Assignment 7 of MATH 2005

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1

$$
\begin{aligned}
G(y) =& F(\sqrt{y}) \\
=& \int_0^{\sqrt{y}} 2se^{-s^2} ds \\
=& -e^{-s^2}|_0^{\sqrt{y}} \\
=& 1 - e^{-y} \\
g(y) =& \frac{d}{dy} G(y) \\
=& e^{-y}
\end{aligned}
$$

when $y > 0$, and $g(y) = 0$ elsewhere.

2

$$
\begin{aligned}
G(y) =& F(e^y) \\
=& \int_0^{e^y} \frac{1}{\theta} e^{-\frac{s}{\theta}} ds \\
=& -e^{-\frac{s}{\theta}}|_0^{e^y} \\
=& 1 - e^{-\frac{e^y}{\theta}} \\
g(y) =& \frac{d}{dy} G(y) \\
=& \frac{1}{\theta} e^{y-\frac{e^y}{\theta}}
\end{aligned}
$$

when $y \in \mathbb{R}$

3

$$\begin{aligned}
G(y) =& F(y^2)\\
=& \int_0^{y^2} dds\\
=& s|_0^{y^2}\\
=& y^2\\
g(y) =& \frac{d}{dy}y^2\\
=& 2y
\end{aligned}$$

when $0 < y < 1$, and $g(y) = 0$ elsewhere.

4

Let $w(y) = \log y$**, thus** $w'(y) = \frac{1}{y} \neq 0$

$$\begin{aligned}
\varphi(y) =& \varphi(w(y); \mu, \sigma)|w'(y)|\\
=& \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}} \frac{1}{y}
\end{aligned}$$

when $y \in \mathbb{R}$

5

We get

$$f(x) = 1$$

when $0 < x < 1$ **and**

$$f(x) = 0$$

elsewhere

5.1

Let $w(y) = e^{-\frac{1}{2}y}$**, and** $w'(y) = -\frac{1}{2}e^{-\frac{1}{2}y} \neq 0$ **when** $y > 0$

$$\begin{aligned}
f(y) =& f(w(y))|w'(y)|\\
=& \frac{1}{2}e^{-\frac{1}{2}y}
\end{aligned}$$

and $f(y) = 0$ **elsewhere, which is a gamma distribution with parameters** $\alpha = 1$ **and** $\beta = 2$

5.2

Let $w(y) = y^{-\gamma}$, then $w'(y) = -\gamma y^{-\gamma-1} \neq 0$ when $y > 1$. Since $\gamma > 0$

$$\begin{aligned} g(y) =& f(w(y))|w'(y)| \\ =& \frac{\gamma}{y^{1+\gamma}} \end{aligned}$$

when $y > 1$, and $g(y) = 0$ elsewhere.

6

$$\begin{aligned} M_X(t) =& \mathbb{E}[e^{tX}] \\ =& \sum_{x=1}^{\infty} 2e^{tx}(\frac{1}{3})^x \\ \mu_1' =& \frac{d}{dt} M_X(t)|_{t=0} \\ =& \sum_{x=1}^{\infty} 2xe^{tx}(\frac{1}{3})^x|_{t=0} \\ =& \sum_{x=1}^{\infty} 2x(\frac{1}{3})^x \\ \mu_2' =& \frac{d^2}{dt^2} M_X(t) \\ =& \frac{d}{dt} \sum_{x=1}^{\infty} 2xe^{tx}(\frac{1}{3})^x|_{t=0} \\ =& \sum_{x=1}^{\infty} 2x^2e^{tx}(\frac{1}{3})^x|_{t=0} \\ =& \sum_{x=1}^{\infty} 2x^2(\frac{1}{3})^x \end{aligned}$$

7

We know that

$$f(x) = 1$$

when $0 < x < 1$, and $f(x) = 0$ elsewhere.

$$\begin{aligned}
M_X(t) =& \mathbb{E}[e^{tX}] \\
=& \int_0^1 e^{tx} 1 dx \\
=& \frac{e^t - 1}{t} \\
\mu_1' =& \frac{d}{dt} M_X(t)|_{t=0} \\
=& \frac{e^t(t-1)+1}{t^2}|_{t=0} \\
=& \lim_{t \to 0} \frac{e^t(t-1)+1}{t^2} \\
=& \frac{1}{2} \\
\mu_2' =& \frac{d^2}{dt^2} M_X(t)|_{t=0} \\
=& \frac{d}{dt} \frac{e^t(t-1)+1}{t^2}|_{t=0} \\
=& \frac{e^t t^3 - 3e^t t + 2e^t - 2}{t^3}|_{t=0} \\
=&
\end{aligned}$$

8

$$\begin{aligned}
R_X(t) =& \log M_X(t) \\
R'_X(t) =& \frac{1}{M_X(t)} \frac{d}{dt} M_X(t) \\
R'_X(0) =& \frac{1}{M_X(0)} \frac{d}{dt} M_X(t)|_{t=0} \\
=& \frac{d}{dt} M_X(t)|_{t=0} \\
=& \mu \\
R''_X(t) =& \frac{d}{dt} R'_X(t) \\
=& \frac{M_X(t) \frac{d^2}{dt^2} M_X(t) - (\frac{d}{dt} M_X(t))^2}{M_X(t)^2} \\
R''_X(0) =& \frac{M_X(0) \frac{d^2}{dt^2} M_X(0) - (\frac{d}{dt} M_X(0))^2}{M_X(0)^2} \\
=& \mathbb{E}[X^2] - \mu^2 \\
=& \sigma^2
\end{aligned}$$

