第二章 解析函数 analytic functions/mappings

函数：$R^n \longrightarrow R^1$

映射：$R^n \longrightarrow R^m$ $(m\geq 1)$

$f(z)=u(x,y)+iv(x,y)$

导数 若 $\lim\limits_{\substack{\Delta z\neq 0\\ \Delta z\to 0}}\dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ 存在且有限，则称$f(z)$在z_0可导，记为$f'(z_0)$

Cauchy-Riemann条件（C-R条件）

$f(z)=u(x,y)+iv(x,y)$ 在$z_0=x_0+iy_0$可导 $\Longleftrightarrow$ 在z_0处：$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$，$\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

证明：$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$，$df=f'(z)dz=du+idv$

设$f'(z)=a+ib$ $(a,b\in R)$ 则 $df=(a+ib)dz=(a+ib)(dx+idy)=(adx-bdy)+i(bdx+ady)$

$=du+idv=(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)+i(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy)$

$$\Rightarrow\begin{cases}adx-bdy=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\\ bdx+ady=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\end{cases}\Rightarrow\begin{cases}(a=)\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ (b=)\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\end{cases}$$ 且有 $f'(z)=a+ib=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

[推广] 若$u,v\in C^n$，则 $f^{(n)}(z)=\dfrac{\partial^n u}{\partial x^n}+i\dfrac{\partial^n v}{\partial x^n}$ （不断求导即可）

例1. 若$f(z)=u+iv$ 处处可导，且$u=u_0$是常数，则$f(z)$是常数

证：
$$\begin{matrix}\frac{\partial u}{\partial x}=\frac{\partial u_0}{\partial x}=0\\ \frac{\partial u}{\partial y}=\frac{\partial u_0}{\partial y}=0\end{matrix}\Rightarrow\begin{cases}\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=0\\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=0\end{cases}\Rightarrow v=v_0=const\Rightarrow f(z)=u_0+iv_0=const$$

[推广] 若$f(z)=u+iv$ 处处可导，且 $\exists a,b,c\in R$ $au+bv+c=0$，则$f(z)=u+iv$是常数

初等函数

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)=u+iv\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=e^x\cos y,\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=e^x\sin y$$

$\Rightarrow f(z)=e^z$ 处处可导，$f'(z)=e^z$

由$\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$，$\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2}$

定义：$\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$ $\quad\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$

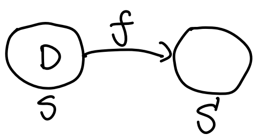
全纯函数/整函数 在C上处处可导的函数

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}=\frac{e^{y}+e^{-y}}{2}\cos x+i\frac{e^{-y}-e^{y}}{2}\sin x$$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}=\frac{e^{y}+e^{-y}}{2}\sin x+i\frac{e^{y}-e^{-y}}{2}\cos x$$

$$\Rightarrow \mathrm{Re}(\cos z)=\frac{e^{y}+e^{-y}}{2}\cos x,\quad \mathrm{Im}(\cos z)=\frac{e^{-y}-e^{y}}{2}\sin x$$

Jacobi矩阵 $J=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}$ 已知f处处可导



$$S'=\iint_{D'}dudv=\iint_{D}|\det(J)|dxdy=\iint_{D}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)dxdy$$

$$=\iint_{D}|f'(z)|^2dxdy\geqslant 0$$ 仅$f'(z)\equiv 0$取等

故 $f=\text{const} \Leftrightarrow f'(z)\equiv 0 \Leftrightarrow S'=0$

对于处处可导函数f，只能把二维映到二维/零维

解析 $f(z)$在z_0解析 $\Longleftrightarrow$ $f(z)$在z_0的一个邻域内处处可导

$\Longleftrightarrow$ $f(z)$在z_0的一个邻域内任意阶可导

定理

$f(z)$在z_0解析 $\Leftrightarrow$ $\exists\delta, |z-z_0|<\delta, f(z)=\sum\limits_{n=0}^{\infty}\frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$

例: $f(x)=\begin{cases}e^{-\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$

