Chapter — Page

18 Warm Up

Revision 1

Reminders

1. Numbers can be grouped into many sets. The major ones are sets of naturals, wholes, integers, rationals and irrationals.

2. There are:

a) three laws of indices

(i) $a^x \times a^y = a^{x+y}$

(ii) $a^x \div a^y = a^{x-y}$

(iii) $(a^x)^y = a^{xy}$

b) three laws of logarithms

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \frac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a M^p = p\log_a M$

c) three laws of surds

(i) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

(ii) $\sqrt{a^2 b} = a\sqrt{b}$

(iii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

3. Area and Volume of common solids

SOLID	SURFACE AREA	VOLUME
Prisms	Sums of areas of faces	Area of cross-section $\times$ height
Sphere	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Cone	πrl	$\frac{1}{3}\pi r^2 h$
Pyramid	Base Area + Area of triangular faces	$\frac{1}{3}$ Base area $\times$ height

4. Give formulae for area and perimeter of plane shapes.

1. Is it True or False that;

a) 5 is a rational number.

b) –3 is a whole number.

c) 2 is an irrational number.

d) 1 is not a prime number.

e) 1 is factor of any number.

f) $3^{-2} = -9$

g) $\dfrac{\log 10}{\log 2} = \log 5$

2. Evaluate a) $-1^0 + (-1)^0$

b) $8^{-\frac{2}{3}}$

c) $\dfrac{\log\sqrt{5}}{\log\frac{1}{5}}$

d) $\sqrt{27} - \sqrt{75} + 3\sqrt{12}$

3. Given that $\log 3 = m$ and $\log 7 = n$, express

a) $\log 49$ in terms of n.

b) $\log 2\frac{1}{3}$ in terms of m and n.

c) $\log 21$ in terms of m and n.

4. Given that $\sqrt{3} = 1{,}73$ and $\sqrt{5} = 2{,}24$, evaluate, to 2 decimal places,

a) $\dfrac{2}{\sqrt{3}}$ b) $\dfrac{1}{2\sqrt{5}}$

5. Take π to be $\frac{22}{7}$ in this question.

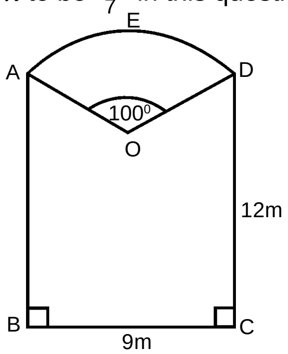


Fig 18.1

Fig 18.1 shows the cross-section of a tobacco barn 9m wide and the vertical sides AB and DC being 12 m high. The roof, AED, is in the form of an arc, 11m long, centre O with $A\hat{O}D = 100^0$.

Calculate a) the radius of the sector AODE.

b) the area of the cross- section.

c) the volume of a barn if it is 25m long.

Revision 2

Reminders

1. Basic geometric facts link
a) angles at a point.
b) angles adding up to 180^0.
c) complementary angles.
d) angles in parallel lines etc.

2. Ratio and scale can be expressed as common fractions e.g. $a : b = \frac{a}{b}$

3. Congruent figures (≡) are identical i.e same shape, same size.

4. Similar figures (III) have the same shape but differ in size. There are three ratios of similar figures.

5. Give as many Theorems about angles in a circle as you can.

6. Inequality signs $>$and $\nleqslant$, $<$ and $\ngeqslant$, $\geqslant$ and $\nless$, $\leqslant$and $\ngtr$ have the same meaning

7. An open circle on a number line and a broken boundary in a Cartesian plane imply a strict inequality whilst a shaded circle on a number line and a solid boundary in a cartesian plane imply the "or equal to" in equality.

1. Is it True or False that:
a) a right –angled isosceles triangle has no lines of symmetry.
b) adjacent angles on a straight line are always supplementary.
c) the sum of the interior angle of any polygon is 360^0.
d) an equilateral triangle is a regular polygon.

2. If the scale of a given map is 1: 30 000, find the actual area of a forest given as 8cm² on the map.

3. Fig 18.2 is made of two similar trapezia with PU//QT//RS, $P\hat{U}T = R\hat{Q}T = 78^0$ and $P\hat{Q}T = 117^0$.
a) Name the angle that is equal to $Q\hat{R}S$.
b) Name, in the correct order, the two similar trapezia.

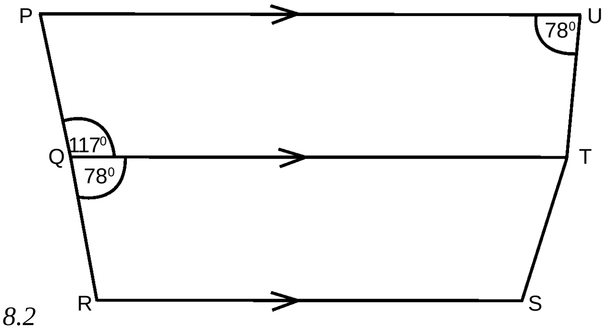


Fig 18.2

4.

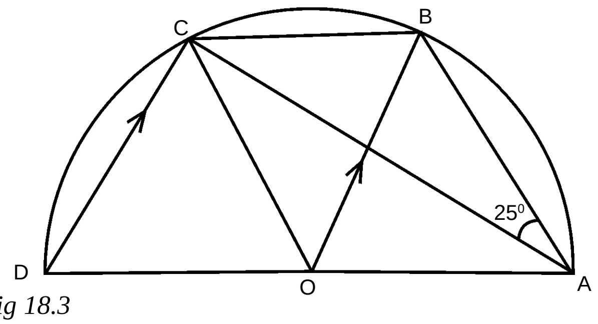


Fig 18.3

Points A, B,C and D are on semi-circle with centre O. Given that DC is parallel to OB and BAC = 25^0.

a) Name all the isosceles triangles in the diagram.

b) Name the triangle that is congruent to triangle AOB.

5. The ratio of volumes of two similar containers is 8:27. Find the height of the bigger container if the smaller one is 10 cm high.

6. Solve the inequality $2 < 3x + 7 \leqslant x + 11$ and list the integer values of x^{-5}.

7.

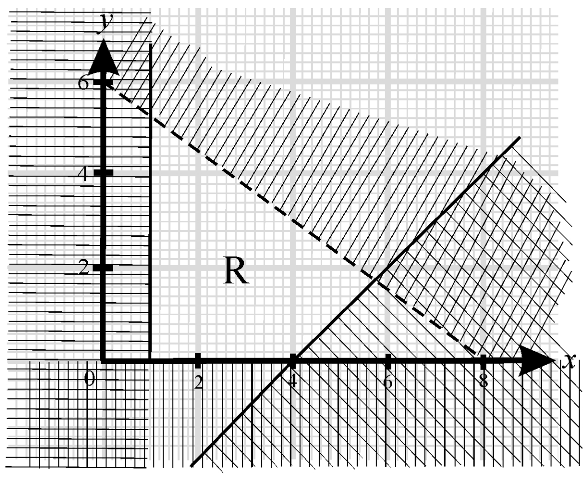


Fig 18.4

Fig 18.4 shows region R.

If $(x{:}y)$ is a point in R, where x and y are integers:

a) give the number of points in the region R.

b) give the four inequalities which define the region R.

Revision 3

Reminders

1. A set is a collection of objects with some common defined characteristics.

2. The basic sets or notation must be understood.

3. Sets can either be represented in notation, by word description or in diagrammatic form.

4. The basic rules in set notation are:
   (i) A^1
   (ii) $A \cap B$
   (iii) $A \cup B$

5. The basic inequality symbols are $>$, $\ngtr$, $<$, $\nless$, $\geqslant$, $\ngeqslant$, $\leqslant$, $\nleqslant$.

6. The inequality sign remains unchanged if the same is added or subtracted from both sides of the inequality.

7. If a positive number is multiplied or divided on sides of the inequality, the inequality remains unchanged.

1. Express the following in set notation:
   a) 6 is a member of set B which is a set of even positive numbers less than ten.
   b) the number of elements in set A and B equals twenty.
   c) 2 and 3 are members of the complement of set C.
   d) universal set ($\mathscr{E}$) is made of x values which are integers between 5 and 25.
   e) the shaded part in fig 18.5.

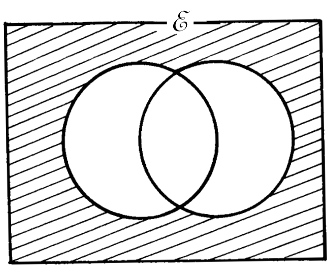


Fig 18.5

2. Supposing

$\mathcal{E}$ = {1, 2, 3, 4, 5, 6, 7, 8 9, 10, 11, 12, 13, 14 }
A = {2, 4, 6, 8, 10 , 12, 10, 14}
B = {2, 4, 8}

Find:
a) $A \cap B$.
b) $A^{I} \cap B$.
c) $(A \cup B)^{I}$.
d) $n(A \cup B)$.
e) Represent the information above on a Venn diagram.

