Assignment 3 of MATH 2005

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October 14, 2020

1

1.1

By the Proposition 3.1.1,

$$\sum_{x=1}^{5} f(x) = 1.$$

Therefore, $k = \frac{1}{15}$**.**

1.2

$$\sum_{x=0}^{5} f(x) = 1.$$

Thus, $k = \frac{1}{32}$**.**

1.3

$$\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} kx^2 = k\sum_{x=1}^{n} x^2 = k\frac{n(x+1)(2n+1)}{6} = 1.$$

Thus, $k = \frac{6}{n(n+1)(2n+1)}$**.**

1.4

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} k(\frac{1}{4})^x = k\sum_{x=1}^{\infty}(\frac{1}{4})^x = k \lim_{x\to\infty} \frac{\frac{1}{4}\times(1-(\frac{1}{4})^x)}{1-\frac{1}{4}} = k\times 3 = 1$$

Thus, $k = \frac{1}{3}$**.**

2

$f(1) = \frac{1}{15}$,
$f(2) = \frac{2}{15}$,
$f(3) = \frac{1}{5}$,
$f(4) = \frac{4}{15}$,
$f(5) = \frac{1}{3}$.
And we can find that $\sum_{x=1}^{5} f(x) = 1$

Therefore,
$F(x) = 0$ **when** $x < 1$**.**
$F(x) = f(1) = \frac{1}{15}$ **when** $1 \leq x < 2$**.**
$F(x) = f(1) + f(2) = \frac{1}{5}$ **when** $2 \leq x < 3$**.**
$F(x) = f(1) + f(2) + f(3) = \frac{2}{5}$ **when** $3 \leq x < 4$**.**
$F(x) = f(1) + f(2) + f(3) + f(4) = \frac{2}{3}$ **when** $4 \leq x < 5$**.**
$F(x) = f(1) + f(2) + f(3) + f(4) + f(5) = 1$ **when** $x \geq 5$**.**

3

3.1

By the definition,

$$P(2 < X \leq 6) = P(x \leq 6) - P(x \leq 2) = F(6) - F(2) = \frac{1}{2}$$

$$P(X = 4) = \frac{1}{6}$$

3.2

By the Proposition 3.1.3, we know that if $X = x_n \in R : n = 1, 2, ...$ **with** $x_1 < x_2 < ... < x_n < ...$**, then** $f(x_k) = F(x_k) - F(x_{k-1})$**.**

Therefore,we can get

$$f(1) = \frac{1}{3}.$$

Similarly, $f(4) = \frac{1}{6}, f(6) = \frac{1}{3}, f(10) = \frac{1}{6}$**. When** $x \neq 1$ **and** $x \neq 4$ **and** $x \neq 6$ **and** $x \neq 10$**,** $f(x) = 0$ **.**

4

4.1

By the Proposition 3.2.2,

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{4} \frac{c}{\sqrt{x}}dx + \int_{4}^{\infty} 0dx = 0 + 2c\sqrt{x}|_0^4 + 0 = 4c = 1$$

Hence, we get $c = \frac{1}{4}$

4.2

By the definition and Proposition 3.2.1,

$$P(X < \frac{1}{4}) = P(X \leq \frac{1}{4}) = \int_{-\infty}^{\frac{1}{4}} f(x)dx = \frac{1}{4}$$

