Assignment 8 of MATH 2003

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By the definition of continuous,$\forall \epsilon > 0, \exists \delta > 0$ **s.t. if** $|x - x_0| < \delta$ **then** $|f(x) - f(x_0)| < \epsilon$ **and** $|g(x) - g(x_0)| < \epsilon$ **i.e.,**

$$\begin{aligned}
-\epsilon < f(x) - f(x_0) <& \epsilon \\
-\epsilon < g(x) - g(x_0) <& \epsilon \\
f(x) - f(x_0) - (g(x) - g(x_0)) <& \epsilon - \epsilon \\
f(x) - g(x) - (f(x_0) - g(x_0)) <& 0 \\
f(x) - g(x) <& f(x_0) - g(x_0) < 0 \\
f(x) - g(x) <& 0
\end{aligned}$$

when $|x - x_0| < \delta$ **which means that** $x \in V_\delta(x_0)$

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Yes.

Proof. Assume that f is not constant. Then is must exist x, $y \in [0, 1]$ such that $f(x)$ and $f(y)$ are rational values and $f(x) \neq f(y)$. By *Bolzano's Intermediate Value Theorem* we know that $\forall k \in (\mathbb{R} \backslash \mathbb{Q})$ satisfies $\inf\{f(x), f(y)\} < k < \sup\{f(x), f(y)\}$,there exists a point $c \in (\inf\{x, y\}, \sup\{x, y\})$ such that $f(c) = k$ whcih contradicts with that $f(x)$ is rational value. Therefore, $f(x)$ is constant. □

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Proof. We know that $\forall x_i \in I, \exists M_i \in \mathbb{R}$, such that $|f(x)| \leq M_i$ where $x \in V_\delta(i)$. Therefore,$|f(x)| \leq \sup\{M_i, M_j\}$ where $x \in V_{\delta_i}(i) \cup V_{\delta_j}(j)$ It is easy to get that

$$I \subset \cup_{x \in I} V_\delta(x)$$

When $x \in I$, then $x \in \cup_{x\in I} V_\delta(x)$. Thus

$$|f(x)| \leq \sup\{M_i : i \in I, x \in V_\delta(i), |f(x)| \leq M_i\}$$

which means $f(x)$ is bounded in I. $\square$

Proof. Assume that f is not bounded on I whcih means that $\exists x_0 \in I$ such that $x \to x_0, |f(x_0)| \to +\infty$. Thus, $\forall x \in V_\delta(x_0)$, there do not exist a real number M such that $|f(x)| \leq M$ which contradicts with f is bounded on a neighborhood $V_{\delta_x}(x), \forall x \in I$ $\square$

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$$g(x) = \frac{1}{x}, x \in (0, 1)$$

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5.1

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two sequences defined as $a_n = n + \frac{1}{n}$ and $b_n = n$. Thus $\lim an - bn = 0$. But $|f(a_n) - f(b_n)| = 2 + \frac{1}{n^2} \geq 2$. Therefore, $f(x) = x^2$ is not uniformly continuous on A $\square$

5.2

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two sequences defined as $a_n = \frac{1}{n\pi}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Thsu $\lim a_n - b_n = 0$. But $|g(a_n) - g(b_n)| = 1$. Therefore, $g(x) = \sin\frac{1}{x}$ is not uniformly continuous on B $\square$

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Proof. $f(x)$ is continuous on $[0, a]$ for some positive constant a. Because $[0, 1]$ is closed bounded interval, $f(x)$ is uniformly continuous on $[0, a]$. We know that $f(x)$ is uniformly continuous on $(a, +\infty)$. Then $f(x)$ is uniformly continuous on $(0, +\infty)$ $\square$

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Proof. $g_\epsilon(x)$ is uniformly continuous on A which means that when $x, u \in A$ and $|x-u| < \delta(\epsilon)$, then

$$|g_\epsilon(x) - g_\epsilon(u)| < \epsilon$$

We know that

$$|f(x) - g_\epsilon(x)| < \epsilon$$

and

$$|f(u) - g_\epsilon(u)| < \epsilon$$

We can get

$$f(x) - g_\epsilon(x) < \epsilon$$

and

$$g_\epsilon(u) - f(u) < \epsilon$$

Thus

$$f(x) - f(u) - (g_\epsilon(x) - g_\epsilon(u)) < 2\epsilon$$

i.e.

$$f(x) - f(u) < 2\epsilon + g_\epsilon(x) - g_\epsilon(u) < 3\epsilon$$

Therefore,

$$f(x) - f(u) < 3\epsilon$$

$$|f(x) - f(u)| < 3\epsilon$$

where ϵ is arbitrary, then 3ϵ is also arbitrary. whcih means that f is also uniformly continuous on A. □

