# Assignment 7 of MATH 2003

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## 1

$\forall x \in \mathbb{R}$**, we can find a sequence** $(x_n)$**, s.t.** $x_n \in \mathbb{Q}, \forall n \in \mathbb{N}$ **and** $\lim(x_n) = x$

$$\begin{aligned} f(x) =& \lim f(x_n) \\ =& \lim g(x_n) \\ =& g(x) \end{aligned}$$

**since** $f$ **and** $g$ **are continuous.**

## 2

*Proof.* Let $a, b$ be real number s.t. $a < b$, we get

$$\begin{aligned} &0 < \frac{1}{n} < b - a \\ \rightarrow &0 < \frac{1}{2^n} < \frac{1}{n} < b - a \\ \rightarrow &1 < 2^n(b-a) = b2^n - a2^n \end{aligned}$$

by Archimedean Property where $n \in \mathbb{N}$ Thus,$\exists m \in \mathbb{Z}$ s.t.

$$a2^n < m < b2^n$$

i.e.

$$a < \frac{m}{2^n} < b$$

Thus, $\{\frac{m}{2^n}\}$ is dense in $\mathbb{R}$. Let $x \in \mathbb{R}$ is arbitrary. We can find a sequence$(x_n)$ s.t $x_n = \frac{i}{2^j}, i, j \in \mathbb{N}$ and

$$x = \lim x_n$$

$$\begin{aligned} h(x) =& \lim h(x_n) \\ =& \lim h(\frac{i}{2^j}) \\ =& 0 \end{aligned}$$

□

## 3

*Proof.* By the definition, let $\epsilon = \frac{f(c)}{2}$, $\exists \delta > 0$, s.t. $\forall x \in \mathbb{R}, |x - c| < \delta$

$$\begin{aligned} & |f(x) - f(c)| < \epsilon \\ \to & -\epsilon < f(x) - f(c) \\ \to & f(c) - \epsilon < f(x) \\ \to & \frac{f(c)}{2} < f(x) \\ \to & 0 < f(x) \end{aligned}$$

Thus we have

$$x \in V_\delta(c) \Rightarrow x \in P$$

i.e.

$$V_\delta(c) \subset P$$

□

## 4

*Proof.*

$$\begin{aligned} (s_n) \subset S \Rightarrow & f(s_n) > g(s_n), \forall n \in \mathbb{N} \\ \Rightarrow & \lim f(s_n) > \lim g(s_n) \\ \Rightarrow & f(s) > g(s) \end{aligned}$$

□

## 5

*Proof.* Let $(x_n)$ be any sequence s.t.

$$\lim x_n = x$$

$$
\begin{aligned}
&\Rightarrow \lim(x_n - x) = 0 \\
&\Rightarrow \lim(x_n - x + x_0) = x_0 \\
&\Rightarrow \lim(f(x_n - x + x_o)) = f(x_0) \\
&\Rightarrow \lim(f(x_n - x) + f(x_0)) = f(x_0) \\
&\Rightarrow \lim f(x_n - x) = 0 \\
&\Rightarrow \lim(f(x_n - x) + f(x)) = f(x) \\
&\Rightarrow \lim f(x_n - x + x) = f(x) \\
&\Rightarrow \lim f(x_0) = f(x)
\end{aligned}
$$

$f$ is continuous. □

# 6

*Proof.* By the Max-Min Theorem, $\exists x \in I$ such that

$$f(x') < f(x)$$

$\forall x \in I$. Since $f(x) > 0$, $f(x') > 0$. Let $\alpha = f(x')$,

$$0 < \alpha \le f(x)$$

□

# 7

*Proof.* Let $(x_n) \in E$ and $\lim x_n = x_0$. Since $(x_n) \in E$, we have $f(x_n) = g(x_n), \forall n \in \mathbb{N}$

$$\lim g(x_n) = f(x_n)$$

$$g(\lim x_n) = f(\lim x_n)$$

$$f(x_0) = g(x_0)$$

Thus,

$$x_0 \in E$$

□

# 8

*Proof.* $\forall x_1 \in I, \exists x_2 \in I$, s.t.

$$|f(x_2)| \le \frac{1}{2} f(x_1)$$

We can find a sequence $(x_n) \in I$ s.t.

$$|f(x_{n+1})| \leq \frac{1}{2}f(x_n) < (\frac{1}{2})^n f(x_1)$$

By Bolzano-Weirstrass Theorem there exists a subsequence $(x_{p(n)})$ of $(x_n)$ which converges to $c \in I$. Since

$$\lim f(x_{p(n)}) = 0$$

since $|f(x_{n+1})| < (\frac{1}{2})^n f(1)$ .Thus

$$f(c = 0)$$

$\square$

## 9

$$f(-8) = 503, f(-7) = -9$$

$$f(1) = -1, f(2) = 63$$

**Thus there exist $c_1 \in (-8, -7)$ and $c_2 \in (1, 2)$ s.t. $f(x_1) = f(c_2) = 0$ By calculator, we know**

$$c_1 \approx -7.02$$

$$c_2 \approx 1.03$$

## 10

**By the property of supermum, there exists a sequence$(x_n) \subset W$ converging to $w$. And $f(x_n) < 0$, we fet**

$$\lim f(x_n) = f(w) \leq 0$$

**S0, $w < b$. So $w$ is an interior point of $I$. So if $f(w) < 0$ then by the continuity, for some $\delta > 0$, $w + \delta \in I$ and $f(w + \delta) < 0$ which contradicts the fact that $w$ is supermum of $W$. Thus**

$$f(w) = 0$$