COSC1125/1127 Artificial Intelligence
School of Computer Science and IT
RMIT University
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Tutorial Sheet 5
Propositional Logic

We use $\Rightarrow$ to denote logical implication and $\Leftrightarrow$ to denote logical equivalence. Other symbols typically used for these logical connectors are $\rightarrow$ and $\supset$ for logical implication, and $\equiv$ for logical equivalence.

1. Use truth tables to show that the following are valid (i.e. that the equivalences hold).

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distribution of $\wedge$
$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	de Morgan's Law
$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	de Morgan's Law
$P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$	Contraposition
$P \Rightarrow Q \Leftrightarrow \neg P \vee Q$	

2. For each of the following sentences, decide whether it is **valid**, **unsatisfiable**, or **neither**. Firstly, trying "guessing" the answer; then evaluate each properly (e.g. using truth tables). How did your guesses match up?

(a) $Smoke \Rightarrow Smoke$

Answer

Basically, just draw the truth table for each sentence - if the whole column for the sentence is T, then it is valud, if the whole column for the sentence is F, then it is unsatisfiable; neither otherwise. Eg.:

S	$S \Rightarrow S$
T	T
F	T

Therefore, it is valid.

(b) $Smoke \Rightarrow Fire$

Answer

S	F	$S \Rightarrow F$
T	T	T
T	F	F
F	T	T
F	F	T

Satisfiable.

(c) $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

Answer

$A : (S \Rightarrow F)$
$B : (\neg S \Rightarrow \neg F)$

S	F	$\neg S$	$\neg F$	A	B	$A \Rightarrow B$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Satisfiable.

(d) $Smoke \lor Fire \lor \neg Fire$

Answer

$A : (S \lor F)$

S	F	$\neg F$	A	$S \lor A$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

Valid.

(e) $((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$

Answer

$A : (S \land H)$
$B : (A \Rightarrow F)$
$C : (S \Rightarrow F)$
$D : (H \Rightarrow F)$
$E : (C \lor D)$

S	F	H	A	B	C	D	E	$B \Rightarrow E$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	T	F	F	F	F	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	T	T	T

Valid.

(f) $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \land Heat) \Rightarrow Fire)$

Answer

S	F	H	$S \Rightarrow F$	$S \wedge H$	$S \wedge H \Rightarrow F$	$(S \Rightarrow F) \Rightarrow (S \wedge H \Rightarrow F)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	F	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

This it is a validity (or tautology) as it holds in every possible interpretation.
To see it from another angle, suppose we want to prove that the implication $(S \Rightarrow F) \Rightarrow (S \wedge H \Rightarrow F)$ is true always, that is, it is a validity/tautology:

- Since it is an implication, the only possible way this is true is if $(S \Rightarrow F)$ is true, but $(S \wedge H \Rightarrow F)$ is false.
- Now for $(S \wedge H \Rightarrow F)$ to be false, $S \wedge H$ has to be true and F false.
- But if $S \wedge H$ is true, then S is true.
- Because we already assumed in the first item that $(S \Rightarrow F)$, then together with S being true, we know that F has to be true, which contradicts our second item.
- Hence, the whole implication cannot actually be made false, that is, it is a validity: always true in every possible interpretation.

(g) $Big \vee Dumb \vee (Big \Rightarrow Dumb)$

Answer

$X : (B \vee D)$
$Y : (B \Rightarrow D)$

B	D	X	Y	$X \vee Y$
T	T	T	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Valid.

(h) $(Big \wedge Dumb) \vee \neg Dumb$

Answer

$X : (B \wedge D)$

B	D	$\neg D$	X	$X \vee \neg D$
T	T	F	T	T
T	F	T	F	T
F	T	F	F	F
F	F	T	F	T

Satisfiable.

3. Represent the following sentences in propositional logic. Can you prove that the unicorn is mythical? What about magical? Horned?

   If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

Answer

You can translate the sentences into the following propositional logic expressions:

(a) $mythical \Rightarrow \neg mortal$

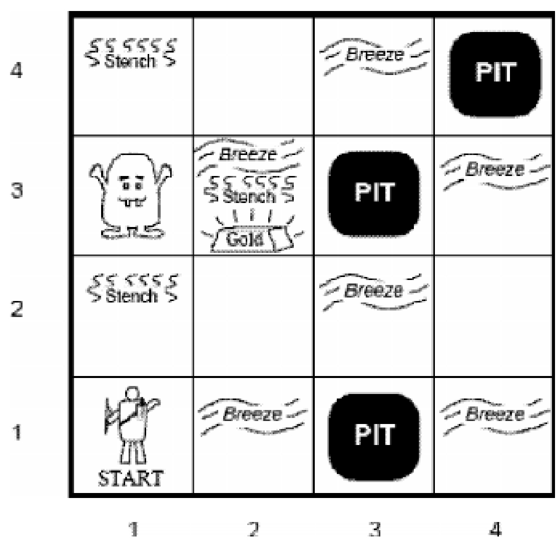
(b) $\neg mythical \Rightarrow mortal \wedge mammal$

