Basic probability theory

Sharon Goldwater
Institute for Language, Cognition and Computation
School of Informatics, University of Edinburgh

DRAFT Version 0.95: 25 Sep 2016. **Do not redistribute without permission.**

Contents

1 Purpose of this tutorial and how to use it

This tutorial is written as an introduction to probability theory aimed at upper-level undergraduates or MSc students who have little or no background in this area of mathematics but who want to take courses on (or even just read up on) computational linguistics, natural language processing, computational cognitive science, or introductory machine learning. Many textbooks in these areas (even introductory ones such as Jurafsky & Martin's *Speech and Language Processing* or Witten et al's *Data Mining* assume either implicitly or explicitly that students are already familiar with basic probability theory. Yet many students with backgrounds in linguistics, psychology, or other social sciences (and even some computer science students) have very little exposure to probability theory.

I have taught students like these in courses on NLP and computational cognitive science, I have tried to find existing materials, preferably free ones, to refer students to. However all the materials I've seen are either too extensive and formal (mainly textbooks aimed at students of mathematics and related subjects) or too basic and incomplete (mainly online tutorials aimed at students learning statistics for social science, which have a very different focus and don't cover all the material needed by my target group). Some online videos (e.g., from Khan Academy) cover relevant material, but don't come with written notes containing definitions and equations for reference.

So, I ultimately decided to write my own tutorial. This tutorial is a work in progress. I have tried to include the most critical topics and to provide a lot of examples and exercises (although the exercises are not finished for all sections yet). Students from non-mathematical backgrounds sometimes have trouble with formal notation, so I have tried to explain the notation carefully and to be consistent in my own use of notation. However, I've also tried to point out variations of the notation that other authors may use, in the hope that when you are reading other materials you will have an easier time relating them back to this tutorial and deciphering the notation even if it isn't exactly the same as what I use here.

Don't expect that you can simply read through this tutorial once and understand everything. Reading mathematics is not like reading a novel: you will need to work at it. You may need to flip back and forth to remind yourself of definitions and equations, and you should ask yourself as you are reading through each example: does this make sense to me? If you find your attention wandering, consider taking a break. You won't learn much if you are just skimming without thinking.

Finally, **I strongly recommend that you actually do the exercises.** Like any area of mathematics, the only way to really understand probability theory and be able to solve problems is to practice! I have included solutions for some of the exercises at the end of the tutorial, but you should only look at them *after* you have worked out your own solution. It is very easy to trick yourself into thinking that you understand something if you look at the solution before working it out fully.

I've done my best to proofread this tutorial, but if you do find any errors, please let me know.

2 Events and Probabilities

2.1 What is probability and why do we care?

Probability theory is a branch of mathematics that allows us to reason about events that are inherently random. However, it can be surprisingly difficult to define what "probability" *is* with respect to the real world, without self-referential definitions. For example, you might try to define probability as follows:

Suppose I perform an action that can produce one of n different possible random OUTCOMES, each of which is equally likely. (For example, I flip a fair coin to produce one of two outcomes: heads or tails. Or, I pick one of the 52 different cards from a deck of playing

OUTCOMES

cards at random.) Then, the probability of each of those outcomes is $1/n$. (So, 1/2 for heads or tails; 1/52 for each of the possible cards.)

The problem with this definition is that it says each random outcome is "equally likely". But what does that mean, if we cannot define it in terms of the probability of different outcomes? Nevertheless, the above definition may begin to give you some intuition about probability.

Another way to think about probability is in terms of repeatable experiments. By "experiment", I mean a STATISTICAL EXPERIMENT. A statistical experiment is an action or occurrence that can have multiple different outcomes, all of which can be specified in advance, but where the particular outcome that will occur cannot be specified in advance because it depends on random chance. Flipping a coin or choosing a card from a deck at random are both statistical experiments. They are also repeatable experiments, because we can perform them multiple times (flip the same coin many times in a row, or return our original card to the deck and choose another card at random).

STATISTICAL EXPERIMENT

So, if we have a repeatable experiment, then one way to think about probability is as follows. Suppose we repeat the experiment an infinite number of times. The probability of a particular outcome is the proportion of times we get that outcome out of our infinite number of trials. For example, if we flip the fair coin an infinite number of times, we should expect to get a head 1/2 of those times.

This definition is not self-referential, but it also has problems. What if our experiment is not repeatable? For example, what if we want to know the probability that it will rain tomorrow? There is certainly some probability associated with that outcome, but no way to determine it (or even approximate it) by a repeatable experiment. So, a third way to think about probability is as a way of talking about and mathematically working with *degrees of belief*. If I say "the probability of rain tomorrow is 75%", that is a way of expressing the fact that my belief in rain tomorrow is fairly strong, and I would be somewhat surprised if it did not rain. On the other hand, if I say "the probability of rain tomorrow is 2%", I have a very strong belief that it will not rain, and I would be very surprised if it did.

Whichever way you want to think about probabilities, they turn out to be extraordinarily useful in many areas of cognitive science and artificial intelligence. Broadly speaking, probabilities can be used for two types of problems:

a) **Generation/prediction.** Reasoning from causes to effects:[1] Given a known set of causes and knowledge about how they interact, what are likely/unlikely outcomes? For example, we could set up a probabilistic system that generates sequences of characters as follows: Roll a fair 6-sided die. If the result is 1 or 2, print an `a`. Otherwise, print a `b`. This isn't a terribly interesting generative process, but we could use probability theory to determine things like: how likely are we to get an `a` for the next character? If we generate a sequence of 10 characters, how likely are we to get at least four `a`'s? And so forth.

   Much of the early work developing probability theory was motivated by answering questions like these in order to be able to predict how often certain events would occur in gambling. However, even if we are not interested in gambling, it's often useful to be able to make these kinds of predictions.

b) **Inference.** Reasoning from effects to causes: Given knowledge about possible causes and how they interact, as well as some observed outcomes, which causes are likely/unlikely? Here are some examples:

   - observe many outcomes of a coin flip $\Rightarrow$ determine if coin is fair.

[1] In machine learning, causes are often referred to as *hidden* or *latent* variables, and effects as *observed* variables. This terminology is more general and often more appropriate, since it doesn't imply causation in every case.

- observe features of an image $\Rightarrow$ determine if it is a cat or dog.
- observe patient's symptoms $\Rightarrow$ determine disease.
- observe words in sentence $\Rightarrow$ infer syntactic parse.

In everyday reasoning, we often do *both* inference and prediction. For example, we might first infer the category of an object we see (cat or dog?) in order to predict its future behavior (purr or bark?). In the following sections, we'll work through some of the mathematical background we need in order to formalize the problems of generation and inference, and then we'll show some examples of how to use these ideas.

2.2 Sample spaces and events

Let's start with some basic definitions:

- The SAMPLE SPACE of a statistical experiment is the set of all possible outcomes (also known as SAMPLE POINTS). SAMPLE SPACE SAMPLE POINTS
- An EVENT is a subset of the sample space. EVENT

Example 2.2.1. Imagine I flip a coin, with two possible outcomes: heads (`H`) or tails (`T`). What is the sample space for this experiment? What about for three flips in a row?

Solution: For the first experiment (flip a coin once), the sample space is just $\{\texttt{H},\texttt{T}\}$. For the second experiment (flip a coin three times), the sample space is $\{\texttt{HHH},\texttt{HHT},\texttt{HTH},\texttt{HTT},\texttt{THH},\texttt{THT},\texttt{TTH},\texttt{TTT}\}$. Order matters: `HHT` is a different outcome than `HTH`.

Example 2.2.2. For the experiment where I flip a coin three times in a row, consider the event that I get exactly one `T`. Which outcomes are in this event?

Solution: The subset of the sample space that contains all outcomes with exactly one `T` is $\{\texttt{HHT},\texttt{HTH},\texttt{THH}\}$.

Example 2.2.3. Suppose I have two bowls, each containing 100 balls numbered 1 through 100. I pick a ball at random from each bowl and look at the numbers on them. How many elements are in the sample space for this experiment?

Solution: Using basic principles of counting (see the Sets and Counting tutorial), since the number of possible outcomes for the second experiment doesn't depend on the outcome of the first experiment, the total number of possible outcomes is 100^2, or 10,000.

Example 2.2.4. Which set of outcomes defines the event that the two balls add up to 200?

Solution: There is only one outcome in this event, namely $\{(100,100)\}$.

Example 2.2.5. Let E be the event that the two balls add up to 201. Which outcomes are elements of E?

Solution: This is no outcome in which the balls can add up to 201, so $E = \emptyset$.

The last example illustrates the idea of an IMPOSSIBLE EVENT, an event which contains no outcomes. In contrast, a CERTAIN EVENT is an event that contains all possible outcomes. For example, in the experiment where I flip a coin three times, the event that we obtain at least one H or one T is a certain event.[2] IMPOSSIBLE EVENT CERTAIN EVENT

[2] In real life, this event would not be certain, since there is some very small chance the coin might land on its edge, or fall down a hole and be lost, or be eaten by a cat before landing. However, in this case I defined the possible outcomes at the outset as only H or T, so these real-life possibilities are not possibilities in my idealized experiment. Certain and impossible events are very rare in real life, but less rare in the idealized world of mathematics.

2.3 The probability of an event

Now, let's consider the probability of an event. By definition, an impossible event has probability zero, and a certain event has probability one. The more interesting cases are events that are neither impossible nor certain. For the moment, let's assume that all outcomes in the sample space S are equally likely. If that is the case, then the probability of an event E, which we write as $P(E)$, is simply the number of outcomes in E divided by number of outcomes in S:

$$P(E) = \frac{|E|}{|S|} \qquad \text{if all outcomes in } S \text{ are equally likely} \tag{1}$$

That is, the probability of an event is the proportion of outcomes in the sample space that are also outcomes in that event.

Example 2.3.1. Imagine I flip a fair coin, with two possible outcomes: heads (`H`) or tails (`T`). What is the probability that I get exactly one `T` if I flip the coin once? What if I flip it three times?

Solution: First, note that I said it's a *fair* coin. This is important, because it means that on any one flip, each outcome is equally likely, so we can use (1) to determine the probabilities we care about. We already determined the relevant events and sample spaces for each experiment in the previous section, so now we just need to divide those numbers. Specifically, if we only flip the coin once, then the event we care about (getting `T`) has one possible outcome, and the sample space has two possible outcomes, so the probability of getting `T` is 1/2. If we flip the coin three times, there are three outcomes with exactly one `T` (see Example 2.2.2), and eight outcomes altogether (see Example 2.2.1), so the probability of getting exactly one `T` is 3/8.

Example 2.3.2. Suppose I have two bowls, each containing 100 balls numbered 1 through 100. I pick a ball at random from each bowl and look at the numbers on them. What is the probability that the numbers add up to 200?

Solution: Again, we already computed that there is only one outcome in this event, and 10,000 outcomes altogether, so the probability of this happening is only 1/10,000

Example 2.3.3. Let E be the event that the numbers on the balls in the previous example add up to exactly 51. What is the probability of E?

Solution: We already know the size of the sample space, but we also need to determine the cardinality of E, i.e., the number of outcomes in this event. Consider what happens if the first ball is 1. For the balls to add up to 51, it must be the case that the second ball is 50. Similarly, if the first ball is 2, then the second must be 49. And so on: for each possible first ball between 1 and 50, there is exactly one second ball that will make the total equal 51. So the total number of outcomes adding up to 51 is the same as the number of ways the first ball can be between 1 and 50, which is to say, 50 ways. Therefore, the probability of E is 50/10,000, or 0.5%.

