

Assignment 6 of MATH 2003

ZHANG Huakang/DB92760

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1

1.1

Let $\delta = \min\{\frac{1}{2}, \frac{\epsilon}{2}\}$ and when $|x - 2| < \delta$, we have

$$\begin{aligned} |x - 2| &< \delta < \frac{1}{2} \\ -\frac{1}{2} &< x - 2 < \frac{1}{2} \\ \frac{1}{2} &< x - 1 < \frac{3}{2} \\ \left| \frac{1}{1-x} + 1 \right| &= \left| \frac{2-x}{1-x} \right| \\ &= \left| \frac{x-2}{x-1} \right| \\ &< 2(x-2) \\ &< 2\delta \\ &\leq \epsilon \end{aligned}$$

By the definition, we know:

$$\lim_{x \rightarrow 2} \frac{1}{1-x} = -1$$

Let $\delta = \epsilon$, for $|x - 0| < \delta$, we have

$$\begin{aligned} |x| &< \delta \\ \left| \frac{x^2}{x} \right| &< \delta \\ \left| \frac{x^2}{|x|} \right| &< \delta \\ \left| \frac{x^2}{|x|} - 0 \right| &< \epsilon \end{aligned}$$

By the definition, we know:

$$\lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0$$

2

f is not continuous

Proof. Without loss of generality, for any rational number $x_0 \in \mathbb{R}$, we can find a sequence (a_n) that converges to x_0 such that $\forall a \in (a_n)$, a is a irrational. We can get a sequence $(f(a_n))$ where

$$f(a_n) = x_0(1 + a_n)$$

Therefore, $(f(a_n))$ is converges to $x_0(1 + x_0)$ when $f(x_0) = x_0(1 - x_0)$. Thus, f is not continuous. \square

3

Proof. For any $x_0 \in [a, b]$, if

$$\inf_{a \leq t \leq x_0} f(t) = f(x_0)$$

because $f(x)$ is continuous, $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ and $\forall x \in [a, x_0]$, $f(x) \geq f(x_0)$. Thus

$$\lim_{x \rightarrow x_0} \inf_{a \leq t \leq x_0} f(t) = f(x_0)$$

If

$$\inf_{a \leq t \leq x_0} f(t) = f(b)$$

where $b \in [a, x_0)$ which means that

$$\lim_{x \rightarrow x_0} \inf_{a \leq t \leq x} f(t) = f(b)$$

Therefore, $\forall x_0 \in [a, b]$,

$$m(x_0) = \inf_{a \leq t \leq x_0} f(t) = \lim_{x \rightarrow x_0} \inf_{a \leq t \leq x} f(t)$$

which means that $m(x)$ is continuous. \square

4

$M = \{\frac{1}{2} \pm \frac{n}{2n+1} : n \in \mathbb{N}\} = \{1 - \frac{1}{4n+2}, \frac{1}{4n+2} : n \in \mathbb{N}\}$. For every $\delta > 0$, when

$$\begin{aligned} n &> \frac{1-2\delta}{4\delta} \Rightarrow \\ \delta &> \frac{1}{4n+2} \end{aligned}$$

which means

$$|\frac{1}{4n+2} - 0| < \delta$$

or

$$|1 - \frac{1}{4n+2} - 1| < \delta$$

By the definition, 1 and 0 are cluster point of M

5

Proof. By the definition,

$$f_2(x) := f(x)$$

for every $x \in J$ f has limit at c , i.e. for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$ where L is the limit of $f(x)$. Because $c \in J$, then $|f_2(x) - L| < \epsilon$ which means that f_2 also has a limit at c \square

Example

Let $f(x) = \operatorname{sgn}(x)$ and $J = (0, +\infty)$. We can get that $\lim_{x \rightarrow 0}$ do not exists but $\lim_{x \rightarrow 0} f_2(x) = 1$

6

Proof. We can get

$$f(0) = f(0 + 0) = 2f(0)$$

which means that

$$f(0) = 0$$

Given that

$$\lim_{x \rightarrow 0} f(x) = L$$

i.e., $\forall \epsilon > 0, \exists \delta > 0, |x - 0| < \delta \rightarrow |f(x) - L| < \epsilon$. When $x \rightarrow 0$,

$$|f(x) - L| = |f(0) - L| < \epsilon$$

i.e., $|L - L| = 0 < \epsilon$. Because $\epsilon > 0$ is arbitrary number. Thus $L = 0$ □

