中 山 大 学 软 件 学 院 **2011** 级 软 件 工 程 专 业 (2012 学年秋季学期)

《SE-205+数值计算方法》期末试题答案(A)

1. Fill in the blankets with proper answers (5 marks each, total 40 marks)

1) For some approximate value $x = 0.0500000000$, if it has 6 significant digits, then the absolute error bound and the relative error bound are about $\underline{0.00000005}$ and $\underline{\dfrac{0.00000005}{0.0500000000} = 10^{-6}}$, respectively.

2) The function $g(x) = \sqrt{6+x}$ has $\underline{\quad 1 \quad}$ attractive fixed point(s): $\underline{x = 3}$.

3) The fast algorithm for evaluate the Newton polynomial

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

is somewhat like Horner's algorithm, and the recursive rule is:

$$\begin{cases} b_n = a_n \\ b_k = a_k + (x - x_k) b_{k+1} \quad k = n-1, n-2, \ldots, 0 \end{cases}$$

$$\underline{b_0 = P(x)}$$

4) Let $f(x)$ be a polynomial of degree 4, and $L_4(x)$ be the Lagrange polynomial based on the nodes x_0, x_1, x_2, x_3, x_4, then the relation between $f(x)$ and $L_4(x)$ is $\underline{f(x) = L_4(x)}$.

5) Chebyshev polynomial can be generated recursively by $T_0(x) = 1, T_1(x) = x$ and $\underline{T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)}$, and it can be represented in trigonometric form as $\underline{T_n(x) = \cos(n\arccos(x))}$.

6) The numerical integration formula

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{2}\left[f(-1) + 2f(0) + f(1)\right] + \frac{1}{12}\left[f'(-1) - f'(1)\right].$$

has $\underline{\quad 2 \quad}$ degrees of precision.

7) For the following linear system,

$$\begin{cases} 3x + y - z = 3 \\ 2x - 5y - z = -4 \\ x + 3y - 6z = -2 \end{cases}$$

starting from $(x_0, y_0, z_0) = (0,0,0)$, the Seidel iteration will generate

$(x_1, y_1, z_1) = \underline{\left(1, \frac{6}{5}, \frac{11}{10}\right)}$

8) Let $y(t)$ be the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

and $\{(t_k, y_k)\}$ is the sequence of approximations generated by Euler's method, then the global discretization error is $\underline{e_k = y(t_k) - y_k = O(h)}$.

2. **（10 Marks）Let's look at the Recursive Simpson rules**

$$S(n) = \frac{4}{3}T(n) - \frac{1}{3}T(n-1)$$

It is easy to see

$$S(n) = T(n) + \frac{1}{3}\left(T(n) - T(n-1)\right)$$

Do you think which is better, and why?

解：从计算量的角度看，后者乘法运算 1 次，而前者两次。

从稳定性的角度看，前者的乘法系数 $\frac{4}{3}$ 大于 1， 使误差放大，而后者没有大于 1 的系数做乘法。更准确地说，假如 $T(n)$ 和 $T(n-1)$ 的误差限都为 ε，则前者的误差限大约为 $\frac{5}{3}\varepsilon$，而后者的误差限大约为 $\frac{4}{3}\varepsilon$， 小于前者。

所以，无论从计算量的角度看，还是从稳定性的角度看，后者更好。

3.　（15 Marks）First find the triangular factorization $A=LU$ for the matrix

$$A=\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix},$$

and then analyze the computational complexity of triangular factorization for general $n\times n$ matrix A.

解：

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -3 & 5 & -8 \\ 5 & -3 & 1 & -18 \\ -3 & 3 & 2 & 18 \end{bmatrix} \quad \text{3次除法,3*3次乘法}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -3 & 5 & -8 \\ 5 & 1 & -4 & -10 \\ -3 & -1 & 7 & 10 \end{bmatrix} \quad \text{2次除法，2*2次乘法}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -3 & 5 & -8 \\ 5 & 1 & -4 & -10 \\ -3 & -1 & -\frac{7}{4} & -\frac{15}{2} \end{bmatrix} \quad \text{1次除法，1*1次乘法}$$

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -\frac{7}{4} & 1 \end{pmatrix}, U=\begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -\frac{15}{2} \end{pmatrix}$$

对于一般的$n\times n$，易知乘除法的总的次数是：

$$\sum_{k=1}^{n-1} k+\sum_{k=1}^{n-1} k^2=\frac{(n-1)n}{2}+\frac{(n-1)n(2n-1)}{6}=\frac{n(n-1)(n+1)}{3}=\frac{n^3-n}{3}$$