但$f^{(n)}(0)=0$, $n=0,1,2,\cdots$ $\Rightarrow \sum\limits_{n=0}^{\infty}\frac{f^{(n)}(0)z^n}{n!}=0$

故 $f(x)\neq\sum\limits_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}\ (x\neq 0)$

$f(z)$在0处甚至不连续 ($\lim\limits_{y\to 0}f(yi)=+\infty$)

Attention

极坐标: $z=re^{i\theta}$ $r=|z|$ $\theta=\arg z$

要求: $z\neq 0$

$z\neq 0$时

$\ln z=\ln(re^{i\theta})=\ln r+i\theta=\ln|z|+i\arg z$

$(\ln z)'=\frac{1}{z}$

$\mathrm{Ln}\, z=\ln z+2k\pi i$

$\begin{cases}\mathrm{Ln}\, z^2=\ln z^2+2k\pi i\\ 2\mathrm{Ln}\, z=\ln z^2+4k\pi i\end{cases}$ $\mathrm{Ln}\, z^2\neq 2\mathrm{Ln}\, z$ 集合不同

例. f可导. $|f|=c$ 或 $\arg f=c$ (c为常数)
则 f 是常函数

证: 若$f\equiv 0$, 则✓　若$f\not\equiv 0$, 则$\forall z$, $f(z)\neq 0$. 令$g(z)=\ln f(z)=\ln|f(z)|+i\arg f(z)=u+iv$

$g'(z)=\dfrac{f'(z)}{f(z)}$ ($f(z)\neq 0$时)　f可导$\Rightarrow g$可导$\Rightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}=0\\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}=0\end{cases}$

$\therefore g'(z)=0\Rightarrow g(z)$常$\Rightarrow f(z)$常

必考题 (cos 或 sin 版本)

求$\cos(x+iy)$的实部、虚部, 并证明$\forall A,B\in R$,
方程$\cos(x+iy)=A+iB$有无穷多解

解: $\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$　令$z=x+iy$　$x,y\in R$

$$\Rightarrow\cos(x+iy)=\frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}=\frac{e^{-y}e^{ix}+e^{y}e^{-ix}}{2}$$

$$=\frac{e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)}{2}$$

$$=\frac{e^{y}+e^{-y}}{2}\cos x+i\frac{e^{-y}-e^{y}}{2}\sin x$$

$$\Rightarrow \mathrm{Re}(\cos(x+iy))=\frac{e^{y}+e^{-y}}{2}\cos x,\quad \mathrm{Im}(\cos(x+iy))=\frac{e^{-y}-e^{y}}{2}\sin x$$

证明: $$\begin{cases}\dfrac{e^{y}+e^{-y}}{2}\cos x=A\\ \dfrac{e^{-y}-e^{y}}{2}\sin x=B\end{cases}\quad(*)$$

(1). $B=0 \Rightarrow \sin x=0$ 或 $y=0$

① $|A|\le 1$，取 $y=0 \Rightarrow \cos x=A \Rightarrow x=x_k=\arccos A+2k\pi \quad k\in Z$

② $|A|>1$，取 $\sin x=0 \Rightarrow \cos x=\pm 1 \Rightarrow \dfrac{e^y+e^{-y}}{2}(\pm 1)=A$

当 $A>1$，取 $\cos x=1$，即 $x=2k\pi=x_k\ (k\in Z) \Rightarrow \dfrac{e^y+e^{-y}}{2}=A>1$

令 $f(y)=\dfrac{e^y+e^{-y}}{2}-A \quad (y>0)$ $\quad \begin{cases} f(0)=1-A<0 \\ f(+\infty)=+\infty-A>0 \end{cases} \Rightarrow$ 彐唯一 $y_A>0$，使 $f(y_A)=0$

$f'(y)=\dfrac{e^y-e^{-y}}{2}>0$ $\qquad \Rightarrow f(-y_A)=0$

$\therefore$ 当 $A>1$，$y=\pm y_A$，$\underset{\substack{\|\\ x_k}}{x=\dfrac{\pi}{2}+2k\pi}\ (k\in Z)$ 无穷多 $z_k=x_k+iy_A$

当 $A<-1$，..................

