

# Assignment 4 of MATH 2005

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November 17, 2020

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$$\begin{aligned} E[X] &= \sum_{n=1}^k n f(n) \\ &= \frac{1}{k} \sum_{n=1}^k n \\ &= \frac{1}{k} \frac{k(k+1)}{2} \\ &= \frac{k+1}{2} \\ \text{var}(X) &= \sum_{n=1}^k (n - E[X])^2 f(n) \\ &= \frac{1}{k} \sum_{n=1}^k \left(n - \frac{k+1}{2}\right)^2 \\ &= \frac{1}{k} \sum_{n=1}^k \left(n^2 - n(k+1) + \frac{(k+1)^2}{4}\right) \\ &= \frac{1}{k} \left(\frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}(k+1) + \frac{k(k+1)^2}{4}\right) \\ &= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{2} + \frac{(k+1)^2}{4} \\ &= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} \\ &= (k+1) \frac{k-1}{12} \\ &= \frac{k^2 - 1}{12} \end{aligned}$$

## 2

$$\begin{aligned}
 b(x; n, \theta) &= C_n^x \theta^x (1 - \theta)^{n-x} \\
 &= C_n^{n-x} (1 - \theta)^{n-x} \theta^x \\
 &= b(n - x; n, 1 - \theta)
 \end{aligned}$$

### 2.1

$$\begin{aligned}
 B(n - x; n, 1 - \theta) - B(n - x - 1; n, 1 - \theta) &= \sum_{y=0}^{n-x} b(y; n, 1 - \theta) \\
 &\quad - \sum_{y=0}^{n-x-1} b(y; n, 1 - \theta) \\
 &= b(n - x; n, 1 - \theta) \\
 &= b(x; n, \theta)
 \end{aligned}$$

### 2.2

$$\begin{aligned}
 B(n; n, 1 - \theta) &= \sum_{y=0}^n b(y; n, \theta) = 1 \\
 B(x; n, \theta) &= \sum_{y=0}^x b(y; n, \theta) \\
 &= \sum_{y=0}^x [B(n - y; n, 1 - \theta) - B(n - y - 1; n, 1 - \theta)] \\
 &= B(n; n, 1 - \theta) + (B(n - 1; n, 1 - \theta) - B(n - 1; n, 1 - \theta) \dots) - B(n - x - 1; n, 1 - \theta) \\
 &= B(n; n, 1 - \theta) - B(n - x - 1; n, 1 - \theta) \\
 &= 1 - B(n - x - 1; n, 1 - \theta)
 \end{aligned}$$

### 3

*Proof.*

$$\begin{aligned} b(x; n, \theta) &= C_n^x \theta^x (1 - \theta)^{n-x} \\ &= \frac{n!}{x!(n-x)!} \theta^x (1 - \theta)^{n-x} \end{aligned}$$

$$\begin{aligned} b(x+1; n, \theta) &= C_n^{x+1} \theta^{x+1} (1 - \theta)^{n-x-1} \\ &= \frac{n!}{(x+1)!(n-x-1)!} \theta^{x+1} (1 - \theta)^{n-x-1} \end{aligned}$$

$$\begin{aligned} \frac{b(x; n, \theta)}{b(x+1; n, \theta)} &= \frac{x+1}{n-x} \frac{1-\theta}{\theta} \\ &= \frac{(x+1)(1-\theta)}{\theta(n-x)} \end{aligned}$$

$$b(x+1; n, \theta) = \frac{\theta(n-x)}{(x+1)(1-\theta)} b(x; n, \theta)$$

□

By the definition, when  $\theta = \frac{1}{2}$

$$\begin{aligned} b(x; n, \frac{1}{2}) &= C_n^x (\frac{1}{2})^n \\ \frac{b(x; n, \theta)}{b(x+1; n, \theta)} &= \frac{(x+1)(1-\theta)}{\theta(n-x)} \\ &= \frac{x+1}{n-x} \end{aligned}$$

