

# Assignment 4 of MATH 2005

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October 26, 2020

**1**

$$\begin{aligned} E(X) &= \sum_{x=-1}^3 xf(x) \\ &= -1 \times \frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 2 \times \frac{0}{7} + 3 \times \frac{1}{7} \\ &= \frac{1}{7} \end{aligned}$$

**2**

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} yh(y)dy \\ &= \int_2^4 \frac{1}{8}(y+1)ydy + 0 \\ &= \frac{79}{3} \end{aligned}$$

**3**

**3.1**

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_1^3 x \frac{1}{x \log 3} dx + 0 \\ &= \frac{2}{\log 3} \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_1^3 \frac{x}{\log 3} dx \\
 &= \frac{4}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 E(X^3) &= \int_{-\infty}^{\infty} x^3 f(x) dx \\
 &= \int_1^3 \frac{x^2}{\log 3} dx \\
 &= \frac{26}{3 \log 3}
 \end{aligned}$$

$$\begin{aligned}
 E(X^3 + 2X^2 - 3X + 1) &= E(X^3) + 2E(X^2) - 3E(X) + E(1) \\
 &= \frac{26}{3 \log 3} + 2 \times \frac{4}{\log 3} - 3 \times \frac{2}{\log 3} + 1 \\
 &= \frac{35}{3 \log 3}
 \end{aligned}$$

### 3.2

$$\begin{aligned}
 \mu'_r &= E[X^r] \\
 &= \int_{-\infty}^{\infty} x^r f(x) dx \\
 &= \int_1^3 x^r \frac{1}{x \log 3} dx \\
 &= \frac{1}{\log 3} \int_1^3 x^{r-1} dx \\
 &= \frac{1}{\log 3} \frac{1}{r} x^r \Big|_1^3 \\
 &= \frac{1}{r \log 3} (3^r - 1)
 \end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \text{var}(X) \\
&= E[X^2] - E[X]^2 \\
&= \frac{4}{\log 3} - \frac{4}{\log 3^2} \\
&= \frac{4 \log 3 - 4}{\log 3^2}
\end{aligned}$$

4

$$\begin{aligned}
E\left(\frac{X}{Y}\right) &= \int_0^1 \int_0^y \frac{x}{y} f(x, y) dx dy \\
&= \int_0^1 \int_0^y \frac{x}{y^2} dx dy \\
&= \int_0^1 \frac{x^2}{2y^2} \Big|_0^y dy \\
&= \int_0^1 \frac{1}{2} dy \\
&= \frac{1}{2} y \Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

5

Let  $\varphi(x)$  is the money he should pay us where  $x$  is the number we get from a balanced die.

$$\begin{aligned}
E(\varphi(X)) &= \sum_{x=1}^6 \varphi(x) f(x) \\
&= \frac{1}{6} \sum_{x=1}^4 \varphi(x) + \frac{10}{3} \\
&= 0
\end{aligned}$$

Thus, we can get,

$$\sum_{x=1}^4 \varphi(x) = -20$$

which means that the total money we should pay that person when we roll a 1, 2, 3, or 4 is equal to \$20. Hence, there are many solutions

for this equation. For example,

$$\begin{aligned}\varphi(1) = \varphi(2) = \varphi(3) &= 0, \\ \varphi(4) &= -20.\end{aligned}$$

## 6

### 6.1

Let

$$\varphi(x; n) = \begin{cases} x - (n - x) \times 0.4 & (0 \leq x \leq n) \\ n & (n \leq x). \end{cases}$$

be the profit when produce  $n$  cake(s) a day.

(a) one of the cakes

$$\begin{aligned}E(\varphi(X; 1)) &= \sum_{n=0}^5 \varphi(x; 1) f(x) \\ &= \frac{1}{6}(-0.4 + 1 + 1 + 1 + 1 + 1) \\ &= \frac{23}{30}\end{aligned}$$

(b) two of the cakes

$$\begin{aligned}E(\varphi(X; 2)) &= \sum_{n=0}^5 \varphi(x; 2) f(x) \\ &= \frac{1}{6}(-0.8 + 0.6 + 2 + 2 + 2 + 2) \\ &= 1.2\end{aligned}$$

(c) three of the cakes

$$\begin{aligned}
 E(\varphi(X; 3)) &= \sum_{n=0}^5 \varphi(x; 3) f(x) \\
 &= \frac{1}{6} (-1.2 + 0.2 + 1.6 + 3 + 3 + 3) \\
 &= 1.6
 \end{aligned}$$

