Calculus A(2) Spring 2021 Midterm Exam

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1. (40 points) For each of the following questions choose **one answer** from A to D.

   (a) A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L. Which of the following statements is true?

   A. There is always n such that $a_n = L$.
   B. There is always n such that $a_n < L$ or $a_n > L$.
   C. There is always n such that $L - 1 < a_n < L + 1$.
   D. None of the above

   C

   (b) Which of the following statements is true about the infinite series $\sum_{n=1}^{\infty} (-1)^n/(n^2+n)$?

   A. The series converges absolutely.
   B. The series converges conditionally.
   C. The series diverges.
   D. The convergence of the series depends on how the summation is performed.

   A

   (c) What is the coefficient of the second (nonvanishing) term in the Taylor series generated by $\sin x$ at $x = \pi/4$?

   A. 1
   B. $1/\sqrt{2}$
   C. $-1/3!$
   D. None of the above

   D

   (d) Halley's Comet has an orbital period of 75.32 years. Which of the following best describes its orbit?

   A. A circle
   B. An ellipse
   C. A parabola
   D. A hyperbola

   B

   (e) A point in a plane has polar coordinates $(r, \theta) = (1, 125\pi/2)$. What is the Cartesian coordinate (x, y) of this point?

   A. $(1, 0)$
   B. $(0, 1)$
   C. $(-1, 0)$
   D. $(0, -1)$

   B

   (f) What is the dot product $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$?

   A. 0
   B. 1
   C. $-(2\cos t)\mathbf{i} - (2\sin t)\mathbf{j} + \mathbf{k}$
   D. None of the above

   A

(g) What is the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} + 9\mathbf{j}$?

A. 0 (scalar)
B. $\mathbf{0}$ (vector)
C. 42
D. None of the above

B

(h) Which of the following equations describes the line in space that is parallel to the vector $\mathbf{i} + \mathbf{j}$ and goes through point $(x, y, z) = (1, 1, 1)$?

$1+t \quad 1+t, 1$

A. $\mathbf{r}(t) = (1+2t)\mathbf{i} + (1+2t)\mathbf{j} + \mathbf{k}, \ -\infty < t < \infty$
B. $\mathbf{r}(t) = (1+t)\mathbf{i} + (1+t)\mathbf{j} + (1+2t)\mathbf{k}, \ -\infty < t < \infty$
C. $\mathbf{r}(t) = (1+t)\mathbf{i} + (1+t)\mathbf{j} + t\mathbf{k}, \ -\infty < t < \infty$
D. None of the above

A

(i) A point P_0 lies in a plane in space, and two different vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are both perpendicular to this plane. Which of the following equations describes this plane (as the set of points P satisfying it)?

A. $\mathbf{n}_1 \cdot \overrightarrow{P_0P} = \mathbf{n}_2 \cdot \overrightarrow{P_0P}$.
B. $\mathbf{n}_1 \times \overrightarrow{P_0P} = \mathbf{n}_2 \times \overrightarrow{P_0P}$.
C. $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \overrightarrow{P_0P} = 0$.
D. None of the above

A

(j) A particle is moving with constant acceleration along the circle $x^2 + y^2 = 1$ in space. Which of the following statements is true about the position vector $\mathbf{r}$ and the velocity $\mathbf{v}$ of the particle?

A. $\mathbf{r} \cdot \mathbf{v} = 0$.
B. $\mathbf{r} \times \mathbf{v} = 0$.
C. $|\mathbf{v}|$ is constant.
D. None of the above is true.