Example 2.3.4. Suppose I choose a PIN containing exactly 4 digits, where each digit is chosen at random and is equally likely to be any of the 10 digits 0-9. What is the probability that my PIN contains four different digits?

Solution: First, consider the sample space S: all possible four-digit PINs. Using our counting techniques (see Sets and Counting), we can compute that the number of outcomes in this sample space is 10^4, or 10,000. Next, consider the event E of interest: the set of PINs with four distinct digits. To compute the size of this set, note that there are 10 possibilities for the first digit. Once that digit is determined, there are only 9 possibilities for the second digit, because it must be different from the first one. Then, 8 possibilities for the third digit (which must be different to both of the first two), and 7 for the final digit. So, the total number of

PINs with four different digits is $10 \cdot 9 \cdot 8 \cdot 7$, or 5040. Therefore $P(E) = 5040/10,000$, or very slightly more than 1/2.

Example 2.3.5. Suppose I have a bowl with 3 green balls (g), 2 blue balls (b), and 1 red ball (r). I draw a single ball at random from the bowl and report its color. What is the probability I got a blue ball?

Solution: The answer may already be obvious to you, but let's make sure we know how to get there using the tools of probability theory that we've seen so far. Notice that if we define the outcomes in our sample space as $\{\mathtt{g}, \mathtt{b}, \mathtt{r}\}$, they are not equally likely, so we cannot use Eq (1). In this problem, the equally likely outcomes are the outcomes of drawing each particular ball, a sample space of size 6. The *events* of interest are still $\{\mathtt{g}, \mathtt{b}, \mathtt{r}\}$, but now we can use Eq (1), which tells us that $P(\mathtt{b}) = 2/6 = 1/3$ because two out of the six balls (outcomes) are blue.

2.4 Probability distributions

Hopefully you have developed some intuition for probabilities at this point. Before going further, we should formalize what makes a probability a probability, mathematically speaking. To do so, we first need to define MUTUALLY EXCLUSIVE (or DISJOINT) events. Two events are mutually exclusive iff ("iff" means "if and only if") they contain no outcomes in common (i.e., both events cannot occur at the same time). For example, if I roll two dice, the events "get a total of 7" and "get a total of 8" are mutually exclusive. On the other hand, "get a total of 7" and "get a 6 on one die" are *not* mutually exclusive, since both could occur on the same roll.

MUTUALLY EXCLUSIVE
DISJOINT

Now, suppose we have a set of n mutually exclusive events that together cover all possible outcomes in our sample space S. Such a set is called a PARTITION of the sample space, and is visualized in Figure 1. A simple example would be flipping a coin, where $S = \{\mathtt{H}, \mathtt{T}\}$ and we define $n = 2$ mutually exclusive events, $E_1 = \{\mathtt{H}\}$ and $E_2 = \{\mathtt{T}\}$. Another example might be rolling two 6-sided dice, where the mutually exclusive events are getting a total of 2 or 3 or 4 or ... or 12 (in this case, $n = 11$ different events).

PARTITION

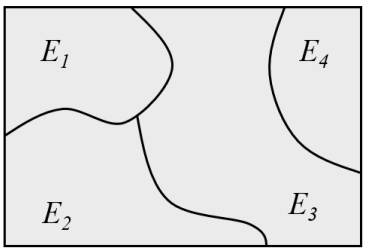


Figure 1: A partition of the sample space into four mutually exclusive events. The rectangle represents all the outcomes in the entire sample space, and each labelled region represents the outcomes in that event.

We then assign a number, $P(E_i)$, to each of the events E_i in the partition. If and only if the following two properties hold, then each $P(E_i)$ can be considered a probability, and the set of values $\{P(E_1) \ldots P(E_n)\}$ can be considered a PROBABILITY DISTRIBUTION.

PROBABILITY DISTRIBUTION

Property 1: $0 \leq P(E_i) \leq 1$

Property 2: $\sum_{i=1}^{n} P(E_i) = 1$

That is, every probability must fall between 0 and 1 (inclusive), and the sum of the probabilities of the mutually exclusive events that cover the the sample space must equal 1.

One way to think of a probability distribution is that altogether the sample space S has one unit of PROBABILITY MASS, and this mass is divided up (distributed) in some way between all the E_i. The amount of mass assigned to each E_i is what we call $P(E_i)$. In our coin flipping example, the two events are equally likely, so the mass is divided evenly and we get $P(E_1) = P(E_2) = 1/2$. This is an example of a UNIFORM DISTRIBUTION, a distribution where all events in a partition are equally likely. Another example of a uniform distribution is the distribution over the number we get when we roll a single fair die.

PROBABILITY MASS

UNIFORM DISTRIBUTION

Since a probability distribution is defined as a function from events to values, it is sometimes also called a PROBABILITY MASS FUNCTION: a function that allocates the probability mass to the different events.[3]

PROBABILITY MASS FUNCTION

Example 2.4.1. I have a jar with four different colored balls and I choose one at random. Is the distribution over the color I get uniform or not?

Solution: If we assume that "choose one at random" means that each ball (and therefore each color) is equally likely, then yes. However the phrase "at random" is ambiguous: technically, it just means there is some randomness involved, not necessarily that all outcomes are equally likely. So, when working with probabilities, we usually try to be more specific. If we mean that the distribution is uniform, then we should say that a ball is chosen UNIFORMLY AT RANDOM.

UNIFORMLY AT RANDOM

Example 2.4.2. What is the distribution over the sum of *two* fair dice? Is it uniform or not?

Solution: We need to figure out the probability of each of the events. As noted above, there are 11 different events, corresponding to sums from 2 to 12. Also, note that we can define our sample space in terms of 36 equally likely outcomes, corresponding to the six possible outcomes on the first die multiplied by the six possible outcomes on the second die:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Now we can use Eq (1) to determine the probabilities of each of the events. For example, $P(\texttt{sum is 2}) = 1/36$ because there is only one outcome in this event, whereas $P(\texttt{sum is 3}) = 2/36$ because there are two equally likely outcomes in this event. We leave it as an exercise for you to complete the probabilities in the rest of the distribution. However, it should already be clear that this distribution is not uniform, since the different events do not all have the same probability.

We'll return to further discussion of distributions a bit later.

2.5 Exercises

Exercise 2.1

Suppose I have a dictionary with 5000 words in it, and I decide to generate a 5-word "sentence" by choosing each word at random from the dictionary (all words are equally likely).

a) What is the size of the sample space in this experiment?

[3] We're assuming that there are a finite number of events in the partition. Later we'll relax this assumption.

b) If E is the event "my sentence starts with the word *the*", how many outcomes are there in E?

c) What is $P(E)$?

d) Let A be the event "my sentence ends with the word *the*". Are A and E mutually exclusive? If so, explain why. If not, give an example of an outcome that belongs to both A and E.

Exercise 2.2

Which of the following are possible probability distributions? For each distribution, state whether it is uniform or not, and whether the distribution includes a certain or impossible event.

Note: We are using a notation that assumes an ordering over events and just lists the probabilities of those events as a vector in the appropriate order. For example, if we had two events, E_1 and E_2, with $P(E_1) = 1/3$ and $P(E_2) = 2/3$, we would write down this distribution as $(1/3,\ 2/3)$.

a) (1.3, 2)

b) (0.2, 0.2, 0.2, 0.2)

c) (0.2, 0.2, 0.2, 0.2, -0.1, 0.3)

d) (0.2, 0.2, 0.2, 0.2, 0.2)

e) (0, 1, 0)

f) (0)

g) (1)

h) (-.5, -.5)

i) (1/2, 1/2)

j) (1/2, 1/4)

k) (1/8, 1/4, 5/8)

l) (3/16, 1/8, 7/16)

Exercise 2.3

Write down the full distribution over the sum of two fair dice. That is, complete the example we started at the end of the last section (Example 2.4.2).

Exercise 2.4

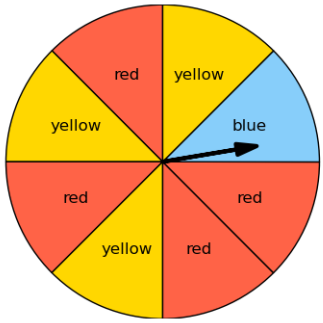
Suppose I have a bowl with 4 green balls (`g`), 2 blue balls (`b`), and 3 red balls (`r`). I draw a single ball uniformly at random from the bowl and report its color. What is the probability distribution over the different colors?

Exercise 2.5

Suppose I have a bowl with 10 balls in it, all of which are either blue, red, or green. I draw a single ball uniformly at random and report its color. The number of balls of each color is such that the probability of getting a blue ball is 0.4 and the probability of getting a green ball is 0.3. What is the probability of getting a red ball? How many balls of each color are there?

Exercise 2.6

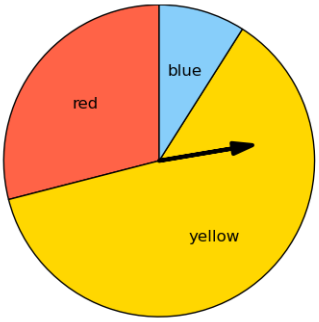
Suppose I have a spinner divided into 8 equal-sized sections, colored as shown:



Assume the arrow on the spinner is carefully balanced so it is not more likely to end up in one place than another. I spin the arrow and report the color of the section it ends up in. What is the sample space of equally likely outcomes? What is the probability of ending up in a red section?

Exercise 2.7

Suppose I replace the colored sections on the spinner in the previous problem with just three sections: a red one that covers 29% of the surface, a yellow one that covers 62%, and a blue one that covers 9%:



How can we use the tools of probability theory introduced so far to compute the probability that, if we spin the arrow, it will end up in the red section?

(You may find it obvious that the probability is 29%. But, can you show this using only the definitions and formulas we've covered so far? Sometimes things that seem obvious in probability theory turn out to be incorrect, so it's important to be able to prove things even if they seem obvious. If you're having trouble, consider how you solved the previous question and try to apply a similar method here.)

Exercise 2.8

Suppose I have a group containing the following first- and second-year university students from various countries. The first 3 are male, and the last 4 female:

Name	Home country	Year
Andrew	UK	1
Sebastian	Germany	1
Wei	China	1
Fiona	UK	1
Lea	Germany	2
Ajitha	UK	1
Sarah	UK	2

I choose a student uniformly at random from the group. For each set of events given below, answer the following questions: (i) Are the events mutually exclusive? (ii) If so, do they cover all possible outcomes in the sample space? (iii) What is the probability of each event? (iv) Do these events and their probabilities, taken together, form a probability distribution?

a) E_1 = the student is male, E_2 = the student is female

b) E_1 = the student is from the UK, E_2 = the student is from China

c) E_1 = the student's name is Andrew or Sebastian, E_2 = the student is from Germany

d) E_1 = the student is a first year student, E_2 = the student is from Germany

e) E_1 = the student's name starts with A, E_2 = the student is from outside the UK, E_3 = the student's name is Fiona or Sarah

3 Combining events

Now let's consider how the definitions in the previous section can be used to determine the probability of some event from the probabilities of other events.

