Assignment 4 of MATH 2005

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1

$$
\begin{aligned}
E[X] &= \sum_{n=1}^{k} n f(n) \\
&= \frac{1}{k} \sum_{n=1}^{k} n \\
&= \frac{1}{k} \, \frac{k(k+1)}{2} \\
&= \frac{k+1}{2} \\
var(X) &= \sum_{n=1}^{k} (n - E[X])^2 f(n) \\
&= \frac{1}{k} \sum_{n=1}^{k} (n - \frac{k+1}{2})^2 \\
&= \frac{1}{k} \sum_{n=1}^{k} (n^2 - n(k+1) - \frac{(k+1)^2}{4}) \\
&= \frac{1}{k} (\frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}(k+1) + \frac{k(k+1)^2}{4}) \\
&= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{2} + \frac{(k+1)^2}{4} \\
&= \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} \\
&= (k+1)\frac{k-1}{12} \\
&= \frac{k^2-1}{12}
\end{aligned}
$$

2

$$\begin{aligned} b(x; n, \theta) =& C_n^x \theta^x (1-\theta)^{n-x} \\ =& C_n^{n-x} (1-\theta)^{n-x} \theta^x \\ =& b(n-x; n, 1-\theta) \end{aligned}$$

2.1

$$\begin{aligned} B(n-x; n, 1-\theta) - B(n-x-1; n, 1-\theta) =& \sum_{y=0}^{n-x} b(y; n, 1-\theta) \\ &- \sum_{y=0}^{n-x-1} b(y; n, 1-\theta) \\ =& b(n-x; n, 1-\theta) \\ =& b(x; n, \theta) \end{aligned}$$

2.2

$$B(n; n, 1-\theta) = \sum_{y=0}^{n} b(y; n, \theta) = 1$$

$$\begin{aligned} B(x; n, \theta) =& \sum_{y=0}^{x} b(y; n, \theta) \\ =& \sum_{y=0}^{x} [B(n-y; n, 1-\theta) - B(n-y-1; n, 1-\theta)] \\ =& B(n; n, 1-\theta) + (B(n-1; n, 1-\theta) - B(n-1; n, 1-\theta)...) - B(n-x-1; n, 1-\theta) \\ =& B(n; n, 1-\theta) - B(n-x-1; n, 1-\theta) \\ =& 1 - B(n-x-1; n, 1-\theta) \end{aligned}$$

3

Proof.

$$\begin{aligned} b(x;n,\theta) =& C_n^x \theta^x (1-\theta)^{n-x} \\ =& \frac{n!}{x!(n-x)!}\theta^x(1-\theta)^{n-x} \end{aligned}$$

$$\begin{aligned} b(x+1;n,\theta) =& C_n^{x+1} \theta^{x+1} (1-\theta)^{n-x-1} \\ =& \frac{n!}{(x+1)!(n-x-1)!}\theta^{x+1}(1-\theta)^{n-x-1} \end{aligned}$$

$$\begin{aligned} \frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{x+1}{n-x}\frac{1-\theta}{\theta} \\ =& \frac{(x+1)(1-\theta)}{\theta(n-x)} \end{aligned}$$

$$b(x+1;n,\theta) = \frac{\theta(n-x)}{(x+1)(1-\theta)}b(x;n,\theta)$$

□

By the definition, when $\theta = \frac{1}{2}$

$$b(x;n,\frac{1}{2}) = C_n^x(\frac{1}{2})^n$$

$$\begin{aligned} \frac{b(x;n,\theta)}{b(x+1;n,\theta)} =& \frac{(x+1)(1-\theta)}{\theta(n-x)} \\ =& \frac{x+1}{n-x} \end{aligned}$$

