0. Definition

1. $\mathbb{S}^3$ is a unit ball in $\mathbb{R}^4$
2. $\mathbb{SO}(3)$ is a group of 3D rotations
3. $\mathbb{SE}(3)$ is the group of 3D rigid transformations

1. Representations of Rotation

1. Euler angle: [roll, pitch, yow]
2. Quaternion: $q = [\omega, \vec{v}]$
   - q and -q is identical in terms of rotation
3. Axis-angle: $[\theta, \vec{\omega}]$
4. Rotation-Matrix

Conversion between representations

1. Quaternion -> Axis-angle
   - $\theta = 2 * arccos(\omega)$
   - $\vec{\omega} = \frac{1}{sin(\theta/2)}\vec{v} \ \ if \ \ \theta \ \ \neq \ \ 0 \ \ else \ \ 0$
2. Axis-angle -> Quaternion
   - $q = [cos(\theta/2), sin(\theta/2)\vec{\omega}]$
3. Rotation-Matrix -> Axis-angle (restrict $\theta \in (0, \pi)$)
   - $\theta = arccos(1/2 * (tr(R) - 1))$
   - $[\vec{\omega}] = \frac{1}{2sin(\theta)}(R - R^T)$
     NOTE: $[\vec{\omega}]$ is the skew-symmetric matrix of $\vec{\omega} = (a, b, c)$, specifically,

$$[\vec{\omega}] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

     $[\vec{\omega}]$ has the following properties:
     - $[\vec{\omega}]^2 = [\vec{\omega}][\vec{\omega}]^T - ||\vec{\omega}||^2 I$
     - $[\vec{\omega}]^3 = -||\vec{\omega}||^2[\vec{\omega}]$
4. Axis-angle -> Rotation-Matrix
   - $R = e^{[\vec{\omega}]\theta} = I + \theta[\vec{\omega}] + \frac{\theta^2}{2!}[\vec{\omega}]^2 + \cdots = I + [\vec{\omega}]sin(\theta) + [\vec{\omega}]^2(1 - cos(\theta))$

Compare: Rotation-Matrix and Quaternion

1. Storage:
   - Quaternion: 4 floating-point
   - Ratation-Matrix: 9 floating-point
2. Multiplication:
   - Quaternion: 16 multiplications and 12 additions
   - Rotation Matrix: 27 multiplications and 18 additions
3. Numerical Stability:
   - Rotation Matrix can accumulate error during operation, including violating orthogonality. **Hard to normalize**.
   - Quaternions: when normalized, maintain unit magnitude.

   NOTE: How to Normalize Rotation Matrix and Quaternion?

   - Rotation Matrix:
     - Schmidt orthogonalization (bad for discarding information about the original axis, and manually privileging one axis over another)
     - Or use SVD, namely, $R = USV$, and change S so that

$$S' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & det(UV^T) \end{bmatrix}$$

in case $||UV^T|| = -1,\ US'V$ is the closest orthogonal matrix to R.

- Quaternion: divide by the magnitude, remember $qq* = w^2 + \vec{v}\vec{v}^T$

2. Applications of Rotation Representations

Definition

1. distance of Rotations:
    - the rotation that transforms R_1 to R_2 is $R_0 = R_2 R_1^{-1}$
    - the distance between R_1 and R_2 is $\theta(R_0)$ which is the angle in Axis-angle representation of R_0, namely, $\theta = arccos(\frac{1}{2}(tr(R_0) - 1))$.
2. distance of Quaternions:
    - the distance between $q_1 = (\omega_1, \vec{v_1})$ and $q_2 = (\omega_2, \vec{v_2})$ is the angle between their corresponding vectors, namely, $dist(q_1, q_2) = arccos(q_1 \cdot q_2) = arccos(w_1 w_2 + \vec{v_1} \cdot \vec{v_2})$ (divided by the magnitudes of the quaternions if not unit-length).

NOTE: The distance of Rotations and distance of corresponding Quaternions are linearly related, only lacking a constant factor of 2. Actually,

$$\begin{aligned} dist(R_1, R_2) &= angle\ of\ q_2 q_1^*\ represented\ in\ Angle\ axis\ form \\ &= 2arccos(Re(q_2 q_1^*)) \\ &= 2arccos(|q_1 \cdot q_2|) \\ &= 2min(dist(q_1, q_2), dist(q_1, -q_2)) \end{aligned}$$

Interpolation

1. Axis-angle: divide angle into N portions
2. Quaternion: Spherical Linear Interplation - Interpolate the Great Circle
    - First find the angle between the two quaternions using the dot product, namely, $\psi = arccos(q_1 \cdot q_2)$
    - the interpolated quaternion should be on the surface of the unit sphere: $q = \omega_1 q_1 + \omega_2 q_2$
    - Assume $\langle q, q_1 \rangle = \beta, \langle q, q_2 \rangle = \alpha$, there are relations ships:

$$\begin{aligned} sin\alpha / \omega_1 = sin\psi \\ sin\beta / \omega_2 = sin\psi \end{aligned}$$

- $$q(t) = \frac{q_1 \sin(1-t)\psi + q_2 \sin t\psi}{sin\psi}\ ,\ t \in [0, 1]$$

Sampling

1. Random sample a variable from $\mathbb{N}(0, \mathbb{I}_{4\times4})$ and normalize it to unit length. This is sampling a random quaternion. And given that distance between quaternions are linearly related to the distance between corresponding rotations, we can use this method to sample random rotations.
2. Uniformly sample from a Euler angle representation is not a uniformly sample from the rotation group.
3. Uniformly sample longitude and latitude on a sphere is not a uniformly sample from the rotation group. Actually the same span of latitude covers different area on the sphere, the closer to the equator, the larger.

3. Balancing all representations:

- rotation matrices to define concepts
- Euler angles to visualize rotations
- angle-axis representation to visualize rotations and calculate derivatives
- quaternion to write fast codes

NOTE: Using rotation matrices to represent rotations is better for training as it changes smoothly for slight transformations.