A

2. (18 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n}-\sqrt{n-1}}{n} x^n .$$

let $u_n = \frac{\sqrt{2n} - \sqrt{n-1}}{n} x^n$

we need $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{\sqrt{2n+2} - \sqrt{n}}{n+1} \cdot \frac{n}{\sqrt{2n}-\sqrt{n-1}} \cdot x \right| < 1$

$\Rightarrow |x| \lim_{n\to\infty} \left| \frac{n+2}{n+1} \cdot \frac{1}{\sqrt{2n+2}+\sqrt{n}} \cdot \frac{n}{n+1} \cdot \frac{\sqrt{2n}+\sqrt{n-1}}{1} \right| < 1$

$\Rightarrow |x| \lim_{n\to\infty} \left| \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \frac{\sqrt{2}+\sqrt{1-\frac{1}{n}}}{\sqrt{2+\frac{2}{n}}+\sqrt{1}} \cdot \frac{1}{1+\frac{1}{n}} \right| < 1$

$\Rightarrow |x| \cdot \left| \frac{1+0}{1+0} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} \cdot \frac{1}{1} \right| < 1 \quad \Rightarrow |x| < 1$

check $x=-1$, the series is diverge.

Check $x=1$, let $b_n = \frac{1}{n^{0.5}}$, the $\lim_{n\to\infty} \frac{\frac{\sqrt{2n}-\sqrt{n-1}}{n}}{\frac{1}{n^{0.5}}} = \lim_{n\to\infty} \sqrt{2} - \sqrt{1-\frac{1}{n}} = \sqrt{2}-1$, as $\frac{1}{n^{0.5}}$ dive

$\frac{\sqrt{2n}-\sqrt{n-1}}{n}$ also diverge

So $-1 < x < 1$ is the interval of convergence.

3. (12 points) Calculate

$$\lim_{x\to 0} \frac{1}{7!} \cdot \frac{[\ln(1+x)]^7}{\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}} .$$

We have $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ and

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

So. $\lim_{x\to 0} \frac{1}{7!} \cdot \frac{[\ln(1+x)]^7}{\sin x - x + \frac{x^3}{6} - \frac{x^5}{120}} = \lim_{x\to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)^7}{7!\left(-\frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)} = \lim_{x\to 0} \frac{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^7}{7!\left(-\frac{1}{7!} + \frac{x^2}{9!} - \frac{x^4}{11!} + \cdots\right)}$

$= \lim_{x\to 0} \frac{(1-0+0-\cdots)^7}{-1+7!(0-0+0\cdots)} = -1$

4. (12 points) Consider the infinite series $\sum_{n=0}^{\infty} 3(-2\cos\theta)^n$.

(a) For what values of θ does the series converge?

when $-1 < -2\cos\theta \le 1$

$\Rightarrow -\frac{1}{2} \le \cos\theta < \frac{1}{2}$

$\Rightarrow 2k\pi + \frac{\pi}{3} < \theta \le 2k\pi + \frac{2}{3}\pi$ or $2k\pi + \frac{4}{3}\pi \le \theta < 2k\pi + \frac{5}{3}\pi$

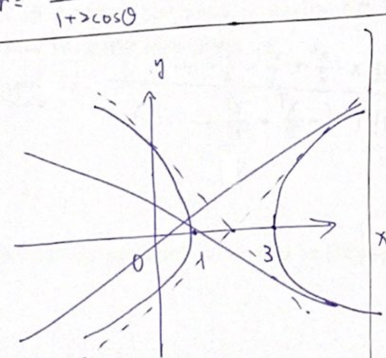
which θ gives the series converge.

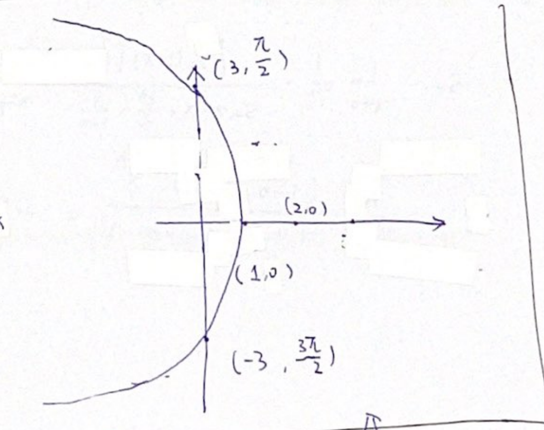
(b) For the values of θ found in (a), let $f(\theta)$ be the sum of the series. Sketch the graph of the equation $r = f(\theta)$ in polar coordinates (r, θ).

