Introduction to Computability in Topological Spaces, with applications to $\mathbb{R}^n$

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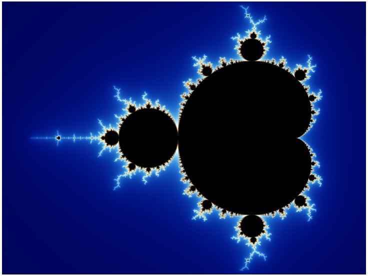
Overview

Motivations

The study of computability is central to computer science. The vanilla version of computability is defined for functions $\mathbb{N} \to \mathbb{N}$. What if we want to study computability on objects with continuum cardinality, say over $\mathbb{R}$? What about general topological spaces?

Generalizing the concept of computability allows us to study interesting questions such as the recursive property of the Mandelbrot set.



Model of Computation

RAM machine

A RAM machine consists of an infinitely long tape of integers and a list of instructions of one of 4 types: zero, increment, swap, jump.

The computation is done by having the RAM machine operate on an initial input on the tape, carrying out instructions sequentially and jump if necessary.

The machine halts when it reaches the end of its list of instructions. Note it may or may not halt. When it does, what is in the start of the tape is defined as the output.

Definition (Computable functions)

A (partial) function $\mathbb{N}^n \to \mathbb{N}$ is computable iff it can be simulated by a RAM machine.

Numbering of Computable Functions

We note that by the famous Church-Turing thesis, every intuitively computable function is equivalent to a RAM machine.

Countability

Computable functions are countable. An enumeration is called a *Gödel numbering*.

Theorem (Universal machine)

There exists a universal machine ϕ_U, s.t. $\phi_U(x, y) = \phi_x(y)$.

Uncomputable functions

Cardinality tells us there exist uncomputable functions. The halting problem is one example: for input x, y, output 1 if machine x halts on input y and 0 if not.

Recursive and Recursive Enumerable Sets

For a set $A \subset \mathbb{N}$,

$$c_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is called its *characteristic function*.

Definition (Recursive set)

$A \subset \mathbb{N}$ is called *recursive* if its characteristic function is computable.

Definition (R.e. set)

$A \subset \mathbb{N}$ is called *recursive enumerable (r.e.)* if it is the domain of a computable function. It can be proved that a set is r.e. iff it is the *range* of a computable function.

Theorem

A set is recursive iff itself and its complement are r.e.

Computability on Sequences

How do we generalize to sets with cardinality greater than $\mathbb{N}$? We consider sequences of $\mathbb{N}$!

Definition (Computable functions of sequences)

Let Σ be a finite alphabet and let Σ^ω denote the set of sequences.

A function $f : \Sigma^\omega \to \Sigma^\omega$ is called computable if there exists a computable function (defined on countable set) that carries finite prefixes of input to finite prefixes of output.

Alternatively, we may characterize computability by a Turing machine that allows infinite input and output, but the output tape may only move in one direction.

Computability on Sequences

Definition (Computable sequences)

A sequence $p \in \Sigma^\omega$ is computable if it is the output of a constant computable function.

Definition (Recursive and r.e. sets)

A subset of Σ^ω is r.e. iff it is the domain of a computable function. A r.e. set is recursive iff its complement is also r.e.

Computable functions are nice with respect to a specific topology one Σ^ω.

Theorem (Computable functions are continuous)

The discrete topology on Σ is defined such that every subset is open. The Cantor topology on Σ^ω is the product topology of discrete topology on Σ. Computable functions are continuous under the Cantor topology.

Sequence Representation

Definition (Representation)

A representation of set M is a (partial) surjection $\delta : \Sigma^\omega \to M$.

Different representations relate to one another via a translation.

Definition (Reducibility)

ϕ is called a translation if the following diagram commutes.

$$\begin{tikzcd} \Sigma^\omega \arrow[r, "\phi"] \arrow[d, "\delta"'] & \Sigma^\omega \arrow[dl, "\delta'"] \\ M & \end{tikzcd}$$

If there is a computable (continuous) translation between δ and δ', say δ is (continuously) reducible to δ', and denote as $\delta < \delta'$ ($\delta <_t \delta'$). Say δ is (continuously) equivalent to δ' if reducibility goes both way and denote $\delta \equiv \delta'$ ($\delta \equiv_t \delta'$). Note that $\equiv$ is an equivalence relation.

Computability on M

Now computability on sequences transfers naturally to M.

Definition (Computability on M)

Let δ be a representation $\Sigma^\omega \to M$.

1. An element $x \in M$ is called δ-computable if for some computable y, $x = \delta(y)$.
2. If exists a diagram:

$$\begin{array}{ccc} \Sigma^\omega & \xrightarrow{g} & \Sigma^\omega \\ \delta \downarrow & & \downarrow \delta \; (\delta') \\ M & \xrightarrow{f} & M \end{array}$$

   s.t. g is computable (continuous), f is called δ-computable (continuous).
3. A set $X \subset M$ is called δ-recursive (r.e., open) if $\delta^{-1}(X)$ is recursive (r.e., open)

Representing $\mathbb{R}$

The first representation that comes to mind is the decimal representation. But multiplying by 3 is uncomputable! We must consider an alternative.