When

$$\frac{x+1}{n-x} > 1$$

we can get

$$x > \frac{n-1}{2}$$

**a**

$n$  is an even number and  $x \in \mathbb{N}$ . Thus when  $x \geq \frac{n}{2}$

$$b(x; n, \theta) > b(x+1; n, \theta)$$

Similarly, when  $x \leq \frac{n}{2}$

$$b(x; n, \theta) < b(x+1; n, \theta)$$

Therefore, we can get a maximum at  $x = \frac{n}{2}$

**b**

$n$  is an odd number and  $x \in \mathbb{N}$ . Thus when  $x \geq \frac{n-1}{2}$

$$b(x; n, \theta) \geq b(x+1; n, \theta)$$

When  $\frac{x+1}{n-x} = 1$ , i.e.  $x = \frac{n-1}{2}$  which means

$$b\left(\frac{n-1}{2}; n, \theta\right) = b\left(\frac{n+1}{2}; n, \theta\right)$$

We can also get that

$$b(x; n, \theta) \leq b(x+1; n, \theta)$$

when  $x < \frac{n-1}{2}$  Therefore, we can get a maximum at  $x = \frac{n-1}{2}$  or  $x = \frac{n+1}{2}$

**4**

Let  $A$  is the event that he get exactly four correct answers.

$$\begin{aligned} \mathbb{P}(A) &= C_8^4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^4 \\ &= \frac{1120}{6561} \\ &\approx 17.0706\% \end{aligned}$$

## 5

Let  $A$  is the event that exactly 6 of 15 mice which have been administered the drug will become very aggressive within 1 minute.

$$\begin{aligned}\mathbb{P}(A) &= C_{15}^6 (0.4)^6 (0.6)^9 \\ &= 5005 \times \frac{64}{15625} \times \frac{19683}{1953125} \\ &\approx 20.6598\%\end{aligned}$$

## 6

Let  $A$  is the event that at least 3 of 5 automobile accidents are due to driver fatigue.

$$\begin{aligned}\mathbb{P}(A) &= C_5^3 (0.1)^3 (0.9)^2 + C_5^4 (0.1)^4 (0.9)^1 + C_5^5 (0.1)^5 (0.9)^0 \\ &= \frac{107}{12500} \\ &= 0.856\%\end{aligned}$$

## 7

By the definition,

$$f(x; n, \theta) = C_n^x \theta^x (1 - \theta)^{1-x}$$

where  $\mathbb{E}[X] = n\theta$  and  $\text{var}(X) = n\theta(1 - \theta)$ .

And  $Y = \frac{X}{n}$ , thus

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}\left[\frac{X}{n}\right] \\ &= \theta \\ \text{var}(Y) &= \text{var}\left(\frac{X}{n}\right) \\ &= \frac{1}{n^2} n\theta(1 - \theta) \\ &= \frac{\theta(1 - \theta)}{n}\end{aligned}$$

8

Let  $Y = \frac{X}{n}$  be the proportion of successes in  $n$  experiments.

$$\mathbb{E}[Y] = \theta, \sigma = \text{var}(Y) = \frac{\theta(1-\theta)}{n}$$

a

We get the equations

$$\begin{cases} \theta - c = 0.4 \\ \theta + c = 0.6 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.1 \\ k = \frac{nc}{\theta(1-\theta)} = 360 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{129599}{129600} \approx 0.9999 > \frac{35}{36}$$

b

We get the equations

$$\begin{cases} \theta - c = 0.47 \\ \theta + c = 0.53 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.03 \\ k = \frac{nc}{\theta(1-\theta)} = 1200 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{1439999}{1440000} \approx 0.9999 > \frac{35}{36}$$

c

We get the equations

$$\begin{cases} \theta - c = 0.497 \\ \theta + c = 0.503 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.003 \\ k = \frac{nc}{\theta(1-\theta)} = 1.2 \times 10^4 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{143999999}{1.44 \times 10^8} \approx 0.9999 > \frac{35}{36}$$

