Assignment 4 of MATH 2005

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1

$$
\begin{aligned}
E(X) &= \sum_{x=-1}^{3} xf(x) \\
&= -1 \times \frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 2 \times \frac{0}{7} + 3 \times \frac{1}{7} \\
&= \frac{1}{7}
\end{aligned}
$$

2

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} yh(y)dy \\
&= \int_{2}^{4} \frac{1}{8}(y+1)ydy + 0 \\
&= \frac{79}{3}
\end{aligned}
$$

3

3.1

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{1}^{3} x\frac{1}{x\log 3}dx + 0 \\
&= \frac{2}{\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) =& \int_{-\infty}^{\infty} x^2 f(x)dx \\
=& \int_1^3 \frac{x}{\log 3}dx \\
=& \frac{4}{\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^3) =& \int_{-\infty}^{\infty} x^3 f(x)dx \\
=& \int_1^3 \frac{x^2}{\log 3}dx \\
=& \frac{26}{3\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^3 + 2X^2 - 3X + 1) =& E(X^3) + 2E(X^2) - 3E(X) + E(1) \\
=& \frac{26}{3\log 3} + 2 \times \frac{4}{\log 3} - 3 \times \frac{2}{\log 3} + 1 \\
=& \frac{35}{3\log 3}
\end{aligned}
$$

3.2

$$
\begin{aligned}
\mu'_r =& E[X^r] \\
=& \int_{-\infty}^{\infty} x^r f(x)dx \\
=& \int_1^3 x^r \frac{1}{x\log 3}dx \\
=& \frac{1}{\log 3} \int_1^3 x^{r-1}dx \\
=& \frac{1}{\log 3} \frac{1}{r} x^r|_1^3 \\
=& \frac{1}{r\log 3}(3^r - 1)
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 =& var(X) \\
=& E[X^2] - E[X]^2 \\
=& \frac{4}{\log 3} - \frac{4}{\log 3^2} \\
=& \frac{4\log 3 - 4}{\log 3^2}
\end{aligned}
$$

4

$$
\begin{aligned}
E(\frac{X}{Y}) =& \int_0^1 \int_0^y \frac{x}{y} f(x,y) dxdy \\
=& \int_0^1 \int_0^y \frac{x}{y^2} dxdy \\
=& \int_0^1 \frac{x^2}{2y^2}|_0^y dy \\
=& \int_0^1 \frac{1}{2} dy \\
=& \frac{1}{2} y|_0^1 \\
=& \frac{1}{2}
\end{aligned}
$$

5

Let $\varphi(x)$ is the money he should pay us where x is the number we get from a balanced die.

$$
\begin{aligned}
E(\varphi(X)) =& \sum_{x=1}^{6} \varphi(x) f(x) \\
=& \frac{1}{6} \sum_{x=1}^{4} \varphi(x) + \frac{5}{3} \\
=& 0
\end{aligned}
$$

Thus, we can get,

$$
\sum_{x=1}^{4} \varphi(x) = -10
$$

which means that the total money we should pay that person when we roll a $1, 2, 3,$ or 4 is equal to \$10. Hence, there are many solutions

for this equation. For example,

$$\varphi(1) = \varphi(2) = \varphi(3) = 0,$$
$$\varphi(4) = -10.$$

6

6.1

Let

$$\varphi(x;n) = \begin{cases} x - (n-x) \times 0.4 & (0 \le x \le n) \\ n & (n \le x). \end{cases}$$

be the profit when produce n cake(s) a day.

(a) one of the cakes

$$\begin{aligned} E(\varphi(X;1)) = & \sum_{n=0}^{5} \varphi(x;1) f(x) \\ = & \frac{1}{6}(-0.4 + 1 + 1 + 1 + 1 + 1) \\ = & \frac{23}{30} \end{aligned}$$

(b) two of the cakes

$$\begin{aligned} E(\varphi(X;2)) = & \sum_{n=0}^{5} \varphi(x;2) f(x) \\ = & \frac{1}{6}(-0.8 + 0.6 + 2 + 2 + 2 + 2) \\ = & 1.2 \end{aligned}$$