Proof. For any $c \in \mathbb{R}$,

$$f(x) = f(x - c) + f(c)$$

Thus,

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} f(x - c) + f(c) \\ &= \lim_{x \rightarrow 0} f(x) + f(c) \\ &= f(c) \end{aligned}$$

Therefore, f has limit at every $c \in \mathbb{R}$ □

7

Let

$$\lim_{x \rightarrow c} f(x) = L$$

which means that for any $\epsilon > 0 \in \mathbb{R}$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

$$|f(x) - L| < \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$0 < |f(x)| < \min\{|L + \epsilon|, |L - \epsilon|\} \leq |L + \epsilon|$$

i.e.,

$$0 < |f(x)| < |L + \epsilon|$$

$$-|L| < |f(x)| - |L| < |L + \epsilon| - |L|$$

$$0 < ||f(x)| - |L|| < \min\{|L| + \epsilon - |L|, |L|\}$$

i.e.,

$$0 < ||f(x)| - |L|| < \min\{|L| + \epsilon - |L|, |L|\}$$

Because $\epsilon > 0$, then $|L| + \epsilon - |L| = \epsilon$

$$0 < ||f(x)| - |L|| < \min\{|L| + \epsilon - |L|, |L|\} \leq \epsilon$$

i.e.,

$$0 < ||f(x)| - |L|| < \epsilon$$

$$0 < ||f(x)| - |\lim_{x \rightarrow c} f(x)|| < \epsilon$$

which means that

$$\lim_{x \rightarrow c} |f|(x) = |\lim_{x \rightarrow c} f(x)|$$

8

Let

$$\lim_{x \rightarrow c} f(x) = L$$

which means that for any $\epsilon > 0 \in \mathbb{R}$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

$$|f(x) - L| < \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$\sqrt{L - \epsilon} < \sqrt{f(x)} < \sqrt{L + \epsilon}$$

$$\sqrt{L - \epsilon} - \sqrt{L} < \sqrt{f(x)} - \sqrt{L} < \sqrt{L + \epsilon} - \sqrt{L}$$

$$\begin{aligned} |\sqrt{f(x)} - \sqrt{L}| &< \min\{|\sqrt{L + \epsilon} - \sqrt{L}|, |\sqrt{L - \epsilon} - \sqrt{L}|\} \\ &\leq |\sqrt{L + \epsilon} - \sqrt{L}| \end{aligned}$$

i.e.,

$$|\sqrt{f(x)} - \sqrt{L}| < |\sqrt{L + \epsilon} - \sqrt{L}|$$

We have $f : (0, \infty) \rightarrow (0, \infty)$ where $f(\epsilon) = |\sqrt{L} - \epsilon| - \sqrt{L}$ is surjective. Then $f(\epsilon)$ is a arbitrary number because ϵ is a arbitrary number. Therefore,

$$\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)}$$

9

Let $c = 0$, $f(x) = \text{sgn}(x)$ and $g(x) = -\text{sgn}(x)$. Then,

$$(f + g)(x) = 0$$

and

$$(f \cdot g)(x) = -1$$

when $x \neq 0$

$$(f \cdot g)(0) = 0$$

Therefore

$$\lim_{x \rightarrow 0} (f + g)(x) = 0$$

$$\lim_{x \rightarrow 0} (f \cdot g)(x) = -1$$

10

10.1

Proof. Let (δ_n) be a sequence such that $\delta_n > 0$ and $\delta_n > \delta_{n+1}$ for all $n \in \mathbb{N}$. Thus $\lim \delta_n$ must exist by monotone convergence theorem. And it is easy to know that $\{x : |x - x_0| < \delta_{n+1}\} \subseteq \{x : |x - x_0| < \delta_n\}$. We can get

$$M_f(x_0, \delta_n) - m_f(x_0, \delta_n) \geq M_f(x_0, \delta_{n+1}) - m_f(x_0, \delta_{n+1})$$

which means that the sequence $(M_f(x_0, \delta_n) - m_f(x_0, \delta_n))$ is monotone sequence. And $M_f(x_0, \delta_n) > m_f(x_0, \delta_n)$, i.e., $(M_f(x_0, \delta_n) - m_f(x_0, \delta_n)) > 0$. We can get $\lim(M_f(x_0, \delta_n) - m_f(x_0, \delta_n))$ must exist by monotone convergence theorem. \square

10.2

if part

Proof.

\square