3. Among 1000 farmers, (750 – x) farmers grow paprika, (500 – x) farmers grow tobacco and the remaining 250 farmers are only live stock producers

a) find the value of x.
b) represent the above information on a Venn diagram.
c) find the number of farmers who grow paprika.
d) find the number of farmers who grow tobacco.

4. In a survey it is found that 38 families either have a gas stove, an electric stove or both of these. If 29 have an electric stove, 15 have both types of stoves and 26 do not have a gas stove. Work out the number of those who:

a) only own an electric stove.
b) have gas stoves.
c) took part in the survey.

Revision 4

Reminders

1. The three rules about surds are

$$\sqrt{m} \times \sqrt{n} = \sqrt{mn}$$

$$\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$$

$$\sqrt{n} \pm \sqrt{m} = \sqrt{n} \pm \sqrt{m}$$

2. The Pythagoras Therorem and the trigonometrical ratios can only be applied or defined using the right-angled triangle Fig 18.6.

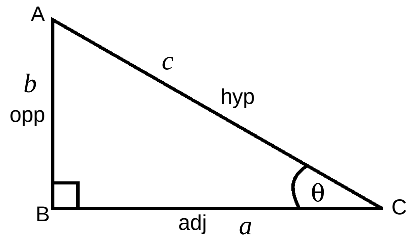


Fig 18.6

3. $c^2 = a^2 + b^2$ (Pythagoras theorem)

4. (i) $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$

(ii) $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$

(iii) $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

1. Simplify the following leaving answer in surd form.

a) $\sqrt{88}$ b) $\sqrt{125}$ c) $\sqrt{512}$

d) $\sqrt{157} - \sqrt{15}$ e) $\sqrt{21} - \sqrt{8} + \sqrt{60}$

2. Simplify the following

a) $(\sqrt{100})^2$ b) $(4\sqrt{3})^2$ c) $3(5\sqrt{3})^2$

d) $\sqrt{64} + \sqrt{36}$ e) $3\left(\dfrac{\sqrt{5}}{\sqrt{2}}\right)^2$

3. Express the following as the square root of a single number and simplify where possible.

a) $(\sqrt{4})^2$ b) $6\sqrt{12}$ c) $\sqrt{6} \times \sqrt{3}$

d) $(\sqrt{5})^2 \times \sqrt{3} \times \sqrt{20}$

4. A trapezium is made of two right angled triangles as illustrated in Fig 18.7.

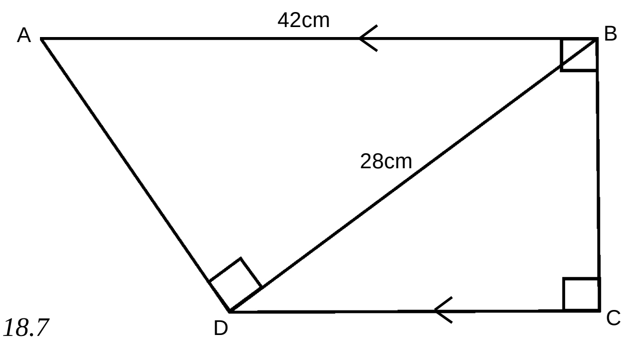


Fig 18.7

Calculate

a) length AD.

b) angle ABD.

c) length DC.

d) the area of the trapezium.

5. One side of the kite is 20cm and the other is 50cm. Work out the length of the diagonals of this kite.

6.

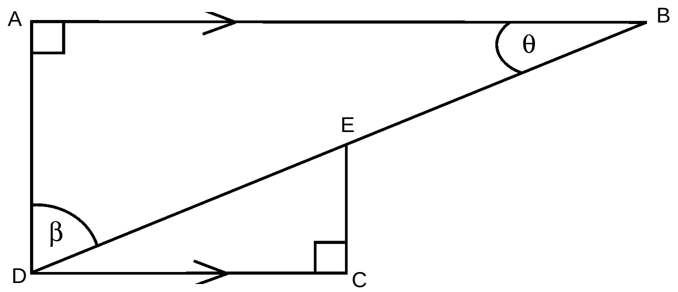


Fig 18.8

From fig 18.8 Given that $\tan\theta = \frac{6}{13}$:

a) write the values of $\sin\beta$ and $\cos\beta$.

b) Given the that BE = 5cm, find EDC.

c) Calculate the length EC.

Revision 5

Reminders

1. A matrix (plural matrices) is a rectangular array of numbers.

2. Addition /subtraction of matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$$

3. multiplication of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

4. Scalar multiplication

If k (a scalar) is a number and A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then kA = k $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

5. A vector is a quantity with both magnitude (size) and direction. A vector AB may be represented as $\overrightarrow{AB}$ or $\underset{\rightarrow}{AB}$ or AB or $\underline{a}$.

6. A vector becomes negative once its direction is reversed.

7. Examples of vectors are:

(i) position vector

(ii) unit vector

(iii) displacement/free vector

(iv) column vector.

8. Vectors are equal if they are equal in both magnitude and direction.

9. Vectors maybe added or subtracted using the triangle of vectors or the parallelogram law of vectors.

1. Given that P = $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ and Q = $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$

Work out the following.

a) P + Q   b) 3P – Q

c) 2P – 3Q   d) 3 (4P + 2Q)

2. Work out

$$\begin{pmatrix} 1 & -2 & 3 \\ 4 & -3 & -2 \end{pmatrix} \begin{pmatrix} -2 & 5 \\ -3 & 0 \\ -1 & -4 \end{pmatrix}$$

3. A = $\begin{pmatrix} 5 & 2 \\ -2 & 3 \end{pmatrix}$ B = $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ C = $(-1 \;\; -3)$

Work out: a) B^2 b) A^3 c) ACB

4. Find the value of x and y.

$$\begin{pmatrix} x & y \\ 3y & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 13 \\ 4y \end{pmatrix}$$

5. Find the inverse of the following matrices, if the inverse exists, otherwise give the answer in singular form.

a) $\begin{pmatrix} -6 & 2 \\ 1 & -2 \end{pmatrix}$ b) $\begin{pmatrix} -3 & -6 \\ 4 & 7 \end{pmatrix}$ c) $\begin{pmatrix} 1\frac{1}{2} & -3\frac{1}{2} \\ \frac{1}{4} & 2 \end{pmatrix}$ d) $\begin{pmatrix} 3 & -6 \\ 4 & -1 \end{pmatrix}$

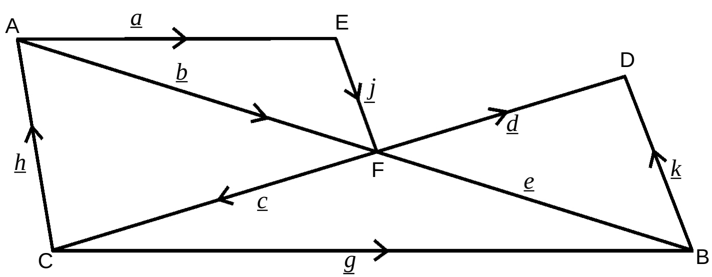


Fig 18.9

1. Using vectors shown in Fig 18.9 express the following as single vectors.

a) $\underline{a} + \underline{j}$
b) $-\underline{c} + \underline{e}$
c) $-\underline{g} + \underline{d} - \underline{c}$
d) $-\underline{b} - \underline{h}$

2. Express the following in terms of the other given vectors.
a) $\overline{AB}$ in terms of $\underline{h}$ and $\underline{g}$
b) $\overline{CD}$ in terms of $\underline{g}$, $\underline{c}$ and $\underline{d}$
c) $\overline{AF}$ in terms of $\underline{c}$ and $\underline{h}$.

3. Given that $m = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ $n = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

a) express the following as column vectors.
(i) $-2m$ (ii) $m - n$ (iii) $3m + 0{,}5n$

b) find the modulus (magnitude) of each of the following vectors.
(i) m (ii) $n - m$ (iii) $3m - 2n$

4. O is the origin and A is the point (5;4). In each of the following cases find the coordinates of a point P which satisfies and given condition.
a) $AP = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b) OP = –3OA

c) $\overline{OP} - OA = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

1. By choosing an appropriate scale ,draw on the same axes the graph of $2x + y = 9$, letting $x = 1, 2, 3, 5$, and the graph of $x + 3y = 7{,}5$ letting $x = 0{,}5; 1{,}5; 2{,}5; 3.5$
Hence solve the two simultaneous equations graphically.

