Assignment 4 of MATH 2005

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1

$$
\begin{aligned}
E(X) &= \sum_{x=-1}^{3} x f(x) \\
&= -1 \times \frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 2 \times \frac{0}{7} + 3 \times \frac{1}{7} \\
&= \frac{1}{7}
\end{aligned}
$$

2

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} y h(y) dy \\
&= \int_{2}^{4} \frac{1}{8}(y+1) y dy + 0 \\
&= \frac{79}{3}
\end{aligned}
$$

3

3.1

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{1}^{3} x \frac{1}{x \log 3} dx + 0 \\
&= \frac{2}{\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_1^3 \frac{x}{\log 3} dx \\
&= \frac{4}{\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^3) &= \int_{-\infty}^{\infty} x^3 f(x)dx \\
&= \int_1^3 \frac{x^2}{\log 3} dx \\
&= \frac{26}{3\log 3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^3 + 2X^2 - 3X + 1) &= E(X^3) + 2E(X^2) - 3E(X) + E(1) \\
&= \frac{26}{3\log 3} + 2 \times \frac{4}{\log 3} - 3 \times \frac{2}{\log 3} + 1 \\
&= \frac{35}{3\log 3}
\end{aligned}
$$

3.2

$$
\begin{aligned}
\mu_r' &= E[X^r] \\
&= \int_{-\infty}^{\infty} x^r f(x)dx \\
&= \int_1^3 x^r \frac{1}{x\log 3} dx \\
&= \frac{1}{\log 3} \int_1^3 x^{r-1} dx \\
&= \frac{1}{\log 3} \frac{1}{r} x^r |_1^3 \\
&= \frac{1}{r\log 3}(3^r - 1)
\end{aligned}
$$

$$\begin{aligned}\sigma^2 =& var(X) \\ =& E[X^2] - E[X]^2 \\ =& \frac{4}{\log 3} - \frac{2}{\log 3} \\ =& \frac{2}{\log 3}\end{aligned}$$

4

$$\begin{aligned}E(\frac{X}{Y}) &= \int_0^1 \int_0^y \frac{x}{y} f(x,y) dx dy \\ &= \int_0^1 \int_0^y \frac{x}{y^2} dx dy \\ &= \int_0^1 \frac{x^2}{2y^2}|_0^y dy \\ &= \int_0^1 \frac{1}{2} dy \\ &= \frac{1}{2} y|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

5

Let $\varphi(x)$ is the money he should pay us where x is the number we get from a balanced die.

$$\begin{aligned}E(\varphi(X)) =& \sum_{x=1}^{6} \varphi(x) f(x) \\ =& \frac{1}{6} \sum_{x=1}^{4} \varphi(x) + \frac{5}{3} \\ =& 0\end{aligned}$$

Thus, we can get,

$$\sum_{x=1}^{4} \varphi(x) = -10$$

which means that the total money we should pay that person when we roll a $1, 2, 3,$ or 4 is equal to \$10. Hence, there are many solutions

for this equation. For example,

$$\begin{aligned}\varphi(1) = \varphi(2) = \varphi(3) = 0,\\ \varphi(4) = -10.\end{aligned}$$

6

6.1

Let

$$\varphi(x; n) = \begin{cases} x - (n-x) \times 0.4 & (0 \le x \le n) \\ n & (n \le x). \end{cases}$$

be the profit when produce n cake(s) a day.

(a) one of the cakes

$$\begin{aligned} E(\varphi(X;1)) = &\sum_{n=0}^{5} \varphi(x;1) f(x) \\ =&\frac{1}{6}(-0.4 + 1 + 1 + 1 + 1 + 1) \\ =&\frac{23}{30} \end{aligned}$$

(b) two of the cakes

$$\begin{aligned} E(\varphi(X;2)) = &\sum_{n=0}^{5} \varphi(x;2) f(x) \\ =&\frac{1}{6}(-0.8 + 0.6 + 2 + 2 + 2 + 2) \\ =&1.2 \end{aligned}$$

