

Assignment 3 of MATH 2005

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October 14, 2020

1

1.1

By the Proposition 3.1.1,

$$\sum_{x=1}^5 f(x) = 1.$$

Therefore, $k = \frac{1}{15}$.

1.2

$$\sum_{x=0}^5 f(x) = 1.$$

Thus, $k = \frac{1}{32}$.

1.3

$$\sum_{x=1}^n f(x) = \sum_{x=1}^n kx^2 = k \sum_{x=1}^n x^2 = k \frac{n(n+1)(2n+1)}{6} = 1.$$

Thus, $k = \frac{6}{n(n+1)(2n+1)}$.

1.4

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} k\left(\frac{1}{4}\right)^x = k \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = k \lim_{x \rightarrow \infty} \frac{\frac{1}{4} \times (1 - (\frac{1}{4})^x)}{1 - \frac{1}{4}} = k \times 3 = 1$$

Thus, $k = \frac{1}{3}$.

2

$$\begin{aligned} f(1) &= \frac{1}{15}, \\ f(2) &= \frac{2}{15}, \\ f(3) &= \frac{1}{5}, \\ f(4) &= \frac{4}{15}, \\ f(5) &= \frac{1}{3}. \end{aligned}$$

And we can find that $\sum_{x=1}^5 f(x) = 1$

Therefore,

$F(x) = 0$ when $x < 1$.

$F(x) = f(1) = \frac{1}{15}$ when $1 \leq x < 2$.

$F(x) = f(1) + f(2) = \frac{1}{5}$ when $2 \leq x < 3$.

$F(x) = f(1) + f(2) + f(3) = \frac{2}{5}$ when $3 \leq x < 4$.

$F(x) = f(1) + f(2) + f(3) + f(4) = \frac{2}{3}$ when $4 \leq x < 5$.

$F(x) = f(1) + f(2) + f(3) + f(4) + f(5) = 1$ when $x \geq 5$.

3

3.1

By the definition,

$$P(2 < X \leq 6) = P(x \leq 6) - P(x \leq 2) = F(6) - F(2) = \frac{1}{2}$$

$$P(X = 4) = \frac{1}{6}$$

3.2

By the Proposition 3.1.3, we know that if $X = x_n \in R : n = 1, 2, \dots$ with $x_1 < x_2 < \dots < x_n < \dots$, then $f(x_k) = F(x_k) - F(x_{k-1})$.

Therefore, we can get

$$f(1) = \frac{1}{3}.$$

Similarly, $f(4) = \frac{1}{6}, f(6) = \frac{1}{3}, f(10) = \frac{1}{6}$. When $x \neq 1$ and $x \neq 4$ and $x \neq 6$ and $x \neq 10$, $f(x) = 0$.

4

4.1

By the Proposition 3.2.2,

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 dx + \int_0^4 \frac{c}{\sqrt{x}} dx + \int_4^{\infty} 0 dx = 0 + 2c\sqrt{x}|_0^4 + 0 = 4c = 1$$

Hence, we get $c = \frac{1}{4}$

4.2

By the definition and Proposition 3.2.1,

$$P(X < \frac{1}{4}) = P(X \leq \frac{1}{4}) = \int_{-\infty}^{\frac{1}{4}} f(x) dx = \frac{1}{4}$$