2. Without drawing the graphs, work out the coordinates of the points where the following graphs meet the x-axis and the y-axis.
a) $2y + 3x = 2$ b) $y = 3x - 4$
c) $5x - 8 = 4y$ d) $6x - 9y = -1$

3. a) Construct and complete a table of values for the equation $y = x + \frac{1}{x}$ for $0{,}25 \leqslant x \leqslant 4{,}5$

b) Using an appropriate scale, draw the graph of $y = x + \frac{1}{x}$
Find:
c) the value of x when $x = 1{,}6$.

d) the two solutions when $y = 3,5$.
e) the maximum value of y.
f) the gradient at the point where $x = 0,8$.

4. A curve $y = (x+3)(x-2)$ cuts the x-axis at A and B and the y-axis at C.
a) Write the co-ordinates of the points A,B and C.
b) Write down the x values for where the line(graph) $y = 2x - 1$ meets the quadratic curve.

5. Given that $f(x) = x^2 + 1$,find
a) $f(-2)$
b) $f(a)$

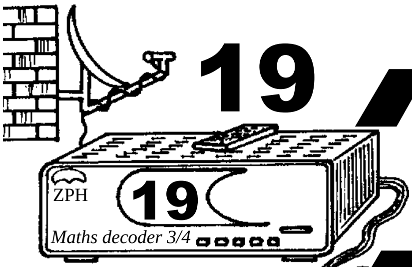
6. Given also that $f(x) = \frac{1}{4}x - \frac{5}{4}$
Find a) $f(-2)$.
b) $f(a)$.

7. Solve the following inequalities
a) $9 - 4x \leqslant 0$
b) $16 - (5 - 4x) \leqslant 10$
c) $2 < \frac{2 - 5x}{3} \leqslant 6$
d) $-2 - x \leqslant 20 < 3 - 5x$

8. Given that $35 - 2x \leqslant 29$,find:
a) The least value of x.
b) The least integral value of x.

9. Draw the diagrams to represent the following inequalities shading the UNWANTED region.
a) $y > \frac{1}{2}x$
b) $3y \leqslant 4x + 2$
c) $\frac{1}{3}y \geqslant \frac{1}{2}x + 2$

10. Beatrice and Bayliss are carpenters. They make x chairs and y tables, each week Beatrice does all the cutting of the planks and Bayliss does the joining and polishing.
To make a chair 3 hours of cutting and 6 hours of joining and polishing are needed. To make a table takes 5 hours of cutting and 12 hours of joining and polishing are needed. Neither carpenter works for more than 50 hours a week.
a) For joining and polishing show that $3y + 6y \leqslant 25$
b) Write down an inequality for cutting.
c) Given that they make at least 6 chairs each week, write down another inequality .
d) By shading the UNWANTED region, show the information in parts a), (b) and (c) graphically.
e) Given that the profit on a chair is \$20.00 and that on a table is \$45.00. Calculate the maximum profit Beatrice and Bayliss can make in a week.



19 Variation

Consider the following set-up (Fig.19.1)
This set-up illustrates how the mass (m) of a liquid varies with respect to its volume (v). If you increase the volume of liquid in the measuring cylinder then naturally its mass also increases see Table (19.1).

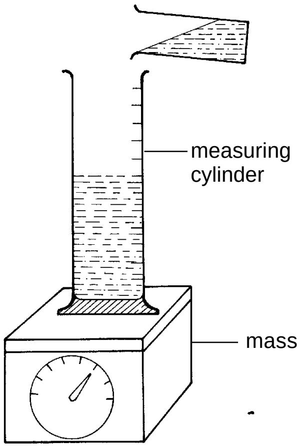


Fig 19.1

Table 19.1

Mass of water (g)	2	4	6	8	10
Volume of water (cm^3)	1	2	3	4	5

We can also say, as mass (m) of liquid increases so does its volume (v) or

(i) mass (m) varies directly to volume (v).
(ii) mass (m)is directly proportional to volume (v).

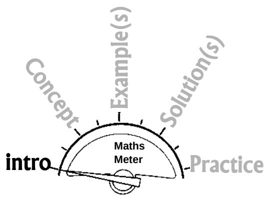
Tip

$m \, \alpha \, v$ means a constant exists between m and v which is always introduced when an equal sign replaces the variation sign. Most constants are given special names e.g. π but, from science, the constant of $\frac{m}{v}$ = density

All the above can be mathematically represented as:
mass α volume where the 'α' is a variation sign.

$$m \;\alpha\; v$$
$$\Rightarrow \quad m = kv$$
$$k = \frac{m}{v}$$

k = constant



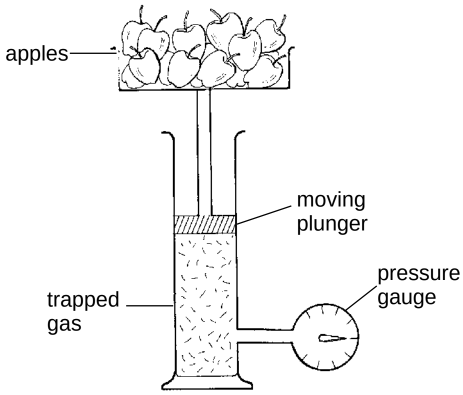


Fig 19.2

The set-up illustrated in Fig 19.2 shows the variation of volume(v) of trapped air and pressure (p) exerted on it. There is also a relationship between gas pressure and its volume. In this case the variation is an inverse variation. As more apples are added to the tray, the pressure (p) of the *trapped* gas increases and, thus, its volume decreases (Table 19.2).

Gas is also trapped under pressure in metal gas bottles.

Table 19.2

Pressure of gas/pa	2	4	6	8	10	12
Volume of gas/cm^3	30	25	20	18	15	12

We summarise the relationship as:
(i) pressure varies inversely to volume
(ii) pressure is inversely proportional to volume.

Mathematically

Hint
Introduce a constant to replace α with =.

$$P \propto \frac{I}{V}$$

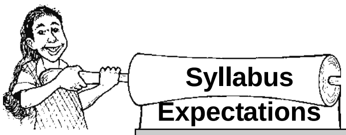
$$P = k \times \frac{I}{V}$$

$$P = \frac{k}{V}$$

$$PV = k$$

Hint
The k is temperature which must be kept constant for the relationship to be true.

In this chapter we look closely at all types of variations.



By the end of this chapter, students should be able to:

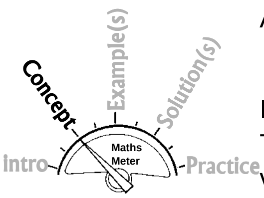
1. define a direct variation and work out related problems.
2. define an inverse variation and work out related problems.
3. define joint variation and work out related problems.
4. define partial variation and work out related problems.
5. draw and interpret graphs showing direct, inverse and partial variation.

ASSUMED KNOWLEDGE

In order to tackle the work in this chapter, it is assumed that candidates are able to:

- plot graphs from a given table of values.
- solve ordinary linear equations.
- solve problems involving changing the subject of a formula.
- find the root of a given number.
- solve simultaneous equations.

A DIRECT VARIATION

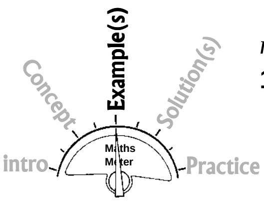


As already noted,

$m \propto v$ **means** $m = kv$.

If $y \propto x$ then $y = kx$
The numerical value of k, can take any form, depending on the values of y and x, i.e., k can be a positive or a negative whole number, or a fraction.

Consider the following examples



m varies directly to v and m = 10g when v = 5cm^3.

1. Find:
  a) the value of k.
  b) the law connecting m and v.
  c) the value of m when v = 20cm^3.
  d) the value of v when m = 4g.

Solution

a) $k = \dfrac{m}{v}$

$k = \dfrac{10}{5} = 2\text{g/cm}^3$

b) since $m \ \alpha \ m$

$\Rightarrow \quad m = kv$

$\therefore$ the required law is $\boldsymbol{m = 2v}$

c) when $v = 20$, $m = 2 \times 20 = 40$.

d) when $m = 4$, $4 = 2v$, $v = 2\text{cm}^3$.

2. The volume (v) of a sphere is directly proportional to the cube of its radius r. If a sphere, of radius 3cm, has a volume of 80cm^3, find:

a) the volume of the sphere of radius 2,5cm.

b) the value of the radius of a sphere of volume 100cm^3.