## 9

By the definition

$$b^*(y; k, \theta) = C_{y-1}^{k-1} \theta^k (1 - \theta)^{y-k}$$

Consider the function

$$f_m(z) = \sum_{k=0}^{\infty} C_{k+m}^m z^k$$

We know that

$$C_{k+m}^m = C_{k+m-1}^{m-1} + C_{k+m-1}^m$$

Thus

$$\begin{aligned} f_m(z) &= \sum_{k=0}^{\infty} C_{k+m-1}^{m-1} z^k + \sum_{k=1}^{\infty} C_{k+m-1}^m z^k \\ &= f_{m-1}(z) + z \sum_{k=1}^{\infty} C_{k+m-1}^m z^{k-1} \\ &= f_{m-1}(z) + z f_m(z) \end{aligned}$$

Thus,

$$f_m(z) = \frac{f_{m-1}(z)}{1 - z}$$

and

$$f_0(z) = \sum_{k=0}^{\infty} C_k^0 z^k = \frac{1}{1 - z}$$

Then

$$\begin{aligned} f_m(z) &= \left( \frac{1}{1 - z} \right)^m \\ \sum_{y=k}^{\infty} b^*(y; k, \theta) &= \sum_{y=k}^{\infty} C_{y-1}^{k-1} \theta^k (1 - \theta)^{y-k} \\ &= \theta^k \theta^{-k} \\ &= 1 \end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{y=k}^{\infty} y b^*(y; k, \theta) \\
&= \sum_{y=k}^{\infty} y C_{y-1}^{k-1} \theta^k (1-\theta)^{y-k} \\
&= \sum_{y=k}^{\infty} y \frac{(y-1)!}{(k-1)!(y-k)!} \theta^k (1-\theta)^{y-k} \\
&= \sum_{y=k}^{\infty} \frac{y!}{(k-1)!(y-k)!} \theta^k (1-\theta)^{y-k} \\
&= \sum_{y=k}^{\infty} k \frac{y!}{k!(y-k)!} \theta^k (1-\theta)^{y-k} \\
&= \sum_{y=k}^{\infty} k C_y^k \theta^k (1-\theta)^{y-k} \\
&= k \sum_{y=k}^{\infty} C_y^k \theta^k (1-\theta)^{y-k} \\
&= \frac{k}{\theta} \sum_{y=k}^{\infty} C_y^k \theta^{k+1} (1-\theta)^{y-k} \\
&= \frac{k}{\theta} \sum_{y=k+1}^{\infty} C_{y+1-1}^{k-1} \theta^{k+1} (1-\theta)^{y-k-1} \\
&= \frac{k}{\theta} \sum_{y=k+1}^{\infty} C_{(y+1)-1}^{k-1} \theta^{k+1} (1-\theta)^{y-(k+1)} \\
&= \frac{k}{\theta} \sum_{y=k+1}^{\infty} b^*(y+1; k, \theta) \\
&= \frac{k}{\theta}
\end{aligned}$$



## 10

*Proof.* By the definition,

$$\begin{aligned}
 h(x; n, N, k) &= \frac{C_k^x C_{N-k}^{n-x}}{C_N^n} \\
 &= \frac{\frac{k!}{x!(k-x)!} \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{\frac{N!}{n!(N-n)!}} \\
 &= \frac{k!(N-k)!n!(N-n)!}{x!(k-x)!(n-x)!(N+x-k-n)!N!}
 \end{aligned}$$

Thus

$$\begin{aligned}
 h(x+1; n, N, k) &= \frac{k!(N-k)!n!(N-n)!}{(x+1)!(k-x-1)!(n-x-1)!(N+x-k-n+1)!N!} \\
 \frac{h(x+1; n, N, k)}{h(x; n, N, k)} &= \frac{x!(k-x)!(n-x)!(N+x-k-n)!}{(x+1)!(k-x-1)!(n-x-1)!(N+x-k-n+1)!} \\
 &= \frac{(k-x)(n-x)}{(x+1)(N+x-k-n+1)}
 \end{aligned}$$