$$P(X > 1) = 1 - P(X \leq 1) = 1 - \int_{-\infty}^1 f(x) dx = \frac{1}{2}$$

4.3

When $x \leq 0$,

$$F(x) = \int_{-\infty}^x 0 dx = 0$$

When $0 < x < 4$,

$$F(x) = F(0) + \int_0^x \frac{1}{4\sqrt{x}} dx = \frac{\sqrt{x}}{2}$$

When $x \geq 4$

$$F(x) = F(4) + \int_4^{\infty} 0 dx = 1$$

5

5.1

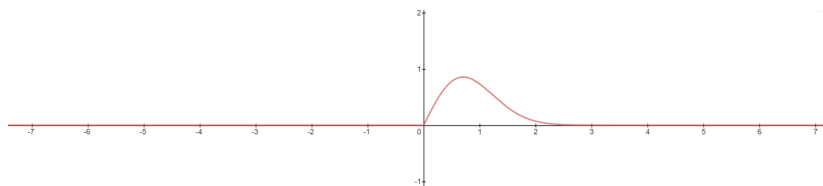


Figure 1: Density Function

By the Proposition 3.2.1,

$$\int_{-\infty}^{\infty} f(z) dz = \int_0^{\infty} k z e^{-z^2} dz = 1$$

Thus, $k = 2$ and $f(x) = 2ze^{-z^2}$ when $z > 0$

5.2

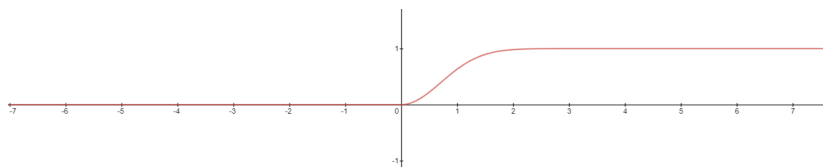


Figure 2: Distribution Function

By the definition,

$$F(z) = P(Z < z) = \int_{-\infty}^z 0 dz = 0 \text{ when } z \leq 0.$$

$$F(z) = F(0) + \int_0^z 2ze^{-z^2} dz = 1 - e^{-z^2} \text{ when } z > 0$$

6

6.1

$$P(X \leq 2) = F(2) = 1 - 3e^{-2}$$

$$P(1 < X < 3) = P(1 \leq X \leq 3) = F(3) - F(1) = 2e^{-1} - 4e^{-3}$$

$$P(X > 4) = 1 - P(X \leq 4) = 5e^{-4}$$

6.2

By the Proposition 3.2.1, $f(x) = \frac{d}{dx}F(x)$. We can get that

$$f(x) = \frac{d}{dx}0 = 0$$

when $x \leq 0$.

$$f(x) = \frac{d}{dx}(1 - (1+x)e^{-x}) = xe^{-x}$$

when $x > 0$.

7**7.1**

$$P(x \leq 6) = \int_{-\infty}^6 f(x)dx = 0 + \int_0^6 \frac{1}{9}xe^{-\frac{1}{3}x}dx = 1 - 3e^{-2}$$

7.2

$$P(x \geq 9) = 1 - P(x < 9) = 1 - P(x \leq 9) = 1 - \int_{-\infty}^9 f(x)dx = 4e^{-3}$$

8**8.1**

$$P(X > 10) = 1 - P(X \leq 10) = 1 - F(10) = 0.25$$

8.2

$$P(X < 8) = P(X \leq 8) = F(8) = \frac{39}{64}$$

8.3

$$P(12 \leq X \leq 15) = F(15) - F(12) = \frac{1}{16}$$

9

9.1

| | $x = 0$ | $x = 1$ | $x = 2$ | $x = 3$ |
|---------|----------------|----------------|----------------|----------------|
| $y = 0$ | 0 | $\frac{1}{30}$ | $\frac{1}{15}$ | $\frac{1}{10}$ |
| $y = 1$ | $\frac{1}{30}$ | $\frac{1}{15}$ | $\frac{1}{10}$ | $\frac{2}{15}$ |
| $y = 2$ | $\frac{1}{15}$ | $\frac{1}{10}$ | $\frac{2}{15}$ | $\frac{1}{6}$ |

9.2

Let $g(x)$ and $h(y)$ be the marginal probability distributions of X and Y respectively. Thus,

| | $x = 0$ | $x = 1$ | $x = 2$ | $x = 3$ | $h(y)$ |
|---------|----------------|----------------|----------------|----------------|----------------|
| $y = 0$ | 0 | $\frac{1}{30}$ | $\frac{1}{15}$ | $\frac{1}{10}$ | $\frac{1}{5}$ |
| $y = 1$ | $\frac{1}{30}$ | $\frac{1}{15}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | $\frac{1}{3}$ |
| $y = 2$ | $\frac{1}{15}$ | $\frac{1}{10}$ | $\frac{2}{15}$ | $\frac{1}{6}$ | $\frac{7}{15}$ |
| $g(x)$ | $\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{3}{10}$ | $\frac{2}{5}$ | |

10

$$\begin{aligned}
 P(X + Y < \frac{1}{2}) &= \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}-y} f(x, y) dx dy \\
 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} f(x, y) dx dy \\
 &= \int_0^{\frac{1}{2}} 12x^2 y|_{x=0}^{\frac{1}{2}-y} dy \\
 &= \int_0^{\frac{1}{2}} 12(\frac{1}{4}y - y^2 + y^3) dy \\
 &= 12(\frac{1}{8}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4)|_{y=0}^{\frac{1}{2}} \\
 &= \frac{1}{16}
 \end{aligned} \tag{1}$$

11

By the Proposition 3.3.3,

$$f(x, y) = \frac{\partial}{\partial x \partial y} F(x, y) = 0$$

when $x \leq 0$ or $y \leq 0$.

$$f(x, y) = \frac{\partial}{\partial x \partial y} F(x, y) = 2xye^{-x^2-y^2}$$

when $x > 0$ and $y > 0$.

