Assignment 6 of MATH 2005

ZHANG Huakang/DB92760

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1

$$\begin{aligned}
\mathbb{P}[X < -\theta \log(1-p)] =& \int_{-\infty}^{-\theta \log(1-p)} f(y)dy \\
=& \int_{0}^{-\theta \log(1-p)} \frac{1}{\theta} e^{-\frac{y}{\theta}} dy \\
=& -e^{-\frac{y}{\theta}}|_{0}^{-\theta \log(1-p)} \\
=& 1-(1-p) \\
=& p
\end{aligned}$$

2

$$\begin{aligned}
\mathbb{E}[X] =& \int_{-\infty}^{\infty} xf(x)dx \\
=& \int_{0}^{\infty} x2\alpha x e^{-\alpha x^2} dx \\
=& 2\alpha \int_{0}^{\infty} x^2 e^{-\alpha x^2} dx
\end{aligned}$$

Let $u = x$ and $v' = xe^{-\alpha x^2}$, thus $u' = 1$ and $v = -\frac{1}{2\alpha}e^{-\alpha x^2}$

$$\begin{aligned}
\int x^2 e^{-\alpha x^2} dx =& -\frac{x}{2\alpha}e^{-\alpha x^2} + \int \frac{1}{2\alpha} e^{-\alpha x^2} dx \\
=& -\frac{x}{2\alpha}e^{-\alpha x^2} + \frac{\sqrt{\pi} erf(\sqrt{\alpha}x)}{2\sqrt{\alpha}}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}[X] =& [-\frac{x}{2\alpha}e^{-\alpha x^2} + \frac{\sqrt{\pi} erf(\sqrt{\alpha}x)}{2\sqrt{\alpha}}]|_{0}^{\infty} \\
=& \frac{\sqrt{\pi}}{2\sqrt{\alpha}}
\end{aligned}$$

And

$$\begin{aligned}\mathbb{E}[X^2] =& \int_{-\infty}^{\infty} x^2 f(x)dx \\ =& \int_{0}^{\infty} x^2 2\alpha x e^{-\alpha x^2} dx \\ =& 2\alpha \int_{0}^{\infty} x^3 e^{-\alpha x^2} dx \\ =& 2\alpha(-\frac{e^{-\alpha x^2}(\alpha x^2+1)}{2\alpha^2})|_0^\infty \\ =& \frac{1}{2\alpha^2}\end{aligned}$$

Therefore,

$$\begin{aligned}var(X) =& \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ =& \frac{1}{2\alpha^2} - (\frac{\sqrt{\pi}}{2\sqrt{\alpha}})^2 \\ =& \frac{4-\pi}{4\alpha}\end{aligned}$$

3

3.1

We know that $\alpha > 0$ and $\beta > 0$.By the definition

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) =& \int_{0}^{\infty} kx^{\beta-1} e^{-\alpha x^\beta} dx \\ =& k \times (-\frac{1}{\alpha\beta}) e^{-\alpha x^\beta} |_0^\infty \\ =& 0 - (-\frac{k}{\alpha\beta}) \\ =& \frac{k}{\alpha\beta} \\ =& 1\end{aligned}$$

Thus,

$$k = \alpha\beta$$

3.2

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{0}^{\infty} \alpha\beta x^{\beta} e^{-\alpha x^{\beta}} dx\end{aligned}$$

Suppose $u = \alpha x^{\beta}$, then $u' = \alpha\beta x^{\beta-1}$.

$$\begin{aligned}\mathbb{E}[X] =&\alpha^{\frac{1}{\beta}} \int_{0}^{\infty} u^{\frac{1}{\beta}} e^{-u} du \\ =&\alpha^{-\frac{1}{\beta}} \Gamma(1+\beta^{-1})\end{aligned}$$

4

We know that $AD = x$, $AC = \frac{a}{2}$, and $BD = a - x$. If they can form a triangle,

$$\begin{aligned}x + a - x >& \frac{a}{2} \\ |x - (a - x)| <& \frac{a}{2}\end{aligned}$$

We have

$$\frac{a}{4} < x < \frac{3a}{4}$$

$$\mathbb{P}(X) = \frac{\frac{3a}{4} - \frac{a}{4}}{a} = \frac{1}{2}$$

5

Because X has gamma distribution with $\alpha = 80\sqrt{n}$ and $\beta = 2$, we have

$$\mathbb{E}[X] = \alpha\beta = 160\sqrt{n}$$

Its profit

$$Profit =160\sqrt{n} - 8n$$

When $Profit' = 0$, $80n^{-\frac{1}{2}} - 8 = 0$

$$n = 100$$

its expected profit is max.