3.1 Event complement and union

Starting simple, suppose we have an event E that has probability $P(E)$. What is the probability that E does *not* happen? Put another way, what is the probability of the COMPLEMENT of E, written as $\neg E$ (or E' or $\bar{E}$)? $\neg E$ is the set of outcomes that are in the sample space S but not in E, and it's easy to see that, since the total probability mass of S is 1, then

COMPLEMENT
$\neg E$
E'
$\bar{E}$

$$P(\neg E) = 1 - P(E). \tag{2}$$

Example 3.1.1. Suppose I have a list of words, and I choose a word uniformly at random. If the probability of getting a word starting with t is 1/7, then what is the probability of getting a word that does not start with t?

Solution: Let E be the event that the word starts with t. Then $P(\neg E)$ is the probability we were asked for, and it is $1 - P(E)$, or 6/7.

Things get slightly trickier when we consider the union of two events, A and B. Since events are just sets of outcomes, taking their union corresponds to considering any outcome that belongs to either A or B. For example, looking at the scenario from Exercise 2.8, let's define A = "the student is female" and B = "the student is from the UK".

Example 3.1.2. What is $P(A \cup B)$, that is, the probability that the student is female *or* from the UK?

Solution: You might imagine that the answer is just $P(A) + P(B)$. Let's see if that is correct. First, we compute $P(A)$, which is 4/7. Next, compute $P(B)$, which is also 4/7. So $P(A) + P(B) = 8/7$. But that clearly can't be correct, since probabilities cannot be greater than one.

So, let's instead consider which outcomes are actually in the set $A \cup B$. They are: {Fiona, Lea, Ajitha, Sarah, Andrew}. Since this set has five elements, we know from Eq (1) that $P(A \cup B)$ must be 5/7.

So what went wrong when we computed $P(A) + P(B)$? Notice that there are three students who belong to both A *and* B: Fiona, Ajitha, and Sarah. So when we counted the outcomes in A, we included these students. And when we counted the outcomes in B, we included these students again. That means when we computed $P(A) + P(B)$, we added in those three students *twice*. To correctly compute $P(A \cup B)$ from $P(A)$ and $P(B)$, we need to subtract off those extra counts. We do so using the following formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{3}$$

Since $A \cap B$ is the set of students that are in both A and B, this is exactly the set that will have been counted twice, so we subtract off that amount from the probability. In our example, we now get $P(A \cup B) = 4/7 + 4/7 - 3/7 = 5/7$, which is exactly what we got when computing $P(A \cup B)$ directly.

One way to help understand/remember this rule is using a Venn diagram to visualize the members of the various sets, as in Figure 2.

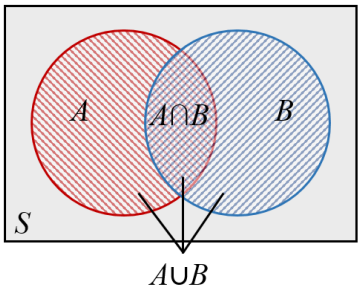


Figure 2: A Venn diagram helping to illustrate that by adding together the number of outcomes in A and B, we count $A \cap B$ twice. This observation provides the intuition for Eq (3).

From Eq (3), we can also see that in the special case where A and B are mutually exclusive, it *is* true that $P(A \cup B) = P(A) + P(B)$, because there are no items in the intersection. However it is important to remember that this is only a special case, whereas Eq (3) is always true.

3.2 Joint probabilities and the law of total probability

Now let's look at event intersections. These come up so often that there is a special term for the probability of the intersection of two events: it is called the JOINT PROBABILITY of A and B, written $P(A \cap B)$.[4]

JOINT PROBABILITY
$P(A \cap B)$

Now, suppose we have a set of events $\{E_1 \ldots E_n\}$ that partition the sample space, and we have some other event B that is also in the sample space. The diagram in Figure 3 illustrates such a situation and provides some intuition for the LAW OF TOTAL PROBABILITY, also known as the SUM RULE:

LAW OF TOTAL PROBABILITY
SUM RULE

$$P(B) = \sum_{i=1}^{n} P(B \cap E_i) \tag{4}$$

The law of total probability tells us that we can compute the probability of B by adding up the joint probability of B with each of the E_i.

[4]It is more common to see joint probability written with just a comma, as $P(A, B)$, and we'll use this simpler notation when we introduce random variables. However until then we use the intersection notation as a reminder of what joint probability really is.

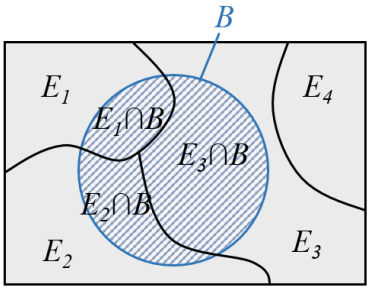


Figure 3: Visualizing the law of total probability. Adding up the outcomes in the intersection of each E_i with B yields all the outcomes in B. So adding the probabilities of each intersection will yield the probability of B.

Example 3.2.1. Consider the scenario from Exercise 2.8. We partition the sample space according to the country that each student comes from, with E_1 = "student is British", E_2 = "student is Chinese", and E_3 = "student is German". Also let B be the event that the student is female. Apply the law of total probability to compute $P(B)$, and check that the result is the same as when computing $P(B)$ directly.

Solution: Using the law of total probability, we have

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + P(B \cap E_3) \\ &= 3/7 + 0/7 + 1/7 \\ &= 4/7 \end{aligned}$$

which is the same result we get by computing $P(B)$ directly (counting the number of female students and dividing by the total number of students).

3.3 Exercises

Exercise 3.1

I have a list of two-character codes, and I choose one uniformly at random. Suppose the probability that the chosen code starts with `s` is .02, and the probability that it ends with `s` is .03. Also, the probability of the code `ss` is .01.

a) What is the probability that I get a code that contains at least one `s`?

b) What is the probability that I get a code that doesn't contain an `s`?

Exercise 3.2

I have a set of balls in a jar, and I choose a ball uniformly at random. Each ball is either black (`b`) or white (`w`), and has a number on it between 1 and 4. Listed below are some of the joint probabilities for this experiment.

$$\begin{aligned} P(\texttt{w},1) &= 1/16 & \qquad P(\texttt{b},1) &= 1/8 \\ P(\texttt{w},2) &= 1/16 \\ P(\texttt{w},3) &= 1/8 \\ P(\texttt{w},4) &= 1/4 \end{aligned}$$

a) What is the probability of getting a 1 of any color?

b) What is the probability of getting a white ball?

c) What is the probability of getting a black ball?

d) What is the probability of getting either a black ball or a ball numbered 1?

Exercise 3.3
Suppose I generate a 6-character password, where each character is chosen uniformly at random from of the 26 lowercase letters of English. What is the probability that my password contains at least one repeated character, i.e., there is at least one character that occurs more than once somewhere in the password? (Hint: extend the solution to Example 2.3.4 to solve this problem.)

4 Conditional probabilities, independence, and Bayes' Rule

4.1 Definition of conditional probability

CONDITIONAL PROBABILITY is one of the most important concepts of probability theory. A conditional probability expresses the probability that some event A will occur, given that (*conditioned on* the fact that) event B occurred. The conditional probability of A given B, written $P(A \mid B)$, where the $\mid$ is pronounced "given", is defined as[5]

CONDITIONAL PROBABILITY

$P(A \mid B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \tag{5}$$

Example 4.1.1. Again let's use the scenario from Exercise 2.8 of Section 2.5, with events A = "the student is male" and B = "the student is from the UK". What is $P(A \mid B)$?

Solution: In this case, $A \cap B$ is the set of male British students, so $P(A \cap B) = 1/7$. $P(B) = 4/7$, so $P(A \mid B) = \frac{1/7}{4/7} = 1/4$.

Hopefully this answer makes sense to you: intuitively, the probability that the chosen student is male given that the student is British is simply the number of male students as a fraction of the number of British students, or 1/4. But again, it's important to learn the formal rules of probability theory since not all problems you will be faced with are so straightforward.

Example 4.1.2. Using the same A and B as in Ex 4.1.1, what is $P(B \mid A)$?

Solution: We have the same $A \cap B$, so $P(A \cap B) = 1/7$. $P(A) = 3/7$, so $P(A \mid B) = \frac{1/7}{3/7} = 1/3$.

This example illustrates the fact that conditional probabilities are *not* commutative: $P(A \mid B)$ is not the same as $P(B \mid A)$, and in general the two will not be equal.

4.2 The product rule and chain rule

If we rearrange the terms in the definition of conditional probability (Eq 5), we obtain the PRODUCT RULE, which allows us to compute joint probabilities from conditional probabilities:

PRODUCT RULE

$$P(A \cap B) = P(A \mid B)\, P(B). \tag{6}$$

Since intersection is commutative, we could also write $P(A \cap B) = P(B \cap A) = P(B \mid A)\, P(A)$.

Example 4.2.1. Suppose I have a jar of marbles, and I choose one uniformly at random. Each marble is either red (R), green (G), or blue (B). In two-thirds of the marbles, these colors are solid (S), while in the remainder the colors are patchy. If I tell you that the probability of getting a red marble given that I have a solid-colored one is 1/2, can you compute the

[5] $P(A \mid B)$ is undefined if $P(B) = 0$.

probability that my chosen marble is solid red?

Solution: Yes: $P(R \cap S) = P(R \mid S)\, P(S)$. We were told that $P(R \mid S) = 1/2$, and $P(S) = 2/3$, so $P(R \cap S) = (1/2)(2/3) = 1/3$.

Example 4.2.2. Now suppose the probability of the marble being blue given that it's patchy is 1/5. What is the probability of getting a patchy blue marble?

Solution: We were not given the probability of a patchy marble, but the previous example stated that all marbles are either solid or patchy, and that $P(S)$, the probability of a solid-color marble, is $2/3$. So we can compute the probability of a patchy marble as $P(\neg S) = 1 - P(S) = 1/3$. Then $P(B \cap \neg S) = P(B \mid \neg S)\, P(\neg S) = (1/5)(1/3) = 1/15$.

One consequence of the product rule is an alternative formulation of the law of total probability. We can use the product rule to replace the joint probabilities in (4) with conditional probabilities, thereby obtaining

$$P(B) = \sum_{i=1}^{n} P(B \mid E_i)\, P(E_i). \tag{7}$$

This version of the law of total probability is often useful in problems involving Bayes' Rule (introduced below in Section 4.6).

The product rule can only handle the intersection of two events, but notice that we can apply it iteratively to expand the joint probability of several events into a product of several conditional probabilities. For example:

$$\begin{aligned} P(A \cap B \cap C \cap D) &= P(A \mid B \cap C \cap D)\, P(B \cap C \cap D) \\ &= P(A \mid B \cap C \cap D)\, P(B \mid C \cap D)\, P(C \cap D) \\ &= P(A \mid B \cap C \cap D)\, P(B \mid C \cap D)\, P(C \mid D)\, P(D) \end{aligned}$$

where in each step, we expanded the final term using the product rule. This is possible because a complex event such as $B \cap C \cap D$ is still just an event: it can take the place of B in Eq (6).