Let

$$M_X(t) = e^{4(e^t - 1)}$$

then

$$\begin{aligned}
R_X(t) =& \log M_X(t) \\
=& 4(e^t - 1) \\
R'_X(t) =& 4e^t \\
R'_X(0) =& 4 \\
R''_X(t) =& 4e^t \\
R''_X(0) =& 4
\end{aligned}$$

Thus, the mean of a random variable X is $\mu = 4$ and the variance of X is $\sigma^2 = 4$

9

$$
\begin{aligned}
M_X(t) =& \int_{-\infty}^{+\infty} e^{tx} f(x) dx \\
=& \int_{-\infty}^{+\infty} e^{tx} \frac{1}{2} e^{-|x|} dx \\
=& \int_{-\infty}^{0} e^{tx} \frac{1}{2} e^{x} dx + \int_{0}^{+\infty} e^{tx} \frac{1}{2} e^{-x} dx \\
=& \frac{1}{2} [\frac{1}{t+1} e^{(t+1)x}|_{-\infty}^{0} + \frac{1}{t-1} e^{(t-1)x}|_{0}^{+\infty}]
\end{aligned}
$$

It is easy to know that when $t \leq -1$ or $t \geq 1$, $M_X(t)$ is not definited. Thus,

$$
M_X(t) = \frac{1}{1-t^2}
$$

when $t \in (-1, 1)$

10

We get $f(x) = \frac{3.3^x}{x!} e^{-3.3}$ when $x \in \mathbb{N}$ and $f(x) = 0$ elsewhere.

10.1

$$
P(X=2) = f(2) = \frac{3.3^2}{2} e^{-3.3} \approx 0.2008
$$

10.2

$$
\begin{aligned}
P =& P_2^2 (f(0)f(5) + f(1)f(4) + f(2)f(3)) \\
=& 2 \times e^{-3.3} \times (\frac{3.3^0}{0!} \frac{3.3^5}{5!} + \frac{3.3^1}{1!} \frac{3.3^4}{4!} + \frac{3.3^2}{2!} \frac{3.3^3}{3!}) \\
=& 3.3^5 \times 2 \times e^{-3.3} \times (\frac{1}{0!} \frac{1}{5!} + \frac{1}{1!} \frac{1}{4!} + \frac{1}{2!} \frac{1}{3!}) \\
\approx& 3.8491
\end{aligned}
$$

11

We get

$$f(x) = \frac{1}{5}e^{-\frac{x}{5}}$$

when $x > 0$ **and** $f(x) = 0$ **elsewhere.**

11.1

$$\begin{aligned}
P =& \int_0^8 \int_0^{8-x} \frac{1}{5}e^{-\frac{x}{5}} \frac{1}{5}e^{-\frac{y}{5}} dydx \\
=& \int_0^8 \frac{1}{5}e^{-\frac{x}{5}}(-e^{-\frac{y}{5}})|_{y=0}^{8-x} dx \\
=& \int_0^8 \frac{1}{5}e^{-\frac{x}{5}}(1 - e^{\frac{x-8}{5}})dx \\
=& \int_0^8 \frac{1}{5}(e^{-\frac{x}{5}} - e^{-\frac{8}{5}})dx \\
=&(-e^{-\frac{x}{5}} - \frac{1}{5}e^{-\frac{8}{5}}x)|_{x=0}^8 \\
=&1 - \frac{13}{5}e^{-\frac{8}{5}} \\
\approx&0.4751
\end{aligned}$$

11.2

$$\begin{aligned}
P =& \int_0^\infty \int_0^\infty \int_{12-x-y}^\infty \frac{1}{125}e^{-\frac{1}{5}(x+y+z)}dzdydx \\
=& \int_0^\infty \int_0^\infty -\frac{1}{25}e^{-\frac{1}{5}(x+y)-\frac{1}{5}z}|_{z=12-x-y}^\infty dydx
\end{aligned}$$

12

We get

$$f(x) = \frac{1}{9}e^{-\frac{x}{9}}$$

when $x > 0$ **and** $f(x) = 0$ **elsewhere**

12.1

$$
\begin{aligned}
P =& \int_0^{20} f(x)dx \\
=& \int_0^{20} \frac{1}{9} e^{-\frac{x}{9}} dx \\
=& -e^{-\frac{x}{9}}|_0^{20} \\
=& 1 - e^{-\frac{20}{9}} \\
\approx& 0.8916
\end{aligned}
$$

12.2

$$
\begin{aligned}
P =& \int_0^{20} \int_0^{20-x} \frac{1}{81} e^{-\frac{1}{9}(x+y)} dydx \\
=& \int_0^{20} -\frac{1}{9} e^{-\frac{1}{9}(x+y)}|_0^{20-x} dx \\
=& \int_0^{20} \frac{1}{9}(e^{-\frac{1}{9}x} - e^{-\frac{20}{9}})dx \\
=& -e^{-\frac{1}{9}x} - \frac{1}{9} e^{-\frac{20}{9}} x|_0^{20} \\
=& 1 - \frac{29}{9} e^{-\frac{20}{9}} \\
\approx& 0.6508
\end{aligned}
$$