

Assignment 7 of MATH 2005

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1

$$\begin{aligned}G(y) &= F(\sqrt{y}) \\&= \int_0^{\sqrt{y}} 2se^{-s^2} ds \\&= -e^{-s^2} \Big|_0^{\sqrt{y}} \\&= 1 - e^{-y} \\g(y) &= \frac{d}{dy} G(y) \\&= e^{-y}\end{aligned}$$

when $y > 0$, and $g(y) = 0$ elsewhere.

2

$$\begin{aligned}G(y) &= F(e^y) \\&= \int_0^{e^y} \frac{1}{\theta} e^{-\frac{s}{\theta}} ds \\&= -e^{-\frac{s}{\theta}} \Big|_0^{e^y} \\&= 1 - e^{-\frac{e^y}{\theta}} \\g(y) &= \frac{d}{dy} G(y) \\&= \frac{1}{\theta} e^{y - \frac{e^y}{\theta}}\end{aligned}$$

when $y \in \mathbb{R}$

3

$$\begin{aligned}
G(y) &= F(y^2) \\
&= \int_0^{y^2} dds \\
&= s \Big|_0^{y^2} \\
&= y^2 \\
g(y) &= \frac{d}{dy} y^2 \\
&= 2y
\end{aligned}$$

when $0 < y < 1$, and $g(y) = 0$ elsewhere.

4

Let $w(y) = \log y$, thus $w'(y) = \frac{1}{y} \neq 0$

$$\begin{aligned}
\varphi(y) &= \varphi(w(y); \mu, \sigma) |w'(y)| \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}} \frac{1}{y}
\end{aligned}$$

when $y \in \mathbb{R}$

5

We get

$$f(x) = 1$$

when $0 < x < 1$ and

$$f(x) = 0$$

elsewhere

5.1

Let $w(y) = e^{-\frac{1}{2}y}$, and $w'(y) = -\frac{1}{2}e^{-\frac{1}{2}y} \neq 0$ when $y > 0$

$$\begin{aligned}
f(y) &= f(w(y)) |w'(y)| \\
&= \frac{1}{2} e^{-\frac{1}{2}y}
\end{aligned}$$

and $f(y) = 0$ elsewhere, which is a gamma distribution with parameters $\alpha = 1$ and $\beta = 2$

5.2

Let $w(y) = y^{-\gamma}$, then $w'(y) = -\gamma y^{-\gamma-1} \neq 0$ when $y > 1$. Since $\gamma > 0$

$$\begin{aligned} g(y) &= f(w(y))|w'(y)| \\ &= \frac{\gamma}{y^{1+\gamma}} \end{aligned}$$

when $y > 1$, and $g(y) = 0$ elsewhere.

6

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ &= \sum_{x=1}^{\infty} 2e^{tx} \left(\frac{1}{3}\right)^x \\ \mu'_1 &= \frac{d}{dt} M_X(t) \Big|_{t=0} \\ &= \sum_{x=1}^{\infty} 2xe^{tx} \left(\frac{1}{3}\right)^x \Big|_{t=0} \\ &= \sum_{x=1}^{\infty} 2x \left(\frac{1}{3}\right)^x \\ \mu'_2 &= \frac{d^2}{dt^2} M_X(t) \\ &= \frac{d}{dt} \sum_{x=1}^{\infty} 2xe^{tx} \left(\frac{1}{3}\right)^x \Big|_{t=0} \\ &= \sum_{x=1}^{\infty} 2x^2 e^{tx} \left(\frac{1}{3}\right)^x \Big|_{t=0} \\ &= \sum_{x=1}^{\infty} 2x^2 \left(\frac{1}{3}\right)^x \end{aligned}$$

7

We know that

$$f(x) = 1$$

when $0 < x < 1$, and $f(x) = 0$ elsewhere.

$$\begin{aligned}
 M_X(t) &= \mathbb{E}[e^{tX}] \\
 &= \int_0^1 e^{tx} 1 dx \\
 &= \frac{e^t - 1}{t} \\
 \mu'_1 &= \frac{d}{dt} M_X(t) \big|_{t=0} \\
 &= \frac{e^t(t-1) + 1}{t^2} \big|_{t=0} \\
 &= \lim_{t \rightarrow 0} \frac{e^t(t-1) + 1}{t^2} \\
 &= \frac{1}{2} \\
 \mu'_2 &= \frac{d^2}{dt^2} M_X(t) \big|_{t=0} \\
 &= \frac{d}{dt} \frac{e^t(t-1) + 1}{t^2} \big|_{t=0} \\
 &= \frac{e^t t^3 - 3e^t t + 2e^t - 2}{t^3} \big|_{t=0} \\
 &=
 \end{aligned}$$

8

$$\begin{aligned}
R_X(t) &= \log M_X(t) \\
R'_X(t) &= \frac{1}{M_X(t)} \frac{d}{dt} M_X(t) \\
R'_X(0) &= \frac{1}{M_X(0)} \frac{d}{dt} M_X(t)|_{t=0} \\
&= \frac{d}{dt} M_X(t)|_{t=0} \\
&= \mu \\
R''_X(t) &= \frac{d}{dt} R'_X(t) \\
&= \frac{M_X(t) \frac{d^2}{dt^2} M_X(t) - (\frac{d}{dt} M_X(t))^2}{M_X(t)^2} \\
R''_X(0) &= \frac{M_X(0) \frac{d^2}{dt^2} M_X(0) - (\frac{d}{dt} M_X(0))^2}{M_X(0)^2} \\
&= \mathbb{E}[X^2] - \mu^2 \\
&= \sigma^2
\end{aligned}$$