(c) $\neg mortal \vee mammal \Rightarrow horned$

(d) $horned \Rightarrow magical$

From statements (a) and (b), we see that if it is mythical, then it is immortal; otherwise it is a mammal. So it must be either immortal or a mammal, and thus horned. That means it is also magical. However we cannot deduce anything about whether it is mythical.

4. For the following Wumpus world:



(a) Develop a notation capturing the important propositions.

Answer

$S_{xy}, W_{xy}, B_{xy}, G_{xy}$

The above denote a sentence, “there is a Stench/Wumpus/Breeze/Gold in square [x,y].

(b) How would you express in a propositional logic sentence:

i. If square $[2, 2]$ has no smell then the Wumpus is not in this square or any of the adjacent squares?

Answer

$$\neg S_{22} \Rightarrow \neg W_{22} \wedge \neg W_{21} \wedge \neg W_{12} \wedge \neg W_{32} \wedge \neg W_{23} \tag{1}$$

ii. If there is stench in square $[1, 2]$ there must be a Wumpus in this square or any of the adjacent squares?

Answer

$$S_{12} \Rightarrow W_{11} \vee W_{12} \vee W_{22} \vee W_{13} \tag{2}$$

(c) How can the agent deduce that the Wumpus is in square $[1, 3]$ using the laws of inference in propositional logic.

Answer

Given that $\neg S_{22} \wedge S_{12}$
Modus Ponens and *And Elimination* applied to the above Equation 1 gives:

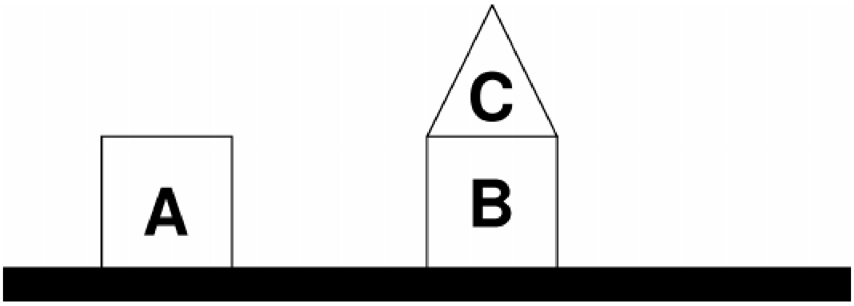
$$\neg W_{22}, \neg W_{21}, \neg W_{12}, \neg W_{32}, \neg W_{23}, \tag{3}$$

Modus Ponens applied to Equation 2 gives:

$$W_{11} \vee W_{12} \vee W_{22} \vee W_{13}, \tag{4}$$

Unit Resolution is applied to Equation 4 based on Equation 3, and also we know the agent has been to square $[1, 1]$, hence $\neg W_{11}$. Therefore, W_{13}

5. Represent the following scene in propositional calculus.



Answer

One example:

$A_{ontable} \wedge B_{ontable} \wedge C_{onB} \wedge A_{square} \wedge B_{square} \wedge C_{triangle}$

6. Consider a knowledge base built of just these three weird implications:

$$\neg A \Rightarrow B$$
$$B \Rightarrow A$$
$$A \Rightarrow (C \wedge D)$$

(a) Prove formula $A \wedge C \wedge D$ using Modus Ponens only, or explain why this is not possible.

Answer

It is not possible using Modus Ponens only: Modus Ponens is not applicable to any pair of formulas in the knowledge base. An example of using Modus Ponens is:
When it is cold, I always wear my jacket $(C \Rightarrow J)$
It is cold (C)
Therefore, I am wearing my jacket (J)

