

## 《数值分析》期中试题(A 卷)

(考试形式：闭 卷 考试时间：2 小时)



《中山大学授予学士学位工作细则》第六条

考试作弊不授予学士学位

方向：\_\_\_\_\_ 姓名：\_\_\_\_\_ 学号：\_\_\_\_\_

1. Fill in the blank with proper answers (5 marks each, total 30 marks)

1) Near an attractive fixed point  $P$ , the fixed point iteration  $\{p_n\}$ , defined by

$p_n = g(p_{n-1})$ , satisfies the error estimation:

$$\boxed{|P - p_n| \leq \frac{K^n}{1-K} |p_1 - p_0|} \text{ 或 } \boxed{|P - p_n| \leq \frac{K}{1-K} |p_n - p_{n-1}|}$$

其中  $K$  是压缩映射定理中的常数，可以取  $K \approx g'(P)$ 。

2) The error term of Lagrange polynomial approximation for the function  $f(x)$  at the

nodes  $a \leq x_0 < x_1 < \dots < x_n \leq b$  is

$$\underline{f(x) - P_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \omega_{N+1}(x)}$$

3) The recursive rule of Newton's methods for solving nonlinear equation  $f(x)=0$  is

$$\underline{p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \quad k=1,2,\dots}$$

and its speed of convergence is 1 near a multiple root.

4) The function  $g(x) = -4 + 4x - \frac{1}{2}x^2$  has 2 fixed points, and the fixed point  $x=4$  is attractive.

5) The fast algorithm for evaluate the polynomial  $y = \sum_{i=0}^n a_i x^i$  is called 秦九韶或

Horner 算法, and the recursive rule is

$$\begin{cases} b_n = a_n \\ b_k = a_k + x_0 b_{k+1} \quad k = n-1, n-2, \dots, 0 \\ b_0 = P(x_0) \end{cases}$$

2. For the linear system,

$$\begin{cases} 3x + y - z = 3 \\ 2x - 5y - z = -4 \\ x + 3y - 6z = -2 \end{cases}$$

Write out the Gauss-Seidel iteration formula in matrix form  $x^{(k+1)} = Bx^{(k)} + f$ .

$$\begin{cases} 3x = -y + z + 3 \\ 2x - 5y = z - 4 \\ x + 3y - 6z = -2 \end{cases}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & -5 & 0 \\ 1 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -5 & 0 \\ 1 & 3 & -6 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 2 & -5 & 0 \\ 1 & 3 & -6 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{15} & -\frac{1}{5} & 0 \\ \frac{11}{90} & -\frac{1}{10} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{15} & -\frac{1}{5} & 0 \\ \frac{11}{90} & -\frac{1}{10} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{15} & -\frac{1}{15} \\ 0 & -\frac{11}{90} & \frac{1}{45} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{6}{5} \\ \frac{11}{10} \end{pmatrix}$$

所以

$$B = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{15} & -\frac{1}{15} \\ 0 & -\frac{11}{90} & \frac{1}{45} \end{pmatrix}, f = \begin{pmatrix} 1 \\ \frac{6}{5} \\ \frac{11}{10} \end{pmatrix}$$

3. (20 marks) Write down the cubic Newton interpolation to the function

$$f(x) = 2x^4 - 5x^3$$

where interpolation is to be exact at the four nodes  $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 3$ .

解:

x	f(x)	一阶均差	二阶均差	三阶均差
-1	7			
0	0	-7		
1	-3	-3	2	
3	27	15	6	1

于是

$$N(x) = 7 - 7(x+1) + 2x(x+1) + x(x+1)(x-3)$$

4. (20 marks) Find the triangular factorization  $A = LU$  for the following matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 6 \\ -5 & 2 & -1 \end{bmatrix}$$

解:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 6 \\ -5 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -3 \\ -5 & 2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -3 \\ -5 & 2 & 20 \end{bmatrix}$$

所以

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 6 \\ -5 & 2 & -1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 20 \end{pmatrix}$$

5. (20 marks) Consider the nonlinear system

$$\begin{cases} x^2 - 2x - y + 0.5 = 0 \\ x^2 + 4y^2 - 4 = 0 \end{cases}$$

Write out the Newton iteration for the system.

解:

$$J = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}$$

$$J^{-1} = \frac{1}{2x-16y+16xy} \begin{pmatrix} 8y & 1 \\ -2x & 2x-2 \end{pmatrix}$$

Newton iteration is

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \frac{1}{2x_k-16y_k+16x_ky_k} \begin{pmatrix} 8y_k & 1 \\ -2x_k & 2x_k-2 \end{pmatrix} \begin{pmatrix} x_k^2-2x_k-y_k+0.5 \\ x_k^2+4y_k^2-4 \end{pmatrix}$$