(d) four of the cakes

$$\begin{aligned}
 E(\varphi(X; 4)) &= \sum_{n=0}^5 \varphi(x; 4) f(x) \\
 &= \frac{1}{6} (-1.6 - 0.2 + 1.2 + 2.6 + 4 + 4) \\
 &= \frac{5}{3}
 \end{aligned}$$

(e) five of the cakes

$$\begin{aligned}
 E(\varphi(X; 5)) &= \sum_{n=0}^5 \varphi(x; 5) f(x) \\
 &= \frac{1}{6} (-2 - 0.6 + 0.8 + 2.2 + 3.6 + 5) \\
 &= \frac{3}{2}
 \end{aligned}$$

## 6.2

By 6.1 we can know that he should bake 4 cakes a day to maximize his expected profit.

7

$$\begin{aligned}
E(Z) &= E\left(\frac{X - \mu}{\sigma}\right) \\
&= \frac{1}{\sigma} (E(X - \mu)) \\
&= \frac{1}{\sigma} (E(X) - E(\mu)) \\
&= \frac{1}{\sigma} (\mu - \mu) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{var}(Z) &= E\{[Z - E(Z)]^2\} \\
&= E(Z^2) - [E(Z)]^2 \\
&= E(Z^2) \\
&= E\left[\left(\frac{X - \mu}{\sigma}\right)^2\right] \\
&= \frac{1}{\sigma^2} E[(X - \mu)^2] \\
&= \frac{1}{\sigma^2} \times \sigma^2 \\
&= 1
\end{aligned}$$

8

By Chebyshevs inequality, we can know that

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

(a) at least 0.95

$$\begin{aligned}
P(|X - \mu| < k\sigma) &\geq 1 - \frac{1}{k^2} \geq 0.95 \\
k &\geq 2\sqrt{5}
\end{aligned}$$

(b) at least 0.99

$$\begin{aligned}
P(|X - \mu| < k\sigma) &\geq 1 - \frac{1}{k^2} \geq 0.99 \\
k &\geq 10
\end{aligned}$$

## 9

$$\begin{aligned}
 P(64 \leq X \leq 184) &= P(|X - \mu| \leq 60) \\
 &= P(|X - \mu| \leq 8\sigma) \\
 &\geq 1 - \frac{1}{8^2} = \frac{63}{64}
 \end{aligned}$$

## 10

From the table we can know that

$$E(X) = \frac{1}{3}$$

$$E(Y) = \frac{3}{4}$$

Therefore,

$$\begin{aligned}
 cov(X, Y) &= E(XY) - E(X) \times E(Y) \\
 &= \sum_x \sum_y xyf(x, y) - \frac{1}{4} \\
 &= \frac{1}{4} - \frac{1}{4} \\
 &= 0
 \end{aligned}$$

But,

$$\begin{aligned}
 f(-1, 0) &= 0 \\
 &\neq g(-1) \times h(0) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}
 \end{aligned}$$

where  $g(x)$  and  $h(y)$  are the marginal probability distribution of  $X$  and  $Y$ , respectively.

## 11

$$\begin{aligned}
 var(X + Y) &= E[((X + Y) - E(X + Y))^2] \\
 &= E[((X - E[X]) + (Y - E[Y]))^2] \\
 &= E[(X - E[X])^2] + E[(Y - E[Y])^2] \\
 &\quad + 2E[(X - E[X])(Y - E[Y])] \\
 &= var(X) + var(Y) + 2cov(X, Y)
 \end{aligned}$$

$$\begin{aligned}
\text{var}(X - Y) &= E[(X - Y) - E[X - Y]]^2 \\
&= E[(X - E[X]) - (Y - E[Y])]^2 \\
&= E[(X - E[X])^2] + E[(Y - E[Y])^2] \\
&\quad - 2E[(X - E[X])(Y - E[Y])] \\
&= \text{var}(X) + \text{var}(Y) - 2\text{cov}(X, Y)
\end{aligned}$$

$$\begin{aligned}
\text{cov}(X + Y, X - Y) &= E[(X + Y)(X - Y)] - E[X + Y] \times E[X - Y] \\
&= E[X^2 - Y^2] - (E[X] - E[Y])(E[X] + E[Y]) \\
&= E[X^2] - E[Y^2] - E[X]^2 + E[Y]^2 \\
&= \text{var}(X) - \text{var}(Y)
\end{aligned}$$