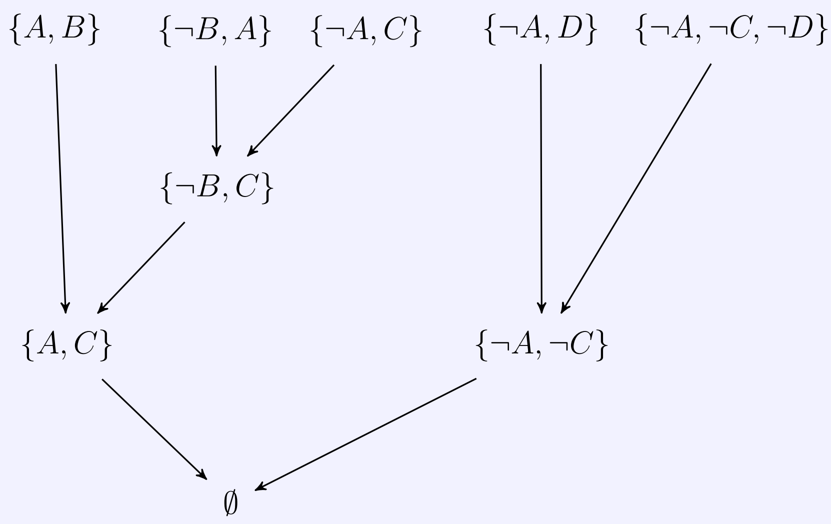
(b) Prove formula $A \wedge C \wedge D$ using resolution.

Answer

Proof by refutation:

1. $A \vee B$ premise.
2. $\neg B \vee A$ premise.
3. $\neg A \vee C$ premise.
4. $\neg A \vee D$ premise.
5. $\neg A \vee \neg C \vee \neg D$ negated thesis.
6. A resolution 1, 2.
7. C resolution 3, 6
8. D resolution 4, 6.
9. $\neg C \vee \neg D$ resolution 5, 6.
10. $\neg D$ resolution 7, 9.
11. [ ] resolution 8, 10

This can also be proved using a resolution tree:



Here we use the clausal set notation where $\{P, \neg Q\} \equiv P \vee \neg Q$

7. Given the following symbols and sentences:

   C to indicate that Gianni is a climber;
   F to indicate that Gianni is fit; L to indicate that Gianni is lucky;
   E to indicate that Gianni climbs mount Everest.

   (a) Formalize the above sentences in propositional logic:

   If Gianni is a climber and he is fit, he climbs mount Everest.
   If Gianni is not lucky and he is not fit, he does not climb mount Everest.
   Gianni is fit.

Answer

$(C \land F) \Rightarrow E$
$(\neg L \land \neg F) \Rightarrow \neg E$
F

(b) Tell if the KB buit in above is consistent, and tell if some of the following sets are models for the above sentences:
$\{\}$; $\{C, L\}$; $\{L, E\}$; $\{F, C, E\}$; $\{L, F, E\}$.

(Recall that for the binary variables A, B and C; the set $S = \{A, C\}$ means A and C are true, and B is false. S is said to be a model of KB iff all statements in KB are true for that given assignment of the variables)

Answer

The KB is consistent: it has at least a model, as the following check shows.

- $\{\}$ is not a model (it models 1 and 2 but not 3.
- $\{C, L\}$ is not a model (it models 1 and 2 but not 3).
- $\{L, E\}$ is not a model (it models 1 and 2 but not 3).
- $\{F, C, E\}$ is a model.
- $\{L, F, E\}$ is a model.

8. Tell whether the propositional formula $[(A \Rightarrow C) \lor (B \Rightarrow C)] \Rightarrow [(A \land B) \Rightarrow C]$ is:

(a) satisfiable;

(b) valid;

(c) a contradiction.

Answer

Try validity first, as if it is valid, you have also proved that it is satisfiable and not contradictory. The formula is valid, and you can prove it with resolution in the following way:

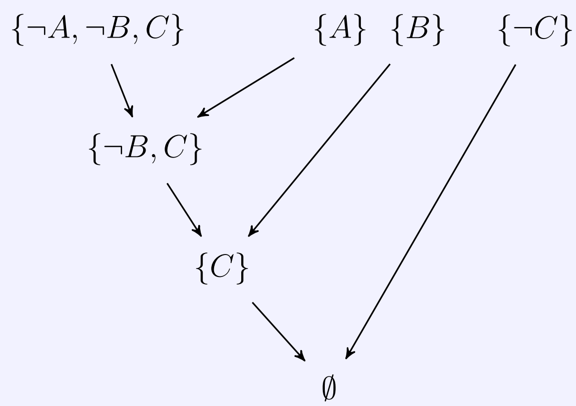
In the over-all implication, we can take the antecedent to be $(A \Rightarrow C) \vee (B \Rightarrow C)$ and the consequent to be $(A \wedge B) \Rightarrow C$.

Putting the antecedent into CNF gives the single clause $(\neg A \vee C \vee \neg B)$.

Putting the consequent into CNF gives $\neg(A \wedge B) \vee C \equiv \neg A \vee \neg B \vee C$.