6

$$\begin{aligned}\mathbb{P}(X > 12) =& 1 - \mathbb{P}(X < 12) \\ =& 1 - \int_0^{12} \frac{1}{2^3 \times 2!} x^2 e^{-\frac{x}{2}} dx \\ \approx& 1 - 0.9380 \\ =& 0.0620\end{aligned}$$

7

7.1

$$\begin{aligned}\mathbb{P}(X < 24) =& \int_0^{24} \frac{1}{120} e^{-\frac{1}{120}x} dx \\ =& e^{-\frac{x}{120}} |_0^{24} \\ \approx& 0.1813\end{aligned}$$

7.2

$$\begin{aligned}\mathbb{P}(X > 180) =& \int_{180}^{\infty} \frac{1}{120} e^{-\frac{1}{120}x} dx \\ =& 1 - \int_0^{180} \frac{1}{120} e^{-\frac{1}{120}x} dx \\ \approx& 0.2231\end{aligned}$$

8

The arrival per hours X_h follow the Poisson distribution with

$$\lambda_h = \frac{\lambda}{24} = 1.2$$

$$\begin{aligned}\mathbb{P}(X_h = 0) =& e^{-\lambda_h} \frac{\lambda_h^0}{0!} \\ =& e^{-1.2} \\ =& 0.3012\end{aligned}$$

9

The bad chech per hours X_h follow the Poisson distribution with

$$\lambda_h = \frac{\lambda}{5} = 0.4$$

$$
\begin{aligned}
\mathbb{P}(X_h > 2) =& \int_2^\infty 0.4e^{-.04x}dx \\
=& 1 - (-e^{-.04x}|_0^2) \\
\approx& 0.4493
\end{aligned}
$$

10

Proof.

$$
\begin{aligned}
cov(X, Y) =& cov(X, X^2) \\
=& \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{X}^{\not{2}} \\
= \mu^3 + 3\mu\sigma^2 - \mu(\sigma^2 + \mu^2) \\
=& 2\mu\sigma^2 \\
=& 0
\end{aligned}
$$

since X is a standard normal distribution with $\mu = 0$ $\square$

11

11.1

$$
\begin{aligned}
\mathbb{P}(X > 1.14) =& 0.5 - \mathbb{P}(X < 1.14) \\
=& 0.5 - 0.3729 \\
=& 0.1271
\end{aligned}
$$

11.2

$$
\begin{aligned}
\mathbb{P}(X < -0.36) =& 0.5 - +\mathbb{P}(X < 0.36) \\
=& 0.5 + 0.1406 \\
=& 0.6406
\end{aligned}
$$

11.3

$$
\begin{aligned}
\mathbb{P}(-0.40 < X < -0.09) =& 0.1554 - 0.0359 \\
=& 0.1195
\end{aligned}
$$

11.4

$$
\begin{aligned}
\mathbb{P}(-0.58 < X < 1.12) =& 0.2190 + 0.3886 \\
=& 0.5876
\end{aligned}
$$

12

12.1

$$\begin{aligned}\mathbb{P}(Z < 1.33) =& 0.5 + 0.4082\\ =& 0.9082\end{aligned}$$

12.2

$$\begin{aligned}\mathbb{P}(Z < -0.79) =& 0.5 - 0.2852\\ =& 0.0.2148\end{aligned}$$

12.3

$$\begin{aligned}\mathbb{P}(0.55 < Z < 1.22) =& 0.388 - 0.2088\\ =& 0.1800\end{aligned}$$

12.4

$$\begin{aligned}\mathbb{P}(-1.90 < Z < 0.44) =& 0.4713 + 0.1700\\ =& 0.0.6413\end{aligned}$$

13

From the table we can get

13.1

1.48

13.2

-0.74

13.3

0.55

13.4

2.17

14

From the table

14.1

$$1.64 < Z_\alpha < 1.65$$

14.2

$$Z_\alpha = 1.96$$

14.3

$$2.32 < Z_\alpha < 2.33$$

14.4

$$2.57 < Z_\alpha < 2.58$$

15

15.1

$$Z = \frac{16 - 15.40}{0.48} = 1.25$$

$$\begin{aligned}\mathbb{P}(Z > 1.25) =& 0.5 - 0.3944\\ =& 0.1056\end{aligned}$$

15.2

$$Z = \frac{14.20 - 15.40}{0.48} = -2.5$$

$$\begin{aligned}\mathbb{P}(Z < -2.5) =& 0.5 - 0.4938\\ =& 0.0062\end{aligned}$$

15.3

$$|Z_1| = |Z_2| = \frac{15.80 - 15.40}{0.48} = 0.83$$

$$\begin{aligned}\mathbb{P}(-0.83 < Z < 0.83) =& 2 \times 0.2967\\ =& 0.5934\end{aligned}$$