This iterative rewriting process is summarized in a general form called the CHAIN RULE: CHAIN RULE

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = \prod_{i=1}^{n} P(E_i \mid E_{i+1} \cap \ldots \cap E_n) \tag{8}$$

4.3 Conditional probability distributions

Suppose we have a set of events $\{E_1 \ldots E_n\}$ that partition the sample space, and we consider the conditional probabilities of these events conditioned on some other event B, where $P(B) > 0$. Then these conditional probabilities $P(E_1 \mid B) \ldots P(E_n \mid B)$ form a distribution—the CONDITIONAL PROBABILITY DISTRIBUTION conditioned on B. We can show that this set forms a distribution by first showing that the sum of the probabilities is one: CONDITIONAL PROBABILITY DISTRIBUTION

$$\begin{aligned} \sum_{i=1}^{n} P(E_i \mid B) &= \sum_{i=1}^{n} \frac{P(E_i \cap B)}{P(B)} && \text{by defn of conditional probability} \\ &= \frac{1}{P(B)} \sum_{i=1}^{n} P(E_i \cap B) \\ &= \frac{1}{P(B)} P(B) && \text{by law of total probability} \\ &= 1 \end{aligned} \tag{9}$$

We also need to show that all the $P(E_i \mid B)$ are between 0 and 1. We know that $P(B)$ and each $P(E_i \cap B)$ is between 0 and 1, because they are probabilities. Also, $P(E_i \cap B)$ cannot be

larger than $P(B)$, because the intersection of B with another set can't be larger than B. But if $P(E_i \cap B) \leq P(B)$, and $P(B) > 0$ and $P(E_i \cap B) \geq 0$, then $\frac{P(E_i \cap B)}{P(B)}$ must be between 0 and 1.

Because a conditional probability distribution is a distribution, the rules for computing probabilities that we discussed already apply just as well to conditional distributions, provided we keep the conditioning event in all parts of the equation. In particular,

$$P(\neg E \,|\, B) = 1 - P(E \,|\, B) \tag{10}$$

$$P(A \cup B \,|\, C) = P(A \,|\, C) + P(B \,|\, C) - P(A \cap B \,|\, C) \tag{11}$$

Example 4.3.1. Using the same jar of marbles as in Examples 4.2.1 and 4.2.2, I give you one more piece of information: the probability of a green marble given a patchy marble is 2/5. What's the probability of a red marble given a patchy marble?

Solution: We now have a lot of different pieces of information to keep straight. Let's first write down all the information that was explicitly stated in each of the examples:

$$P(S) = 2/3 \qquad P(R \,|\, S) = 1/2 \qquad P(B \,|\, \neg S) = 1/5$$
$$P(G \,|\, \neg S) = 2/5$$

Notice that I have arranged the information so that each column forms part of a conditional distribution, conditioned on a different thing (either nothing, or S, or $\neg S$). We want to know $P(R \,|\, \neg S)$, which would belong in the third column, and we also know that adding up all the items in that distribution should equal one. But $P(R \,|\, \neg S)$ is the only item missing from the distribution. Therefore, $P(R \,|\, \neg S) = 1 - (P(B \,|\, \neg S) + P(G \,|\, \neg S))$, or 2/5.

4.4 Conditional and joint probability tables

Another way of representing a conditional probability distribution is using a CONDITIONAL PROBABILITY TABLE, or CPT. A CPT for the distribution conditioned on A has one column for each event E_i in the partition. Each column is labelled with E_i and lists the value of $P(E_i \,|\, A)$. So, the CPT for $\neg S$ before solving the previous example would be:

CONDITIONAL PROBABILITY TABLE

R	G	B
	2/5	1/5

This table makes it completely clear that R is the only missing value, and therefore can be filled in with the value that makes the row sum to one.

Depending on what problem we are trying to solve, it may be more useful to construct a table listing joint probabilities rather than conditional probabilities (a JOINT PROBABILITY TABLE, or JPT). This kind of table assumes we have two different partitions of the sample space, $\{D_1 \ldots D_n\}$, and $\{E_1 \ldots E_m\}$ (such as our partitions into solid/patchy and red/blue/green in the marble example).[6] Each row in the table represents a D_i, and each column represents an E_j. The value in cell (i, j) is $P(D_i, E_j)$.

JOINT PROBABILITY TABLE

Example 4.4.1. Construct a joint probability table for the scenario from Example 4.3.1, where the rows represent the S and $\neg S$ events, and the columns represent the color events. Fill in the cells for which we have computed values already in previous examples, then fill in as many additional cells as possible given the other information we have so far.

Solution: The table initially looks like this, with the two values we computed in Examples 4.2.1 and 4.2.2:

	R	G	B
S	1/3		
$\neg S$			1/15

[6] It it is perfectly possible to have a joint distribution using more than two different partitions, but in that case it is difficult to use the table respresentation.

Using the information from Example 4.3.1, we can then compute two additional cells as $P(R|\neg S)\,P(\neg S)$ and $P(G|\neg S)\,P(\neg S)$. The table now looks like this:

	R	G	B
S	1/3		
$\neg S$	2/15	2/15	1/15

Without further information, the two remaining cells remain unknown.

Notice that adding up the values in the $\neg S$ row gives a value of 1/3, which is the same value we computed earlier for $P(\neg S)$. This is not a coincidence! It is the law of total probability in action: look again at Eq 4 and you'll see that adding up the values in any row or column of a JPT will give the probability of the event labelling that row or column. It is often useful to write down the summed probability of each row and column in the margins of the JPT as an aid to problem solving. The probabilities we end up with in the margins are sometimes called MARGINAL PROBABILITIES, and using the law of total probability to compute one is called MARGINALIZATION. That is, if we are given $P(S\cap R)$, $P(S\cap G)$, and $P(S\cap B)$, then computing $P(S)$ by summing the three joint probabilities would be "marginalizing over" (or "marginalizing out") the other events to obtain the marginal probability of S.

MARGINAL PROBABILITIES
MARGINALIZATION

Example 4.4.2. In our running example, suppose that $P(B) = 1/5$. Is this enough information to fill in the rest of the JPT and its margins?

Solution: Yes. Let's start by adding in the marginal probabilities that we know so far. Recall that $P(S)$ was given as 2/3, and $P(\neg S)$ was then computed as 1/3. So we have:

	R	G	B	
S	1/3			2/3
$\neg S$	2/15	2/15	1/15	1/3
			1/5	

Now we can fill in $P(R) = 1/3 + 2/15 = 7/15$. Also, $P(B\cap S) = P(B) - P(B\cap\neg S) = 2/15$:

	R	G	B	
S	1/3		2/15	2/3
$\neg S$	2/15	2/15	1/15	1/3
	7/15		1/5	

We can now compute $P(G\cap S)$ as $P(S) - P(R\cap S) - P(B\cap S)$ and then $P(G)$ as $(G\cap S) + P(G\cap\neg S)$. Or we could find $P(G)$ first, as $1 - P(R) - P(B)$, and compute $P(G\cap S)$ from there. Either way, we end up with:

	R	G	B	
S	1/3	1/5	2/15	2/3
$\neg S$	2/15	2/15	1/15	1/3
	7/15	1/3	1/5	1

I added a 1 in the bottom right corner as a reminder that the values in the row margin, as well as the values in the column margin, as well as the values in the non-marginal part of the table, all add up to 1.

4.5 Independent events

In general, the conditional probability of an event is not the same as the unconditional probability of that event. We can see this from our marbles example, where we computed $P(R) = 7/15$ while $P(R|S) = 1/2$ and $P(R|\neg S) = 2/5$.

However, in some cases the conditional probability $P(A \mid B)$ *is* equal to the unconditional probability $P(A)$. That is, whether or not B occurs has no effect on the probability of A occurring. This is one way of defining the notion of INDEPENDENCE of two events. Two events A and B are INDEPENDENT EVENTS[7] iff

INDEPENDENCE
INDEPENDENT EVENTS

$$P(A \mid B) = P(A). \tag{12}$$

By substituting in the definition of conditional probability from Eq (5) and rearranging the terms, we can equivalently state that events A and B are independent iff

$$P(A \cap B) = P(A)P(B). \tag{13}$$

Sometimes, it's clear intuitively that two events are independent. For example, if I flip a coin twice, the event "I get a head on the first flip" is independent of the event "I get a head on the second flip". Or, it may be clear intuitively that two events are *not* independent. For example, the event that I arrive at work on time is clearly not independent of whether I oversleep that morning. However, sometimes whether two events are independent is not intuitively obvious, and we need to use the definition of independence to determine the answer.

Example 4.5.1. Using Eq (13) and the joint probability table computed in Example 4.4.2, determine which of the possible pairs of events are independent.

Solution: To solve this problem, for each pair of events we check whether the joint probability is equal to the product of the probabilities of the individual events. Put another way, is the value in cell (i, j) equal to the product of the marginal value in row i times the marginal value in column j? We find that $P(S \cap R) \neq P(S)P(R)$, so S and R are not independent. Similarly for $(\neg S \cap R)$, $(S \cap G)$, and $(\neg S \cap G)$. However, $P(S \cap B) = P(S)P(B)$, so S and B *are* independent; and similarly for $\neg S$ and B.

In fact, if we look back at the CPT we computed for $\neg S$ at the beginning of Section 4.4, we see that $P(B \mid \neg S) = 1/5$, which is equal to $P(B)$ in the joint probability table margin. So we also could have determined these variables were independent using Eq (12).

4.6 Bayes' Rule

We started off this tutorial by saying that probability theory is useful for *prediction* and *inference*. Perhaps the most important tool we'll use for inference is BAYES' RULE, also known as BAYES' THEOREM. Bayes' Rule is derived by replacing the $P(A \cap B)$ in the definition of conditional probability with $P(B \mid A)\, P(A)$ (using the product rule). We obtain:

BAYES' RULE
BAYES' THEOREM

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}. \tag{14}$$

Example 4.6.1. Suppose I have some playing cards in my hand from a standard deck, and it turns out that 1/4 of them are red cards (hearts or diamonds) and 3/8 are face cards (jack, queen, or king). Also, 1/2 of the red cards are face cards. If I choose one of the cards in my hand uniformly at random, and get a face card, what's the probability that it is also a red card?

Solution: Let R be the event that the card is red, and F be the event that it is a face card. We are told that $P(R) = 1/4$, $P(F) = 3/8$, and $P(F \mid R) = 1/2$. (Notice which way the conditioning goes! The problem said 1/2 of the red cards are faces, i.e., if we know the card is red, the

[7] Don't confuse this definition with the related, but not identical definition of independent random variables, introduced later in Section 5.4.

probability that it's a face is 1/2.) So,

$$\begin{aligned} P(R\,|\,F) &= \frac{P(F\,|\,R)\,P(R)}{P(F)} \\ &= \frac{(1/2)(1/4)}{3/8} \\ &= 1/3 \end{aligned}$$

In the previous example, the probability in the denominator $P(F)$ was provided directly. However it's very common (especially in real-life problems) that we aren't given the value in the denominator and must compute it from the conditional probabilities that we do know. Typically this can be done with the law of total probability (usually the version in Eq 7), as we'll see in the following example. This example will also give us our first taste of an INFERENCE problem: a problem where we need to infer something that can't be observed directly (in this case, the presence or absence of a genetic marker) using information about something that can be observed directly (a test result) and the probabilistic relationship between the two.

INFERENCE

Example 4.6.2. In the general population, 1/1000 infants have a particular genetic marker that is linked to cancer in later life. There is a test for this marker, but it isn't perfect. If an infant has the genetic marker, the test will always show a positive result. But it also has a *false positive* rate of 1% (that is, 1% of infants who do not have the marker will still show a positive test result). If we test a particular infant and the test comes out positive, what's the probability that infant actually has the genetic marker?

