Assignment 4 of MATH 2005

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1

$$
\begin{aligned}
E[X] =& \sum_{n=1}^{k} nf(n) \\
=& \frac{1}{k}\sum_{n=1}^{k} n \\
=& \frac{1}{k}\ \frac{k(k+1)}{2} \\
=& \frac{k+1}{2} \\
var(X) =& \sum_{n=1}^{k} (n-E[X])^2 f(n) \\
=& \frac{1}{k}\sum_{n=1}^{k} (n-\frac{k+1}{2})^2 \\
=& \frac{1}{k}\sum_{n=1}^{k} (n^2 - n(k+1) - \frac{(k+1)^2}{4}) \\
=& \frac{1}{k}(\frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}(k+1) + \frac{k(k+1)^2}{4}) \\
=& \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{2} + \frac{(k+1)^2}{4} \\
=& \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} \\
=& (k+1)\frac{k-1}{12} \\
=& \frac{k^2-1}{12}
\end{aligned}
$$

2

$$\begin{aligned} b(x;n,\theta) =& C_n^x\theta^x(1-\theta)^{n-x} \\ =& C_n^{n-x}(1-\theta)^{n-x}\theta^x \\ =& b(n-x;n,1-\theta) \end{aligned}$$

2.1

$$\begin{aligned} B(n-x;n,1-\theta)-B(n-x-1;n,1-\theta) =& \sum_{y=0}^{n-x} b(y;n,1-\theta) \\ & - \sum_{y=0}^{n-x-1} b(y;n,1-\theta) \\ =& b(n-x;n,1-\theta) \\ =& b(x;n,\theta) \end{aligned}$$

2.2

$$B(n;n,1-\theta)=\sum_{y=0}^{n} b(y;n,\theta)=1$$

$$\begin{aligned} B(x;n,\theta) =& \sum_{y=0}^{x} b(y;n,\theta) \\ =& \sum_{y=0}^{x}[B(n-y;n,1-\theta)-B(n-y-1;n,1-\theta)] \\ =& B(n;n,1-\theta)+(B(n-1;n,1-\theta)-B(n-1;n,1-\theta)...)-B(n-x-1;n,1-\theta) \\ =& B(n;n,1-\theta)-B(n-x-1;n,1-\theta) \\ =& 1-B(n-x-1;n,1-\theta) \end{aligned}$$

3

Proof.

$$\begin{aligned} b(x;n,\theta) =& C_n^x \theta^x (1-\theta)^{n-x} \\ =& \frac{n!}{x!(n-x)!} \theta^x (1-\theta)^{n-x} \end{aligned}$$

$$\begin{aligned} b(x+1;n,\theta) =& C_n^{x+1} \theta^{x+1} (1-\theta)^{n-x-1} \\ =& \frac{n!}{(x+1)!(n-x-1)!} \theta^{x+1} (1-\theta)^{n-x-1} \end{aligned}$$

$$\begin{aligned} \frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{x+1}{n-x} \frac{1-\theta}{\theta} \\ =& \frac{(x+1)(1-\theta)}{\theta(n-x)} \end{aligned}$$

$$b(x+1;n,\theta) = \frac{\theta(n-x)}{(x+1)(1-\theta)} b(x;n,\theta)$$

□

By the definition, when $\theta = \frac{1}{2}$

$$b(x;n,\frac{1}{2}) = C_n^x (\frac{1}{2})^n$$

$$\begin{aligned} \frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{(x+1)(1-\theta)}{\theta(n-x)} \\ =& \frac{x+1}{n-x} \end{aligned}$$

When

$$\frac{x+1}{n-x} > 1$$

we can get

$$x > \frac{n-1}{2}$$

a

n is an even number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n}{2}$

$$b(x; n, \theta) > b(x+1; n, \theta)$$

Similarily, when $x \leq \frac{n}{2}$

$$b(x; n, \theta) < b(x+1; n, \theta)$$

Therefore, we can get a maximum at $x = \frac{n}{2}$

b

n is an odd number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n-1}{2}$

$$b(x; n, \theta) \geq b(x+1; n, \theta)$$

When $\frac{x+1}{n-x} = 1$,i.e. $x = \frac{n-1}{2}$ which means

$$b(\frac{n-1}{2}; n, \theta) = b(\frac{n+1}{2}; n, \theta)$$

We can also get that

$$b(x; n, \theta) \leq b(x+1; n, \theta)$$

when $x < \frac{n-1}{2}$ Therefore, we can get a maximum at $x = \frac{n-1}{2}$ or $x = \frac{n+1}{2}$

4

Let A is the event that he get exactly four correct answers.

$$\begin{aligned}\mathbb{P}(A) =& C_8^4(\frac{1}{3})^4(\frac{2}{3})^4 \\ =& \frac{1120}{6561} \\ \approx& 17.0706\%\end{aligned}$$

5

Let A is the event that exactly 6 of 15 mice which have been administered the drug will become very aggressive within 1 minute.

$$\begin{aligned}\mathbb{P}(A) =& C_{15}^{6}(0.4)^6(0.6)^9\\ =& 5005\times\frac{64}{15625}\times\frac{19683}{1953125}\\ \approx& 20.6598\%\end{aligned}$$