$$r = f(\theta) = \begin{cases} \frac{3}{1+2\cos\theta} & \text{where } \cos\theta \neq -\frac{1}{2} \\ 3 & \text{where } \cos\theta = -\frac{1}{2} \end{cases}$$

as $\begin{cases} ke = 3 \\ e = 2 \\ ke = a(1-e^2) \end{cases} \Rightarrow \begin{cases} k = 1.5 \\ e = 2 \end{cases}$

with $r = \frac{3}{1+2\cos\theta} \Rightarrow r + 2r\cos\theta = 3 \Rightarrow \sqrt{x^2+y^2} + 2x = 3 \Rightarrow (x-2)^2 - \frac{y^2}{3} = 1$





this.

5. (18 points) Alice is in a spaceship that is moving with velocity $\mathbf{v}(t) = -2\sin(2t)\mathbf{i} + \mathbf{j} + 4\cos(2t)\mathbf{k}$ at time t.

(a) Alice is at point $(x, y, z) = (1, 1, 1)$ at $t = 0$. Find the acceleration $\mathbf{a}(t)$ and the position $\mathbf{r}(t)$ of Alice at time t.

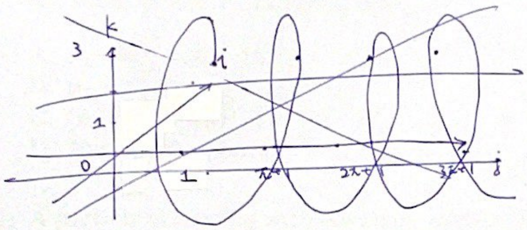
$$\vec{a}(t) = \frac{V(t)}{dt} = -4\cos(2t)\,\vec{i} - 8\sin(2t)\,\vec{k}$$

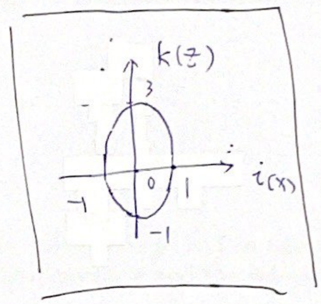
$$\frac{d\vec{r}}{dt} = \vec{V}, \quad \text{so } \vec{r} = \cos(2t)\vec{i} + t\vec{j} + 2\sin(2t)\vec{k} + C$$

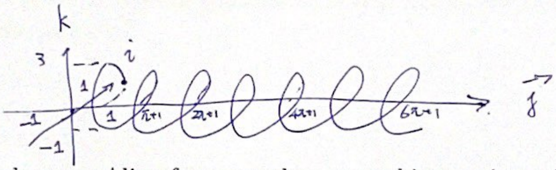
$$\text{with } \vec{r}(0) = \vec{i} + \vec{j} + \vec{k} = \cos 0\,\vec{i} + 0\,\vec{j} + 2\sin 0\,\vec{k} + C = \vec{i} + C. \Rightarrow C = \vec{j} + \vec{k}$$

$$\text{so } \vec{r}(t) = (\cos 2t)\vec{i} + (t+1)\vec{j} + (1 + 2\sin 2t)\vec{k}.$$

(b) Sketch the trajectory of Alice, and sketch its projection to the xz-plane.







(c) Bob observes Alice from another spaceship moving with velocity $\mathbf{w}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{k}$. Find the arc length that Alice travels from $t = 0$ to $t = 1$, as measured by Bob (that is, measured in a coordinate system whose origin is Bob's location).

$$\cancel{\text{Arc length: } L = \int_0^1 |V|\,dt = \int_0^1 \sqrt{\cos^2 2t + (t+1)^2 + (1+\sin 2t)^2}}\,dt.$$

From Bob:

$$L' = \int_0^1 |V(t) - w(t)|\,dt.$$

$$= \int_0^1 \left| -3\sin(2t)\,\vec{i} + \vec{j} + 3\cos(2t)\,\vec{k} \right| dt.$$

$$= \int_0^1 \sqrt{10}\,dt$$

$$= \sqrt{10}.$$