(c) three of the cakes

$$\begin{aligned}E(\varphi(X;3)) =& \sum_{n=0}^{5} \varphi(x;3)f(x)\\ =& \frac{1}{6}(-1.2+0.2+1.6+3+3+3)\\ =& 1.6\end{aligned}$$

(d) four of the cakes

$$\begin{aligned}E(\varphi(X;4)) =& \sum_{n=0}^{5} \varphi(x;4)f(x)\\ =& \frac{1}{6}(-1.6-0.2+1.2+2.6+4+4)\\ =& \frac{5}{3}\end{aligned}$$

(e) five of the cakes

$$\begin{aligned}E(\varphi(X;5)) =& \sum_{n=0}^{5} \varphi(x;5)f(x)\\ =& \frac{1}{6}(-2-1.6+0.8+2.2+3.6+5)\\ =& \frac{4}{3}\end{aligned}$$

6.2

By 6.1 **we can know that he should bake 3 cakes a day to maximize his expected profit.**

7

$$\begin{aligned}
E(Z) =& E(\frac{X-\mu}{\sigma}) \\
=& \frac{1}{\sigma}(E(X-\mu)) \\
=& \frac{1}{\sigma}(E(X) - E(\mu)) \\
=& \frac{1}{\sigma}(\mu - \mu) \\
=& 0
\end{aligned}$$

$$\begin{aligned}
var(Z) =& E\{[Z - E(Z)]^2\} \\
=& E(Z^2) - [E(Z)]^2 \\
=& E(Z^2) \\
=& E[(\frac{X-\mu}{\sigma})^2] \\
=& \frac{1}{\sigma^2}E[(X-\mu)^2] \\
=& \frac{1}{\sigma^2} \times \sigma^2 \\
=& 1
\end{aligned}$$

8

By Chebyshevs inequality, we can know that

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

(a) at least 0.95

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \geq 0.95$$
$$k \geq 2\sqrt{5}$$

(b) at least 0.99

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \geq 0.99$$
$$k \geq 10$$

9

$$\begin{aligned}P(64 \leq X \leq 184) =& P(|X-\mu| \leq 60)\\ =& P(|X-\mu| \leq 8\sigma)\\ \geq& 1-\frac{1}{8^2}=\frac{63}{64}\end{aligned}$$

10

From the table we can know that

$$E(X)=\frac{1}{3}$$

$$E(Y)=\frac{3}{4}$$

Therefore,

$$\begin{aligned}cov(X,Y) =& E(XY)-E(X)\times E(Y)\\ =& \sum_x\sum_y xyf(x,y)-\frac{1}{4}\\ =& \frac{1}{4}-\frac{1}{4}\\ =& 0\end{aligned}$$

But,

$$\begin{aligned}f(-1,0) =& 0\\ \neq& g(-1)\times h(0)=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}\end{aligned}$$

where $g(x)$ and $h(y)$ are the marginal probability distribution of X and Y, respectively.

11

$$\begin{aligned}var(X+Y) =& E[((X+Y)-E(X+Y))^2]\\ =& E[((X-E[X])+(Y-E[Y]))^2]\\ =& E[(X-E[X])^2]+E[(Y-E[Y])^2]\\ &+2E[(X-E[X])(Y-E[Y])]\\ =& var(X)+var(Y)+2cov(X,Y)\end{aligned}$$

$$\begin{aligned}
var(X-Y) =& E[((X-Y)-E[X-Y])^2] \\
=& E[((X-E[X])-(Y-E[Y]))^2] \\
=& E[(X-E[X])^2]+E[(Y-E[Y])^2] \\
& -2E[(X-E[X])(Y-E[Y])] \\
=& var(X)+var(Y)-2cov(X,Y)
\end{aligned}$$

$$\begin{aligned}
cov(X+Y,X-Y) =& E[(X+Y)(X-Y)]-E[X+Y]\times E[X-Y] \\
=& E[X^2-Y^2]-(E[X]-E[Y])(E[X]+E[Y]) \\
=& E[X^2]-E[Y^2]-E[X]^2+E[Y]^2 \\
=& var(X)-var(Y)
\end{aligned}$$

