

Notes of MATH 2005

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1 Discrete Uniform Distributions

Definition

A discrete random variable X is said to have a *discrete uniform distribution*, and it is called a *discrete uniform variable*, if it can take on k different values: x_1, x_2, \dots, x_k , and its probability distribution $f(x)$ is given by

$$f(x_i) = \frac{1}{k}$$

where $i = 1, 2, \dots, k$.

Mean and Variance

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^k x_i f(x_i) \\ &= \frac{1}{k} \sum_{i=1}^k x_i \\ \mathbb{E}[X^2] &= \sum_{i=1}^k x_i^2 f(x_i) \\ &= \frac{1}{k} \sum_{i=1}^k x_i^2 \\ \text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{k} \sum_{i=1}^k x_i^2 - \left(\frac{1}{k} \sum_{i=1}^k x_i\right)^2\end{aligned}$$

2 Bernoulli Distributions

Definition

Support that an experiment has two possible outcomes: success and failure, and their probability are respectively, θ and $1 - \theta$. Then, this experiment is called a *Bernoulli Distributions*. Let X be the number of successes of a Bernoulli experiment, i.e. $X = 1$ or $X = 0$. Then, X is called a random variable having the Bernoulli probability distribution, which is given by

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x}$$

where $x = 0, 1$ and $0 < \theta < 1$ is a parameter.

Mean and Variance

$$\begin{aligned}\mathbb{E}[X] &= \theta \\ \mathbb{E}[X^2] &= \theta \\ \text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \theta - \theta^2 \\ &= \theta(1 - \theta)\end{aligned}$$

3 Binomial Distributions

Definition

Let n be a natural number, and let $0 < \theta < 1$. Then, a discrete random variable X is said to have a *binomial distribution*, and X is called a binomial random variable, if its probability distribution $b(x; n, \theta)$ is given by

$$b(x; n, \theta) = C_n^x \theta^x (1 - \theta)^{n-x}$$

where $x = 1, 2, \dots, n$ and θ are two parameters, and

$$C_n^x = \frac{n!}{x!(n-x)!}$$

is the total number of combinations of n distinct numbers taken x numbers at a time.

Remark

We consider n independent Bernoulli experiments, in which the parameter θ (the probability of a success) is the same for each experiment. Let X be the total number of successes in this sequence of n independent Bernoulli experiments. Then, we can see that X is a random variable having a binomial distribution with parameters n and θ , i.e., we have the following result.

Let X_1, X_2, \dots, X_n be n independent Bernoulli random variables with the same parameter θ . Then, the random variable $X = X_1 + X_2 + \dots + X_n$ has a binomial distribution with parameters n and θ .

Mean and Variance

$$\begin{aligned}\mathbb{E}[X] &= n\theta \\ \text{var}(X) &= \text{var}(x_1 + X_2 + \dots + X_n) \\ &= \text{var}(X_1 + X_2 + \dots + X_{n-1}) + \text{var}(X_n) - 2\text{cov}(X_1 + X_2 + \dots + X_{n-1}, X_n) \\ &\quad \dots \\ &= \text{var}(X_1) + \text{var}(X_2) + \dots + \text{var}(X_n) \\ &= n\theta(1 - \theta)\end{aligned}$$

Theorem

$$\begin{aligned}b(x; n, \theta) &= C_n^x \theta^x (1 - \theta)^{n-x} \\ &= C_n^{n-x} (1 - \theta)^{1-\theta} \theta^x \\ &= b(n - x; n, 1 - \theta)\end{aligned}$$

Since a binomial random variable X with parameters n and θ is the total number of successes in n independent Bernoulli experiments. $Y = \frac{X}{n}$ is the proportion of successes in n independent Bernoulli experiments.

$$\begin{aligned}\mathbb{E}[Y] &= \theta \\ \text{var}(Y) &= \frac{\theta(1 - \theta)}{n}\end{aligned}$$