Notes of MATH 2005

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1 Discrete Uniform Distributions

Definition

A discrete random variable X is said to have a *discrete uniform distribution*, and it is called a *discrete uniform variable*, if it can take on k different values: $x_1, x_2, ..., x_k$, and its probability distribution $f(x)$ is given by

$$f(x_i) = \frac{1}{k}$$

where $i = 1, 2, ..., k$.

Mean and Variance

$$\begin{aligned}
\mathbb{E}[X] =& \sum_{i=1}^{k} x_i f(x_i) \\
=& \frac{1}{k} \sum x_i \\
\mathbb{E}[X^2] =& \sum_{i=1}^{k} {x_i}^2 f(x_i) \\
=& \frac{1}{k} \sum x_i^2 \\
var(X) =& \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
=& \frac{1}{k} \sum x_i^2 - (\frac{1}{k} \sum x_i)^2
\end{aligned}$$

2 Bernoulli Distributions

Definition

Support that and experiment has two possible outcomes: success and failure, and their probability are respectively, θ and $1 - \theta$. Then, this experiment is called a *Bernoulli Distributions*. Let X be the number of successes of a Bernoulli experiment, i.e. $X = 1$ or $X = 0$. Then, X is called a random variable having the Bernoulli probability distribution, which is given by

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x}$$

where $x = 0, 1$ and $0 < \theta < 1$ is a parameter.

Mean and Variance

$$\begin{aligned}
\mathbb{E}[X] =&\theta \\
\mathbb{E}[X^2] =&\theta \\
var(X) =&\mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
=&\theta - \theta^2 \\
=&\theta(1-\theta)
\end{aligned}$$

3 Binomial Distributions

Definition

Let n be a nutural number, and let $0 < \theta < 1$. Then, a discrete random variable X is said to have a *binomial distribution*, and X is called a binomial random variable, if its probability distribution $b(x; n, \theta)$ is given by

$$b(x; n, \theta) = C_n^x \theta^x (1-\theta)^{n-x}$$

where $x = 1, 2, ..., n$,n and θ are two parameters, and

$$C_n^x = \frac{n!}{x!(n-x)!}$$

is the total number of combinations of n distinct numbers taken x numbers at a time.

Remark

We consider n independent Bernoullis experiments, in which the parameter θ (the probability of a success) is the same for each experiment. Let X be the total number of successes in this sequence of n independent Bernoullis experiments. Then, we can see that X is a random variable having a binomial distribution with parameters n and θ, i.e., we have the following result.
Let $X_1, X_2, ..., X_n$ be n independent Bernoulli random variables with the same parameter θ. Then, the random variable $X = X_1 + X_2 + ... + X_n$ has a binomial distribution with parameters n and θ.

Mean and Variance

$$\begin{aligned}
\mathbb{E}[X] =&n\theta \\
var(X) =&var(x_1 + X_2 + ... + X_n) \\
=&var(X_1 + X_2 + ... + X_{n-1}) + var(X_n) - 2cov(X_1 + X_2 + ... + X_{n-1}, X_n) \\
&... \\
=&var(X_1) + var(X_2) + ... + var(X_n) \\
=&n\theta(1-\theta)
\end{aligned}$$

Theorem

$$
\begin{aligned}
b(x;n,\theta) =& C_n^x \theta^x (1-\theta)^{n-x}\\
=& C_n^{n-x}(1-\theta)^{1-\theta}\theta^x\\
=& b(n-x;n,1-\theta)
\end{aligned}
$$

Since a binomial random variable X with parameters n and θ is the total number of successes in n independent Bernoullis experiments. $Y = \frac{X}{n}$ is the proportion of successes in n independent Bernoullis experiments.

$$\mathbb{E}[Y] = \theta$$

$$var(Y) = \frac{\theta(1-\theta)}{n}$$

4 Negative Binomial Distributions

Definition

Let k be a nutural number and let $0 < \theta < 1$. Then, a discrete random variable Y is said ti have a (Pascal) negative binomial distribution, and it is called a (Pascal) negative binomial random variable, if its probability distribution $b^*(y;k,\theta)$ is given by

$$b^*(y;k,\theta) = C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k}$$

where k and θ are two parameters.

Mean and Variance

$$
\begin{aligned}
\mathbb{E}[Y] =& \sum_{i=k}^{\infty} ib^*(i;k.\theta)\\
=& \sum_{i=k}^{\infty} iC_{i-1}^{k-1}\theta^k(1-\theta)^{i-k}\\
&...\\
=& \frac{k}{\theta}\\
var(Y) =& \frac{k}{\theta}(\frac{1}{\theta}-1)
\end{aligned}
$$

Theorem

Let Y be a negative binomial random variable with parameters k and θ. Then for each $y = k, k+1...$,

$$b^*(y;k,\theta) = \frac{k}{y}b(k;y,\theta)$$

Proof. By the definition, we have

$$\begin{aligned} b^*(y;k,\theta) =& C_{y-1}^{k-1}\theta^k(1-\theta)^{y-k} \\ =& \frac{(y-1)!}{(k-1)!(y-k)!}\theta^k(1-\theta)^{y-k} \\ =& \frac{k}{y}\frac{y!}{k!(y-k)!}\theta^k(1-\theta)^{y-k} \\ =& \frac{k}{y}b(k;y,\theta) \end{aligned}$$

□

5 Geometric Distributions

Definition

If X is a (Pascal) negative binomial random variable with parameters $k=1$ and θ, we say that this random variable X has a geometric distribution, and we also call this random variable as a geometric random variable. By the definition of negative binomial distribution, we see that the probability distribution $g(x;\theta) = b^*(x;1,\theta)$ of geometric distribution is given by

$$g(x;\theta) = \theta(1-\theta)^{x-1}$$

where θ is a parameter.