

《SE-205+数值计算方法》期末试题答案(A)

1. Fill in the blankets with proper answers (5 marks each, total 40 marks)

- 1) Suppose the approximate value 3.141601 has 5 significant digits, then the relative

error bound is about $\frac{0.000050}{3.141601} = \frac{\frac{1}{2} \times 10^{-4}}{3.141601}$.

- 2) For the numerical integration formula $\int_a^b f(x)dx \approx \sum_{i=1}^n A_i f(x_i)$, if $\sum_{i=1}^n |A_i| > (b-a)$,

then we think it is not stable.

- 3) The error term of Lagrange polynomial approximation for the function f at the nodes

$$a \leq x_0 < x_1 < \dots < x_n \leq b$$

is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x), \text{ 其中 } \xi \in (a, b)。$$

- 4) The Chebyshev polynomial of degree 3 is

$$T_3(x) = 4x^3 - 3x,$$

and it can be represented in trigonometric form as $T_3(x) = \cos(3 \arccos(x))$.

- 5) The recursive rule of Newton's methods for solving nonlinear equation $f(x)=0$ is

$$x_{k+1} = x - \frac{f(x_k)}{f'(x_k)}, \text{ and its speed of convergence is 2阶 for single root.}$$

- 6) Let $X = (1, 2, 3)$ and $A = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$, then $\|X\|_2 = \sqrt{14}$, $\|A\|_\infty = 4$.

- 7) Consider the Euler's methods for solving the initial value problem $y' = f(t, y)$ with

$y(t_0) = y_0$, the global discretization error is $O(h)$.

- 8) For the following linear system,

$$\begin{cases} 3x + y = 3 \\ 2x - 3y = -4 \end{cases}$$

Seidel iteration formula in matrix form $X_{k+1} = BX_k + f$ is

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The iteration will **converge** (converge or not converge), because $\|B\|_{\infty} = \frac{1}{3} < 1$.

2 (12 marks) Consider the quadrature rule

$$\int_0^h f(x) dx \approx \frac{h}{2} [f(0) + f(h)] + \lambda h^2 [f'(0) - f'(h)],$$

where λ is some unknown parameter. Find the value of λ so that the quadrature rule has as high degree of precision as possible.

解:

$$f(x) = 1: \int_0^h f(x) dx = \frac{h}{2} [f(0) + f(h)] + \lambda h^2 [f'(0) - f'(h)] = h$$

$$f(x) = x: \int_0^h f(x) dx = \frac{h}{2} [f(0) + f(h)] + \lambda h^2 [f'(0) - f'(h)] = \frac{h^2}{2}$$

$$f(x) = x^2: \int_0^h f(x) dx = \frac{h}{2} [f(0) + f(h)] + \lambda h^2 [f'(0) - f'(h)] = \frac{h^3}{2} - 2h^3\lambda$$

$$\therefore \lambda = \frac{1}{12}$$

3. (13 marks) Given 4 end points (1,0), (2,1), (3,0) and (5,0), find the three moment equation (三弯矩方程) for the cubic spline S(x) that satisfies the third, i.e. the periodic boundary condition.

解: 待定的参数有三个 m_1, m_2, m_3 , 它们是样条函数在后三个节点的二阶导数。 g_1, g_2, g_3

可以按如下的方法列表计算, 其中给出了 g_3 的两种计算方法。特别注意, g_3 的第一种计算方法是考虑到周期性而虚拟出了一个点!

x_k	$f(x_k)$	一阶均差	二阶均差
1	0		
2	1	$f[x_0, x_1] = 1$	
3	0	$f[x_1, x_2] = -1$	$\frac{g_1}{6} = f[x_0, x_1, x_2] = -1$
5	0	$f[x_2, x_3] = 0$	$\frac{g_2}{6} = f[x_1, x_2, x_3] = \frac{1}{3}$
6	1	1	$\frac{g_3}{6} = \frac{1}{3}$