(c) three of the cakes

$$
\begin{aligned}
E(\varphi(X;3)) =& \sum_{n=0}^{5} \varphi(x;3)f(x) \\
=& \frac{1}{6}(-1.2 + 0.2 + 1.6 + 3 + 3 + 3) \\
=& 1.6
\end{aligned}
$$

(d) four of the cakes

$$
\begin{aligned}
E(\varphi(X;4)) =& \sum_{n=0}^{5} \varphi(x;4)f(x) \\
=& \frac{1}{6}(-1.6 - 0.2 + 1.2 + 2.6 + 4 + 4) \\
=& \frac{5}{3}
\end{aligned}
$$

(e) five of the cakes

$$
\begin{aligned}
E(\varphi(X;5)) =& \sum_{n=0}^{5} \varphi(x;5)f(x) \\
=& \frac{1}{6}(-2 - 1.6 + 0.8 + 2.2 + 3.6 + 5) \\
=& \frac{4}{3}
\end{aligned}
$$

6.2

By 6.1 **we can know that he should bake 3 cakes a day to maximize his expected profit.**

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$$
\begin{aligned}
E(Z) =& E(\frac{X-\mu}{\sigma}) \\
=& \frac{1}{\sigma}(E(X-\mu)) \\
=& \frac{1}{\sigma}(E(X)-E(\mu)) \\
=& \frac{1}{\sigma}(\mu-\mu) \\
=& 0
\end{aligned}
$$

$$
\begin{aligned}
var(Z) =& E\{[Z-E(Z)]^2\} \\
=& E(Z^2)-[E(Z)]^2 \\
=& E(Z^2) \\
=& E[(\frac{X-\mu}{\sigma})^2] \\
=& \frac{1}{\sigma^2}E[(X-\mu)^2] \\
=& \frac{1}{\sigma^2}\times\sigma^2 \\
=& 1
\end{aligned}
$$

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By Chebyshevs inequality, we can know that

$$P(|X-\mu|<k\sigma)\geq 1-\frac{1}{k^2}$$

(a) at least 0.95

$$P(|X-\mu|<k\sigma)\geq 1-\frac{1}{k^2}\geq 0.95$$
$$k\geq 2\sqrt{5}$$

(b) at least 0.99

$$P(|X-\mu|<k\sigma)\geq 1-\frac{1}{k^2}\geq 0.99$$
$$k\geq 10$$

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$$\begin{aligned} P(64 \leq X \leq 184) =& P(|X-\mu| \leq 60) \\ =& P(|X-\mu| \leq 8\sigma) \\ \geq& 1 - \frac{1}{8^2} = \frac{63}{64} \end{aligned}$$

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From the table we can know that

$$E(X) = \frac{1}{3}$$

$$E(Y) = \frac{3}{4}$$

Therefore,

$$\begin{aligned} cov(X,Y) =& E(XY) - E(X) \times E(Y) \\ =& \sum_x \sum_y xyf(x,y) - \frac{1}{4} \\ =& \frac{1}{4} - \frac{1}{4} \\ =& 0 \end{aligned}$$

But,

$$\begin{aligned} f(-1,0) =& 0 \\ \neq& g(-1) \times h(0) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \end{aligned}$$

where $g(x)$ and $h(y)$ are the marginal probability distribution of X and Y, respectively.

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$$\begin{aligned} var(X+Y) =& E[((X+Y) - E(X+Y))^2] \\ =& E[((X - E[X]) + (Y - E[Y]))^2] \\ =& E[(X - E[X])^2] + E[(Y - E[Y])^2] \\ &+ 2E[(X - E[X])(Y - E[Y])] \\ =& var(X) + var(Y) + 2cov(X,Y) \end{aligned}$$

$$\begin{aligned}
var(X-Y) =& E[((X-Y)-E[X-Y])^2] \\
=& E[((X-E[X])-(Y-E[Y]))^2] \\
=& E[(X-E[X])^2]+E[(Y-E[Y])^2] \\
& -2E[(X-E[X])(Y-E[Y])] \\
=& var(X)+var(Y)-2cov(X,Y)
\end{aligned}$$

$$\begin{aligned}
cov(X+Y,X-Y) =& E[(X+Y)(X-Y)]-E[X+Y]\times E[X-Y] \\
=& E[X^2-Y^2]-(E[X]-E[Y])(E[X]+E[Y]) \\
=& E[X^2]-E[Y^2]-E[X]^2+E[Y]^2 \\
=& var(X)-var(Y)
\end{aligned}$$