(2) $B\ne 0$

由 (*) 和 $\sin^2 x+\cos^2 x=1 \Rightarrow \dfrac{4A^2}{(e^y+e^{-y})^2}+\dfrac{4B^2}{(e^y-e^{-y})^2}=1 \quad (y\ne 0)$

令 $g_{AB}(y)=\dfrac{4A^2}{(e^y+e^{-y})^2}+\dfrac{4B^2}{(e^y-e^{-y})^2} \quad (y\ne 0)$ 偶函数

$B\ne 0 \Rightarrow \lim\limits_{y\to 0^+} g_{AB}(y)=+\infty$，$\lim\limits_{y\to +\infty} g_{AB}(y)=0$

由 $g_{AB}(y)$ 在 $(0,+\infty)$ 连续 $\Rightarrow$ 彐 $y_{AB}>0$，使 $g_{AB}(y_{AB})=1 \Rightarrow y=\pm y_{AB}$

$\Rightarrow \dfrac{e^{y_{AB}}+e^{-y_{AB}}}{2}\cos x=A \Rightarrow x=x_k$ 有无穷多解

证毕

$a^b \quad a\neq 0\ (a\neq e),\ a,b\in C$

$$a^b = e^{b\operatorname{Ln} a} = e^{b(\ln a+2k\pi i)} \quad k\in Z$$

注: $a=e$时, 定义$e^b = e^{x+yi}$
$= e^x(\cos y+i\sin y)$

例: $1^{\frac{1}{n}} = e^{\frac{1}{n}\operatorname{Ln}1} = e^{\frac{2k\pi}{n}i} = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n} \quad k=0,1,\cdots,n-1$

$1^{\sqrt{2}} = e^{\sqrt{2}\operatorname{Ln}1} = e^{2\sqrt{2}k\pi i} = \cos(2\sqrt{2}k\pi)+i\sin(2\sqrt{2}k\pi) \quad k\in Z$

记 $z_k=e^{2\sqrt{2}k\pi i}$ 若$z_k=z_m$, 则 $e^{2\sqrt{2}(k-m)\pi i}=1=e^{2n\pi i}$

$\Rightarrow \exists n\in Z$, 使 $2\sqrt{2}(k-m)\pi i=2n\pi i \Rightarrow \sqrt{2}(k-m)=n \Rightarrow k=m,\ n=0$

$\therefore z_k$无周期, 完全互异 $|z_k|=1$ $\{z_k\}$为可数集 (与N一一对应)

$\{z_k\}$在$|z|=1$上构成可数稠密集合

$$i^i = e^{i\operatorname{Ln}i} = e^{i(\ln i+2k\pi i)} = e^{i(\frac{\pi}{2}+2k\pi)i} = e^{-(\frac{\pi}{2}+2k\pi)} \in R$$

$$e^z=\sum_{n=0}^{+\infty}\frac{z^n}{n!}$$

$$\sin z=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}=z\prod_{n=1}^{+\infty}\left(1-\frac{z^2}{n^2\pi^2}\right) \qquad \sin(k\pi)=0 \quad k\in Z$$

$$\cos z=\sum_{n=0}^{+\infty}\frac{(-1)^n z^{2n}}{(2n)!}=\prod_{n=1}^{+\infty}\left(1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right) \qquad \cos(k\pi+\frac{\pi}{2})=0 \quad k\in Z$$

$$\Rightarrow \frac{\sin z}{z}=\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\cdots=1-\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots\right)z^2+(\cdots)z^4-\cdots$$

$$=1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\cdots$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ 一般地, $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=C_k\pi^{2k}$ $(C_k\in Q^+ \quad k=1,2,\cdots)$