或

$$g_3 = 6 \frac{f[x_0, x_1] - f[x_2, x_3]}{h_0 + h_2} = 6 \cdot \frac{1-0}{1+2} = 2$$

列方程组

$$\begin{bmatrix} 2 & \lambda_1 & \mu_1 \\ \mu_2 & 2 & \lambda_2 \\ \lambda_3 & \mu_3 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1/2 & 1/2 \\ 1/3 & 2 & 2/3 \\ 1/3 & 2/3 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$$

4. (10 marks) Given the function constrain table

x	-1	1	3	4
f(x)	-4	-2	-8	1

first construct the divided difference table, and then find the Newton interpolation polynomial.

解：均差表为

X _k	f(x _k)	一阶均差	二阶均差	三阶均差
-1	-4			
1	-2	1		
3	-8	-3	-1	
4	1	9	4	1

所以

$$P_3(x) = -4 + (x+1) - (x+1)(x-1) + (x+1)(x-1)(x-3) = x^3 - 4x^2 + 1$$

5. Given the 3×3 matrix



$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

1、(10 marks) Find out the PA=LU factorization for A. The factorization should be correspondent to Gauss elimination.

2、(5 marks) Using the factorization to solve the linear system.

$$\begin{cases} -3x_2 - x_3 = 3 \\ x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \end{cases}$$

解 1、三阶矩阵做高斯消去法需要两次消去的过程，具体如下：

$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, P = (1, 2, 3)$$

$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1/2 & 3/2 & 1/2 \\ 2 & 1 & 1 \end{bmatrix}, P = (3, 2, 1)$$

$$A = \begin{bmatrix} 0 & -3 & -1 \\ 1/2 & -1/2 & 0 \\ 2 & 1 & 1 \end{bmatrix}, P = (3, 1, 2)$$

于是可以得到 $PA=LU$ 分解中的各个矩阵:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

2、在已知 $PA=LU$ 分解的基础上, 求解方程组的基本思想是:

$$Ax = b$$

$$PAx = Pb$$

$$LUx = Pb$$

$$\begin{cases} Ly = Pb \\ Ux = y \end{cases}$$

也就是我们要解两个方程组

$$\begin{cases} y_1 = 5 \\ y_2 = 3 \\ \frac{1}{2}y_1 - \frac{1}{2}y_2 + y_3 = 1 \end{cases} \quad \begin{cases} 2x_1 + x_2 + x_3 = y_1 \\ -3x_2 - x_3 = y_2 \\ 0 \cdot x_3 = y_3 \end{cases}$$

解方程得

$$y_1 = 5, y_2 = 3, y_3 = 0$$

$$x_3 \text{ 取任意值}, x_2 = -1 - \frac{x_3}{3}, x_1 = 3 - \frac{x_3}{3}$$

6. (10 marks) Given the differential equation,

$$\begin{cases} \frac{dy}{dt} = f(t) \\ y(a) = 0 \end{cases}$$

and the step size is chosen to be $h = \frac{b-a}{n}$, where n is some positive number.

1、 Find the Euler recursive rule for numerically solving the equation;

2、 Find out the approximate solution at $t = b$.

解： 1、微分方程离散化

$$\frac{y_{k+1} - y_k}{h} = f(t_k) = f(a + kh)$$

即可得到 Euler 递推法则

$$y_{k+1} = y_k + hf(a + kh) \quad k = 0, 1, 2, \dots, n-1$$

2、 \therefore

$$\sum_{k=0}^{n-1} y_{k+1} = \sum_{k=0}^{n-1} y_k + h \sum_{k=0}^{n-1} f(a + kh)$$

$$y_1 + y_2 + \dots + y_n = y_0 + y_1 + \dots + y_{n-1} + h \sum_{k=0}^{n-1} f(a + kh)$$

$$y_n = y_0 + h \sum_{k=0}^{n-1} f(a + kh)$$

$$y(b) \approx y_n = y(a) + h \sum_{k=0}^{n-1} f(a + kh) = h \sum_{k=0}^{n-1} f(a + kh)$$