x^2-x-6=0\\&(x-3)(x+2)=0\\&x=3,x=-2(舍去)\end{aligned}$$

所以方程 $x=g(x)$ 有唯一一个解 $x=3$，且

$$g'(x)=\frac{1}{2\sqrt{x+6}},\left|g'(3)\right|=\frac{1}{6}<1$$

于是 $x=3$ 是 $g(x)$ 的吸性不动点，可以用不动点迭代求解到这个根。

令 $[a,b]=[0,b]$，其中 b 是任何一个不小于 3 的数，则 $g(x)$ 是 $[a,b]$ 上的压缩映射：

1. $\forall x\in[a,b]$， $g(x)\in[a,b]$；
2. $\forall x\in[a,b]$， $\left|g'(x)\right|=\frac{1}{2\sqrt{6}}<1$

根据压缩映射定理，对于任何一个不小于 0 的初值，迭代收敛于 $x=3$。

又由于对于任何一个不小于 -6 的初值 x_0，迭代一次 $x_1\ge 0$ 。由前所述，之后的跌代必收敛于 $x=3$。所以对于任何一个不小于 -6 的初值，迭代收敛于 $x=3$。

7. (10 Marks)Find the nature cubic spline that passes through the end points $(-1,-2),(0,0),(1,3)$ with the free boundary conditions $S''(-1)=S''(1)=0$。

解：对于唯一的内部节点 $x=0$，可以列两个方程

$$\frac{1}{2}m_0+2m_1+\frac{1}{2}m_2=g_1$$

其中 $m_0=m_2=0$， $g_1=6f[-1,0,1]=3\left(f[0,1]-f[-1,0]\right)=6\left(\frac{3-2}{2}\right)=3$，所以 $m_1=\frac{3}{2}$。最后用待定系数法求出三次样条中的各个系数

$$S(x)=\begin{cases}\frac{1}{4}x^3+\frac{3}{4}x^2+\frac{5}{2}x & \text{当}x\in[-1,0]\\ -\frac{1}{4}x^3+\frac{3}{4}x^2+\frac{5}{2}x & \text{当}x\in[0,1]\end{cases}$$