When

$$\frac{x+1}{n-x} > 1$$

we can get

$$x > \frac{n-1}{2}$$

a

n is an even number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n}{2}$

$$b(x; n, \theta) > b(x + 1; n, \theta)$$

Similarily, when $x \leq \frac{n}{2}$

$$b(x; n, \theta) < b(x + 1; n, \theta)$$

Therefore, we can get a maximum at $x = \frac{n}{2}$

b

n is an odd number and $x \in \mathbb{N}$. Thus when $x \geq \frac{n-1}{2}$

$$b(x; n, \theta) \geq b(x + 1; n, \theta)$$

When $\frac{x+1}{n-x} = 1$,i.e. $x = \frac{n-1}{2}$ which means

$$b(\frac{n-1}{2}; n, \theta) = b(\frac{n+1}{2}; n, \theta)$$

We can also get that

$$b(x; n, \theta) \leq b(x + 1; n, \theta)$$

when $x < \frac{n-1}{2}$ Therefore, we can get a maximum at $x = \frac{n-1}{2}$ or $x = \frac{n+1}{2}$

4

Let A is the event that he get exactly four correct answers.

$$\begin{aligned}\mathbb{P}(A) =& C_8^4(\frac{1}{3})^4(\frac{2}{3})^4 \\ =& \frac{1120}{6561} \\ \approx& 17.0706\%\end{aligned}$$

5

Let A is the event that exactly 6 of 15 mice which have been administered the drug will become very aggressive within 1 minute.

$$\begin{aligned}\mathbb{P}(A) =& C_{15}^{6}(0.4)^6(0.6)^9 \\ =& 5005 \times \frac{64}{15625} \times \frac{19683}{1953125} \\ \approx& 20.6598\%\end{aligned}$$

6

Let A is the event that at least 3 of 5 automobile accidents are due to driver fatigue.

$$\begin{aligned}\mathbb{P}(A) =& C_5^3(0.1)^3(0.9)^2 + C_5^4(0.1)^4(0.9)^1 + C_5^5(0.1)^5(0.9)^0 \\ =& \frac{107}{12500} \\ =& 0.856\%\end{aligned}$$

7

By the definition,

$$f(x; n, \theta) = C_n^x \theta^x (1-\theta)^{1-x}$$

where $\mathbb{E}[X] = n\theta$ and $var(X) = n\theta(1-\theta)$.
And $Y = \frac{X}{n}$, thus

$$\begin{aligned}\mathbb{E}[Y] =& \mathbb{E}[\frac{X}{n}] \\ =& \theta \\ var(Y) =& var(\frac{X}{n}) \\ =& \frac{1}{n^2} n\theta(1-\theta) \\ =& \frac{\theta(1-\theta)}{n}\end{aligned}$$

8

Let $Y = \frac{X}{n}$ be the proportion of successes in n experiments.

$$\mathbb{E}[Y] = \theta, \sigma = var(Y) = \frac{\theta(1-\theta)}{n}$$

a

We get the equations

$$\begin{cases} \theta - c = 0.4 \\ \theta + c = 0.6 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.1 \\ k = \frac{nc}{\theta(1-\theta)} = 360 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{129599}{129600} \approx 0.9999 > \frac{35}{36}$$

b

We get the equations

$$\begin{cases} \theta - c = 0.47 \\ \theta + c = 0.53 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.03 \\ k = \frac{nc}{\theta(1-\theta)} = 1200 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{1439999}{1440000} \approx 0.9999 > \frac{35}{36}$$

c

We get the equations

$$\begin{cases} \theta - c = 0.497 \\ \theta + c = 0.503 \\ k\sigma = c \end{cases} \Rightarrow \begin{cases} \theta = 0.5 \\ c = 0.003 \\ k = \frac{nc}{\theta(1-\theta)} = 1.2 \times 10^4 \end{cases}$$

$$\mathbb{P} \geq 1 - \frac{1}{k^2} = \frac{143999999}{1.44 \times 10^8} \approx 0.9999 > \frac{35}{36}$$

9

By the definition

$$\begin{aligned} b^*(y;k,\theta) =& C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k} \\ =& \frac{k}{y} \times b(k;y,\theta) \end{aligned}$$

$$\begin{aligned} \mathbb{E}[Y] &= \sum_{y=k}^{\infty} y b^*(y;k,\theta) \\ &= \sum_{y=k}^{\infty} y C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k} \\ &= \sum_{y=k}^{\infty} y \frac{(y-1)!}{(k-1)!(y-k)!}\theta^k(1-\theta)^{y-k} \\ &= \sum_{y=k}^{\infty} \frac{y!}{(k-1)!(y-k)!}\theta^k(1-\theta)^{y-k} \end{aligned}$$