Solution

a) $v \ \alpha \ r^3$

$\Rightarrow \quad v = kr^3$ where k is a constant

$v = 80$ when $r = 3$

$\therefore \quad 80 = k \times 3^3$

$80 = 27k$

$k = \dfrac{80}{27}$

$\therefore \quad v = \dfrac{80}{27} \times r^3$

when $r = 2,5$

$v = \dfrac{80}{\cancel{27}} \times 2,5^3 = 46,30\text{cm}^3$ (3 s.f)

b) When $v = 100$ then radius (r) is found as:

$100 = \dfrac{80\,r^3}{27}$

$80r^3 = 2700$

$r^3 = \dfrac{270\cancel{0}}{8\cancel{0}}$

$r^3 = 33,75$

$r = \sqrt[3]{33,75}$

$r = 3,23\text{cm}$

1. A α B and A = 20 when B = 15

a) Find the law connecting A and B.

b) Find A when B = 2,5.

c) Find B when A = 6.

2. Given that v varies directly to the square of t and v = 8 when $t = \frac{1}{2}$,

a) find the relationship between v and t.

b) find t when v = 122.

3. Given that $y \alpha x$ and y = 60 when x = 10.

a) find the relationship between y and x.

b) find y when $x = 1\frac{2}{5}$.

c) find x when y = 84.

4. Copy and complete table 19.3 by observation.

Table 19.3

x	3	5	7	9		13
y	4		10	13	16	

a) Sketch the graph of x against y.

b) Comment whether this is a direct relationship or otherwise.

B. INVERSE VARIATION

As already mentioned, some quantities may vary inversely to each other, that is, as one quantity is increasing the other is decreasing or vice versa.
Consider the efficiency (E) of a working machine and its age (y). (assume the machine is initially perfect).
Efficiency (E) is inversely proportional to age (y).
This physically implies that as the machine is ageing its efficiency decreases.

We can represent that as: $E \; \alpha \; \frac{1}{y}$

$$E = \frac{k}{y}$$

$$\therefore \; Ey = k$$

This indicates that the product of E and y is a constant number k.

Supposing $k = 50$, then,

$$Ey = 50 \text{ or } E = \frac{50}{y}.$$

The graphs of (i) E against y and (ii) E against $\frac{1}{y}$ are shown in Fig. 19.3, respectively.

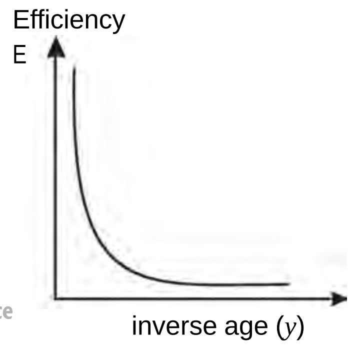


Fig. 19.3(a)

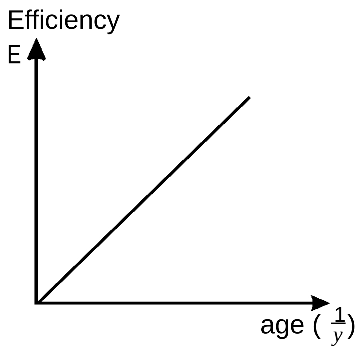


Fig. 19.3(b)

The graph of E against age (y) is not a straight line, it is a curve called a **hyperbola**, Fig 19.3(a).

Points to note about Inverse Variation

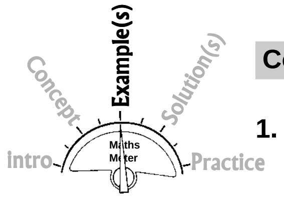
(i) In general if y is inversely proportional to x, then, it is represented as:

$$y \; \alpha \; \frac{1}{x} \quad \text{or} \quad x \; \alpha \; \frac{1}{y}$$

(ii) $yx = k$ where k is a constant number.

(iii) In its general form $y \; \alpha \; \frac{1}{x}$

Note that x can take any value e.g. $2x$, $(x+2)$, x^2, x^3 and in all aspects the graph of y against the **inverse** of $2x$ or $(x + 2)$ or x^2 or x^3 is a straight line through the origin.



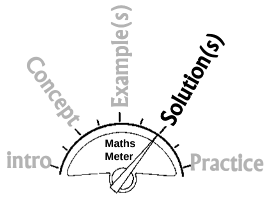
Consider the following example:

1. Given that y is inversely proportional to $(x + 2)$ and that $y = 6$ when $x = 2$ express y in terms of x.

Solution

$$y \;\alpha\; \frac{1}{(x+2)}$$

$$y = \frac{k}{x+2}$$



$$y(x+2) = k$$

But $y = 6$ when $x = 2$

$$6(2+2) = k$$

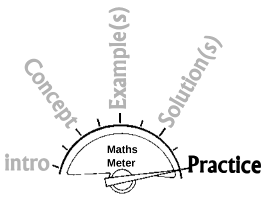
$$6 \times 4 = k$$

$$24 = k$$

$$\therefore \; y = \frac{k}{(x+2)}$$

$$y = \frac{24}{(x+2)}$$

1. Say whether the following relationships are inverse or not.



a) The number of sweets you buy in relation to the money in your purse.

b) The length of stretched rubber band with force applied.

c) The thickness of the tyres on a car in relation to the time, if the vehicle is always moving.

d) The number of years left of your life in relation to your age.

e) The height of a newly planted gum-tree in relation to its age.

2. Given that p is inversely proportional to q^3 and that $p = 27$ when $q = 3$ find the value of p when $q = 4$.

3. a) Given that $y \propto \frac{1}{x^2}$ and $y = 4$ when $x = 10a$, find the value of y when $x = 3$.

b) Sketch the graph of y in relation to the inverse proportion of x^2 and that of y against x.

4. Given that y is inversely proportional to $x - 2$ and that $y = 8$ when $x = 10$, express y in terms of x.

5. Given that p is inversely proportional to q and that $p = 40$ when $q = 4$, find the value of p when $q = 10$.

6. Given that y varies inversely as x^2 and $y = 8$ when $x = 6$
(i) express y in terms of x
(ii) calculate the values of x when $y = 20$.

7. Make separate graphs of y in relation to x, to show the following relationships:
(i) $y \propto x$ (ii) $y \propto \frac{1}{x}$ (iii) $y \propto x^2$

C. PARTIAL VARIATION

Some Examination Centres charge a fixed amount (centre fee) and an amount which varies with the number of subjects registered. Hence the candidates's total examination fee is partly constant and partly varies with the number of subjects registered. Mathematically, the variation sign is not used in partial variation.

Consider the following example

2. The total cost, T dollars, of registering a candidate for Cambridge examinations at a particular examination centre is partly constant and partly varies with the number of subjects.
The cost of registering 6 subjects is \$80 while the cost of registering 10 subjects is \$100.

Find the law connecting the total cost, T, to the number of subjects, x.

Hint

*Partial variation is about the **sum** of two parts.*

Solution

$T = a + bx$
Constants a and b have been used here.
$80 = a + 6b$ ....(1)
$100 = a + 10b$ ....(2)

Solving the two simultaneous equations.

$-20 = -4b$
$b = 5$

$\therefore\ 80 = a + 6 \times 5$
$80 = a + 30$
$a = 50$

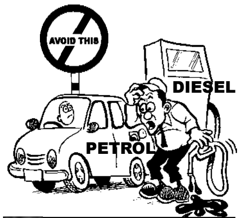
Hint

Partial variation have an addition or subtraction sign between them.

Hence the law is:
$T = 50 + 5b$

In general, in a relationship of the type $y = k_1 x + k_2 z$, y is said to vary partly with x and partly with z.

D. JOINT VARIATION



Common Error

Students treat the word "**and**" as a addition which is wrong.

When more than two variables are involved in a relationship then it is referred to as a joint variation. y varies jointly as x and z if there exists a real number, k, such that:

$$\boxed{y = kxz}$$

Hint

*In the expression "y varies jointly as x **and** z" translates to the product, y = kxz and **not** addition. The word "jointly" can be omitted without any loss of clarity hence we can safely say "y varies as x and z"*

A joint variation is a relationship or formula where a variable, y, may be dependent on two or more other variables. A typical example of a joint variation formula is the one to find the curved surface area of a cone, $A = \pi rl$, where A is dependent **jointly** on r, the base radius, and l, the slant height of the cone.
Consider the following formulae in Mathematics and Science which show joint variation with the variables in brackets.

a) $A = \frac{1}{2} bh$ (A, b, h)

b) $V = \pi r^2 h$ (V r h)

c) $A = \frac{1}{3} \pi r^2 h$ (A r h)

d) $F = \frac{Gm_1 m_2}{r^2}$ (Fm_1 m_2 r)

The corresponding relationship of all the above, respectively, is as follows:

a) $A \propto bh$

b) $V \propto r^2 h$

c) $A \propto r^2 h$

d) $F \propto \frac{m_1 m_2}{r^2}$

Consider the examples below:

1. You are given that E varies directly as m and as the square of v. Find:

a) the formula relating the three.

b) Given that the constant is 1, calculate E, if m = 2kg and $c = 3{,}0 \times 10^8$.