Solution: Before answering this question, let's determine the sample space of this statistical experiment.[8] It's probably easiest to do that by working backwards. First, note that the relevant events we'll need to work with are G (infant has genetic marker) and T (test is positive).[9] We will also want to talk about conditional and joint probabilities involving G and T, which implies that G and T are different subsets of the same sample space. The easiest way, then, to define the sample space is as a set of four outcomes: $\{(\texttt{pos},\texttt{yes}),(\texttt{pos},\texttt{no}),(\texttt{neg},\texttt{yes}),(\texttt{neg},\texttt{no})\}$ where `pos`/`neg` indicates the test result and `yes`/`no` indicates the presence of the marker. G and T each pick out two of those outcomes. We are no longer working with equally likely outcomes here, but that's ok because we need not compute probabilities directly by counting outcomes, instead we have been given some probabilities and need to compute other probabilities from those.

So, with the pedantic part out of the way, let's work through the rest of the problem. We are told that $P(G) = .001$, that $P(T\,|\,\neg G) = .01$ (false positive rate), and that $P(T\,|\,G) = 1$ (perfect detection). Now we want to compute $P(G\,|\,T)$. We begin by applying Bayes' Rule:

$$P(G\,|\,T) = \frac{P(T\,|\,G)\,P(G)}{P(T)}$$

We have been given the probabilities in the numerator, but not $P(T)$. However, we can apply the law of total probability in the denominator to split up $P(T)$ into other probabilities that

[8] This part is not strictly necessary to solve the problem and might seem pedantic, but I have found that when students make errors in computing probabilities, it's often because they haven't thought about the sample space. So it's good to get in the habit of considering the sample space for the problem you are working on.

[9] We could equally well define G and T as their complement events, "doesn't have marker" and "test is negative". Try working through the solution on your own with that starting point and notice the result is the same.

we have already or can easily compute:

$$\begin{aligned} P(G \mid T) &= \frac{P(T \mid G)\, P(G)}{P(T \mid G)\, P(G) + P(T \mid \neg G)\, P(\neg G)} \\ &= \frac{(1)(.001)}{(1)(.001) + (.01)(.999)} \qquad \text{where } P(\neg G) = 1 - P(G) \\ &\approx .09 \end{aligned}$$

In other words, the probability that an infant with a positive test result actually has the genetic marker is less than 10%.

Are you surprised that the probability is so low? It's because even though the false positive rate is low, there are far more infants who don't have the genetic marker than those who do, and altogether those infants generate quite a few false positives. However, it's a well-known result from psychology that people are bad at estimating the answers to problems like this, so if you were surprised, you are in good company!

In inference problems like this one, we often refer to $P(G)$ as the PRIOR PROBABILITY of G: the probability of that event *before* we make any observations. In contrast, $P(G \mid T)$ is the POSTERIOR PROBABILITY of G: the probability of that event *after* observing some data (in this case, the test result).

PRIOR PROBABILITY

POSTERIOR PROBABILITY

4.7 Exercises

Exercise 4.1

In Example 4.6.1, we said that $P(F \mid R) = 1/2$ and $P(R) = 1/4$.

a) What is $P(F \cap R)$? What does this probability express (describe it in words)?

b) Are F and R independent? Why or why not?

Exercise 4.2

Suppose we have events A, B, C, D, and are given the following probabilities:

$$\begin{aligned} P(A) &= 0.5 & P(A \cap B) &= 0.25 \\ P(B) &= 0.3 & P(B \mid C) &= 0.2 \\ P(C) &= 0.6 & P(A \mid B \cap C) &= 0.1 \\ P(D) &= 0.4 & P(B \mid A \cap C) &= 0.7 \end{aligned}$$

Compute the following probabilities.

a) $P(A \mid B)$

b) $P(B \cap C)$

c) $P(C \mid B)$

d) $P(A \cap B \cap C)$

Exercise 4.3

Consider again the group of students from Exercise 2.8 and repeated below, and assume we choose a student from this group uniformly at random. Let F be "the student is in first year", U = "the student is from the UK", C = the student is from China", and G = "the student is from Germany".

Name	Home country	Year
Andrew	UK	1
Sebastian	Germany	1
Wei	China	1
Fiona	UK	1
Lea	Germany	2
Ajitha	UK	1
Sarah	UK	2

a) What is $P(F \mid U)$?

b) What is $P(\neg F \mid U)$?

c) What is $(U|F)$?

d) Are F and U independent? Why or why not?

e) What is the conditional probability table giving the year of the student conditioned on U?

f) Fill in the joint probability table shown here, including the marginal probabilities:

	U	C	G	
F				
$\neg F$				

Exercise 4.4

Suppose we have a *different* group of first- and second-year students from the same three countries (UK, Germany, China). Again we choose a student uniformly at random, with F = "the student is in first year", U = "the student is from the UK", C = the student is from China", and G = "the student is from Germany". For this group of students, $P(U) = 0.6$, $P(G) = 0.3$, $P(C) = 0.1$, $P(F \mid U) = 0.7$, $P(F \mid G) = 0.5$, $P(F \mid C) = 0.6$.

a) What is the probability that we've chosen a first-year student?

b) What is the probability that the student is from the UK if it is a first-year student?

c) Now fill in the entire JPT for the events defined in the problem.

Exercise 4.5

In an experiment on human memory, participants have to memorize a random sequence of items. The experiment is designed so that each item is either a word (W) or image (I), with $P(W) = 0.6$ and $P(I) = 0.4$. After the items are presented, participants then try to recall each one. Let R be the event that an item is correctly recalled. Results show that $P(R \mid W) = 0.4$, and $P(R \mid I) = 0.7$. If we know that the first participant correctly recalled a particular item, what is the probability that this item is an image?

Exercise 4.6
Suppose I want to choose a two-word name for a rock band via a random process. The first word will be chosen with uniform probability from the set {`yellow`, `purple`, `sordid`, `twisted`}. The second word will be either `quake`, with probability 1/3, or `revolution`, with probability 2/3.

a) What's the sample space for this experiment? How many outcomes are in it?

b) If I choose the first and second words independently, what is the probability that my rock band will be called `yellow revolution`?

c) Instead, I decide to condition the second word on the first. If the first word is a color, then I'll pick `quake` as the second word with probability 1/6. Assuming I still want the overall probability of a name with `quake` to be 1/3, what is the probability of `quake` if the first word is *not* a color? (Hint: use a joint probability table to help you answer this question. You may not need to fill in the entire table.)

5 Random variables and discrete distributions

So far, we have been using distinct labels to refer to every event of interest. However, this gets cumbersome, especially when we need to simultaneously refer to all the events in a particular partition of the sample space. In practice, it's much more common to work with random variables, as introduced in this section.

5.1 Definition of a random variable

A RANDOM VARIABLE (or RV) is a variable that represents all the possible events in some partition of the sample space. Put another way, an RV has several possible values, with each value being one event in a partition (and where the values cover all events in the partition). We will write random variables with uppercase letters, and their possible values with lowercase letters or numbers.[10]

RANDOM VARIABLE
RV

Example 5.1.1. Define a random variable X to represent the outcome flipping a fair coin, where this variable can take on two possible values (h or t) representing heads or tails. What is the distribution over X?

Solution: The distribution over an RV simply tells us the probability of each value, so the distribution over X is $P(X = h) = P(X = t) = 1/2$.

We use the notation $P(X)$ as a shorthand meaning "the entire distribution over X", in contrast to $P(X = x)$, which means "the probability that X takes the value x".

$P(X)$
$P(X = x)$

Example 5.1.2. Suppose I roll a fair six-sided die, with one face colored red, two colored blue, and three colored green. Let X be the color I get when I roll. What is $P(X)$?

Solution: Probably the simplest way to represent $P(X)$ is with a table. Here's one way to write it:

$X =$	`red`	`blue`	`green`
$P(X = x)$	1/6	1/3	1/2

[10] This convention is fairly standard for introductory materials on probability, but like a lot of mathematical notation, more advanced materials often use less explicit notation because (for someone who is familiar with the area) it's possible to disambiguate what's meant from context. We'll start with very explicit notation here and gradually introduce less explicit notation.

This table simply lists the probabilities for three events: $X = \texttt{red}$, $X = \texttt{blue}$, and $X = \texttt{green}$. So we could have just called those events R, B, and G, as we did in the previous sections of the tutorial. However, previously we didn't have a good way of referring to the entire distribution over the partition created by R, B, and G. Now that we have defined the RV X, we can refer to that distribution as $P(X)$. We can also talk about SAMPLING A VALUE FROM A DISTRIBUTION $P(X)$, which just means choosing a value for X at random according to the distribution $P(X)$.

SAMPLING A VALUE FROM A DISTRIBUTION

It's worth pointing out that if we repeatedly sample values from $P(X)$, then as the number of samples grows, the fraction of outcomes matching each value of X gets closer and closer to the probability of that value under $P(X)$. For example, if we repeatedly roll the die in Example 5.1.2, initially the fraction of outcomes that are `red` may not be very close to 1/6, but as we roll more and more times, the fraction of `red` outcomes will get closer and closer to 1/6. This observation is a deep theoretical result known as the LAW OF LARGE NUMBERS; we refer the reader elsewhere (e.g., to a statistics textbook) for its formal definition, proof, and many consequences.

LAW OF LARGE NUMBERS

5.2 The geometric distribution

The following example illustrates another advantage of random variables: in some cases, we can use them to specify a distribution very compactly, using an equation rather than a table of probabilities.

Example 5.2.1. Suppose we flip a fair coin several times in a row until we get a head. Let X be the total number of flips. What is the distribution over X?

Solution: Notice that in principle there is no limit on the number of flips that could be required. In other words, there is an infinite number of possible values that X can take on, though most of them have vanishingly low probabilities.

Luckily, we need not explicitly list the probability of every possible value. Instead, we use a formula to express the probabilities compactly. To find this formula, first consider $P(X = 1)$. In this case, we must get a head on the first flip and then stop. Since we have a fair coin, $P(X = 1) = 1/2$. Next, for $X = 2$, we must first get a tail (with probability 1/2), and then a head (also with probability 1/2). So, $P(X = 2) = (1/2)(1/2) = 1/4$. Similarly, $X = 3$ means we get exactly two tails followed by a head, and in general $X = n$ means $n-1$ tails and then one head. Since each tail and each head have probability 1/2, we conclude that $P(X = n) = (1/2)^n$ for positive integer n.

This example shows that RVs can be used to express the distribution over an infinite number of possible outcomes using a very simple formula. In the next example, we will see that the formula we just derived is a special case of a more general family of distributions.

Example 5.2.2. Suppose we perform the same type of experiment as in the previous example, but our coin isn't fair: the probability of getting a head is some value p. Now what is the distribution over X, the number of flips required to get a head?

Solution: By the same line of reasoning as in the previous example, if $X = n$ we will need to obtain $n-1$ tails, each with probability $1-p$, followed by a single head, with probability p. Therefore $P(X = n) = (1-p)^{n-1}p$ for positive integer n.

A random variable whose distribution follows the formula we just derived is said to have a GEOMETRIC DISTRIBUTION:

GEOMETRIC DISTRIBUTION

$$P(X = n) = (1-p)^{n-1}p \qquad \text{for positive integer } n. \tag{15}$$

I referred to this as a "family" of distributions because the formula contains a PARAMETER p. Depending on the specific value of p (which must fall between 0 and 1), we will get different

PARAMETER

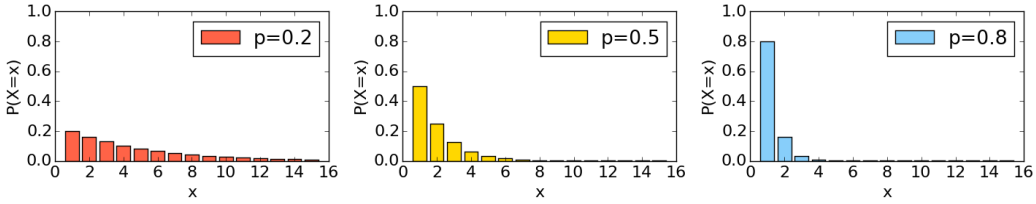


Figure 4: Three examples of geometric distributions, with parameter values of 0.2, 0.5, and 0.8. Possible values of X extend to infinity, but we only show values up to 15.

specific formulas. However, they are all considered to be geometric distributions. Figure 5.2 plots a few examples with different values of p.