As we are trying to prove the consequent, it must be negated: $A \wedge B \wedge \neg C$

The formula can now be shown to be valid using a resolution tree:



9. Let A, B, C be propositional symbols. Given $KB = \{A \Rightarrow C, B \Rightarrow C, A \vee B\}$, tell whether C can be derived from KB or not. Use resolution.

Answer

C can be derived with Resolution:

(a) $\neg A \vee C$.

(b) $\neg B \vee C$.

(c) $A \vee B$.

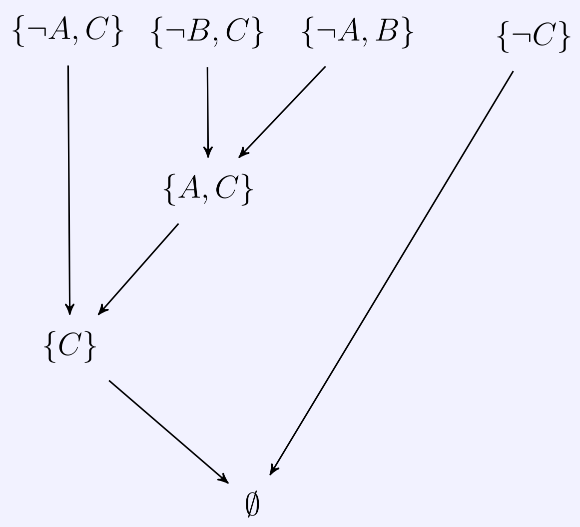
(d) $\neg C$ negated thesis

(e) $B \vee C$ from 1 and 3.

(f) C from 2 and 5.

(g) $\{\}$ from 4 and 6.

This can also be proved using a resolution tree:



10. Heads, I win. Tails, you lose. Use propositional resolution to prove that I always win.

Answer

We use H and T to signal heads or tails, resp. Also I_W and I_L to denote I win or lose, respectively, and Y_W and Y_L to denote you win or lose, resp. To formalize the problem we have:

(a) I win iff I don't lose: $I_W \Leftrightarrow \neg I_L$.

(b) You win iff you don't lose: $Y_W \Leftrightarrow \neg Y_L$.

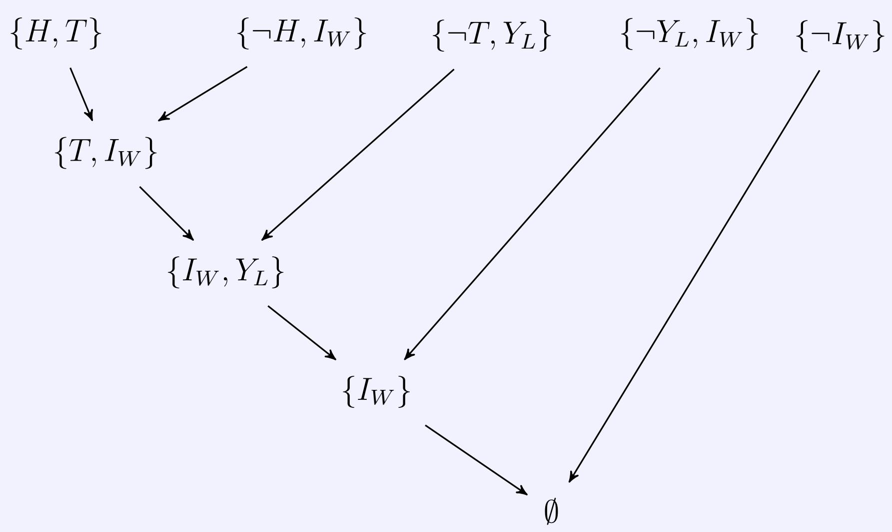
(c) Coin either tails or heads: $H \vee T$.

(d) Zero-sum game: $I_W \Leftrightarrow Y_L$.

(e) "Heads, I win": $H \Rightarrow I_W$.

(f) "Tails, you lose": $T \Rightarrow Y_L$.

We need to prove that I always win, that is, that I_W is entailed by the above formulas. We convert all the above to clausal form and then do resolution with those clauses plus $\neg I_W$ and arrive to empty clause.



Biconditional statements such as $P \Leftrightarrow Q \equiv (P \Rightarrow Q \wedge Q \Rightarrow P)$ render the clauses $\{\neg P, Q\}$ and $\{\neg Q, P\}$. It is clear that these two clauses will annihilate each other when resolved. Therefore, in the interest of keeping the diagram as simple as possible, we can choose to only use one of the implication directions from these biconditionals. All other, non-biconditional, clauses must be included and resolved fully.