Equivalence of Stiffness Matrix Calculated in Current and Reference Configuration

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We use $\mathbf{F}$ to denote the deformation gradient, $\mathbf{P}$ to denote the First Piola Kirchhoff stress, $\mathbf{S}$ to denote the Second Piola Kirchhoff stress, $\boldsymbol{\sigma}$ to denote the Cauchy stress, $\mathbf{x}$ to denote the current position, $\mathbf{X}$ to denote the reference position, N to denote the shape function in the FEM discretization, $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ to denote the Green-Lagrangian strain tensor, and $C_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}}$ to denote the elasticity tensor in the reference configuration. We use subscripts i, j, k, l, m, n, p, q and their upper case counterparts to denote spacial dimension indices which can take values from 1, 2 and 3. We use superscripts a, b to denote local element indexes, ranging from 1 to the number of element nodes.

The stiffness matrix given by [1] without the external force component is

$$[K_{ab}]_{mn} = [K^c_{ab}]_{mn} + [K^\sigma_{ab}]_{mn}. \tag{1}$$

$[K^c_{ab}]_{mn}$ is given by (9.35) in [1]:

$$[K^c_{ab}]_{mn} = \int \frac{\partial N^a}{\partial x_i} c_{minj} \frac{\partial N^b}{\partial x_j} d\mathbf{x},$$

in which c_{ijkl} is the current configuration elasticity tensor that satisfies $Jc_{ijkl} = F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL}$ ((8.12d) in [1]).

$[K^\sigma_{ab}]_{mn}$ is given by (9.44c) in [1]

$$[K^\sigma_{ab}]_{mn} = \int \frac{\partial N^a}{\partial x_i} \sigma_{ij} \frac{\partial N^b}{\partial x_j} \delta_{mn} d\mathbf{x}.$$

The stiffness matrix in our implementation is given by

$$[K_{ab}]_{mn} = \int \frac{\partial F_{iI}}{\partial x^a_m} \frac{\partial P_{iI}}{\partial F_{jJ}} \frac{\partial F_{jJ}}{\partial x^b_n} d\mathbf{X}. \tag{2}$$

We will show that the stiffness matrix we implement in Equation 2 is analytically equivalent to the one in Equation 1.

To see the equivalence, we make use of the identity $\mathbf{P} = \mathbf{FS}$, which implies

$$\begin{aligned}
\frac{\partial P_{iI}}{\partial F_{jJ}} &= \frac{\partial F_{iK} S_{KI}}{\partial E_{PQ}} \frac{\partial E_{PQ}}{\partial F_{jJ}} \\
&= \frac{\partial F_{iK}}{\partial E_{PQ}} S_{KI} \frac{\partial E_{PQ}}{\partial F_{jJ}} + F_{iK} C_{KIPQ} \frac{\partial E_{PQ}}{\partial F_{jJ}} \\
&= \delta_{ij}\delta_{KJ} S_{KI} + F_{iK} C_{KIPQ} \left(\frac{1}{2} \left(\delta_{PJ} F_{jQ} + F_{jP} \delta_{QJ} \right) \right) \\
&= \delta_{ij} S_{IJ} + F_{iK} C_{KIJL} F_{jL}
\end{aligned} \tag{3}$$

where the last equality uses the symmetries $S_{IJ} = S_{JI}$ and $C_{IJKL} = C_{IJLK}$.

Plugging Equation 3 into Equation 2 and using the fact that $\frac{\partial F_{kL}}{\partial x_j^a} = \delta_{jk} \frac{\partial N^a}{\partial X_L}$, we get:

$$
\begin{aligned}
[K_{ab}]_{mn} &= \int \frac{\partial F_{iI}}{\partial x_m^a} \frac{\partial P_{iI}}{\partial F_{jJ}} \frac{\partial F_{jJ}}{\partial x_n^b} d\mathbf{X}. \\
&= \int \delta_{im} \frac{\partial N^a}{\partial X_I} \frac{\partial P_{iI}}{\partial F_{jJ}} \delta_{jn} \frac{\partial N^b}{\partial X_J} d\mathbf{X} \\
&= \int \delta_{im} \frac{\partial N^a}{\partial X_I} \delta_{ij} S_{IJ} \delta_{jn} \frac{\partial N^b}{\partial X_J} d\mathbf{X} \int \delta_{im} \frac{\partial N^a}{\partial X_I} F_{iK} C_{KIJL} F_{jL} \delta_{jn} \frac{\partial N^b}{\partial X_J} d\mathbf{X} \\
&= \int \delta_{im} \frac{\partial N^a}{\partial x_i} F_{iI} \delta_{ij} S_{IJ} \delta_{jn} \frac{\partial N^b}{\partial x_j} F_{jJ} d\mathbf{X} \\
&\quad + \int \delta_{im} \frac{\partial N^a}{\partial x_p} F_{pI} F_{iK} C_{KIJL} F_{jL} \delta_{jn} \frac{\partial N^b}{\partial x_q} F_{qJ} d\mathbf{X} \\
&= \int \delta_{mn} \frac{\partial N^a}{\partial x_i} F_{iI} S_{IJ} \frac{\partial N^b}{\partial x_j} F_{jJ} d\mathbf{X} \\
&\quad + \int \frac{\partial N^a}{\partial x_p} F_{pI} F_{mK} C_{KIJL} F_{nL} F_{qJ} \frac{\partial N^b}{\partial x_q} d\mathbf{X} \\
&= \int \frac{\partial N^a}{\partial x_i} \sigma_{ij} \frac{\partial N^b}{\partial x_j} \delta_{mn} d\mathbf{x} + \int \frac{\partial N^a}{\partial x_p} c_{mpqn} \frac{\partial N^b}{\partial x_q} d\mathbf{x} \\
&= \int \frac{\partial N^a}{\partial x_i} \sigma_{ij} \frac{\partial N^b}{\partial x_j} \delta_{mn} d\mathbf{x} + \int \frac{\partial N^a}{\partial x_i} c_{minj} \frac{\partial N^b}{\partial x_j} d\mathbf{x} \\
&= [K_{ab}^c]_{mn} + [K_{ab}^\sigma]_{mn}
\end{aligned},
$$

where the sixth equality uses the identity $\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T$ and the seventh equality uses the symmetry $c_{ijkl} = c_{ijlk}$ in the elasticity tensor.

References

[1] Bonet, J., Gil, A.J. and Wood, R.D *Nonlinear solid mechanics for finite element analysis: statics.* Cambridge University Press, 2016.