CMSC 303 Introduction to Theory of Computation, VCU
Spring 2017, Assignment 6
Due: Thursday, April 14, 2017 in class

Total marks: 59 marks + 6 marks bonus for typing your solutions in LaTeX.

Unless otherwise noted, the alphabet for all questions below is assumed to be $\Sigma = \{0, 1\}$. This assignment will get you primarily to practice reductions in the context of decidability.

1. [10 marks] We begin with some mathematics regarding uncountability. Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denote the set of natural numbers.

   (a) [5 marks] Prove that the set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$ has the same size as $\mathbb{N}$ by giving a bijection between $\mathbb{Z}$ and $\mathbb{N}$. First list all of the integers beginning with 0 and then alternating between positive and negative like this $\{0, 1, -1, 2, -2, 3, -3...\}$ create the assignment $1 \to 0, 2 \to 1, 3 \to -1...$ in this way every integer will be assigned to exactly 1 natural number and every natural number will be assigned to exactly one integer.

   (b) [5 marks] Let B denote the set of all infinite sequences over $\{0, 1\}$. Show that B is uncountable using a proof by diagonalization.

   Proof. For the proof we will assume that B is countable then construct a string x which has not been counted thus showing that B is in fact uncountable. Begin by listing out every possible string $w \in B$ contruct x in for following way:
   $x_1, x_2, ..., x_n$ etc will indicate the 1st, 2nd, and nth character in x.
   $w_1, w_2, ..., w_n$ will indicate the 1st, 2nd, and nth string in the listing of all strings in B.
   w_{n_i} will indicate the ith character in the nth string of B choose x_1 s.t. $x_1 \neq w_{1_1}$ continue this so that $x_n \neq w_{n_n}$ that is the nth character of x is not equal to the nth character in the nth string of our enumeration of B. In this way we guarantee that x is not in our enumeration of B $\therefore$ B is uncountable. □

2. [9 marks] We next move to a warmup question regarding reductions.

   (a) [2 marks] Intuitively, what does the notation $A \leq B$ mean for problems A and B? This notation means that the ability to solve B implies the ability to solve A.

   (b) [2 marks] What is a mapping reduction $A \leq_m B$ from language A to language B? Give both a formal definition, and a brief intuitive explanation in your own words.

   Language A is *mapping reducible* to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

   $$w \in A \Leftrightarrow f(w) \in B \tag{1}$$

   The function f is called the *reduction* from A to B
   (Sipser pg. 235)

   Essentially, if there is a programmatic way to transform problem A into one or more cases of problem B then you have an example of mapping reducibility. Mapping mulitplication into a series of addition problems or alphabetizing a list of words into comparing 2 integers are examples of such mappings.

(c) [2 marks] What is a computable function? Give both a formal definition, and a brief intuitive explanation in your own words.

> A funciton $f : \Sigma^* \to \Sigma^*$ is a *computable function* is some Turing machine M, on every input w, halts with just $f(w)$ on its tape. (Sipser, pg. 234)

A computable function is a mapping from a string to a string which is Turing decidable.

(d) [3 marks] Suppose $A \leq_m B$ for languages A and B. Please answer each of the following with a brief explanation.

i. If B is decidable, is A decidable?
ii. If A is undecidable, is B undecidable?
iii. If B is undecidable, is A undecidable?

3. [40 marks] Prove using reductions that the following languages are undecidable.

(a) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^*\}$.

(b) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM and } \{000, 111\} \subseteq L(M)\}$.

(c) [8 marks] $L = \{\langle M \rangle \mid M \text{ is a TM which accepts all strings of even parity}\}$. (Recall the *parity* of a string $x \in \{0, 1\}$ is the number of 1's in x.)

(d) [8 marks] $L = \left\{\langle M \rangle \mid M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\right\}$. Recall here that w^R is the string w written in reverse, i.e. $011^R = 110$.

(e) [8 marks] Consider the problem of determining whether a TM M on an input w ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.