Solution

Hint

In Joint Variation, the two parts are joined by multiplication. In partial variation there is either a plus or a minus between the terms.

a) $E \propto m$
$E \propto c^2$

$\boxed{\mathbf{E = kmc^2}}$

b) $E = 1 \times 2 \times 3{,}0 \times 10^8$
$= 6{,}0 \times 10^8$

2. The interest on an investment is given as $I = prt$, where
p = principal
r = interest rate
t = time in years

If the investment earns $100 interest at 5% for 2 years, how much interest will the same principal earn at 4,5% for 3 years?

Solution

From $I = prt$ (where p is the constant of variation because it is the same for both investments).

Substituting

$100 = p(0{,}05)(2)$

$100 = 0{,}1p$

$p = 1\,000$

Finding I when $p = 1000$, $r = 0{,}045$(4,5%) and t = 3 years.

Substituting $I = 1\,000(0{,}045)(3)$

$I = 135$

The interest will be $135.

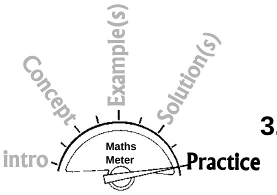
1. Given that y varies as x^a, write down the value of 'a' in the following:

a) y is the area of a circle of radius x.

b) y is the volume of a cylinder of base area A and height h.

2. $T \,\alpha\, \dfrac{R + r}{R}$

If T = 10 when R = 3 and r = 3
Find the formula for T in terms of R and r, and hence find the value of T when r = 2 and R = 4.



3. $g \;\alpha\; \dfrac{m}{r^2}$ If g = 10, m = 2000 and r = 100,

find the formula for g in terms of m and r, then find the value of g when m = 3 000 and r = 200.

4. Express each as a formula with constants a and b:

a) y is the difference of two parts, one is directly proportional to x and the other directly proportional to x^3.

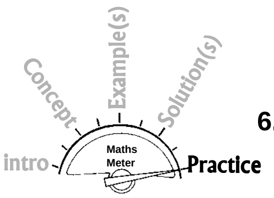
b) V is the sum of two parts, one part is directly proportional to u and the other is inversely proportional to t.

5. In the sum of two parts it is given that:

$S = ut + \frac{1}{2} at^2$.

a) State the relationship between S and t in the two parts.

b) When t = 3 S = 9 and u = 1,
find the value of a.



6. x is partly constant and partly varies in relation to y when x = 40 y = 4 and when x = 42 y = 7.

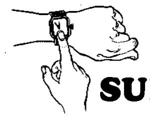
a) Find x when y = 30.

b) Find y when x = 100.

7. $S = p + qx^2$, where *p* and *q* are constants. Given that, S is 93 when x = 6 and S = 643, when x = 2.

a) find the constants p and q.

b) find x given S = 363.



SUMMARY

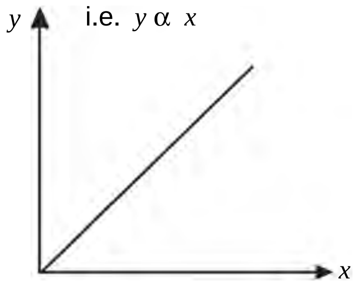
1. If y increases as x increases then we say y is directly proportional to x, or y varies directly as x, i.e.
$y \propto x$
There must be a constant usually represented by k.

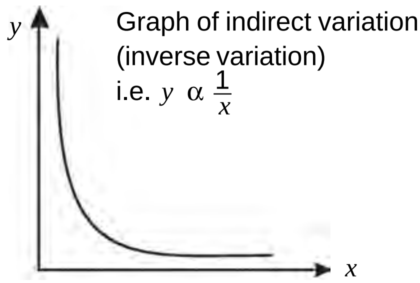
$$\therefore\ y = kx \ \Rightarrow k = \frac{y}{x}$$

2. If y increases as x decreases, then we say, y is inversely proportional to x, or y varies indirectly to x.

$$y \propto \frac{1}{x} \Rightarrow yx = k$$

3. Graph of direct variation





4. Partial variation refers to a situation where y varies according to a fixed value as well as to a variable constant.
The two parts are **added**.

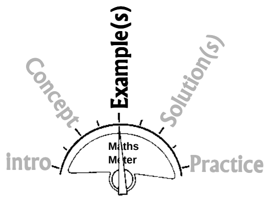
5. Joint variation is when three or more variables are involved in a relationship.
The variables are joined by **multiplication**.

6. Types of Variations

- Fixed (Direct) variation
- Indirect (Inverse) variation
- Joint Variation
- Partial Variation

EXAM PRACTICE 19

Consider the following example:



1. The total surface area (A) of any given cube of length (l) is given by:

$A = 6l^2$

where l = length of one side, 6 is the number of faces.
For values of l = 2, 4, 6, 8, 10.

a) Find the corresponding values of l^2.

b) Find the corresponding values of A.

c) Plot the graph of A against l and also that of A against l^2.

d) Comment on the shape of the graphs

Solution

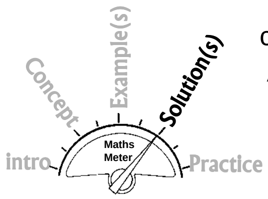
Hint

Note that in the table, $\frac{A}{l^2}$ gives us a constant value = 6.
Thus $A \alpha l^2$

1. Table of values

Table 19.4

l	2	4	6	8	10
l^2	4	16	36	64	100
A	24	96	216	384	600



c)

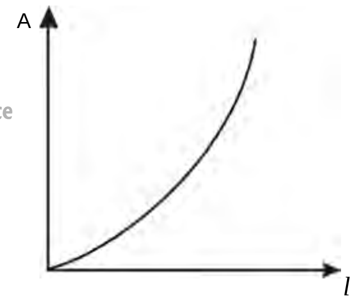


Fig. 19.4(a)

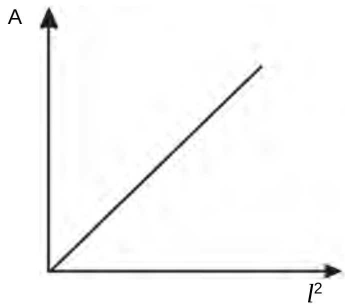


Fig. 19.4(b)

Hint

Appreciate that the graph of $A = 6l^2$ is that of A in relation to l, which is a parabola. A graph of $y = x^3$ is y in relation to the values of x not y against the values of x^3.

d) We can deduce that A is directly proportional to l^2.

$A \alpha l^2$ or that A varies directly with l^2, for fig 19.4(b).

However, the first graph, Fig 19.4(a), shows that A in relationship to l is not a straight line.

Now do the following:

1. Given that p varies directly to the cube of q and $p = 10$ when $q = 30$.

a) find the law connecting p and q.

b) find q when $p = 16$

c) find p when $q = \frac{1}{8}$

2. Given that $M \propto \frac{1}{N}$ and

$M = 10$ when $N = \frac{1}{2}$,

a) find the law connecting M and N.

b) find N when M = 11.

c) find N when M = 23.

3. $C \propto \frac{MV}{x^2}$

Given that C = 150 when M = 25, V = 8. Find x in terms of k.

4. A is proportional to B and C. A = 64 when B = 4 and C = 32

a) Find C when A is 140 and B = 5

5. M is partly constant and partly varies as the square of x. Given that M = 200 when $x = 4$ and M = 120 when $x = 2$,

a) find the law that connects M and x.

b) find M when $x = 136$.

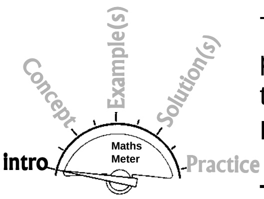
6. The cost of printing a newspaper is partly constant and also varies partly as the number printed. If the cost of printing 500 newspapers is $320,00 and the cost of 1 000 newspapers is $540,00. Find the cost of printing 750 newspapers.

7. The sum of two parts is y. One part varies as p and the other part as the cube of p ($y = 100$ when $p = 4$ and $y = 60$ when $p = 2$).

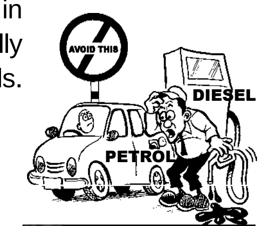
a) Deduce the formula to express y in terms of p.

b) Find the value of y when $p = 6$.

20 Transformations



Transformation is the process that involves the change of a figure in position, shape and/or size. Examples of figures that are usually transformed are: triangles, rectangles, squares and quadrilaterals. Lines and points can also be transformed.