## 12

### 12.1

$$\begin{aligned}
E[U] &= E[2X - 3Y + 4Z] \\
&= 2E[X] - 3E[Y] + 4E[Z] \\
&= -7
\end{aligned}$$

$$\begin{aligned}
E[V] &= E[X + 2Y - Z] \\
&= E[X] + 2E[Y] - E[Z] \\
&= 19
\end{aligned}$$

$$\begin{aligned}
\text{var}(U) &= \text{var}(2X - 3Y + 4Z) \\
&= \text{var}(2X) + \text{var}(-3Y) + \text{var}(4Z) \\
&= 4\text{var}(X) + 9\text{var}(Y) + 16\text{var}(Z) \\
&= 155
\end{aligned}$$

$$\begin{aligned}
\text{var}(V) &= \text{var}(X + 2Y - Z) \\
&= \text{var}(X) + 4\text{var}(Y) + \text{var}(Z) \\
&= 36
\end{aligned}$$

### 12.2

*Claim:*

$$\text{cov}(X + Y, Z) = \text{cov}(X, Z) + \text{cov}(Y, Z)$$



*Proof.*

$$\begin{aligned}
 \text{cov}(X + Y, Z) &= E[(X + Y)Z] - E[X + Y]E[Z] \\
 &= E[XZ + YZ] - (E[X] + E[Y])E[Z] \\
 &= E[XZ] - E[X]E[Z] + E[YZ] - E[Y]E[Z] \\
 &= \text{cov}(X, Z) + \text{cov}(Y, Z)
 \end{aligned}$$

□

*Claim:*

$$\text{cov}(nX, Y) = n \text{cov}(X, Y)$$

where  $n$  is a real number.

*Proof.*

$$\begin{aligned}
 \text{cov}(nX, Y) &= E[nXY] - E[nX]E[Y] \\
 &= nE[XY] - nE[X]E[Y] \\
 &= n(E[XY] - E[X]E[Y]) \\
 &= n \text{cov}(X, Y)
 \end{aligned}$$

□

It is easy to know that

$$\begin{aligned}
 \text{cov}(aX, bY) &= a \text{cov}(X, bY) \\
 &= ab \text{cov}(X, Y)
 \end{aligned}$$

$$\begin{aligned}
 \text{var}(U) &= \text{var}((2X - 3Y) + 4Z) \\
 &= \text{var}(2X - 3Y) + \text{var}(4Z) + 2\text{cov}(2X - 3Y, 4Z) \\
 &= \text{var}(2X - 3Y) + \text{var}(4Z) + 2\text{cov}(2X, 4Z) + 2\text{cov}(-3Y, 4Z) \\
 &= \text{var}(2X) + \text{var}(-3Y) + \text{var}(4Z) + 16\text{cov}(X, Z) - 24\text{cov}(Y, Z) \\
 &= 155
 \end{aligned}$$

$$\begin{aligned}
 \text{var}(V) &= \text{var}(X + 2Y - Z) \\
 &= \text{var}(X) + 4\text{var}(Y) + \text{var}(Z) + 2\text{cov}(X + 2Y, -Z) \\
 &= 36 + 2\text{cov}(X, -Z) + 2\text{cov}(2Y, -Z) \\
 &= 50
 \end{aligned}$$

$$\begin{aligned}
E[U] &= E[2X - 3Y + 4Z] \\
&= 2E[X] - 3E[Y] + 4E[Z] \\
&= -7
\end{aligned}$$

$$\begin{aligned}
E[V] &= E[X + 2Y - Z] \\
&= E[X] + 2E[Y] - E[Z] \\
&= 19
\end{aligned}$$

### 13

We can get the joint probability distribution  $f(z, w)$

	$z = 0$	$z = 1$
$w = 0$	0.36	0
$w = 1$	0.24	0.24
$w = 2$	0	0.16

Thus,

$$\begin{aligned}
E[Z] &= 0.4 \\
E[W] &= 0.8 \\
cov(Z, W) &= E[ZW] - E[Z]E[W] \\
&= \sum_z \sum_w zwf(z, w) - E[Z]E[W] \\
&= -0.4
\end{aligned}$$

### 14

The probability distribution

$$f(x, y, z) = \frac{C_3^x C_2^y C_3^z}{C_8^2}$$

where  $x + y + z = 2$ ,  $x$  is the number of statistics texts,  $y$  is the number of mathematics texts and  $z$  is the number of physics texts.

$$\begin{aligned}
E(Y; X = 0) &= \sum_y yf(0, y, z) \\
&= \frac{2}{7}
\end{aligned}$$