12

12.1

$$\begin{aligned}
E[U] =& E[2X-3Y+4Z] \\
=& 2E[X]-3E[Y]+4E[Z] \\
=& -7
\end{aligned}$$

$$\begin{aligned}
E[V] =& E[X+2Y-Z] \\
=& E[X]+2[Y]-E[Z] \\
=& 19
\end{aligned}$$

$$\begin{aligned}
var(U) =& var(2X-3Y+4Z) \\
=& var(2X)+var(-3Y)+var(4Z) \\
=& 4var(X)+9var(Y)+16var(Z) \\
=& 155
\end{aligned}$$

$$\begin{aligned}
var(V) =& var(X+2Y-Z) \\
=& var(X)+4var(Y)+var(Z) \\
=& 22
\end{aligned}$$

12.2

Claim:

$$cov(X+Y,Z)=cov(X,Z)+cov(Y,Z)$$

Proof.

$$\begin{aligned}cov(X+Y,Z) =&E[(X+Y)Z]-E[X+Y]E[Z]\\ =&E[XZ+YZ]-(E[X]+E[Y])E[Z]\\ =&E[XZ]-E[X]E[Z]+E[YZ]-E[Y]E[Z]\\ =&cov(X,Z)+cov(Y,Z)\end{aligned}$$

□

Claim:

$$cov(nX,Y)=n\ cov(X,Y)$$

where n is a real number.

Proof.

$$\begin{aligned}cov(nX,Y) =&E[nXY]-E[nX]E[Y]\\ =&nE[XY]-nE[X]E[Y]\\ =&n(E[XY]-E[X]E[Y])\\ =&n\ cov(X,Y)\end{aligned}$$

□

It is easy to know that

$$\begin{aligned}cov(aX,bY) =&a\ cov(X,bY)\\ =&ab\ cov(X,Y)\end{aligned}$$

$$\begin{aligned}var(U) =&var((2X-3Y)+4Z)\\ =&var(2X-3Y)+var(4Z)+2cov(2X-3Y,4Z)\\ =&var(2X-3Y)+var(4Z)+2cov(2X,4Z)+2cov(-3Y,4Z)\\ =&var(2X)+var(-3Y)+var(4Z)+16cov(X,Z)-24cov(Y,Z)\\ =&155\end{aligned}$$

$$\begin{aligned}var(V) =&var(X+2Y-Z)\\ =&var(X)+4var(Y)+var(Z)+2cov(X+2Y,-Z)\\ =&22+2cov(X,-Z)+2cov(2Y,-Z)\\ =&36\end{aligned}$$

$$\begin{aligned} E[U] =& E[2X - 3Y + 4Z] \\ =& 2E[X] - 3E[Y] + 4E[Z] \\ =& -7 \end{aligned}$$

$$\begin{aligned} E[V] =& E[X + 2Y - Z] \\ =& E[X] + 2[Y] - E[Z] \\ =& 19 \end{aligned}$$

13

We can get the joint probability distribution $f(z, w)$

	$z = 0$	$z = 1$
$w = 0$	0.36	0
$w = 1$	0.24	0.24
$w = 2$	0	0.16

Thus,

$$\begin{aligned} E[Z] = 0.4 \\ E[W] = 0.8 \\ cov(Z, W) =& E[ZW] - E[Z]E[W] \\ =& \sum_z \sum_w zwf(z, w) - E[Z]E[W] \\ =& -0.4 \end{aligned}$$

14

The probability distribution

$$f(x, y, z) = \frac{C_3^x C_2^y C_3^z}{C_8^2}$$

where $x + y + z = 2$, x is the number of statistics texts, y is the number of mathematics texts and z is the number of physics texts.

$$\begin{aligned} E(Y; X = 0) =& \sum_y yf(0, y, z) \\ =& \frac{2}{7} \end{aligned}$$