Assignment 8 of MATH 2003

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By the definition of continuous,$\forall \epsilon > 0, \exists \delta > 0$ **s.t. if** $|x - x_0| < \delta$ **then** $|f(x) - f(x_0)| < \epsilon$ **and** $|g(x) - g(x_0)| < \epsilon$ **i.e.,**

$$\begin{aligned}
-\epsilon < f(x) - f(x_0) <& \epsilon \\
-\epsilon < g(x) - g(x_0) <& \epsilon \\
f(x) - f(x_0) - (g(x) - g(x_0)) <& \epsilon - \epsilon \\
f(x) - g(x) - (f(x_0) - g(x_0)) <& 0 \\
f(x) - g(x) <& f(x_0) - g(x_0) < 0 \\
f(x) - g(x) <& 0
\end{aligned}$$

when $|x - x_0| < \delta$ **which means that** $x \in V_\delta(x_0)$

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Yes.

Proof. Assume that f is not constant. Then is must exist x, $y \in [0, 1]$ such that $f(x)$ and $f(y)$ are rational values and $f(x) \neq f(y)$. By *Bolzano's Intermediate Value Theorem* we know that $\forall k \in (\mathbb{R} \backslash \mathbb{Q})$ satisfies $\inf\{f(x), f(y)\} < k < \sup\{f(x), f(y)\}$,there exists a point $c \in (\inf\{x, y\}, \sup\{x, y\})$ such that $f(c) = k$ whcih contradicts with that $f(x)$ is rational value. Therefore, $f(x)$ is constant. □

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Proof. We know that $\forall x_i \in I, \exists M_i \in \mathbb{R}$, such that $|f(x)| \leq M_i$ where $x \in V_\delta(i)$. Therefore,$|f(x)| \leq \sup\{M_i, M_j\}$ where $x \in V_{\delta_i}(i) \cup V_{\delta_j}(j)$ It is easy to get that

$$I \subset \cup_{x \in I} V_\delta(x)$$

When $x \in I$, then $x \in \cup_{x \in I} V_\delta(x)$. Thus

$$|f(x)| \leq \sup\{M_i : i \in I, x \in V_\delta(i), |f(x)| \leq M_i\}$$

which means $f(x)$ is bounded in I. $\square$

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