Definition (Cauchy Representation)

The Cauchy representation $\rho_C : \Sigma^\omega \to \mathbb{R}$ is defined s.t.:
Let $p \in \Sigma^\omega$ be a sequence of (representation of) rational numbers w_i.
$\rho_C(p) = x$ if and only if $|w_i - w_k| < 2^{-i}$ for $k > i$ and $x = \lim w_i$.

ρ_C has all the expected properties.

Example

Under ρ_C:
e, π, $\sqrt{2}$, $\log_3 5$ are computable numbers.
$x + y$, xy, $1/x$, e^x, $\sin x$ are computable functions.

ρ_C is considered the standard representation of $\mathbb{R}$. Note that the naïve version of representation via unconstraint Cauchy sequence has some similar properties but is not considered standard. We shall soon see why.

Admissible Representation

We have seen that not all representations are useful. Of particular interest are representations induced by a topology.

Definition (Computable topology)

Let (M, τ) be a T_0-space with a countable subbasis σ. Let $\rho : \Sigma^* \to \sigma$ be a (finite) notation for σ. Call the tuple (M, τ, ρ) a **computable topological space** if the equivalence problem $\{(u, v) | \rho(u) = \rho(v)\}$ is r.e.

Definition (Standard Representation)

Let $\delta_S : \Sigma^\omega \to M$ be defined as:

$$\delta_S(p) = x \iff \{A \in \sigma \mid x \in A\} = \{\rho(w) | w \text{ a subword of } p\}$$

δ_s is called the standard representation of M.

In English, we are simply saying "p denotes x if and only if p includes all names for $A \in \sigma$ that contains x".

Admissible Representation

The standard representation has a number of great properties which we shall not belabor, but only attempt to convince your of its importance by some frantic hand-waving.

Definition (Admissible Representation)

δ is said to be admissible w.r.t. τ if $\delta \equiv_t \delta_S$.

There is one crucial property for admissible representations that justifies the previous hand-waving.

Theorem (Continuity agreement)

Let δ_1, δ_2 be admissible representations of $(M_1, \tau_1), (M_2, \tau_2)$.
Let $f : M_1 \to M_2$. Then:
f is continuous (as defined by τ_1, τ_2) $\iff$ f is (δ_1, δ_2)-continuous.

Corollary (Punchline)

Under admissible representations, computable functions are continuous.

$\mathbb{R}$ Revisited

Now we present the real reason why ρ_C is selected as the standard representation of $\mathbb{R}$.

Theorem (Cauchy is standard)

ρ_C is admissible w.r.t. the standard topology on $\mathbb{R}$.

Proof.

δ_S translates to nested interval translates to ρ_C. Yupyup. (Handwave or elaborate depending on time.) $\square$

Corollary (Mega-punchline)

All computable functions in $\mathbb{R}$ are continuous.

If we recall more esoteric topologies on $\mathbb{R}$, such as $\tau_>$, $\tau_<$ generated by infinite open intervals (open rays), we obtain two more representations of $\mathbb{R}$, $\rho_>$ and $\rho_<$, that are of some theoretical interest.

Open and Closed Subsets

Defining representations for open and closed subsets help us investigate their recursive properties.

Definition (Representing open sets)

The standard representation of open sets, δ_o, is defined as follows. Let p encode rational open intervals $\{(u_i, v_i)\}$. $\delta_o(p) = X$ if and only if $X = \cup(u_i, v_i)$.

Representation of closed sets can be defined by simply looking at complements.

Theorem (Properties of δ_o)

1. *X is δ_o-computable $\iff$ X is ρ_C-r.e.*
2. *Union and intersection are δ_o-computable.*

Compact Subsets

Definition (Representations of compact sets)

1. (closed and bounded) κ_{cb}, which takes a representation q of open sets together with a rational number r. $\kappa_{cb}(r, q) = X$ if and only if $X = \mathbb{R} \setminus \delta_o(q)$ and $X \subset [-r, r]$.
2. (weak covering) κ_w, which takes $\{w_i\}$, sequence of notations for (rational) **finite open covers**. $\kappa_w(p) = X$ if and only if p is exactly all open covers of X.
3. (strong covering) κ, defined similar to κ_w, but p is only allowed to contain 'minimal' open covers, i.e. for open cover $\cup I_i$ of X, $X \cap I_i \neq \emptyset$ for all I_i.

Theorem (Strong covering is nice)

A compact set is recursive if and only if it is κ-computable.

Theorem (Computational Heine-Borel)

$$\kappa_w \equiv \kappa_{cb}$$

Further Studies

- Computable analysis; Limit of sequences and sequences of functions, power series, continuous functions, differentiation and integration; complexity.
- Recursive properties of dynamical systems; Julia and Mandelbrot sets.
- Representations and computability of other topological structures; manifolds
- Alternative models of computability; real RAM.

That's all, folks! Questions?