Common Error

Students often fail to appreciate or visualise that a point $(x;y)$ can be moved. Similarly a line can be transformed.

Types of transformations

1. Translation (T)
2. Reflection (M)
3. Rotation (R)
4. Enlargement (E)
5. Shear (H)
6. Stretch (S)
   a) One way stretch
   b) Two way stretch

The first three transformations are **isometric** (the original shape maintains its dimensions hence, it will have the same shape and size after transformation). The last three are **non-isometric** (the original shape changes its dimensions hence also its shape and size).

A full description of the above transformations will be made later in this chapter.

As an introduction, let us consider rectangle X, with its vertices (corners) A, B, C and D as illustrated in Fig 20.1.

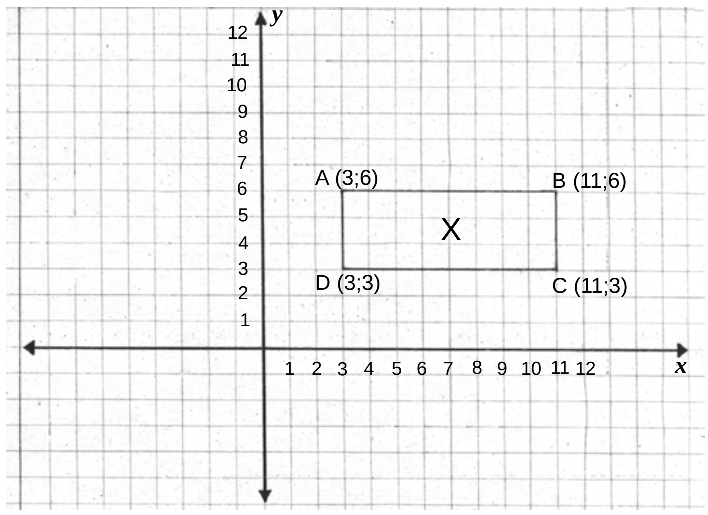


Fig 20.1.

The matrix of the rectangle X is a numeric mode of the graphic rectangle as illustrated in Fig 20.1

$$\text{Matrix of X} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 3 & 11 & 11 & 3 \\ 6 & 6 & 3 & 3 \end{pmatrix} \end{array}$$

Supposing we want to change the position of rectangle X, then naturally its matrix will also change. We say the shape **transforms**. Similarly, we may transform the shape by varying its original size to make it bigger or smaller. Again this is referred to as transformation.

Fig 20.2 illustrates how the rectangle X may be transformed to take either a new position or change in appearance (measurement/matrix) or both. This branch of geometry is often referred to as **motion geometry** or simply geometry of transformation.

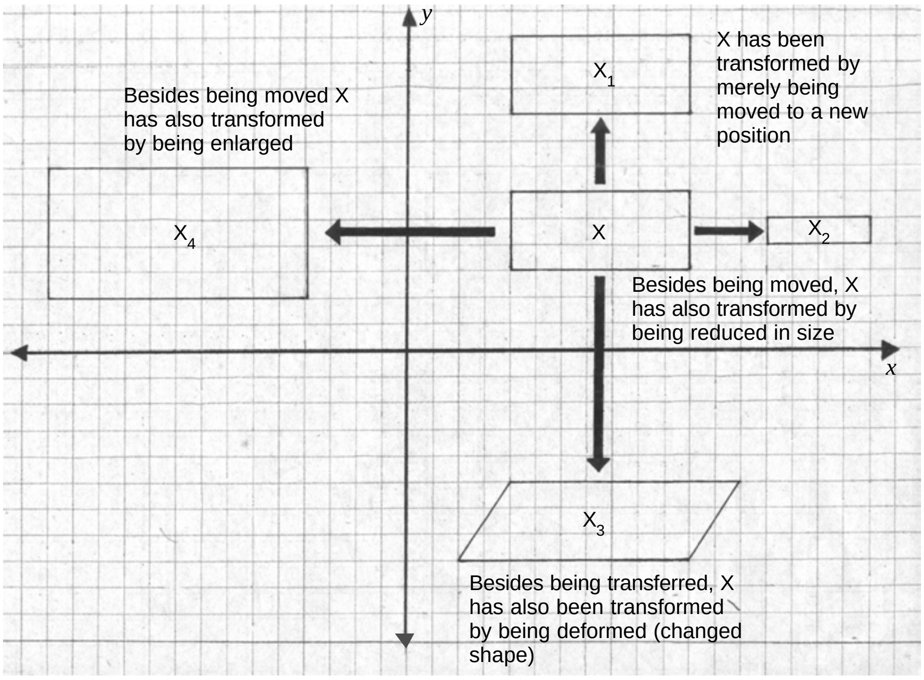
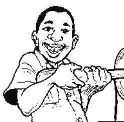


Fig. 20.2 Basic transformations



Syllabus Expectations

By the end of this chapter, students should be able to:

1. define transformation and the corresponding transformations in the x–y plane.
2. translate simple plane figures using a given vector.

3. reflect simple plane figures in the axes and in any line.

4. give the centre of rotation, direction and angle, rotate the given figure about a given point.

5. enlarge, using a rational enlargement scale factor.

6. stretch figures, given the stretch factor and an invariant line.

7. shear figures, given the shear factor and an invariant line.

8. identify all the above transformations and describe fully the transformations between given objects and images.

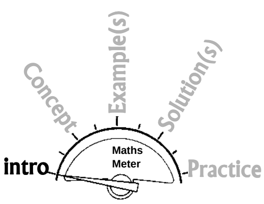
9. use matrices to effect transformations.

10. identify the matrix used in a given transformation.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- add and subtract matrices.
- multiply matrices.
- configure shapes with vertices of their shapes given in matrix form.
- understand the concept of a column vector.
- locate and plot a point on the cartesian plane.
- list properties of plane shapes.
- write the matrix of the vertices of a plane shape on a cartesian plane.
- draw linear graphs.
- draw a perpendicular bisector of a line.



As already mentioned, a transformation is a change or variation in position or measurement or both, for a given figure. We describe all variations on original shape, or its matrix, as transformations. There are six forms of transformation as illustrated in Fig 20.3.

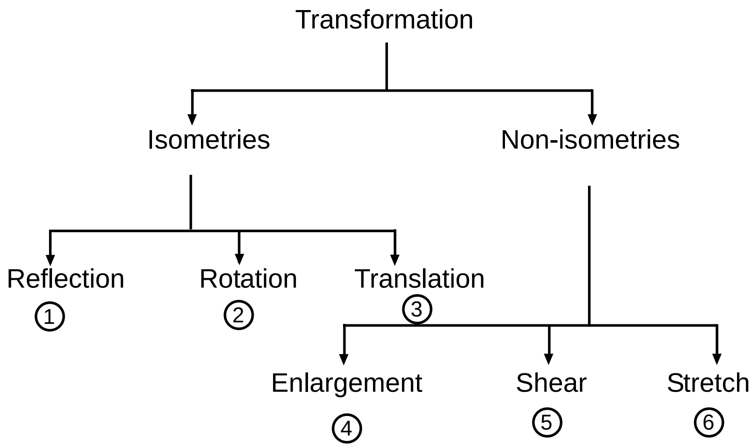


Fig. 20.3 Aspects of transformations

Isometry transformation is when the original shape and image remain unchanged i.e. they are **congruent**. The other transformations give rise to non-congruent figures as we shall discover in this chapter.

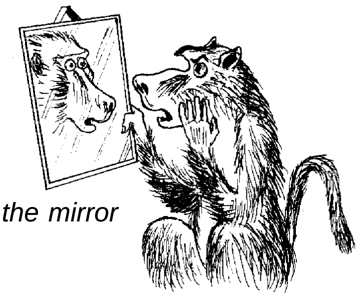
Invariant point or line
A point or a line which does not change under a transformation is said to be invariant under that transformation.

A. REFLECTION

A reflection is a transformation in which a shape/figure is changed from one position to another with the same dimensions.

For one to understand reflection you must first define a line which represents an imaginary mirror. We call such a line a mirror line/ axis of reflection. Images of reflection may be located by the use of either:

a) **Geometrical facts.**
b) **Matrix of reflection.**



A full description of reflection should include:

1. *stating that the transformation is a reflection.*
2. *stating the reflection line (giving the equation of the mirror line).*

It should be appreciated that mirror lines are derived from basic, linear equations. The equations which will be commonly used throughout this chapter include:

Tip

In equations $y = x$ and $y = -x$, the gradients are $+1$ and -1 respectively.