Let

$$M_X(t) = e^{4(e^t - 1)}$$

then

$$\begin{aligned}
R_X(t) &= \log M_X(t) \\
&= 4(e^t - 1) \\
R'_X(t) &= 4e^t \\
R'_X(0) &= 4 \\
R''_X(t) &= 4e^t \\
R''_X(0) &= 4
\end{aligned}$$

Thus, the mean of a random variable X is $\mu = 4$ and the variance of X is $\sigma^2 = 4$

9

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{+\infty} e^{tx} f(x) dx \\
&= \int_{-\infty}^{+\infty} e^{tx} \frac{1}{2} e^{-|x|} dx \\
&= \int_{-\infty}^0 e^{tx} \frac{1}{2} e^x dx + \int_0^{+\infty} e^{tx} \frac{1}{2} e^{-x} dx \\
&= \frac{1}{2} \left[\frac{1}{t+1} e^{(t+1)x} \Big|_{-\infty}^0 + \frac{1}{t-1} e^{(t-1)x} \Big|_0^{+\infty} \right]
\end{aligned}$$

It is easy to know that when $t \leq -1$ or $t \geq 1$, $M_X(t)$ is not defined. Thus,

$$M_X(t) = \frac{1}{1-t^2}$$

when $t \in (-1, 1)$

10

We get $f(x) = \frac{3.3^x}{x!} e^{-3.3}$ when $x \in \mathbb{N}$ and $f(x) = 0$ elsewhere.

10.1

$$P(X=2) = f(2) = \frac{3.3^2}{2} e^{-3.3} \approx 0.2008$$

10.2

$$\begin{aligned}
P &= P_2^2(f(0)f(5) + f(1)f(4) + f(2)f(3)) \\
&= 2 \times e^{-3.3} \times \left(\frac{3.3^0}{0!} \frac{3.3^5}{5!} + \frac{3.3^1}{1!} \frac{3.3^4}{4!} + \frac{3.3^2}{2!} \frac{3.3^3}{3!} \right) \\
&= 3.3^5 \times 2 \times e^{-3.3} \times \left(\frac{1}{0!} \frac{1}{5!} + \frac{1}{1!} \frac{1}{4!} + \frac{1}{2!} \frac{1}{3!} \right) \\
&\approx 3.8491
\end{aligned}$$

11

We get

$$f(x) = \frac{1}{5}e^{-\frac{x}{5}}$$

when $x > 0$ and $f(x) = 0$ elsewhere.

11.1

$$\begin{aligned} P &= \int_0^8 \int_0^{8-x} \frac{1}{5}e^{-\frac{x}{5}} \frac{1}{5}e^{-\frac{y}{5}} dy dx \\ &= \int_0^8 \frac{1}{5}e^{-\frac{x}{5}} (-e^{-\frac{y}{5}})|_{y=0}^{8-x} dx \\ &= \int_0^8 \frac{1}{5}e^{-\frac{x}{5}} (1 - e^{-\frac{8-x}{5}}) dx \\ &= \int_0^8 \frac{1}{5}(e^{-\frac{x}{5}} - e^{-\frac{8}{5}}) dx \\ &= (-e^{-\frac{x}{5}} - \frac{1}{5}e^{-\frac{8}{5}}x)|_{x=0}^8 \\ &= 1 - \frac{13}{5}e^{-\frac{8}{5}} \\ &\approx 0.4751 \end{aligned}$$

11.2

$$\begin{aligned} P &= \int_0^\infty \int_0^\infty \int_{12-x-y}^\infty \frac{1}{125}e^{-\frac{1}{5}(x+y+z)} dz dy dx \\ &= \int_0^\infty \int_0^\infty -\frac{1}{25}e^{-\frac{1}{5}(x+y)-\frac{1}{5}z}|_{z=12-x-y}^\infty dy dx \end{aligned}$$

12

We get

$$f(x) = \frac{1}{9}e^{-\frac{x}{9}}$$

when $x > 0$ and $f(x) = 0$ elsewhere

12.1

$$\begin{aligned}P &= \int_0^{20} f(x) dx \\&= \int_0^{20} \frac{1}{9} e^{-\frac{x}{9}} dx \\&= -e^{-\frac{x}{9}} \Big|_0^{20} \\&= 1 - e^{-\frac{20}{9}} \\&\approx 0.8916\end{aligned}$$

12.2

$$\begin{aligned}P &= \int_0^{20} \int_0^{20-x} \frac{1}{81} e^{-\frac{1}{9}(x+y)} dy dx \\&= \int_0^{20} -\frac{1}{9} e^{-\frac{1}{9}(x+y)} \Big|_0^{20-x} dx \\&= \int_0^{20} \frac{1}{9} (e^{-\frac{1}{9}x} - e^{-\frac{20}{9}}) dx \\&= -e^{-\frac{1}{9}x} - \frac{1}{9} e^{-\frac{20}{9}} x \Big|_0^{20} \\&= 1 - \frac{29}{9} e^{-\frac{20}{9}} \\&\approx 0.6508\end{aligned}$$