$$P(X > 1) = 1 - P(X \leq 1) = 1 - \int_{-\infty}^{1} f(x)dx = \frac{1}{2}$$

4.3

When $x \leq 0$**,**

$$F(x) = \int_{-\infty}^{x} 0dx = 0$$

When $0 < x < 4$**,**

$$F(x) = F(0) + \int_{0}^{x} \frac{1}{4\sqrt{x}}dx = \frac{\sqrt{x}}{2}$$

When $x \geq 4$

$$F(x) = F(4) + \int_{4}^{\infty} 0dx = 1$$

5

5.1

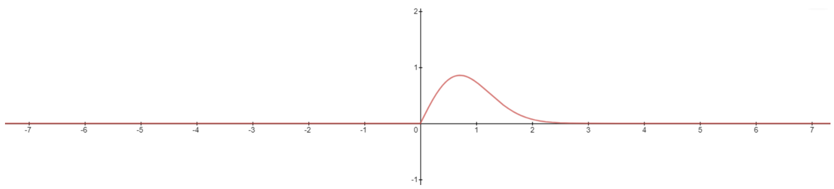

Figure 1: Density Function

By the Proposition 3.2.1,

$$\int_{-\infty}^{\infty} f(z)dz = \int_{0}^{\infty} kze^{-z^2} = 1$$

Thus, $k = 2$ **and** $f(x) = 2ze^{-z^2}$ **when** $z > 0$

5.2

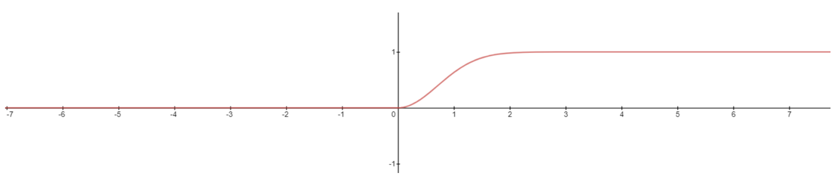

Figure 2: Distribution Function

By the definition,
$F(z) = P(Z < z) = \int_{-\infty}^{z} 0dz = 0$ **when** $z \leq 0$**.**
$F(z) = F(0) + \int_0^z 2ze^{-z^2}dz = 1 - e^{-z^2}$ **when** $z > 0$

6

6.1

$$P(X \leq 2) = F(2) = 1 - 3e^{-2}$$

$$P(1 < X < 3) = P(1 \leq X \leq 3) = F(3) - F(1) = 2e^{-1} - 4e^{-3}$$

$$P(X > 4) = 1 - P(X \leq 4) = 5e^{-4}$$

6.2

By the Proposition 3.2.1, $f(x) = \frac{d}{dx}F(x)$**. We can get that**

$$f(x) = \frac{d}{dx}0 = 0$$

when $x \leq 0$**.**

$$f(x) = \frac{d}{dx}(1 - (1+x)e^{-x}) = xe^{-x}$$

when $x > 0$**.**

7

7.1

$$P(x \leq 6) = \int_{-\infty}^{6} f(x)dx = 0 + \int_{0}^{6} \frac{1}{9}xe^{-\frac{1}{3}x}dx = 1 - 3e^{-2}$$

7.2

$$P(x \geq 9) = 1 - P(x < 9) = 1 - P(x \leq 9) = 1 - \int_{-\infty}^{9} f(x)dx = 4e^{-3}$$

8

8.1

$$P(X > 10) = 1 - P(X \leq 10) = 1 - F(10) = 0.25$$

8.2

$$P(X < 8) = P(X \leq 8) = F(8) = \frac{39}{64}$$

8.3

$$P(12 \le X \le 15) = F(15) - F(12) = \frac{1}{16}$$

9

9.1

	$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	0	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$
$y=1$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$
$y=2$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{1}{6}$

9.2

Let $g(x)$ and $h(y)$ be the marginal probability distributions of X and Y respectively. Thus,

	$x=0$	$x=1$	$x=2$	$x=3$	$h(y)$
$y=0$	0	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{5}$
$y=1$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{1}{3}$
$y=2$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{1}{6}$	$\frac{7}{15}$
$g(x)$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	

10

$$\begin{aligned} P(X+Y<\frac{1}{2}) &= \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}-y} f(x,y)dxdy \\ &= \int_{0}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}-y} f(x,y)dxdy \\ &= \int_{0}^{\frac{1}{2}} 12x^2y|_{x=0}^{\frac{1}{2}-y} dy \\ &= \int_{0}^{\frac{1}{2}} 12(\frac{1}{4}y - y^2 + y^3)dy \\ &= 12(\frac{1}{8}y^2 - \frac{1}{3}y^3 + \frac{1}{4}y^4)|_{y=0}^{\frac{1}{2}} \\ &= \frac{1}{16} \end{aligned} \tag{1}$$

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By the Proposition 3.3.3,

$$f(x,y) = \frac{\partial}{\partial x \partial y} F(x,y) = 0$$

when $x \leq 0$ **or** $y \leq 0$**.**

$$f(x,y) = \frac{\partial}{\partial x \partial y} F(x,y) = 2xye^{-x^2-y^2}$$

when $x > 0$ **and** $y > 0$**.**

Let $g(x)$ **and** $h(y)$ **be the marginal densities of** X **and** Y **respectively. Thus,**