The geometric distribution characterizes not just coin flips, but any situation in which:

- We repeat some random experiment until we succeed, and we are only interested in how long it will take to succeed.
- Each attempt is independent and has the same probability of success: the attempts are INDEPENDENT AND IDENTICALLY DISTRIBUTED or IID.

INDEPENDENT AND IDENTICALLY DISTRIBUTED
IID

Although the geometric distribution can be described by an equation, not all random variables that take on numeric values can be. Although we didn't call it a random variable at the time, we have already seen an example of a numerically valued random variable that can't easily be described by a single equation, namely the random variable corresponding to the sum of two fair 6-sided dice. In Example 2.4.2, we had to write down the distribution in terms of events, for example $P(\texttt{sum is 2}) = 1/36$, $P(\texttt{sum is 3}) = 2/36$, and so forth. But, using the random variable notation, we could define an RV S to be the sum of the two dice, and write instead $P(S = 2) = 1/36$, $P(S = 3) = 2/36$, and so forth.

5.3 Other discrete distributions

The geometric distribution is not the only discrete distribution with its own equation, though it's one of the most commonly used. In a future version of this tutorial I might go into more detail about some others, but here is a very quick overview. If you need to understand these distributions in any detail or want the equations, you can find them in any probability textbook or online. There are no exercises based on this section.

The Bernoulli distribution This is just a fancy name for a distribution we've already seen, namely a single binary variable (such as a coin flip, where $X = 1$ means 'head' and $X = 0$ means 'tail'). There is a single parameter p which determines the probability that $X = 1$.

The binomial distribution If we have a Bernoulli variable X and we repeatedly sample values from $P(X)$ (e.g., flip a coin many times), we will get some number of 1's (heads) and some number of 0's (tails). The binomial distribution with parameters p and n describes the probability of getting a certain number of 1's out of a total of n samples when the probability of a 1 on a single sample is p.

The multinomial distribution This distribution is an extension of the binomial distribution to the case where the "coin" has more than two outcomes. So, maybe it is a die instead of a coin. But again we want to determine the probability of getting different numbers of each outcome given a fixed total number of outcomes.

The Poisson distribution This distribution has a single parameter, usually called λ (lambda), the "rate". One way to think about this distribution is as follows: we are counting how often a particular thing happens. On average, that thing happens λ times within a certain fixed period of time, but the intervals between the occurrences are random. The Poisson distribution tells us the probability of that thing happening exactly x times with the fixed period. An example would be: we are counting the number of typos in each 1000 words of text. On average, there are 5 typos per 1000 words. What is the probability that there are exactly 4 typos in the next 1000 words?

5.4 Restating the probability rules using random variables

As we've seen, there is a close relationship between random variables and partitions of the sample space into events, but the notation is slightly different. Since the random variable notation is more commonly used, it's worth going through each of the rules of probability and other tools we presented earlier to show how they apply when we're using random variables.

Let's assume we are working with a random variable X whose possible values are $\{x_1, x_2, \ldots, x_n\}$. These values could be numeric (e.g., integers) or categorical (e.g., different letters or different colors), and n could be a finite number or infinity. However, we assume that the x_i values are ENUMERABLE, which means that it is possible to create a one-to-one mapping between these values and the integers (think of it as assigning a unique integer to each x_i value). This assumption means that X is a DISCRETE RANDOM VARIABLE, i.e., it has a set of discrete possible values. Later we will consider continuous random variables, which can take on real-number values.

ENUMERABLE

DISCRETE RANDOM VARIABLE

We can now (re)state the requirements for a DISCRETE PROBABILITY DISTRIBUTION or PROBABILITY MASS FUNCTION as follows:

DISCRETE PROBABILITY DISTRIBUTION

PROBABILITY MASS FUNCTION

$$0 \leq P(X = x_i) \leq 1 \tag{16}$$

$$\sum_{i=1}^{n} P(X = x_i) = 1 \tag{17}$$

As a shorthand notation, instead of writing the values of an RV explicitly in a sum, such as $\sum_{i=1}^{n} P(X = x_i)$, we will often just write $\sum_x P(X = x)$, which should be interpreted in the same way: that is, as a sum in which x ranges over all possible values of X. In more advanced materials, you will often find that an even more concise notation is used: to indicate the probability of an RV taking a particular value x, authors may just write $P(x)$ rather than $P(X = x)$ as we have been doing. In this case, the above sum would be written as $\sum_x P(x)$.

Before going through the rest of the rules of probability, we introduce one more notational variant, this time for the joint probability of two random variables X and Y. We previously used the intersection symbol for joint probability, but it is more common (especially when using random variables) to use a comma instead, as in $P(X = 1, Y = \texttt{blue})$. And, just as $P(X)$ refers to the distribution over X, we refer to the entire JOINT PROBABILITY DISTRIBUTION over X and Y as $P(X,Y)$: you can think of it as referring to the entire JPT.

JOINT PROBABILITY DISTRIBUTION $P(X,Y)$

Example 5.4.1. Rewrite the JPT from Example 4.4.2 using random variable notation, with X representing the color of the marble, and Y representing the pattern (solid or patchy).

Solution:

	$X = \texttt{r}$	$X = \texttt{g}$	$X = \texttt{b}$	
$Y = \texttt{solid}$	1/3	1/5	2/15	2/3
$Y = \texttt{patchy}$	2/15	2/15	1/15	1/3
	7/15	1/3	1/5	

With all the notation out of the way, we now restate several of the other rules of probability, using both the verbose and concise notations.

verbose	concise	
$P(X=x) = 1 - P(X \neq x)$	$P(x) = 1 - P(\neg x)$	complement (18)
$P(X=x) = \sum_y P(X=x, Y=y)$	$P(x) = \sum_y P(x,y)$	law of total prob (19)
$P(X=x \mid Y=y) = \frac{P(X=x,Y=y)}{P(Y=y)}$	$P(x \mid y) = \frac{P(x,y)}{P(y)}$	defn of cond prob (20)
$P(X=x,Y=y) = P(X=x \mid Y=y)P(y)$	$P(x,y) = P(x \mid y)P(y)$	product rule (21)
$P(X=x \mid Y=y) = \frac{P(Y=y \mid X=x)P(X=x)}{P(Y=y)}$	$P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)}$	Bayes' Rule (22)

So far, all of these rules are straightforwardly equivalent to the ones we saw earlier. However, there are a couple of places where we need to be more careful. The first of these is when referring to a CONDITIONAL PROBABILITY DISTRIBUTION. We can refer to a (marginal) distribution as $P(X)$ and a joint distribution as $P(X,Y)$. But it is *incorrect* to write $P(X \mid Y)$, because a conditional distribution has to be conditioned on *a particular event*. Valid ways to refer to a conditional distribution include $P(X \mid Y = y)$ or $P(X \mid y)$.

CONDITIONAL PROBABILITY DISTRIBUTION

$P(X \mid Y = y)$

The second case we need to be careful about is the definition of INDEPENDENT RANDOM VARIABLES. We say that two RVs X and Y are independent iff, for *all* values of x and y,

INDEPENDENT RANDOM VARIABLES

$$P(X=x, Y=y) = P(X=x)P(Y=y). \tag{23}$$

This is a more general statement than independence of two events. We can say that the *events* $X = x$ and $Y = y$ are independent iff Eq (23) is true for those particular events, regardless of whether other outcomes of X and Y obey Eq (23). But for the *random variables* X and Y to be independent, *all* possible outcomes of X and Y must obey Eq (23).

Example 5.4.2. Are the RVs X and Y from Example 5.4.1 independent?

Solution: No. Recall that in Example 4.5.1 we found that whether the marble is blue is independent of whether it's solid or patchy. So those particular events are independent. However, we also found that the other two colors were not independent of the solid/patchy events. So overall, X and Y are not independent.

If we had not already determined the independence of the various events in the JPT in Example 4.5.1, we would only need to find a single case where Eq (23) fails to know that the RVs are not independent. For example, $P(X = \texttt{r}, Y = \texttt{patchy}) \neq P(X = \texttt{r})P(Y = \texttt{patchy})$.

5.5 Working with more than two variables

In the previous sections, we mostly assumed probability distributions are only over one or two variables. However, we may want to define joint distributions over more than two variables, such as $P(X,Y,Z)$. And if we do that, then we can also condition on one or more of those variables, for example looking at $P(X,Y \mid Z = z)$ or $P(X \mid Y = y, Z = z)$.

Example 5.5.1. Returning to our marble example, assume we start with the same jar of marbles as in 5.4.1, but we add some new marbles to the jar. These new marbles have the same joint distribution over X (color) and Y (pattern) as the existing ones, but there are

twice as many new marbles as old ones, and the new ones are a larger size. Let Z be the size of a marble (small or large). If we pull a marble out uniformly at random, what are $P(X,Y \mid Z = \texttt{small})$ and $P(X,Y \mid Z = \texttt{large})$? What is $P(X,Y,Z)$?

Solution: $P(X,Y \mid Z = \texttt{small})$ is the joint distribution over X and Y conditioned on the marble being a small one. The small marbles are exactly the ones that were in the jar before, so we just need to give the joint distribution over those marbles, i.e., the table in the solution to Example 5.4.1. Moreover, since the statement of the problem says that the new (large) marbles have the same joint distribution over X and Y as the old (small) ones, $P(X,Y \mid Z = \texttt{large})$ is *also* given by the solution to Example 5.4.1.

As for $P(X,Y,Z)$, the question says there are twice as many large as small marbles, so we know the marginal probabilities $P(Z = \texttt{small}) = 1/3$ and $P(Z = \texttt{large}) = 2/3$. We then apply the product rule to get each value in $P(X,Y,Z)$, computed as $P(X = x, Y = y, Z = z) = P(X = x, Y = y \mid Z = z)P(Z = z)$. The actual numbers are shown in the following two tables (with $P(X = x, Y = y, Z = \texttt{small})$ on the left, and $P(X = x, Y = y, Z = \texttt{large})$ on the right; marginals aren't shown). As with any distribution, adding up all possible combinations of values for the RVs (i.e., all the numbers in both tables) sums to 1.