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12.1

$$\begin{aligned}
E[U] =& E[2X-3Y+4Z] \\
=& 2E[X]-3E[Y]+4E[Z] \\
=& -7
\end{aligned}$$

$$\begin{aligned}
E[V] =& E[X+2Y-Z] \\
=& E[X]+2[Y]-E[Z] \\
=& 19
\end{aligned}$$

$$\begin{aligned}
var(U) =& var(2X-3Y+4Z) \\
=& var(2X)+var(-3Y)+var(4Z) \\
=& 4var(X)+9var(Y)+16var(Z) \\
=& 155
\end{aligned}$$

$$\begin{aligned}
var(V) =& var(X+2Y-Z) \\
=& var(X)+4var(Y)+var(Z) \\
=& 22
\end{aligned}$$

12.2

Claim:

$$cov(X+Y,Z)=cov(X,Z)+cov(Y,Z)$$

Proof.

$$\begin{aligned}cov(X+Y,Z) =& E[(X+Y)Z]-E[X+Y]E[Z]\\ =& E[XZ+YZ]-(E[X]+E[Y])E[Z]\\ =& E[XZ]-E[X]E[Z]+E[YZ]-E[Y]E[Z]\\ =& cov(X,Z)+cov(Y,Z)\end{aligned}$$

□

Claim:

$$cov(nX,Y)=n\ cov(X,Y)$$

where n is a real number.

Proof.

$$\begin{aligned}cov(nX,Y) =& E[nXY]-E[nX]E[Y]\\ =& nE[XY]-nE[X]E[Y]\\ =& n(E[XY]-E[X]E[Y])\\ =& n\ cov(X,Y)\end{aligned}$$

□

It is easy to know that

$$\begin{aligned}cov(aX,bY) =& a\ cov(X,bY)\\ =& ab\ cov(X,Y)\end{aligned}$$

$$\begin{aligned}var(U) =& var((2X-3Y)+4Z)\\ =& var(2X-3Y)+var(4Z)+2cov(2X-3Y,4Z)\\ =& var(2X-3Y)+var(4Z)+2cov(2X,4Z)+2cov(-3Y,4Z)\\ =& var(2X)+var(-3Y)+var(4Z)+16cov(X,Z)-24cov(Y,Z)\\ =& 155\end{aligned}$$

$$\begin{aligned}var(V) =& var(X+2Y-Z)\\ =& var(X)+4var(Y)+var(Z)+2cov(X+2Y,-Z)\\ =& 22+2cov(X,-Z)+2cov(2Y,-Z)\\ =& 36\end{aligned}$$

$$\begin{aligned} E[U] =& E[2X - 3Y + 4Z] \\ =& 2E[X] - 3E[Y] + 4E[Z] \\ =& -7 \end{aligned}$$

$$\begin{aligned} E[V] =& E[X + 2Y - Z] \\ =& E[X] + 2[Y] - E[Z] \\ =& 19 \end{aligned}$$

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We can get the joint probability distribution $f(z, w)$

	$z = 0$	$z = 1$
$w = 0$	0.36	0
$w = 1$	0.24	0.24
$w = 2$	0	0.16

Thus,

$$\begin{aligned} E[Z] &= 0.4 \\ E[W] &= 0.8 \\ cov(Z, W) =& E[ZW] - E[Z]E[W] \\ =& \sum_z \sum_w zwf(z, w) - E[Z]E[W] \\ =& -0.4 \end{aligned}$$

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The probability distribution

$$f(x, y, z) = \frac{C_3^x C_2^y C_3^z}{C_8^2}$$

where $x + y + z = 2$, x is the number of statistics texts, y is the number of mathematics texts and z is the number of physics texts.

$$\begin{aligned} E(Y; X = 0) =& \sum_y yf(0, y, z) \\ =& \frac{2}{7} \end{aligned}$$