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{\infty} f(x,y)dy = 2xe^{-x^2-y^2}(1-2y),$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{\infty} f(x,y)dx = 2ye^{-x^2-y^2}(1-2x)$$

$$\begin{aligned} g(x)h(y) &= 2xe^{-x^2-y^2}(1-2y) \times 2ye^{-x^2-y^2}(1-2x) \\ &= 4xye^{-2x^2-2y^2}(1-2x)(1-2y) \\ &\neq f(x,y) \end{aligned} \tag{2}$$

Therefore, X **and** Y **are not independent.**

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12.1

$$\begin{aligned} P(X \leq 0.3, S > 2) &= 1 - P(X \leq 0.3, S \leq 2) \\ &= 1 - \int_{-\infty}^{0.3} \int_{-infty}^{2} f(x,s)dsdx \\ &= 1 - \int_{0}^{0.3} \int_{0}^{2} f(x,s)dsdx \\ &\approx 0.62798 \end{aligned} \tag{3}$$

12.2

Let $g(x)$ be the marginal distributions of X. By the definition,

$$g(x) = 0.$$

when $x \leq 0.20$ or $x \geq 0.40$,

$$g(x) = \int_{-\infty}^{\infty} f(x,s)ds = 5$$

when $0.20 < x < 0.40$
By the definition, when $s \leq 0$,

$$f(s|x) = \frac{f(x,y)}{g(x)} = 0.$$

when $s > 0$,

$$f(s|x) = xe^{-xs}.$$

12.3

$$\begin{aligned} P(S \leq 3|x = 25) &= \int_{-\infty}^{3} f(s|25)ds \\ &= \int_{0}^{3} 25e^{-25s}ds \\ &= 1 - e^{-75} \\ &\approx 1 \end{aligned} \tag{4}$$

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13.1

Let $f(X,W)$ be the joint probability distribution of Z and W.

$$f(0,0) = \frac{48}{52} \times \frac{47}{51} = \frac{188}{221}$$

$$f(1,1) = \frac{4}{52} \times \frac{48}{51} = \frac{16}{221}$$

$$f(0,1) = \frac{48}{52} \times \frac{4}{51} = \frac{16}{221}$$

$$f(1,2) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

13.2

Let $g(z)$ be the marginal distribution of Z,

$$g(0) = \frac{12}{13}$$

$$g(1) = \frac{1}{13}$$

13.3

By the definition,

$$f(w|z) = \frac{f(z,w)}{g(z)}$$

Thus, $f(w=1|z=1) = \frac{f(1,1)}{g(1)} = \frac{16}{17}$ **and** $f(w=2|z=1) = \frac{f(2,1)}{g(1)} = \frac{1}{17}$

14

14.1

Let $g(x)$ and $h(x)$ be the marginal densities of X and Y, respectively.

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{1-x} f(x,y)dy = 4(1-x)^3$$

and

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{1-y} f(x,y)dx = 12(1-y)^2 y$$

14.2

If $x > 0$, $y > 0$ and $x + y < 1$

$$\begin{aligned} g(x)h(x) - f(x,y) &= 48(1-x)^3(1-y)^2 y - 24y(1-x-y) \\ &\neq 0 \end{aligned} \tag{5}$$

Therefore, X and Y are not independent

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15.1

By the definition of Independence, if $0 < x < 2$ and $0 < y < 3$,

$$f(x,y) = g(x)h(y) = \frac{1}{6}$$

If $x \leq 0$ or $x \geq 2$ or $y \leq 0$ or $y \geq 3$

$$f(x,y) = 0$$

15.2

$$\begin{aligned}
P(X^2+Y^2>1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx \\
&= \int_0^1 \int_{\sqrt{1-x^2}}^3 \frac{1}{6}dydx + \int_1^2 \int_0^3 \frac{1}{6}dydx \\
&= \int_0^1 \frac{1}{6}y|_{\sqrt{1-x^2}}^3 dx + \int_1^2 \frac{1}{6}y|_0^3 dx \\
&= \int_0^1 \frac{1}{2} - \frac{\sqrt{1-x^2}}{6}dx + \frac{1}{2} \\
&\approx 0.8691
\end{aligned} \tag{6}$$