递推公式: $C_n=\frac{2}{2n+1}\sum_{k=1}^{n-1}C_kC_{n-k}$

$$C_n-\frac{C_{n-1}}{3!}+\frac{C_{n-2}}{5!}-\cdots+\frac{(-1)^{n-1}C_1}{(2n-1)!}+\frac{(-1)^n n}{(2n+1)!}=0 \quad (n\geq 2)$$

$$\ln(\sin z) = \ln z + \sum_{n=1}^{+\infty} \ln\left(1 - \frac{z^2}{n^2\pi^2}\right) \qquad \ln(1-x) = -\sum_{k=1}^{+\infty} \frac{x^k}{k}$$

$$= \ln z - \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \frac{z^{2k}}{k n^{2k} \pi^{2k}}$$

$$= \ln z - \sum_{k=1}^{+\infty}\left(\sum_{n=1}^{+\infty} \frac{1}{n^{2k}}\right)\frac{z^{2k}}{k\pi^{2k}}$$

$$= \ln z - \sum_{k=1}^{+\infty} \frac{z^{2k}}{k\pi^{2k}} \zeta(2k) \qquad \zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}$$ Rieman Zeta 函数

$$\xrightarrow{\text{求导}} \frac{\cos z}{\sin z} = \frac{1}{z} - 2\sum_{k=1}^{+\infty} \frac{z^{2k-1}}{\pi^{2k}} \zeta(2k)$$

$$\Rightarrow z\cos z = \sin z \left(1 - 2\sum_{k=1}^{+\infty} \frac{z^{2k}}{\pi^{2k}} \zeta(2k)\right)$$ 大伯有点伤了

$$\prod_{n=1}^{+\infty}\left(1 + \frac{1}{n^2}\right) = \frac{\sin(\pi i)}{\pi i} = \frac{1}{\pi i} \cdot \frac{e^{-\pi} - e^{\pi}}{2i} = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

[例] $z \neq 1$ 求 $\ln(1-z)$

解: $\ln(1-z) = \ln|1-z| + i\arg(1-z)$

1/ 令 $z = x + iy$ $\quad \ln(1-z) = \frac{1}{2}\ln[(1-x)^2 + y^2] + i\,\mathrm{arctg}\left(\frac{-y}{1-x}\right)$

2/ 令 $z = re^{i\theta}$ $\quad \ln(1-z) = \frac{1}{2}\ln(1 - 2r\cos\theta + r^2) - i\,\mathrm{arctg}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right)$

令 $r = 1$ $\quad \ln(1 - e^{i\theta}) = \ln\left(2\sin\frac{\theta}{2}\right) - i\left(\frac{\pi - \theta}{2}\right) \quad \theta \in (0, 2\pi)$

应用: 当 $|z| < 1$. $-\ln(1-z) = \int_0^z \frac{1}{1-z}dz = \sum_{k=0}^{+\infty}\int_0^z t^k dt = \sum_{k=0}^{+\infty} \frac{z^k}{k}$

令 $z = re^{i\theta}$, $r < 1$, $\theta \in [0, 2\pi)$

$$-\ln(1 - re^{i\theta}) = \sum_{n=0}^{+\infty} \frac{r^n \cos(n\theta)}{n} + i\sum_{n=0}^{+\infty} \frac{r^n \sin(n\theta)}{n}$$

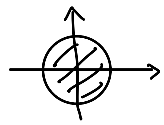
结论:

$$\begin{cases} \sum_{n=0}^{+\infty} \frac{r^n \cos(n\theta)}{n} = -\frac{1}{2}\ln(1 - 2r\cos\theta + r^2) \\ \sum_{n=0}^{+\infty} \frac{r^n \sin(n\theta)}{n} = \mathrm{arctg}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) \end{cases} \qquad 0 \leq r < 1 \quad \theta \in [0, 2\pi)$$

$$\begin{cases} \sum_{n=0}^{+\infty} \frac{\cos(n\theta)}{n} = -\ln\left(2\sin\frac{\theta}{2}\right) \\ \sum_{n=0}^{+\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2} \end{cases} \qquad r \to 1^- \quad \theta \in (0, 2\pi)$$

$\theta = \pi,\ r = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -\ln 2$

$-\ln(1-z) = \sum_{n=1}^{+\infty} \frac{z^n}{n}$



只在 $z=1$ 不收敛