Mega Homework 4

Write out solutions to the following questions neatly by hand or typed (but if you type them, make sure to use proper math notation). Clearly label each question, include your answer to each, including an explanation of why it is correct. Submit your solutions (as a pdf) on Canvas.

1. Suppose you own 4 fezzes and 13 bow ties.

(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Carefully explain.

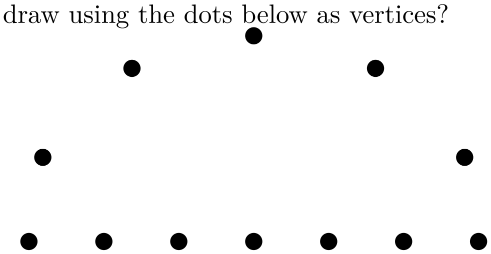
(b) Find this same answer in another way: subtract two binomial coefficients from a binomial coefficient. Explain why it makes sense to count the combinations this way too.

(c) Generalize what you noticed above to give a combinatorial proof of the identity

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy.$$

Make sure your explanation uses x and y instead of 4 and 13.

2. How many triangles can you draw using the dots below as vertices?



(a) Find an expression for the answer which is the sum of three terms involving binomial coefficients. Carefully explain why your answer makes sense.

(b) Find an expression for the answer which is the difference of two binomial coefficients. Again, explain why your answer makes sense.

(c) Generalize the above to state and prove a binomial identity using a combinatorial proof. Hint: Say you have x points on the horizontal axis and y points in the semi-circle.

3. BONUS: Consider the binomial identity

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(a) Give a combinatorial proof of this identity. Hint: The right-hand side is simpler, so start there. That looks like the multiplicative principle where you pick from n things, then decide yes/no for the remaining things.

(b) Give an alternate proof by multiplying out $(1+x)^n$ and taking derivatives of both sides.