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Proof. Assume that $f : [0, p] \to \mathbb{R}$. $[0, p]$ is closed bounded interval and f is continuous, then $f([0, p])$ is also closed bounded interval. Then f is bounded. and f is uniformly continuous since $[0, p]$ is closed bounded interval. Because $f([0, p]) = f([np.(n + 1)p]), n \in \mathbb{Z}$, we can get that $f : [np, (n + 1)p] \to \mathbb{R}$ is also bounded and uniformly continuous. Therefor f is bounded and uniformly continuous on $\mathbb{R}$ □

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9.1

$$\begin{aligned}
f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c} \frac{x^3-c^3}{x-c} \\
&= \lim_{x\to c} x^2+xc+c^2 \\
&= 3c^2
\end{aligned}$$

9.2

$$\begin{aligned}
g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} \\
&= \lim_{x\to c} \frac{c-x}{xc(x-c)} \\
&= \lim_{x\to c} -\frac{1}{cx} \\
&= -\frac{1}{c^2}
\end{aligned}$$

9.3

$$\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} \\
&= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} \\
&= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{c}+\sqrt{x})} \\
&= \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} \\
&= \frac{1}{2\sqrt{c}}
\end{aligned}$$

9.4

$$
\begin{aligned}
k'(c) &= \lim_{x\to c} \frac{k(x)-k(c)}{x-c} \\
&= \lim_{x\to c} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}}{x-c} \\
&= \lim_{x\to c} \frac{\sqrt{c}-\sqrt{x}}{\sqrt{cx}(x-c)} \\
&= \lim_{x\to c} \frac{\sqrt{c}-\sqrt{x}}{\sqrt{xc}(\sqrt{x}-\sqrt{c})(\sqrt{c}+\sqrt{x})} \\
&= \lim_{x\to c} -\frac{1}{\sqrt{cx}(\sqrt{x}+\sqrt{c})} \\
&= -\frac{1}{2c\sqrt{c}}
\end{aligned}
$$

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Claim. If $f : \mathbb{R} \to \mathbb{R}$ is an even function and has a derivative at every point, then the derivative f' is an odd function.

Proof.

$$
\begin{aligned}
f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
f'(-c) &= \lim_{x\to -c} \frac{f(x)-f(-c)}{x+c} \\
&= \lim_{x\to -c} \frac{f(x)-f(c)}{x+c} \\
&= \lim_{x\to c} \frac{f(-x)-f(c)}{-x+c} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{c-x} \\
&= -\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= -f'(c)
\end{aligned}
$$

$f'(x)$ is odd function. □

Claim. If $f : \mathbb{R} \to \mathbb{R}$ is an odd function and has a derivative at every point, then the derivative f' is an even function.

Proof.

$$
\begin{aligned}
g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
g'(-c) &= \lim_{x\to -c} \frac{g(x)-g(-c)}{x+c} \\
&= \lim_{x\to -c} \frac{g(x)+g(c)}{x+c} \\
&= \lim_{x\to c} \frac{g(-x)+g(c)}{-x+c} \\
&= \lim_{x\to c} \frac{-g(x)+g(c)}{c-x} \\
&= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= g'(c)
\end{aligned}
$$

$f'(x)$ is even function. □

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Claim. g is differentiable for all $x \in \mathbb{R}$

Proof. When $c \neq 0$

$$
\begin{aligned}
g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{x^2 \sin \frac{1}{x^2} - c^2 \sin \frac{1}{c^2}}{x-c} \\
&= 2c \sin \frac{1}{c^2} - \frac{2}{c} \cos \frac{1}{c^2}
\end{aligned}
$$

and when $c = 0$

$$
\begin{aligned}
g'(0) &= \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} \\
&= \lim_{x\to 0} \frac{g(x)}{x} \\
&= \lim_{x\to 0} x \sin \frac{1}{x^2} \\
&= 0
\end{aligned}
$$

Therefore, f is differentiable on $\mathbb{R}$ □

Claim. The derivative g' is not bounded on the interval $[-1, 1]$

Proof. When $x \to 0$, $2c\sin\frac{1}{c^2} \to 0$, $\frac{2}{c} \to \infty$and $|\cos\frac{1}{x^2}| \leq 1$ which means that g' can not be bounded on the neighborhood of 0 and therefore g' is not bounded on $[-1, 1]$ □

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Proof.

$$\begin{aligned} f'(0) &= \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0}\frac{x^r\sin\frac{1}{x}}{x} \\ &= \lim_{x\to 0} x^{r-1}\sin\frac{1}{x} \end{aligned}$$

When $r > 1$,

$$\lim_{x\to 0} x^{r-1}\sin\frac{1}{x} = 0$$

since $\lim_{x\to 0} x = 0$ and $|\sin\frac{1}{x}| < 1$.
When $r = 1$,

$$\lim_{x\to 0} x^{r-1}\sin\frac{1}{x} = \lim_{x\to 0}\sin\frac{1}{x}$$

Limit do not exist.
When $0 < r < 1$,

$$\frac{1}{x^{1-r}}$$

is not bounded on neighborhood of 0 and therefore limit do not exist.
Thus, $f'(0)$exists when $r > 1$. □