

Assignment 4 of MATH 2005

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1

$$\begin{aligned} E[X] &= \sum_{n=1}^k n f(n) \\ &= \frac{1}{k} \sum_{n=1}^k n \\ &= \frac{1}{k} \frac{k(k+1)}{2} \\ &= \frac{k+1}{2} \\ \text{var}(X) &= \sum_{n=1}^k (n - E[X])^2 f(n) \\ &= \frac{1}{k} \sum_{n=1}^k \left(n - \frac{k+1}{2}\right)^2 \\ &= \frac{1}{k} \sum_{n=1}^k \left(n^2 - n(k+1) + \frac{(k+1)^2}{4}\right) \\ &= \frac{1}{k} \left(\frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}(k+1) + \frac{k(k+1)^2}{4}\right) \\ &= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{2} + \frac{(k+1)^2}{4} \\ &= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} \\ &= (k+1) \frac{k-1}{12} \\ &= \frac{k^2-1}{12} \end{aligned}$$

2

$$\begin{aligned}
 b(x; n, \theta) &= C_n^x \theta^x (1 - \theta)^{n-x} \\
 &= C_n^{n-x} (1 - \theta)^{n-x} \theta^x \\
 &= b(n - x; n, 1 - \theta)
 \end{aligned}$$

2.1

$$\begin{aligned}
 B(n - x; n, 1 - \theta) - B(n - x - 1; n, 1 - \theta) &= \sum_{y=0}^{n-x} b(y; n, 1 - \theta) \\
 &\quad - \sum_{y=0}^{n-x-1} b(y; n, 1 - \theta) \\
 &= b(n - x; n, 1 - \theta) \\
 &= b(x; n, \theta)
 \end{aligned}$$

2.2

$$\begin{aligned}
 B(n; n, 1 - \theta) &= \sum_{y=0}^n b(y; n, \theta) = 1 \\
 B(x; n, \theta) &= \sum_{y=0}^x b(y; n, \theta) \\
 &= \sum_{y=0}^x [B(n - y; n, 1 - \theta) - B(n - y - 1; n, 1 - \theta)] \\
 &= B(n; n, 1 - \theta) + (B(n - 1; n, 1 - \theta) - B(n - 1; n, 1 - \theta) \dots) - B(n - x - 1; n, 1 - \theta) \\
 &= B(n; n, 1 - \theta) - B(n - x - 1; n, 1 - \theta) \\
 &= 1 - B(n - x - 1; n, 1 - \theta)
 \end{aligned}$$

3

Proof.

$$\begin{aligned} b(x; n, \theta) &= C_n^x \theta^x (1 - \theta)^{n-x} \\ &= \frac{n!}{x!(n-x)!} \theta^x (1 - \theta)^{n-x} \end{aligned}$$

$$\begin{aligned} b(x+1; n, \theta) &= C_n^{x+1} \theta^{x+1} (1 - \theta)^{n-x-1} \\ &= \frac{n!}{(x+1)!(n-x-1)!} \theta^{x+1} (1 - \theta)^{n-x-1} \end{aligned}$$

$$\begin{aligned} \frac{b(x; n, \theta)}{b(x+1; n, \theta)} &= \frac{x+1}{n-x} \frac{1-\theta}{\theta} \\ &= \frac{(x+1)(1-\theta)}{\theta(n-x)} \end{aligned}$$

$$b(x+1; n, \theta) = \frac{\theta(n-x)}{(x+1)(1-\theta)} b(x; n, \theta)$$

□

By the definition, when $\theta = \frac{1}{2}$

$$\begin{aligned} b(x; n, \frac{1}{2}) &= C_n^x (\frac{1}{2})^n \\ \frac{b(x; n, \theta)}{b(x+1; n, \theta)} &= \frac{(x+1)(1-\theta)}{\theta(n-x)} \\ &= \frac{x+1}{n-x} \end{aligned}$$

When

$$\frac{x+1}{n-x} > 1$$

we can get

$$x > \frac{n-1}{2}$$

a

n is an even number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n}{2}$

$$b(x; n, \theta) > b(x+1; n, \theta)$$

Similarly, when $x \leq \frac{n}{2}$

$$b(x; n, \theta) < b(x+1; n, \theta)$$

Therefore, we can get a maximum at $x = \frac{n}{2}$

b

n is an odd number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n-1}{2}$

$$b(x; n, \theta) \geq b(x+1; n, \theta)$$

When $\frac{x+1}{n-x} = 1$, i.e. $x = \frac{n-1}{2}$ which means

$$b\left(\frac{n-1}{2}; n, \theta\right) = b\left(\frac{n+1}{2}; n, \theta\right)$$

We can also get that

$$b(x; n, \theta) \leq b(x+1; n, \theta)$$

when $x < \frac{n-1}{2}$ Therefore, we can get a maximum at $x = \frac{n-1}{2}$ or $x = \frac{n+1}{2}$

4

Let A is the event that he get exactly four correct answers.

$$\begin{aligned} \mathbb{P}(A) &= C_8^4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^4 \\ &= \frac{1120}{6561} \\ &\approx 17.0706\% \end{aligned}$$

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Let A is the event that exactly 6 of 15 mice which have been administered the drug will become very aggressive within 1 minute.

$$\begin{aligned}\mathbb{P}(A) &= C_{15}^6 (0.4)^6 (0.6)^9 \\ &= 5005 \times \frac{64}{15625} \times \frac{19683}{1953125} \\ &\approx 20.6598\%\end{aligned}$$

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Let A is the event that at least 3 of 5 automobile accidents are due to driver fatigue.

$$\begin{aligned}\mathbb{P}(A) &= C_5^3 (0.1)^3 (0.9)^2 + C_5^4 (0.1)^4 (0.9)^1 + C_5^5 (0.1)^5 (0.9)^0 \\ &= \frac{107}{12500} \\ &= 0.856\%\end{aligned}$$

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By the definition,

$$f(x; n, \theta) = C_n^x \theta^x (1 - \theta)^{1-x}$$

where $\mathbb{E}[X] = n\theta$ and $\text{var}(X) = n\theta(1 - \theta)$.

And $Y = \frac{X}{n}$, thus

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}\left[\frac{X}{n}\right] \\ &= \theta \\ \text{var}(Y) &= \text{var}\left(\frac{X}{n}\right) \\ &= \frac{1}{n^2} n\theta(1 - \theta) \\ &= \frac{\theta(1 - \theta)}{n}\end{aligned}$$

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