Assignment 6 of MATH 2003

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1.1

Let $\delta = min\{\frac{1}{2}, \frac{\epsilon}{2}\}$ and when $|x-2| < \delta$, we have

$$\begin{aligned} |x-2| < \delta < &\frac{1}{2} \\ -\frac{1}{2} < x-2 < &\frac{1}{2} \\ \frac{1}{2} < x-1 < &\frac{3}{2} \end{aligned}$$

$$\begin{aligned} |\frac{1}{1-x}+1| =& |\frac{2-x}{1-x}| \\ =& |\frac{x-2}{x-1}| \\ <& 2(x-2) \\ <& 2\delta \\ \leq& \epsilon \end{aligned}$$

By the definition, we know:

$$\lim_{x\to 2} \frac{1}{1-x} = -1$$

Let $\delta = \epsilon$, for $|x - 0| < \delta$, we have

$$\begin{aligned} |x| &< \delta \\ |\frac{xx}{x}| &< \delta \\ |\frac{x^2}{|x|}| &< \delta \\ |\frac{x^2}{|x|} - 0| &< \epsilon \end{aligned}$$

By the definition, we know:

$$\lim_{x \to 0} \frac{x^2}{|x|} = 0$$

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f is not continuous

Proof. Without loss of generality, for any rational number $x_0 \in \mathbb{R}$, we can find a sequence (a_n) that converges to x_n such that $\forall a \in (a_n)$, a is a irrational. We can get a sequence $(f(a_n))$ where

$$f(a_n) = x(1 + x)$$

Therefore, $(f(a_n))$ is converges to $x_0(1 + x_0)$ when $f(x_0) = x_0(1 - x_0)$. Thus, f is not continuous. $\square$

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Proof. For any $x_0 \in [a, b]$, if

$$\inf_{a \le t \le x_0} f(t) = f(x_0)$$

because $f(x)$ is continuous, $\lim_{x \to x_0} f(x) = f(x_0)$ and $\forall x \in [a, x_0]$, $f(x) \ge f(x_0)$. Thus

$$\lim_{x \to x_0} \inf_{a \le t \le x_0} f(t) = f(x_0)$$

If
$$\inf_{a\le t\le x_0} f(t) = f(b)$$
where $b \in [a, x_0)$ which means that
$$\lim_{x\to x_0} \inf_{a\le t\le x} f(t) = f(b)$$
Therefore, $\forall x_0 \in [a, b]$,
$$m(x_0) = \inf_{a\le t\le x_0} f(t) = \lim_{x\to x_0} \inf_{a\le t\le x} f(t)$$
which means that $m(x)$ is continuous. □

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$M = \{\frac{1}{2} \pm \frac{n}{2n+1} : n \in \mathbb{N}\} = \{1 - \frac{1}{4n+2}, \frac{1}{4n+2} : n \in \mathbb{N}\}$**.For every** $\delta > 0$**, when**
$$n > \frac{1-2\delta}{4\delta} \Rightarrow$$
$$\delta > \frac{1}{4n+2}$$
which means
$$|\frac{1}{4n+2} - 0| < \delta$$
or
$$|1 - \frac{1}{4n+2} - 1| < \delta$$
By the definition, 1 **and** 0 **are cluster point of** M

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Proof. By the definition,
$$f_2(x) := f(x)$$
for every $x \in J$ f has limit at c ,i.e. for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then$|f(x) - L| < \epsilon$ where L is the limit of $f(x)$. Because $c \in J$, then $|f_2(x) - L| < \epsilon$ which means that f_2 also has a limit at c □

Example

Let $f(x) = sgn(x)$ **and** $J = (0, +\infty)$**. We can get that** $\lim_{x\to 0}$ **do not exists but** $\lim_{x\to 0} f_2(x) = 1$

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Proof. We can get

$$f(0) = f(0+0) = 2f(0)$$

which means that

$$f(0) = 0$$

Given that

$$\lim_{x \to 0} = L$$

i.e., $\forall \epsilon > 0, \exists \delta > 0, |x - 0| < \delta \to |f(x) - L| < \epsilon$. When $x \to 0$,

$$|f(x) - L| = |f(0) - L| < \epsilon$$

i.e.,$| - L| = |L| < \epsilon$. Because $\epsilon > 0$ is arbitrary number. Thus $L = 0$ □

Proof. For any $c \in \mathbb{R}$,

$$f(x) = f(x - c) + f(c)$$

Thus,

$$\begin{aligned}\lim_{x \to c} f(x) =& \lim_{x \to c} f(x - c) + f(c)\\ =& \lim_{x \to 0} f(x) + f(c)\\ =& f(c)\end{aligned}$$