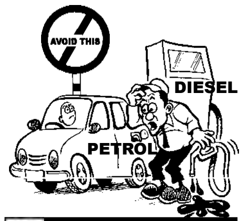
a) $y = 0$ The x-axis.

b) $x = 0$ The y-axis.

c) $y = x$ Straight line passing through the origin with a positive gradient.

d) $y = -x$ Straight line passing through the origin with negative gradient.

e) $y = mx + c$ Any other linear graph which is not described in the above four equations with m and c being either positive or negative.

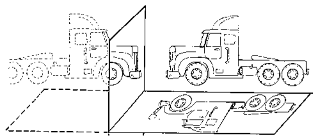


Common Error

Students fail to appreciate that m can be a fraction e.g. $y = 0{,}5x$ is a linear equation

Reflection using Geometrical facts

Consider the y-axis and x-axis to be plane mirrors (Fig 20.4) The reflection of the shape F would be F_1 in the Y-axis and F_2 in the X-axis. You may even put a **real mirror** along these axes and verify this for yourself.



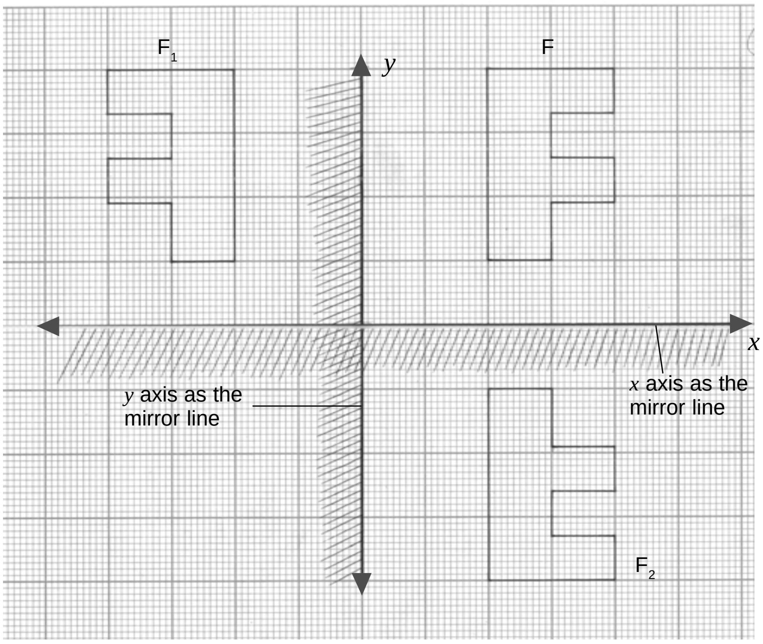


Fig. 20.4 Reflection

The following **rules** are used when locating the reflection of an image in a plane mirror.

(i) First, clearly define and draw the mirror line (any of the above linear graphs).
(ii) The image is the same size as that of the object.
(iii) The image is as far behind the mirror as the object is in front of it, but it is laterally inverted.
(iv) The image is directly opposite the object, i.e., a straight line joining corresponding points of the object and the image is always perpendicular to the mirror line or axis of reflection.
(v) The axis of reflection is the invariant line.

Matrix of reflection

Let us examine rectangle X, in Fig 20.5 once more. We are given that its matrix is:

$$X = \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{pmatrix} 3 & 11 & 11 & 3 \\ 6 & 6 & 3 & 3 \end{pmatrix} \end{array}$$

Supposing we reflect rectangle X:

a) along the x-axis.

b) along the y-axis.

using the geometrical facts, we produce:

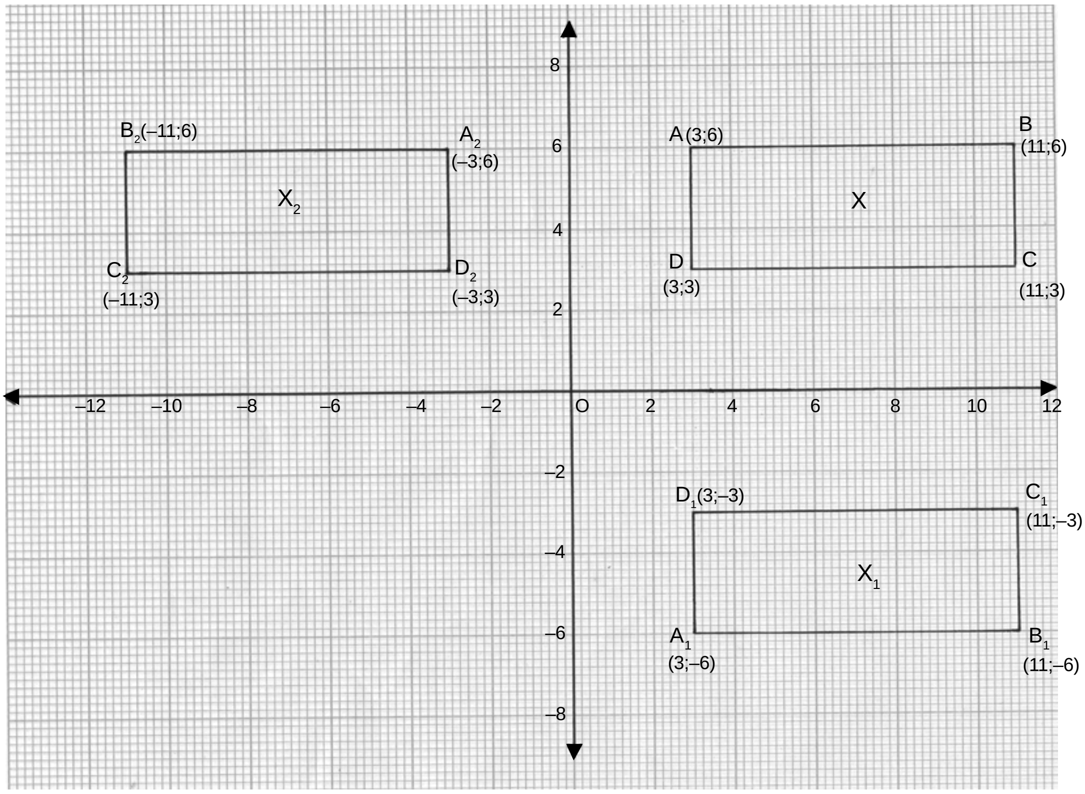


Fig 20.5

It can be deduced, from Fig 20.5 that the matrix of

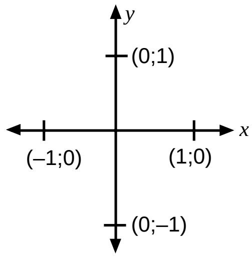
$$X_1 = \begin{array}{c} \begin{matrix} A_1 & B_1 & C_1 & D_1 \end{matrix} \\ \begin{pmatrix} 3 & 11 & 11 & 3 \\ -6 & -6 & -3 & -3 \end{pmatrix} \end{array}$$

$$X_2 = \begin{array}{c} \begin{matrix} A_2 & B_2 & C_2 & D_2 \end{matrix} \\ \begin{pmatrix} -3 & -11 & -11 & -3 \\ 6 & 6 & 3 & 3 \end{pmatrix} \end{array}$$

In fact, if you want to get the transformation of X, when reflected about the x-axis or the y-axis, without using geometrical facts, we may do so by using the two matrices.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . . . for x-axis

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . . . for y-axis



All you need to do is to **multiply** by these matrices.

$$\text{Therefore: } X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 11 & 11 & 3 \\ 6 & 6 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ 3 & 11 & 11 & 3 \\ -6 & -6 & -3 & -3 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 11 & 11 & 3 \\ 6 & 6 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} A_2 & B_2 & C_2 & D_2 \\ -3 & -11 & -11 & -3 \\ 6 & 6 & 3 & 3 \end{pmatrix}$$

The following matrices of reflection transformation should be memorized.

Table 20.1

Transformation	**Matrix used to multiply the matrix of the object**
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Tip

At the end of this chapter we discuss how to devise the matrix used to multiply the matrix of the object. For now just memorise the matrix.

Study the following examples carefully:

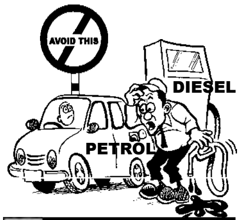
1. On the cartesian plane, draw triangle ABC, A(1; 6) B(1; 3) C(3; 6). Also draw the line $y = x$. Draw the image of triangle ABC after reflection in:

a) $y = x \quad \Rightarrow A_1B_1C_1$

b) $x = 0$ (y–axis) $\Rightarrow A_2B_2C_2$

c) The image of the reflection of $A_1B_1C_1$ in $y = 0$ (x-axis).
$\Rightarrow A_3B_3C_3$

Note *Use either the Geometrical facts method or the matrices method (or both just for extra practice).*



Common Error

Students fail to identify or sketch the proper mirror line.