□

## 11

*Proof.* By the definition

$$p(x; \lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Thus

$$\begin{aligned}
 p(x+1; \lambda) &= \frac{\lambda^{x+1}}{(x+1)!} e^{-\lambda} \\
 \frac{p(x+1; \lambda)}{p(x; \lambda)} &= \frac{\frac{\lambda^{x+1}}{(x+1)!} e^{-\lambda}}{\frac{\lambda^x}{x!} e^{-\lambda}} \\
 &= \frac{\lambda}{x+1}
 \end{aligned}$$

□

## 12

### 12.1

Let  $A$  be the event that a family's fourth child is their first son.

$$\begin{aligned}\mathbb{P}(A) &= C_3^0 0.5^3 \times 0.5 \\ &= \frac{1}{16}\end{aligned}$$

### 12.2

Let  $B$  be the event that a family's seventh child is their second daughter.

$$\begin{aligned}\mathbb{P}(B) &= C_6^1 0.5^5 \times 0.5^2 \\ &= \frac{3}{64}\end{aligned}$$

### 12.3

Let  $C$  be the event that a family's tenth child is their fourth or fifth son.

$$\begin{aligned}\mathbb{P}(C) &= C_9^3 0.5^6 \times 0.5^4 + C_9^4 0.5^5 \times 0.5^5 \\ &= \frac{105}{512}\end{aligned}$$

## 13

### 13.1

Let  $A$  be the event that the eighth person to hear the rumor will be the fifth to believe it.

$$\begin{aligned}\mathbb{P}(A) &= C_7^4 0.75^5 \times 0.25^3 \\ &= \frac{8505}{65536} \\ &\approx 0.1298\end{aligned}$$

**13.2**

Let  $B$  be the event that 2) the fifteenth person to hear the rumor will be the tenth to believe it.

$$\begin{aligned}\mathbb{P}(B) &= C_{14}^9 0.75^{10} \times 0.25^5 \\ &\approx 0.1101\end{aligned}$$

**14**

Let  $A$  be the event that at the switch will not fail during the first 800 times it is turned on or off.

$$\begin{aligned}\mathbb{P}(A) &= C_{800}^8 0.001^8 \times (1 - 0.001)^{792} \\ &\approx 0.4491\end{aligned}$$

**15****15.1**

$$\begin{aligned}\mathbb{P} &= \frac{C_{14}^2}{C_{18}^2} \\ &= \frac{91}{153} \\ &\approx 0.5948\end{aligned}$$

**15.2**

$$\begin{aligned}\mathbb{P} &= \frac{C_{10}^2}{C_{18}^2} \\ &= \frac{5}{17} \\ &\approx 0.2941\end{aligned}$$

**15.3**

$$\begin{aligned}
 \mathbb{P} &= \frac{C_6^2}{C_{18}^2} \\
 &= \frac{5}{51} \\
 &\approx 0.0980
 \end{aligned}$$

**16****16.1**

$$\begin{aligned}
 \mathbb{P} &= \frac{C_4^1 C_{76}^2}{C_{80}^3} \\
 &= \frac{285}{2054} \\
 &\approx 0.1388
 \end{aligned}$$

**16.2**

$$\begin{aligned}
 \theta &= \frac{4}{80} = \frac{1}{20} \\
 \mathbb{P} &= C_3^1 \theta^1 (1 - \theta)^2 \\
 &= \frac{1083}{8000} \\
 &\approx 0.1354
 \end{aligned}$$

**17**

By the definition,

$$\mathbb{P}(X) = \frac{\lambda^x}{x!} e^{-\lambda}$$

where  $\lambda = np = 1.2$

$$\begin{aligned}\mathbb{P}(X \leq 2) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) \\ &= e^{-1.2} + 1.2e^{-1.2} + \frac{1.2^2}{2}e^{-1.2} \\ &= \frac{73}{25}e^{-1.2} \\ &\approx 0.8795\end{aligned}$$

18

By the definition,

$$\mathbb{P}(X) = \frac{\lambda^x}{x!}e^{-\lambda}$$

where  $\lambda = 0.25$

$$\begin{aligned}\mathbb{P}(X \leq 1) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) \\ &= e^{-0.25} + 0.25e^{-0.25} \\ &\approx 0.9735\end{aligned}$$