4. （10 Marks）**Suppose that** $\left\{\left(x_k, y_k\right)\right\}_{k=1}^{N}$ **are N points, where the abscissas are distinct. Find the normal equation for the coefficients of the least-squares parabola**

$$y = f(x) = ax^2 + bx + c$$

解： $M = 3,\ f_1(x) = 1,\ f_2(x) = x,\ \ f_3(x) = x^2$

$$F = \begin{pmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \\ \cdots & \cdots & \cdots \\ f_1(x_N) & f_2(x_N) & f_3(x_N) \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ . & \cdots & \cdots \\ 1 & x_N & x_N^2 \end{pmatrix}, C = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_N \end{pmatrix}$$

$$F^T F = \begin{pmatrix} N & x_1 + x_2 + \ldots + x_N & x_1^2 + x_2^2 + \ldots + x_N^2 \\ x_1 + x_2 + \ldots + x_N & x_1^2 + x_2^2 + \ldots + x_N^2 & x_1^3 + x_2^3 + \ldots + x_N^3 \\ x_1^2 + x_2^2 + \ldots + x_N^2 & x_1^3 + x_2^3 + \ldots + x_N^3 & x_1^4 + x_2^4 + \ldots + x_N^4 \end{pmatrix}$$

$$F^T Y = \begin{pmatrix} y_1 + y_2 + \ldots + y_N \\ x_1 y_1 + \ldots + x_N y_N \\ x_1^2 y_1 + \ldots + x_N^2 y_N \end{pmatrix}$$

正规方程是：

$$F^T F C = F^T Y$$

即

$$\begin{pmatrix} N & x_1 + x_2 + \ldots + x_N & x_1^2 + x_2^2 + \ldots + x_N^2 \\ x_1 + x_2 + \ldots + x_N & x_1^2 + x_2^2 + \ldots + x_N^2 & x_1^3 + x_2^3 + \ldots + x_N^3 \\ x_1^2 + x_2^2 + \ldots + x_N^2 & x_1^3 + x_2^3 + \ldots + x_N^3 & x_1^4 + x_2^4 + \ldots + x_N^4 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} y_1 + y_2 + \ldots + y_N \\ x_1 y_1 + \ldots + x_N y_N \\ x_1^2 y_1 + \ldots + x_N^2 y_N \end{pmatrix}$$

5. （**15 Marks**）**Show that when Heun's method is used to solve the following ordinary differential equation over** $[a,b]$**:**

$$\begin{cases} \dfrac{dy}{dt} = f(t) \\ y(a) = 0 \end{cases}$$

The result is

$$y(b) \approx \frac{h}{2}\sum_{\mathrm{k}=0}^{N-1}\left(f(t_k) + f(t_{k+1})\right)$$

Where $h = \dfrac{b-a}{N}$**.**

证明：Heun's 递推公式为

$$y_0 = 0$$

$$\begin{cases} p_{k+1} = y_k + hf\left(t_k, y_k\right) = y_k + hf\left(t_k\right) & \text{这个式子下面没有用到} \\ y_{k+1} = y_k + \dfrac{h}{2}\left(f\left(t_k, y_k\right) + f\left(t_{k+1}, p_{k+1}\right)\right) = y_k + \dfrac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right) \end{cases}$$

所以

$$\sum_{\mathrm{k}=0}^{N-1} y_{k+1} = \sum_{\mathrm{k}=0}^{N-1} y_k + \sum_{\mathrm{k}=0}^{N-1}\frac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right)$$

$$y(b) \approx y_N = y_0 + \sum_{\mathrm{k}=0}^{N-1}\frac{h}{2}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right) = \frac{h}{2}\sum_{\mathrm{k}=0}^{N-1}\left(f\left(t_k\right) + f\left(t_{k+1}\right)\right)$$

6. **（10 Marks）Write out the three-moment equation(三弯矩) of the cubic spline that pass through $(-1,1)$, $(0,2)$, $(2,-2)$ and $(3,1)$, and satisfies periodic condition:**

$$S(-1)=S(3),\ S'(-1)=S'(3)\ \textbf{and}\ S''(-1)=S''(3).$$

解：设 m_0,m_1,m_2,m_3 分别表示样条函数在四个节点上的二阶导数，则 $m_0=m_3$。三弯矩方程是关于 m_1,m_2,m_3 的一个线性方程组。

根据周期性，虚拟一个点 $(4,2)$，并列均差表如下

x_k	y_k	一阶均差	二阶均差
-1	1		
0	2	1	
2	-2	-2	$-1=\frac{g_1}{6}$
3	1	3	$\frac{5}{3}=\frac{g_2}{6}$
4	2	1	$-1=\frac{g_3}{6}$

所以三弯矩方程为

$$\begin{pmatrix} 2 & \lambda_1 & \mu_1 \\ \mu_2 & 2 & \lambda_2 \\ \lambda_3 & \mu_3 & 2 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}=\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

即

$$\begin{pmatrix} 2 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 2 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}=\begin{pmatrix} -6 \\ 10 \\ -6 \end{pmatrix}$$