Therefore, f has limit at every $c \in \mathbb{R}$ □

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Let

$$\lim_{x \to c} f(x) = L$$

which means that for any $\epsilon > 0 \in \mathbb{R}$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

$$|f(x) - L| < \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$0 < |f(x)| < min\{|L + \epsilon|, |L - \epsilon|\} \leq |L + \epsilon|$$

i.e.,

$$0 < |f(x)| < |L + \epsilon|$$

$$-|L| < |f(x)| - |L| < |L + \epsilon| - |L|$$

$$0 < ||f(x)| - |L|| < min\{||L| + \epsilon| - |L|, | - |L||\}$$

i.e.,

$$0 < ||f(x)| - |L|| < min\{||L| + \epsilon| - |L|, |L|\}$$

Because $\epsilon > 0$**, then** $|L| + \epsilon| - |L| = \epsilon$

$$0 < ||f(x)| - |L|| < min\{||L| + \epsilon| - |L|, |L|\} \leq \epsilon$$

i.e.,

$$0 < ||f(x)| - |L|| < \epsilon$$

$$0 < ||f(x)| - |\lim_{x \to c} f(x)|| < \epsilon$$

which means that

$$lim_{x \to}|f|(x) = |\lim_{x \to c} f(x)|$$

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Let

$$\lim_{x \to c} f(x) = L$$

which means that for any $\epsilon > 0 \in \mathbb{R}$**, there exists** $\delta > 0$ **such that if** $0 < |x - c| < \delta$**, then** $|f(x) - L| < \epsilon$**.**

$$|f(x) - L| < \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$\sqrt{L - \epsilon} < \sqrt{f(x)} < \sqrt{L + \epsilon}$$

$$\sqrt{L - \epsilon} - \sqrt{L} < \sqrt{f(x)} - \sqrt{L} < \sqrt{L + \epsilon} - \sqrt{L}$$

$$\begin{aligned} |\sqrt{f(x)} - \sqrt{L}| &< min\{|\sqrt{L + \epsilon} - \sqrt{L}|, |\sqrt{L - \epsilon} - \sqrt{L}|\} \\ &\leq |\sqrt{L + \epsilon} - \sqrt{L}| \end{aligned}$$

i.e.,

$$|\sqrt{f(x)} - \sqrt{L}| < |\sqrt{L + \epsilon} - \sqrt{L}|$$

We have $f : (0, \infty) \to (0, \infty)$ **where** $f(\epsilon) = |\sqrt{L} - \epsilon| - \sqrt{L}$ **is surjective. Then** $f(\epsilon)$ **is a arbitrary number because** ϵ **is a arbitrary number. Therefore,**

$$\lim_{x \to c} \sqrt{f}(x) = \sqrt{\lim_{x \to c} f(x)}$$

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Let $c = 0$, $f(x) = sgn(x)$ and $g(x) = -sgn(x)$. Then,

$$(f + g)(x) = 0$$

and

$$(f \cdot g)(x) = -1$$

when $x \neq 0$

$$(f \cdot g)(0) = 0$$

Therefore

$$\lim_{x \to 0} (f + g)(x) = 0$$

$$\lim_{x \to 0} (f \cdot g)(x) = -1$$

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10.1

Proof. Let (δ_n) be a sequence such that $\delta_n > 0$ and $\delta_n > \delta_{n+1}$ for all $n \in \mathbb{N}$. Thus $\lim \delta_n$ must exists by monotone convergence theorem. And it is easy to know that $\{x : |x - x_0| < \delta_{n+1}\} \subseteq \{x : |x - x_0| < \delta_n\}$. We can get

$$M_f(x_0, \delta_n) - m_f(x_0, \delta_n) \geq M_f(x_0, \delta_{n+1}) - m_f(x_0, \delta_{n+1})$$

which means that the sequence $(M_f(x_0, \delta_n) - m_f(x_0, \delta_n))$ is monotone sequence. And $M_f(x_0, \delta_n) > m_f(x_0, \delta_n))$, i.e., $(M_f(x_0, \delta_n) - m_f(x_0, \delta_n)) > 0$ We can get $\lim(M_f(x_0, \delta_n) - m_f(x_0, \delta_n))$ must exists by monotone convergence theorem. □

10.2

if Part

Cliam. If $\lim_{\delta \to o^+}(M_f(x_0, \delta) - m_f(x_0, \delta)) = 0$, then $f(x)$ is continuous.

Proof. $\lim_{\delta \to o^+}(M_f(x_0, \delta) - m_f(x_0, \delta)) = 0$ which means that $\lim_{\delta \to o^+} M_f(x_0, \delta) = \lim_{\delta \to o^+} m_f(x_0, \delta)$. Thus, $\lim_{\delta \to o^+} sup(A) = \lim_{\delta \to o^+} inf(A)$ where $A = \{f(x) : x \in [a, b], |x - x_0| < \delta\}$. Therefore, A has only one element $f(x_0)$. Thus,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

□

***only if* Part**

Cliam. If $f(x)$ is continuous, then $\lim_{\delta\to o^+}(M_f(x_0,\delta) - m_f(x_0,\delta)) = 0$.

Proof. $f(x)$ is continuous which means that $\lim_{x\to x_0} = f(x_0)$. When $\delta \to 0^+$, $x \to x_0$, $M_f(x_0,\delta) = sup\{\lim_{x\to x_0} f(x)\} = f(x_0)$
,and $m_f(x_0,\delta) = inf\{\lim_{x\to x_0} f(x)\} = f(x_0)$ Therefore,$\lim_{\delta\to o^+}(M_f(x_0,\delta) - m_f(x_0,\delta)) = 0$ $\square$