6

Let A is the event that at least 3 of 5 automobile accidents are due to driver fatigue.

$$\begin{aligned}\mathbb{P}(A) =& C_5^3(0.1)^3(0.9)^2 + C_5^4(0.1)^4(0.9)^1 + C_5^5(0.1)^5(0.9)^0\\ =& \frac{107}{12500}\\ =& 0.856\%\end{aligned}$$

7

By the definition,

$$f(x;n,\theta) = C_n^x\theta^x(1-\theta)^{1-x}$$

where $\mathbb{E}[X] = n\theta$ and $var(X) = n\theta(1-\theta)$.
And $Y = \frac{X}{n}$, thus

$$\begin{aligned}\mathbb{E}[Y] =& \mathbb{E}[\frac{X}{n}]\\ =& \theta\\ var(Y) =& var(\frac{X}{n})\\ =& \frac{1}{n^2}n\theta(1-\theta)\\ =& \frac{\theta(1-\theta)}{n}\end{aligned}$$

8

Let $Y = \frac{X}{n}$ be the proportion of successes in n experiments.

$$\mathbb{E}[Y] = \theta, \sigma = var(Y) = \frac{\theta(1-\theta)}{n}$$

a

We get the equations

$$\begin{cases} \theta - c = 0.4 \\ \theta + c = 0.6 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.1 \\ k = \frac{nc}{\theta(1-\theta)} = 360 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{129599}{129600} \approx 0.9999 > \frac{35}{36}$$

b

We get the equations

$$\begin{cases} \theta - c = 0.47 \\ \theta + c = 0.53 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.03 \\ k = \frac{nc}{\theta(1-\theta)} = 1200 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{1439999}{1440000} \approx 0.9999 > \frac{35}{36}$$

c

We get the equations

$$\begin{cases} \theta - c = 0.497 \\ \theta + c = 0.503 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.003 \\ k = \frac{nc}{\theta(1-\theta)} = 1.2 \times 10^4 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{143999999}{1.44 \times 10^8} \approx 0.9999 > \frac{35}{36}$$

9

By the definition

$$b^*(y;k,\theta) = C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k}$$

Consider the function

$$f_m(z) = \sum_{k=0}^{\infty} C_{k+m}^{m} z^k$$

We know that

$$C_{k+m}^{m} = C_{k+m-1}^{m-1} + C_{k+m-1}^{m}$$

Thus

$$\begin{aligned} f_m(z) =& \sum_{k=0}^{\infty} C_{k+m-1}^{m-1} z^k + \sum_{k=1}^{\infty} C_{k+m-1}^{m} z^k \\ =& f_{m-1}(z) + z\sum_{k=1}^{\infty} C_{k+m-1}^{m} z^{k-1} \\ =& f_{m-1}(z) + z f_m(z) \end{aligned}$$

Thus,

$$f_m(z) = \frac{f_{m-1}(z)}{1-z}$$

and

$$f_0(z) = \sum_{k=0}^{\infty} C_k^0 z^k = \frac{1}{1-z}$$

Then

$$f_m(z) = (\frac{1}{1-z})^m$$

$$\begin{aligned} \sum_{y=k}^{\infty} b^*(y;k,\theta) =& \sum_{y=k}^{\infty} C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k} \\ =& \theta^k\theta^{-k} \\ =& 1 \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[Y] =& \sum_{y=k}^{\infty} y b^*(y;k,\theta) \\
=& \sum_{y=k}^{\infty} y C_{y-1}^{k-1} \theta^k (1-\theta)^{y-k} \\
=& \sum_{y=k}^{\infty} y \frac{(y-1)!}{(k-1)!(y-k)!} \theta^k (1-\theta)^{y-k} \\
=& \sum_{y=k}^{\infty} \frac{y!}{(k-1)!(y-k)!} \theta^k (1-\theta)^{y-k} \\
=& \sum_{y=k}^{\infty} k \frac{y!}{k!(y-k)!} \theta^k (1-\theta)^{y-k} \\
=& \sum_{y=k}^{\infty} k C_y^k \theta^k (1-\theta)^{y-k} \\
=& k \sum_{y=k}^{\infty} C_y^k \theta^k (1-\theta)^{y-k} \\
=& \frac{k}{\theta} \sum_{y=k}^{\infty} C_y^k \theta^{k+1} (1-\theta)^{y-k} \\
=& \frac{k}{\theta} \sum_{y=k+1}^{\infty} C_{y+1-1}^{k-1} \theta^{k+1} (1-\theta)^{y-k-1} \\
=& \frac{k}{\theta} \sum_{y=k+1}^{\infty} C_{(y+1)-1}^{k-1} \theta^{k+1} (1-\theta)^{y-(k+1)} \\
=& \frac{k}{\theta} \sum_{y=k+1}^{\infty} b^*(y+1;k,\theta) \\
=& \frac{k}{\theta}
\end{aligned}
$$

