Assignment 8 of MATH 2003

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By the definition of continuous,$\forall \epsilon > 0, \exists \delta > 0$ **s.t. if** $|x - x_0| < \delta$ **then** $|f(x) - f(x_0)| < \epsilon$ **and** $|g(x) - g(x_0)| < \epsilon$ **i.e.,**

$$
\begin{aligned}
-\epsilon < f(x) - f(x_0) &< \epsilon \\
-\epsilon < g(x) - g(x_0) &< \epsilon \\
f(x) - f(x_0) - (g(x) - g(x_0)) &< \epsilon - \epsilon \\
f(x) - g(x) - (f(x_0) - g(x_0)) &< 0 \\
f(x) - g(x) &< f(x_0) - g(x_0) < 0 \\
f(x) - g(x) &< 0
\end{aligned}
$$

when $|x - x_0| < \delta$ **which means that** $x \in V_\delta(x_0)$

2

Yes.

Proof. Assume that f is not constant. Then is must exist x, $y \in [0, 1]$ such that $f(x)$ and $f(y)$ are rational values and $f(x) \neq f(y)$. By *Bolzano's Intermediate Value Theorem* we know that $\forall k \in (\mathbb{R}\backslash\mathbb{Q})$ satisfies $\inf\{f(x), f(y)\} < k < \sup\{f(x), f(y)\}$,there exists a point $c \in (\inf\{x, y\}, \sup\{x, y\})$ such that $f(c) = k$ whcih contradicts with that $f(x)$ is rational value. Therefore, $f(x)$ is constant. $\square$

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Proof. We know that $\forall x_i \in I, \exists M_i \in \mathbb{R}$, such that $|f(x)| \leq M_i$ where $x \in V_\delta(i)$. Therefore,$|f(x)| \leq \sup\{M_i, M_j\}$ where $x \in V_{\delta_i}(i) \cup V_{\delta_j}(j)$ It is easy to get that

$$I \subset \cup_{x \in I} V_\delta(x)$$

When $x \in I$, then $x \in \cup_{x \in I} V_\delta(x)$. Thus

$$|f(x)| \leq \sup\{M_i : i \in I, x \in V_\delta(i), |f(x)| \leq M_i\}$$

which means $f(x)$ is bounded in I. $\square$

Proof. Assume that f is not bounded on I whcih means that $\exists x_0 \in I$ such that $x \to x_0, |f(x_0)| \to +\infty$. Thus, $\forall x \in V_\delta(x_0)$, there do not exist a real number M such that $|f(x)| \leq M$ which contradicts with f is bounded on a neighborhood $V_{\delta_x}(x), \forall x \in I$ $\square$

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$$g(x) = \frac{1}{x}, x \in (0, 1)$$

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5.1

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two sequences defined as $a_n = n + \frac{1}{n}$ and $b_n = n$. Thus $\lim an - bn = 0$. But $|f(a_n) - f(b_n)| = 2 + \frac{1}{n^2} \geq 2$. Therefore, $f(x) = x^2$ is not uniformly continuous on A $\square$

5.2

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two sequences defined as $a_n = \frac{1}{n\pi}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Thsu $\lim a_n - b_n = 0$. But $|g(a_n) - g(b_n)| = 1$. Therefore, $g(x) = \sin \frac{1}{x}$ is not uniformly continuous on B $\square$

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Proof. $f(x)$ is continuous on $[0, a]$ for some positive constant a. Because $[0, 1]$ is closed bounded interval, $f(x)$ is uniformly continuous on $[0, a]$. We know that $f(x)$ is uniformly continuous on $(a, +\infty)$. Then $f(x)$ is uniformly continuous on $(0, +\infty)$ $\square$

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Proof. $g_\epsilon(x)$ is uniformly continuous on A which means that when $x, u \in A$ and $|x - u| < \delta(\epsilon)$, then

$$|g_\epsilon(x) - g_\epsilon(u)| < \epsilon$$

We know that

$$|f(x) - g_\epsilon(x)| < \epsilon$$

and

$$|f(u) - g_\epsilon(u)| < \epsilon$$

We can get

$$f(x) - g_\epsilon(x) < \epsilon$$

and

$$g_\epsilon(u) - f(u) < \epsilon$$

Thus

$$f(x) - f(u) - (g_\epsilon(x) - g_\epsilon(u)) < 2\epsilon$$

i.e.

$$f(x) - f(u) < 2\epsilon + g_\epsilon(x) - g_\epsilon(u) < 3\epsilon$$

Therefore,

$$f(x) - f(u) < 3\epsilon$$

$$|f(x) - f(u)| < 3\epsilon$$

where ϵ is arbitrary, then 3ϵ is also arbitrary. whcih means that f is also uniformly continuous on A. □

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9

9.1

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{x^3 - c^3}{x - c} \\ &= \lim_{x \to c} x^2 + xc + c^2 \\ &= 3c^2 \end{aligned}$$

9.2

$$
\begin{aligned}
g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\
&= \lim_{x \to c} \frac{c - x}{xc(x - c)} \\
&= \lim_{x \to c} -\frac{1}{cx} \\
&= -\frac{1}{c^2}
\end{aligned}
$$

9.3

$$
\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\
&= \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} \\
&= \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{c} + \sqrt{x})} \\
&= \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} \\
&= \frac{1}{2\sqrt{c}}
\end{aligned}
$$

9.4

$$
\begin{aligned}
k'(c) &= \lim_{x \to c} \frac{k(x) - k(c)}{x - c} \\
&= \lim_{x \to c} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}}{x - c} \\
&= \lim_{x \to c} \frac{\sqrt{c} - \sqrt{x}}{\sqrt{cx}(x - c)} \\
&= \lim_{x \to c} \frac{\sqrt{c} - \sqrt{x}}{\sqrt{xc}(\sqrt{x} - \sqrt{c})(\sqrt{c} + \sqrt{x})} \\
&= \lim_{x \to c} -\frac{1}{\sqrt{cx}(\sqrt{x} + \sqrt{c})} \\
&= -\frac{1}{2c\sqrt{c}}
\end{aligned}
$$