Assignment 4 of MATH 2005

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1

$$
\begin{aligned}
E[X] &= \sum_{n=1}^{k} n f(n) \\
&= \frac{1}{k} \sum_{n=1}^{k} n \\
&= \frac{1}{k} \frac{k(k+1)}{2} \\
&= \frac{k+1}{2} \\
var(X) &= \sum_{n=1}^{k} (n - E[X])^2 f(n) \\
&= \frac{1}{k} \sum_{n=1}^{k} (n - \frac{k+1}{2})^2 \\
&= \frac{1}{k} \sum_{n=1}^{k} (n^2 - n(k+1) - \frac{(k+1)^2}{4}) \\
&= \frac{1}{k} (\frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}(k+1) + \frac{k(k+1)^2}{4}) \\
&= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{2} + \frac{(k+1)^2}{4} \\
&= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} \\
&= (k+1)\frac{k-1}{12} \\
&= \frac{k^2-1}{12}
\end{aligned}
$$

2

$$\begin{aligned}b(x;n,\theta) =& C_n^x \theta^x (1-\theta)^{n-x}\\ =& C_n^{n-x}(1-\theta)^{n-x}\theta^x\\ =& b(n-x;n,1-\theta)\end{aligned}$$

2.1

$$\begin{aligned}B(n-x;n,1-\theta) - B(n-x-1;n,1-\theta) =& \sum_{y=0}^{n-x} b(y;n,1-\theta)\\ &- \sum_{y=0}^{n-x-1} b(y;n,1-\theta)\\ =& b(n-x;n,1-\theta)\\ =& b(x;n,\theta)\end{aligned}$$

2.2

$$B(n;n,1-\theta) = \sum_{y=0}^{n} b(y;n,\theta) = 1$$

$$\begin{aligned}B(x;n,\theta) =& \sum_{y=0}^{x} b(y;n,\theta)\\ =& \sum_{y=0}^{x}[B(n-y;n,1-\theta) - B(n-y-1;n,1-\theta)]\\ =& B(n;n,1-\theta) + (B(n-1;n,1-\theta) - B(n-1;n,1-\theta)...) - B(n-x-1;n,1-\theta)\\ =& B(n;n,1-\theta) - B(n-x-1;n,1-\theta)\\ =& 1 - B(n-x-1;n,1-\theta)\end{aligned}$$

3

Proof.

$$\begin{aligned}b(x;n,\theta) =& C_n^x \theta^x (1-\theta)^{n-x}\\ =& \frac{n!}{x!(n-x)!}\theta^x(1-\theta)^{n-x}\end{aligned}$$

$$\begin{aligned}b(x+1;n,\theta) =& C_n^{x+1}\theta^{x+1}(1-\theta)^{n-x-1}\\ =& \frac{n!}{(x+1)!(n-x-1)!}\theta^{x+1}(1-\theta)^{n-x-1}\end{aligned}$$

$$\begin{aligned}\frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{x+1}{n-x}\frac{1-\theta}{\theta}\\ =& \frac{(x+1)(1-\theta)}{\theta(n-x)}\end{aligned}$$

$$b(x+1;n,\theta) = \frac{\theta(n-x)}{(x+1)(1-\theta)}b(x;n,\theta)$$

□

By the definition, when $\theta = \frac{1}{2}$

$$b(x;n,\frac{1}{2}) = C_n^x(\frac{1}{2})^n$$

$$\begin{aligned}\frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{(x+1)(1-\theta)}{\theta(n-x)}\\ =& \frac{x+1}{n-x}\end{aligned}$$

When

$$\frac{x+1}{n-x} > 1$$

we can get

$$x > \frac{n-1}{2}$$

a

n **is an even number and** $x \in \mathbb{N}$**. Thus when** $x \geq \frac{n}{2}$

$$b(x; n, \theta) > b(x+1; n, \theta)$$

Similarily, when $x \leq \frac{n}{2}$

$$b(x; n, \theta) < b(x+1; n, \theta)$$

Therefore, we can get a maximum at $x = \frac{n}{2}$

b

n **is an odd number and** $x \in \mathbb{N}$**. Thus when** $x \geq \frac{n-1}{2}$

$$b(x; n, \theta) \geq b(x+1; n, \theta)$$

When $\frac{x+1}{n-x} = 1$**,i.e.** $x = \frac{n-1}{2}$ **which means**

$$b(\frac{n-1}{2}; n, \theta) = b(\frac{n+1}{2}; n, \theta)$$

We can also get that

$$b(x; n, \theta) \leq b(x+1; n, \theta)$$

when $x < \frac{n-1}{2}$ **Therefore, we can get a maximum at** $x = \frac{n-1}{2}$ **or** $x = \frac{n+1}{2}$