10

Proof. By the definition,

$$\begin{aligned}h(x;n,N,k) =& \frac{C_k^x C_{N-k}^{n-x}}{C_N^n} \\ =& \frac{\frac{k!}{x!(k-x)!}\frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{\frac{N!}{n!(N-n)!}} \\ =& \frac{k!(N-k)!n!(N-n)!}{x!(k-x)!(n-x)!(N+x-k-n)!N!}\end{aligned}$$

Thus

$$h(x+1;n,N,k) = \frac{k!(N-k)!n!(N-n)!}{(x+1)!(k-x-1)!(n-x-1)!(N+x-k-n+1)!N!}$$

$$\begin{aligned}\frac{h(x+1;n,N,k)}{h(x;n,N,k)} =& \frac{x!(k-x)!(n-x)!(N+x-k-n)!}{(x+1)!(k-x-1)!(n-x-1)!(N+x-k-n+1)!} \\ =& \frac{(k-x)(n-x)}{(x+1)(N+x-k-n+1)}\end{aligned}$$

□

11

Proof. By the definition

$$p(x;\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$$

Thus

$$p(x+1;\lambda) = \frac{\lambda^{x+1}}{(x+1)!}e^{-\lambda}$$

$$\begin{aligned}\frac{p(x+1;\lambda)}{p(x;\lambda)} =& \frac{\frac{\lambda^{x+1}}{(x+1)!}e^{-\lambda}}{\frac{\lambda^x}{x!}e^{-\lambda}} \\ =& \frac{\lambda}{x+1}\end{aligned}$$

□

12

12.1

Let A be the event that a family's fourth child is their first son.

$$\begin{aligned}\mathbb{P}(A) =& 0.5^3 \times 0.5\\ =& \frac{1}{16}\end{aligned}$$

12.2

Let B be the event that a family's seventh child is their second daught.

$$\begin{aligned}\mathbb{P}(B) =& C_6^1 0.5^5 \times 0.5^2\\ =& \frac{7}{128}\end{aligned}$$

12.3

Let C be the event that a family's tenth child is their fourth or fifth son.

$$\begin{aligned}\mathbb{P}(C) =& C_9^4 0.5^6 \times 0.5^4 + C_9^5 0.5^5 \times 0.5^5\\ =& \frac{63}{512}\end{aligned}$$

13

13.1

Let A be the event that the eighth person to hear the rumor will be the fifth to believe it.

$$\begin{aligned}\mathbb{P}(A) =& C_7^4 0.75^5 \times 0.25^3\\ =& \frac{8505}{65536}\\ \approx& 0.1298\end{aligned}$$

13.2

Let B be the event that 2) the fifteenth person to hear the rumor will be the tenth to believe it.

$$\begin{aligned}\mathbb{P}(B) =& C_{14}^{9} 0.75^{10} \times 0.25^{5} \\ \approx& 0.1101\end{aligned}$$

14

Let A be the event that at the switch will not fail during the first 800 times it is turned on or off.

$$\begin{aligned}\mathbb{P}(A) =& C_{800}^{8} 000.001^{0} \times (1-0.001)^{8}00 \\ \approx& 0.4491\end{aligned}$$

15

15.1

$$\begin{aligned}\mathbb{P} =& \frac{C_{14}^{2}}{C_{18}^{2}} \\ =& \frac{91}{153} \\ \approx& 0.5948\end{aligned}$$

15.2

$$\begin{aligned}\mathbb{P} =& \frac{C_{10}^{2}}{C_{18}^{2}} \\ =& \frac{5}{17} \\ \approx& 0.2941\end{aligned}$$

15.3

$$
\begin{aligned}
\mathbb{P} =& \frac{C_6^2}{C_{18}^2} \\
=& \frac{5}{51} \\
\approx& 0.0980
\end{aligned}
$$

16

16.1

$$
\begin{aligned}
\mathbb{P} =& \frac{C_4^1 C_{76}^2}{C_{80}^3} \\
=& \frac{285}{2054} \\
\approx& 0.1388
\end{aligned}
$$

16.2

$$
\theta = \frac{4}{80} = \frac{1}{20}
$$

$$
\begin{aligned}
\mathbb{P} =& C_3^1 \theta^1 (1-\theta)^2 \\
=& \frac{1083}{8000} \\
\approx& 0.1354
\end{aligned}
$$

17

By the definition,

$$
\mathbb{P}(X) = \frac{\lambda^x}{x!} e^{-\lambda}
$$

where $\lambda = np = 1.2$

$$\begin{aligned}\mathbb{P}(X \leq 2) =&\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)\\ =&e^{-1.2} + 1.2e^{-1.2} + \frac{1.2^2}{2}e^{-1.2}\\ =&\frac{73}{25}e^{-1.2}\\ \approx&0.8795\end{aligned}$$

18

By the definition,

$$\mathbb{P}(X) = \frac{\lambda^x}{x!}e^{-\lambda}$$

where $\lambda = 0.25$

$$\begin{aligned}\mathbb{P}(X \leq 1) =&\mathbb{P}(X = 0) + \mathbb{P}(X = 1)\\ =&e^{-0.25} + 0.25e^{-0.25}\\ \approx&0.0.9735\end{aligned}$$