Assignment 5 of MATH 2003

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1.1

Let $(a_n) = (x_{2n})$ **and** $(b_n) = (x_{2n-1})$ **be the subsequences of** (x_n)**.**

We can get the subsequences:

$$\begin{aligned} a_n =& x_{2n} \\ =& 1 - (-1)^{2n} + \frac{1}{2n} \\ =& \frac{1}{2n} \\ b_n =& x_{2n-1} \\ =& 2 + \frac{1}{2n-1} \end{aligned}$$

Thus

$$\lim(a_n) = 0 \neq 2 = \lim(b_n)$$

Therefore, we know the sequence is divergent since Divergence criteria.

1.2

Let$(a_n) = (x_{4n})$ **and** $(b_n) = (x_{8n-1})$ **be the subsequences of** (x_n)**.**

Thus

$$\begin{aligned}a_n =& x_{4n}\\ =& \sin(\frac{4n\pi}{4})\\ =& \sin(n\pi)\\ =& 0\\ b_n =& x_{8n-1}\\ =& \sin(\frac{(8n+1)\pi}{4})\\ =& \frac{\pi}{4}\\ =& \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore,

$$\lim(a_n) = 0 \neq \frac{\sqrt{2}}{2} = \lim(b_n)$$

which means that the sequence is divergent.

2

Because x_n **is unbouned sequence, then there exist a subsequence** (x_{n_k}) **where**

$$|x_{n_k}| > k$$

i.e.

$$0 \leq \frac{1}{|x_{n_k}|} \leq \frac{1}{k}$$

By the sandwich theorem

$$0 = \lim 0 \leq \lim \frac{1}{|x_{n_k}|} \leq \lim \frac{1}{k} = 0$$

i.e.

$$\lim \frac{1}{|x_{n_k}|} = 0$$

3

By the definition of supermum,$\forall \epsilon > 0, \exists x_n$ **that**

$$s - \epsilon < x_n < s$$

There must exist a subsequence (x_{n_k}) **that**

$$s - \frac{1}{k} < x_{n_k} < s$$

Therefore,by the sandwich theorem,

$$s = \lim_{k\to\infty}(s - \frac{1}{k}) < \lim_{k\to\infty} x_{n_k} < \lim_{k\to\infty} s = s$$

Thus,

$$\lim(x_{n_k}) = s$$

4

$$x^3 - 5x + 1 = 0$$

i.e.

$$x = \frac{x^3 + 1}{5}$$

Let (x_n) **be a sequence that:**

$$x_{n+1} = \frac{x_n^3 + 1}{5}$$

where $0 < x_1 < 5$

Because $0 < x_1 < 1$**, then** $x_1^3 + 1 < 2$ **and** $x_2 = \frac{x_1^3+1}{5} < \frac{2}{5} < 1$
And assume that $0 < x_n < 1$**,then** $x_n^3 + 1 < 2$ **and** $x_{n+1} = \frac{x_n^3+1}{5} < \frac{2}{5} < 1$
Thus, $x_n < 1, \forall n$ **and it is easy to know that** $0 < x_n$**.**
It is easy to know that

$$\frac{x_{n+1}^2 + x_{n+1}x_n + x_n^2}{5} < \frac{3}{5} < 1$$

$$\begin{aligned}|x_{n+2}-x_{n+1}| =&|\frac{x_{n+1}^3+1}{5}-\frac{x_n^3+1}{5}|\\ =&|\frac{x_{n+1}^3-x_n^3}{5}|\\ =&|\frac{(x_{n+1}-x_n)(x_{n+1}^2+x_{n+1}x_n+x_n^2)}{5}|\\ <&\frac{3}{5}|x_{n+1}-x_n|\end{aligned}$$

which shows that (x_n) is contractive and the constant of contractive sequence $C=\frac{3}{5}$.
We know that

$$\begin{aligned}|r-x_n| \le&\frac{C}{1-C}|x_n-x_{n-1}|\\ =&\frac{3}{2}|x_n-x_{n-1}|\end{aligned}$$

Let $x_1 = 0.5$

$$\begin{aligned}x_2 =&\frac{9}{40} = 0.2250000\\ |r-x_2| \le&\frac{3}{2}|x_2-x_1| = \frac{33}{80} = 0.4125000\\ x_3 \approx&0.2022781\\ |r-x_3| \le&\frac{3}{2}|x_3-x_2| \approx 0.0340828\\ x_4 \approx&0.2016553\\ |r-x_4| \le&\frac{3}{2}|x_4-x_3| \approx 0.0009342\\ x_5 \approx&0.2016401\\ |r-x_5| \le&\frac{3}{2}|x_5-x_4| \approx 0.0000229\end{aligned}$$

0.2016 **is the approximation of r within** 10^{-4}

5

Proof. □

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(a)

Proof.

$$\begin{aligned}\lim a_n =& \sum_{k=1}^{\infty} \frac{1}{k} \\ \geq& 1 + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4})... \\ =& 1 + 1 + 1 + ... + 1 \\ =& \infty\end{aligned}$$

Thus (a_n) is divergence. □

b

Proof. It is easy to know that

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

by Taylor expansion. When $x = 1$,

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Thus,

$$\lim a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$$

a_n is convergence. □

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$$x_n = \frac{1}{2}((1 + (-1)^n)n + (1 + (-1)^{n+1})\frac{1}{n})$$