$Z = \texttt{small}$	$X = \texttt{r}$	$X = \texttt{g}$	$X = \texttt{b}$
$Y = \texttt{solid}$	1/9	1/15	2/45
$Y = \texttt{patchy}$	2/45	2/45	1/45

$Z = \texttt{large}$	$X = \texttt{r}$	$X = \texttt{g}$	$X = \texttt{b}$
$Y = \texttt{solid}$	2/9	2/15	4/45
$Y = \texttt{patchy}$	4/45	4/45	2/45

We used the product rule in this example although we never explicitly stated it for more than two variables. In fact, all the rules of probability can be applied to multiple variables. If you get confused about how to apply them, here are some handy rules of thumb (with examples using the concise notation):

- When multiple conditioning variables are present, the rules can be applied to the two variables immediately adjacent to the '|' sign, with the additional conditioning variables just hanging around on the end, as in:

$$P(x \mid y,z) = \frac{P(y \mid x,z)P(x \mid z)}{P(y \mid z)} \qquad \text{Bayes' Rule} \tag{24}$$

$$P(x \mid z) = \sum_y P(x \mid y,z)P(y \mid z) \qquad \text{law of tot prob} \tag{25}$$

- Alternatively, you can treat some or all of the variables on one side of the '|' as a single "compound" variable. For instance, the value of the pair (X,Y) can be thought of as a single random variable A whose outcomes are all of the outcomes in the cross product of X and Y. Below are two examples of Bayes' Rule with multiple variables. In the first case we treat Y and Z as a single variable, and in the second case we treat B,C as a single variable but leave D as a conditioning variable.

$$P(x \mid y,z) = \frac{P(y,z \mid x)P(x)}{P(y,z)} \tag{26}$$

$$P(a \mid b,c,d) = \frac{P(b,c \mid a,d)P(a \mid d)}{P(b,c \mid d)} \tag{27}$$

- Remember that the ordering of variables is irrelevant as long as they stay on the same side of the '|' sign:

$$P(x,y \mid z) = P(y,x \mid z) \tag{28}$$

$$P(x \mid y,z) = P(x \mid z,y) \tag{29}$$

$$P(x \mid y,z) \neq P(x,y \mid z)!! \tag{30}$$

Finally, we introduce the notion of CONDITIONAL INDEPENDENCE, which relies on multiple variables. Informally, two RVs X and Y are conditionally independent given Z iff once we know the value of Z, X and Y are independent. More formally, X and Y are conditionally independent given Z iff for all values of x, y, and z,

CONDITIONAL INDEPENDENCE

$$P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z)P(Y = y \mid Z = z) \tag{31}$$

Example 5.5.2. Let X and Y be variables representing whether I stay up late and whether I show up on time for my 9 a.m. class. Let Z represent whether I left my house on time. Intuitively, are X and Y independent? Are they conditionally independent given Z?

Solution: They are not independent: presumably I'm less likely to arrive on time if I stayed up late. But, if we know whether I left the house on time, they are independent: knowing if I stayed up late or not is now irrelevant. So, X and Y are conditionally independent given Z.

Conditional independence (or the assumption of conditional independence) comes up a lot in machine learning and NLP. One place where it can be useful is if we are trying to classify items into different categories based on the features of those items, another example of probabilistic inference.

Example 5.5.3. Suppose we are studying two closely related species of birds, the azure-breasted nuthatch (species a) and the blue-throated nuthatch (species b). Unfortunately, it isn't possible to determine with complete accuracy which species a particular individual belongs to just by looking at it. However, certain features are more common among one or the other species. In particular, 70% of individuals in species a have red eyes, while only 20% of those in species b do. On the other hand, 20% of a individuals have yellow feet, while 40% of b individuals do. While bird-watching in an area in which the azure-breasted and blue-throated nuthatches are equally common, we see a bird with red eyes and yellow feet. Assuming that the eye color and foot color are conditionally independent given the species of bird, what is the probability that this bird is from species a?

Solution: I will use the concise notation here so the equations fit, with a and b to refer to the species, r for red feet, and y for yellow eyes. We apply Bayes' Rule, followed by the law of total probability, and then the definition of conditional independence:

$$\begin{aligned}
P(a \mid r, y) &= \frac{P(r, y \mid a)P(a)}{P(r, y)} && (32)\\
&= \frac{P(r, y \mid a)P(a)}{P(r, y \mid a)P(a) + P(r, y \mid b)P(b)} && (33)\\
&= \frac{P(r \mid a)P(y \mid a)P(a)}{P(r \mid a)P(y \mid a)P(a) + P(r \mid b)P(y \mid b)P(b)} && (34)\\
&= \frac{(0.7)(0.2)(0.5)}{(0.7)(0.2)(0.5) + (0.2)(0.4)(0.5)} && (35)\\
&\approx 0.63 && (36)
\end{aligned}$$

This probabilistic method for classifying items into categories by assuming that features are conditionally independent given the category is known in machine learning as NAIVE BAYES CLASSIFICATION. The "naive" part is because in reality the features usually are not conditionally independent given the category. However, this method can still be effective in many situations.

NAIVE BAYES CLASSIFICATION

5.6 Exercises

Exercise 5.1
Suppose I decide to generate a "word" as follows: I start by generating the character `a`. Then, with probability q, I generate a single `b` and stop, otherwise I generate another `a` and keep going. I continue this process, always either generating a single `b` with probability q and stopping, or generating an `a` and continuing.

a) What are the possible words I might generate using this process?

b) If L is a random variable representing the length of a word, give an equation for $P(L = n)$. What kind of distribution does L have?

c) If $q = 0.3$, what is the probability that my generated word has 4 characters? What is the probability of generating the word `aaab`?

Exercise 5.2
Now suppose I have five different characters that can be in my word (`a`, `b`, `c`, `d`, `e`). As before, I start by generating the character `a`. Then, with probability q, I generate a `b` and stop, otherwise I choose one of the other four characters uniformly at random and keep going. I continue this process, always either generating a `b` with probability q and stopping, or choosing one of the other four characters uniformly at random and continuing.

a) What are the possible words I might generate using this process?

b) If L is a random variable representing the length of a word, give an equation for $P(L = n)$. How does it differ from the equation found in Exercise 5.1?

c) If $q = 0.3$, what is the probability that my generated word has 4 characters? What is the probability of generating the word `aaab`? Are your answers different to those in Exercise 5.1? Why?

d) Again assuming $q = 0.3$, what is the probability of generating the word `aaab` given that I generate a 4-character word?

[More exercises to be added]

6 Expectation and variance

6.1 Definitions

When a random variable takes on numeric values, it can be useful to consider what the *average* or *mean* value of that variable is. That is: suppose we sample many values for an RV X (so that, by the Law of Large Numbers, the fraction of samples with value x matches the probability of x). Adding up all the values we got and dividing by the number of samples yields the average value we got for X.

Example 6.1.1. Suppose I offer to play a game with you. You roll two dice, and I'll pay you £4 if you get a 6 or 11, but you pay me £1 otherwise. If we played this game many times in a row, who would end up winning more money? On average, how much would they win per game?

Solution: Let X be the number of pounds you win on a single dice roll, so X can be either 4 or -1. $P(X = 4)$ is equal to the probability of rolling either 6 or 11. In Exercise 2.3 you computed these as 5/36 and 2/36, respectively (you did do the excercise, didn't you?), and they are disjoint events, so $P(X = 4) = 5/36 + 2/36 = 7/36$. That leaves $P(X = -1) = 1 - 7/36 = 29/36$. So over many games, you'd expect to win £4 7/36 of the time,

and lose £1 29/36 of the time. Altogether, your winnings over N games would be about $(4)(7N/36)-(1)(29N/36) = -N/36$ pounds. So I win more money, by an average of about 3 pence for each game.

In the previous example, we literally took the average over a large number of samples of X. However, even if we don't take lots of samples, we can still say that, on average, we'd *expect* the value of a single sample to be the same as the actual average over lots of samples. This idea leads to the definition of the EXPECTED VALUE or EXPECTATION of a random variable X, written $E[X]$ and formally defined as

EXPECTED VALUE
EXPECTATION

$$E[X] = \sum_x x \cdot P(X = x) \tag{37}$$

Since $E[X]$ is really just the MEAN value for X, if it's clear which X is being referred to, $E[X]$ may also be wrtten simply as μ (pronounced "myoo")—an abbreviation for "mean".

MEAN
μ

Regardless of notation, Eq (37) defines the expected value of X as the sum of the probability of each possible X value times the value itself. If we run an experiment N times, we can expect the sum of the X values to be about $N \cdot E[X]$.

Example 6.1.2. A contestant on a game show wins \$1000. Then the host presents two boxes. In one box is a label marked "double" and in the other "nothing". The contestant may, if she wishes, choose one of the boxes without seeing their contents. If she chooses the "double" box, she'll get another \$1000, but if she chooses "nothing", she'll lose the \$1000 she already won. If X is the additional amount she gets by taking the choice, what is $E[X]$? Should she choose a box, or just keep the money she has?

Solution: We have $P(X = 1000) = 1/2$ and $P(X = -1000) = 1/2$. So $E[X] = 1000(1/2) - 1000(1/2) = 0$. Of course, if she doesn't choose a box, her expected additional winnings are also 0. So, if the only consideration is her expected total winnings, it makes no difference if she decides to choose a box or not. Of course there are other psychological factors that might be important, such as whether the contestant likes to take risks. There are also mathematical differences between the two choices, one of which we'll return to shortly.

In addition to computing the expectation of an RV X, sometimes it's useful to compute the expectation of a function of X, $f(X)$, to give us the average value of the function:

$$E[f(X)] = \sum_x f(x) \cdot P(X = x). \tag{38}$$

Example 6.1.3. I offer to let you roll a single die, and will give you a number of pounds equal to the square of the number that comes up. How much would you expect to win by playing this game?

Solution: Solution: If X is the number that comes up, the expected value of your winnings, in pounds, is

$$\begin{aligned} E[X^2] &= (1/6)(1^2) + (1/6)(2^2) + (1/6)(3^2) \qquad (39) \\ &\quad +(1/6)(4^2) + (1/6)(5^2) + (1/6)(6^2) \\ &= 91/6 \qquad (40) \end{aligned}$$

or about £15.17.

Let's now return to the example of the contestant on the game show. We said there was no difference in expected winnings if she decides to choose a box, or sticks with her \$1000. But clearly there's *something* different in these two cases. One way to describe that something is using the concept of VARIANCE. The variance of an RV tells us, for any given sample, how far away from the expected value is the result likely to be? The variance of X is defined as

VARIANCE

$$Var[X] = E[(X - E[X])^2]. \tag{41}$$

That is, it's the expected value of the squared difference between X's actual value and expected value. The variance can also be written as

$$Var[X] = E[X^2] - (E[X])^2, \tag{42}$$

a version which is often easier to compute.

Informally, RVs that have a vary large range of possible values will have higher variance than those whose range of values is smaller.

Example 6.1.4. We've seen two different RVs with expectations at or close to zero: the value of your winnings in Example 6.1.1, and the value of the contestant's additional winnings in Example 6.1.2 if she decides to choose a box. Intuitively, which of these has higher variance? What are their variances?

Solution: In the dice game, the outcomes are to win £4 or lose £1, whereas in the game show, the outcomes are to win or lose \$1000. The latter case clearly has a larger range of outcomes, so the variance should be bigger. Letting D be the winnings in the dice game and G be the winnings in the game show, we use Eq (42) and the expected values already computed in the previous examples. We find $Var[D] = (4^2)(7/36) + (1^2)(29/36) - (1/36)^2 \approx 3.92$ and $Var[G] = (1000^2)(1/2) + (-1000^2)(1/2) - (0)^2 = 1,000,000$. So indeed, the variance for the game show contestant is (much) larger than for the dice game. (Of course, if the contestant decides not to choose a box, then the variance of that choice is 0: there is no randomness!)

In statistics, it's also common to use a related measure of variation known as the STANDARD DEVIATION. The standard deviation, written as $SD[X]$ or just σ (sigma) if it's clear which X is being referred to, is simply the square root of the variance:

STANDARD DEVIATION
σ

$$SD[X] = \sqrt{Var[X]} \tag{43}$$

In fact, you may also see the variance written as σ^2.