Let $g(x)$ and $h(y)$ be the marginal densities of X and Y respectively. Thus,

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^{\infty} f(x, y) dy = 2xe^{-x^2-y^2}(1-2y),$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^{\infty} f(x, y) dx = 2ye^{-x^2-y^2}(1-2x)$$

$$\begin{aligned} g(x)h(y) &= 2xe^{-x^2-y^2}(1-2y) \times 2ye^{-x^2-y^2}(1-2x) \\ &= 4xye^{-2x^2-2y^2}(1-2x)(1-2y) \\ &\neq f(x, y) \end{aligned} \tag{2}$$

Therefore, X and Y are not independent.

12

12.1

$$\begin{aligned} P(X \leq 0.3, S > 2) &= 1 - P(X \leq 0.3, S \leq 2) \\ &= 1 - \int_{-\infty}^{0.3} \int_{-\infty}^2 f(x, s) ds dx \\ &= 1 - \int_0^{0.3} \int_0^2 f(x, s) ds dx \\ &\approx 0.62798 \end{aligned} \tag{3}$$

12.2

Let $g(x)$ be the marginal distributions of X . By the definition,

$$g(x) = 0.$$

when $x \leq 0.20$ or $x \geq 0.40$,

$$g(x) = \int_{-\infty}^{\infty} f(x, s) ds = 5$$

when $0.20 < x < 0.40$

By the definition, when $s \leq 0$,

$$f(s|x) = \frac{f(x, s)}{g(x)} = 0.$$

when $s > 0$,

$$f(s|x) = xe^{-xs}.$$

12.3

$$\begin{aligned} P(S \leq 3|x = 25) &= \int_{-\infty}^3 f(s|25) ds \\ &= \int_0^3 25e^{-25s} ds \\ &= 1 - e^{-75} \\ &\approx 1 \end{aligned} \tag{4}$$

13

13.1

Let $f(X, W)$ be the joint probability distribution of Z and W .

$$f(0, 0) = \frac{48}{52} \times \frac{47}{51} = \frac{188}{221}$$

$$f(1, 1) = \frac{4}{52} \times \frac{48}{51} = \frac{16}{221}$$

$$f(0, 1) = \frac{48}{52} \times \frac{4}{51} = \frac{16}{221}$$

$$f(1, 2) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

13.2

Let $g(z)$ be the marginal distribution of Z ,

$$g(0) = \frac{12}{13}$$

$$g(1) = \frac{1}{13}$$

13.3

By the definition,

$$f(w|z) = \frac{f(z, w)}{g(z)}$$

Thus, $f(w = 1|z = 1) = \frac{f(1,1)}{g(1)} = \frac{16}{17}$ and $f(w = 2|z = 1) = \frac{f(2,1)}{g(1)} = \frac{1}{17}$

14

14.1

Let $g(x)$ and $h(y)$ be the marginal densities of X and Y , respectively.

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^{1-x} f(x, y) dy = 4(1-x)^3$$

and

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^{1-y} f(x, y) dx = 12(1-y)^2 y$$

14.2

If $x > 0$, $y > 0$ and $x + y < 1$

$$\begin{aligned} g(x)h(y) - f(x, y) &= 48(1-x)^3(1-y)^2y - 24y(1-x-y) \\ &\neq 0 \end{aligned} \tag{5}$$

Therefore, X and Y are not independent

15

15.1

By the definition of Independence, if $0 < x < 2$ and $0 < y < 3$,

$$f(x, y) = g(x)h(y) = \frac{1}{6}$$

If $x \leq 0$ or $x \geq 2$ or $y \leq 0$ or $y \geq 3$

$$f(x, y) = 0$$

15.2

$$\begin{aligned}
 P(X^2 + Y^2 > 1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx \\
 &= \int_0^1 \int_{\sqrt{1-x^2}}^3 \frac{1}{6} dy dx + \int_1^2 \int_0^3 \frac{1}{6} dy dx \\
 &= \int_0^1 \frac{1}{6} y|_{\sqrt{1-x^2}}^3 dx + \int_1^2 \frac{1}{6} y|_0^3 dx \\
 &= \int_0^1 \frac{1}{2} - \frac{\sqrt{1-x^2}}{6} dx + \frac{1}{2} \\
 &\approx 0.8691
 \end{aligned} \tag{6}$$