Example 6.1.5. Compute the standard deviations of the D and G variables from Example 6.1.4.

Solution: We computed the variances already, so $SD[D] = \sqrt{3.92} \approx 1.98$ and $SD[G] = \sqrt{1,000,000} = 1,000$.

6.2 Exercises

To be added.

7 Continuous random variables

The tools of probability theory as applied to discrete random variables are very powerful, but there are some kinds of random processes they cannot capture. Consider, for example, the amount of time I need to wait at the bus stop before a bus arrives. Although we sometimes divide it up into discrete chunks like minutes and seconds, time is continuous, so to model the probability of how long I need to wait, we must use a CONTINUOUS RANDOM VARIABLE: an RV whose possible values are a continuous range of real numbers. Another example of a continuous random variable might be: when I aim for a target, how far to the the left or right of the target do I hit?

CONTINUOUS RANDOM VARIABLE

Figure 5 illustrates two possible distributions for these two continuous random variables. Each plot is an example of a PROBABILITY DENSITY FUNCTION, or PDF. Recall that the distribution of a discrete random variable is called its probability mass function; the analogous function for a continuous random variable is a density function.

PROBABILITY DENSITY FUNCTION
PDF

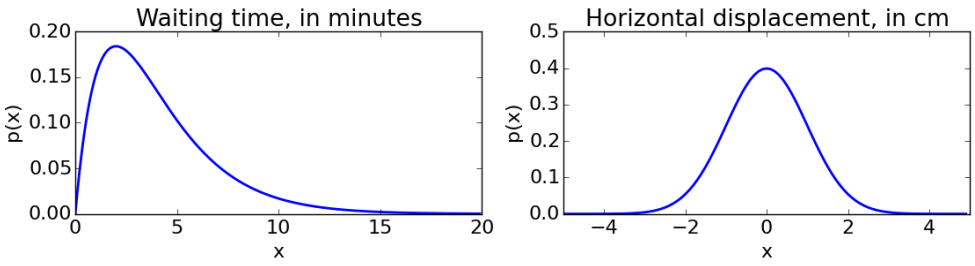


Figure 5: Two probability density functions: (left) how long I wait for a bus if I arrive shortly before its scheduled time, and (right) the distance away from a target my shot is if I aim for the target (negative values are displaced to the left and positive values to the right).

In the plots, we see that the x-axis of each plot indicates the possible values for the random variable. The y-axis of each plot indicates—what? Well, as in a discrete distribution, values of x with higher y values (higher $p(x)$) are in some sense more probable than those with lower y values.

I say "in some sense" because in fact the probability of any particular value in a continuous distribution is 0: there are just so many different possible values that a continuous RV can have, that if we choose any particular one, we are never going to get exactly that value. Technically we can only talk about the probability of getting a value within some *range* of a given value, like $p(2.9 \leq X \leq 3.1)$. The probability of this event is equal to the *area* under the density function between $X = 2.9$ and $X = 3.1$. So what it really means to say that $p(x_1) > p(x_2)$ is that the probability of X taking a value within a very small (infinitesimally small!) range of x_1 is higher than the probability of X taking a value within an equally tiny range of x_2.

To emphasize the distinction between continuous and discrete RVs, I use $P(X)$ for discrete distributions and $p(X)$ for continuous distributions. This is a somewhat standard convention, although many authors don't bother to follow it.

Although there are these technical differences between continuous and discrete random variables, in practice we often don't need to worry about them. Almost all of the rules of probability and other tools we've seen already apply just as well to continuous distributions, provided we replace all sums with integrals. For example:

$$p(x) = \int_{-\infty}^{\infty} p(x,y)\, dy \qquad \text{law of tot prob, continuous version} \tag{44}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\, dx \qquad \text{expectation, continuous version} \tag{45}$$

The product rule, definitions of conditional probability and independence, Bayes' rule, etc. have no sums in them and can be applied as usual.

One place we need to be careful is with the two fundamental properties of probability: summing to one and being between 0 and 1. The first of these translates directly to the continuous case:

$$\int_{-\infty}^{\infty} p(x)\, dx = 1 \tag{46}$$

In other words, the total area under the curve of a continuous distribution adds up to 1, just as the sum of all different possible outcomes of a discrete distribution adds up to 1. However, while $p(x) \geq 0$ in the continuous case (as in discrete), it is *not* the case that $p(x) \leq 1$ for a continuous RV. To illustrate why, consider an RV X that is uniformly distributed between 0.5 and 1, as shown in Figure 6. Since the area under the pdf has size 1, but the width is only 0.5, the height of the pdf must be 2.

At some point I may add some information to this tutorial about common continuous distributions (such as the normal or Gaussian distribution), but for the moment you will need to look elsewhere if you need that information.

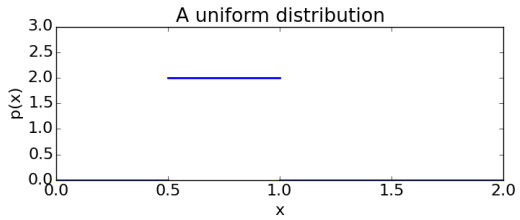


Figure 6: The probability density function of a continuous RV that is uniformly distributed between .5 and 1. The area under the pdf has size 1.

8 A note about estimating probabilities from data

This tutorial has aimed to provide both intuition and mathematical tools to work with probability distributions. However, throughout the tutorial we have assumed that the probabilities of different events and outcomes are known: we worked with fair dice, specified the number of marbles of different colors and patterns, and chose students from a group with fixed size and known properties. While these kinds of examples are useful for developing mathematical facility with probabilities, they are actually rare in real life. Most of the time, we *don't* know the probabilities of different events ahead of time. Instead, we typically observe a sequence of outcomes of some random process and must use these observations to estimate the probabilities of the different outcomes in order to *generalize* beyond the data we observed.

For example, we might have access to a large corpus of text, and we want to estimate the probability of seeing the word `gallant` after the word `the`. This might seem straightforward at first glance: can't we simply use the definition of conditional probability, and count the number of times that `gallant` occurs after `the`, divided by the number of times `the` occurs? Indeed, if what we want to know is the probability of `gallant` following `the` in *this particular corpus*, then that would be a correct solution. However, usually what we really want to know is the probability of `gallant` following `the` *in general* (in other corpora like this one, or in English). Suppose that `the gallant` never happens to occur in the corpus we observe. Does that mean its probability is 0, i.e., that it will never occur in any corpus? Hopefully you will agree that the answer is no: instead we'll need to figure out a better way to estimate probabilities that doesn't tell us the probability is 0.

Large parts of natural language processing, machine learning, and computational cognitive science are concerned with how computers can (or humans do) estimate probabilities from observed data in such a way as to best predict future outcomes. Although this tutorial does not cover the topic of probability estimation, if you have carefully read through and understood everything in the tutorial and worked through the exercises, you should be well placed to begin learning about probability estimation and other probabilistic methods for NLP, machine learning, and cognitive science.

9 Solutions to selected exercises

Solution 2.1

a) Each word in the sentence has 5000 possibilities, so there are 5000^5 different sentences in the sample space.

b) Given a fixed first word, there are now 5000^4 different four-word continuations.

c) $P(E) = \frac{5000^4}{5000^5}$, or 1/5000. (We can also get the same answer by considering only the outcomes for the first word: out of 5000 possible outcomes, only 1 is *the*. The rest of the sentence doesn't matter.)

d) They are not mutually exclusive: one outcome that belongs to both A and E is *the big a yellow the*.

Solution 2.3

$P(\texttt{sum is 2}) = 1/36$
$P(\texttt{sum is 3}) = 2/36$
$P(\texttt{sum is 4}) = 3/36$
$P(\texttt{sum is 5}) = 4/36$
$P(\texttt{sum is 6}) = 5/36$
$P(\texttt{sum is 7}) = 6/36$
$P(\texttt{sum is 8}) = 5/36$
$P(\texttt{sum is 9}) = 4/36$
$P(\texttt{sum is 10}) = 3/36$
$P(\texttt{sum is 11}) = 2/36$
$P(\texttt{sum is 12}) = 1/36$

Solution 2.4

$P(\texttt{g}) = 4/9$, $P(\texttt{b}) = 2/9$, $P(\texttt{r}) = 3/9$.

Solution 2.6

The equally likely outcomes are the eight equally sized sections. Since there are four red sections out of eight, the probability of landing on a red section is 4/8, or 1/2.

Solution 2.8

a) These events are mutually exclusive and cover all possible outcomes, therefore the probabilities form a distribution: $P(E_1) = 3/7$ and $P(E_2) = 4/7$.

b) These events are mutually exclusive but do not cover all possible outcomes so they do not form a distribution. $P(E_1) = 4/7$ and $P(E_2) = 1/7$.

c) These events are not mutually exclusive and do not cover all possible outcomes, so the probabilities will not form a distribution. $P(E_1) = 2/7$ and $P(E_2) = 2/7$.

d) This is a tricky exmple. $P(E_1) = 5/7$ and $P(E_2) = 2/7$: the probabilities add up to one, which means that they can in principle form a distribution. However the stated events are not mutually exclusive and do not cover all possible outcomes, so the probabilities do not form a distribution with respect to those particular events.

e) These events are mutually exclusive and cover all possible outcomes, therefore the probabilities form a distribution: $P(E_1) = 2/7$, $P(E_2) = 3/7$, and $P(E_3) = 2/7$.

Solution 3.1

a) Let A = "starts with S" and B = "ends with S". Then "at least one S" is $A \cup B$. And $P(A \cup B) = P(A) + P(B) - P(A,B) = .02 + .03 - .01 = .04$.

b) This event is the complement of the previous one, so its probability is $1 - .04$, or .96.

Solution 4.1

a) $P(F \cap R) = P(F \,|\, R)\; P(R) = 1/8$. This is the probability that the chosen card is a red face card.

b) They are not independent because $P(R)P(F) \neq P(R \cap F)$.

Solution 4.3

a) $P(F \,|\, U) = \frac{P(F \cap U)}{P(U)} = \frac{3/7}{4/7} = 3/4$

b) $P(\neg F \,|\, U) = 1 - P(F \,|\, U) = 1/4$

c) 3/5

d) No, because $P(F \,|\, U) \neq P(F)$.

e)

	F	$\neg F$
	3/4	1/4

f)

	U	C	G	
F	3/7	1/7	1/7	5/7
$\neg F$	1/7	0	1/7	2/7
	4/7	1/7	2/7	

Solution 4.5

$$\begin{aligned} P(I \,|\, R) &= \frac{P(R \,|\, I)\, P(I)}{P(R)} \\ &= \frac{P(R \,|\, I)\, P(I)}{P(R \cap I) + P(R \cap W)} \\ &= \frac{P(R \,|\, I)\, P(I)}{P(R \,|\, I)P(I) + P(R \,|\, W)P(W)} \\ &= \frac{(0.7)(0.4)}{(0.7)(0.4) + (0.4)(0.6)} \\ &\approx .54 \end{aligned}$$

Solution 5.1

a) The words are sequences of characters consisting of one or more `a`'s followed by exactly one `b`.

b) The equation is $P(L = n) = (1-q)^{n-2}(q)$. The distribution is nearly a geometric distribution, except the exponent is $n-2$ instead of $n-1$ because we always start with an `a`. Put another way, the distribution over lengths *after the first character* (i.e., over $L-1$) is geometric with parameter q.

c) Both are equal to $(0.7)^2(0.3) \approx 0.15$