Solution

1. Geometrical Method

Draw the object ABC on the cartesian plane (Refer to Fig 20.6) throughout

a) Draw the mirror line.
Use geometrical facts to locate the coordinates of vertices of the image $A_1B_1C_1$. Finally, construct image $A_1B_1C_1$ by joining the vertices.

b) Draw the graph $x = 0$ as the mirror line. Apply geometrical facts as in (i) above to draw image $A_2B_2C_2$.

c) Draw the graph $y = 0$ and apply geometrical facts to configure image $A_3B_3C_3$.

Tip

– The fact that: Image distance equals object distance and a perpendicular line joining the image of object to the mirror is used when making geometrical representations of reflections.
– The key concept is to draw perpendicular lines from the vertices of the original image and extend the perpendicular lines for equal distances, thereby locating the vertices of the image.

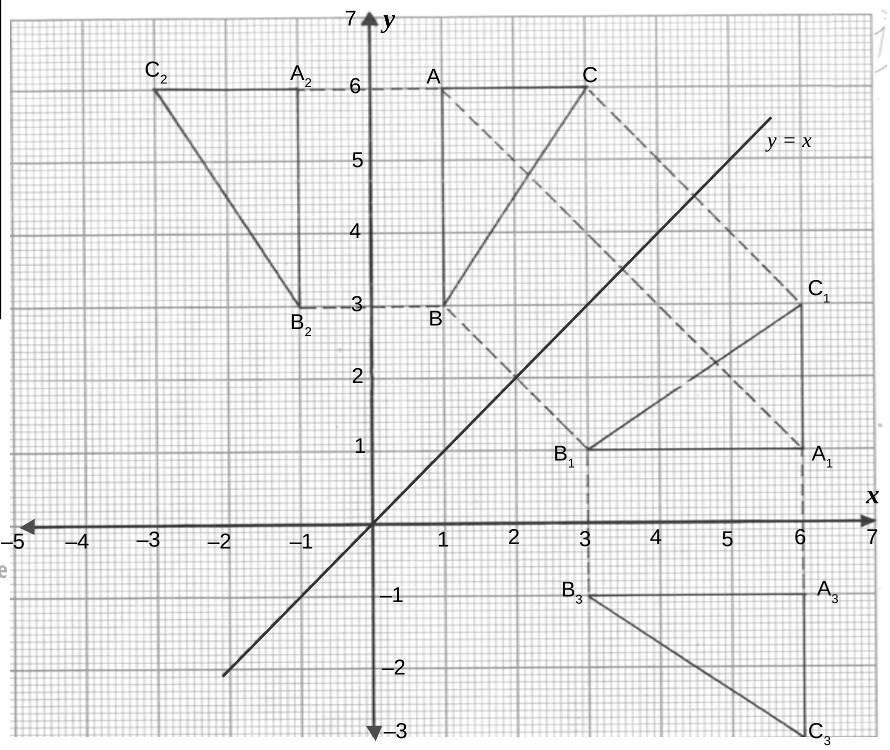
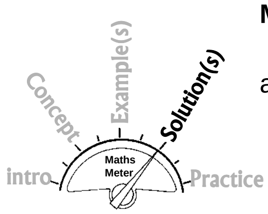


Fig. 20.6 Reflections on a Cartesian plane

Matrix Method



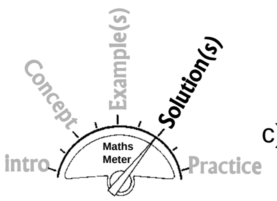
a) $\triangle$ ABC with A(1; 6), B(1; 3) and C(3; 6), reflected on $\boldsymbol{y = x}$ is given by:

$$\underset{\text{matrix of reflection}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \times \begin{array}{c} \text{matrix of triangle} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 6 & 3 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \text{matrix of image} \\ \begin{array}{ccc} A_1 & B_1 & C_1 \end{array} \\ \begin{pmatrix} 6 & 3 & 6 \\ 1 & 1 & 3 \end{pmatrix} \end{array}$$

Therefore $A_1(6; 1)$, $B_1(3; 1)$ and $C_1(6; 3)$.

b) $\triangle$ $A_1B_1C_1$ with $A_1(6; 1)$ $B_1(3; 1)$ and $C_1(6; 3)$, reflected on the $\boldsymbol{y = 0}$ (x-axis) is given by:

$$\underset{\text{matrix of reflection}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \times \begin{array}{c} \text{matrix of triangle} \\ \begin{array}{ccc} A_1 & B_1 & C_1 \end{array} \\ \begin{pmatrix} 6 & 3 & 6 \\ 1 & 1 & 3 \end{pmatrix} \end{array} = \begin{array}{c} \text{matrix of image} \\ \begin{array}{ccc} A_3 & B_3 & C_3 \end{array} \\ \begin{pmatrix} 6 & 3 & 6 \\ -1 & -1 & -3 \end{pmatrix} \end{array}$$

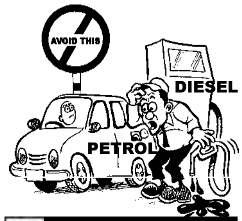


Therefore $A_3(6; -1)$, $B_3(3; -1)$ and $C_3(6; -3)$

c) $\triangle$ ABC with A(1; 6), B(1; 3) and C(3; 6) reflected on $\boldsymbol{x = 0}$ (y-axis) is given by:

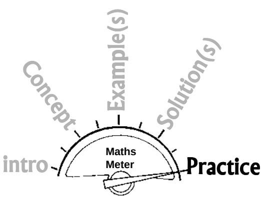
$$\underset{\text{matrix of reflection}}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \times \begin{array}{c} \text{matrix of triangle} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 6 & 3 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \text{matrix of image} \\ \begin{array}{ccc} A_2 & B_2 & C_2 \end{array} \\ \begin{pmatrix} -1 & -1 & -3 \\ 6 & 3 & 6 \end{pmatrix} \end{array}$$

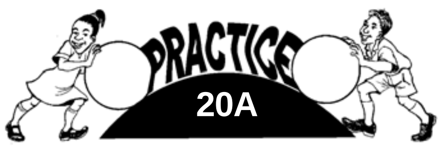
Therefore $A_2(-1;6)$ $B_2(-1; 3)$ and $C_2(-3; 6)$.



Common Error

Students multiply with the wrong matrix of reflection e.g. multiplying by the matrix used when x-axis is the mirror line instead of the matrix used for $y = x$ as the mirror line.





In this section use either the geometrical method or matrices (where possible) or to gain experience. Draw images of objects in Fig 20.7 given that the objects are reflected along the dotted lines.

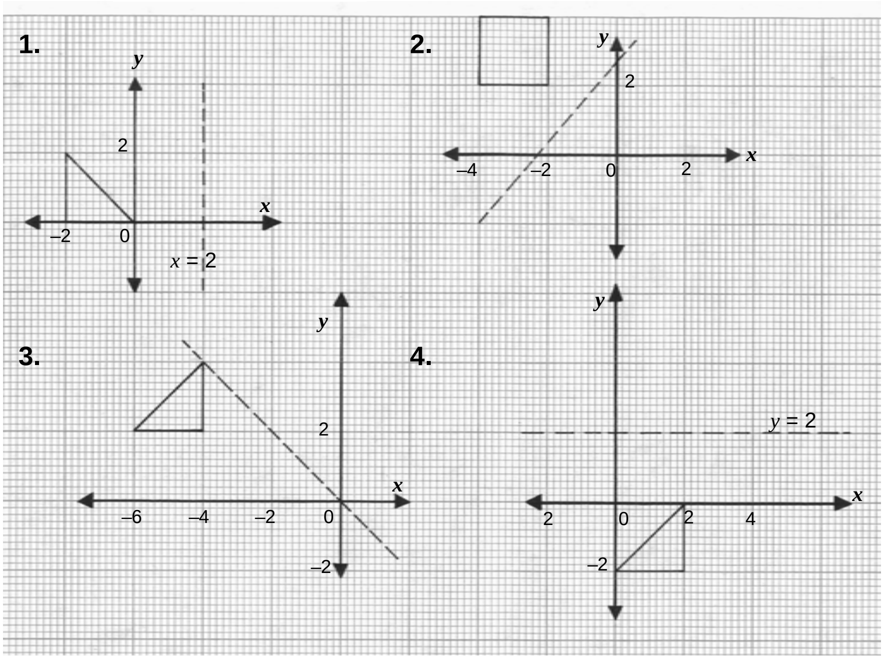


Fig. 20.7

5. Draw the image of $\triangle$ ABC (Fig 20.8) when reflected in line M_1. Label it $A_1B_1C_1$ and also draw the image of $A_1B_1C_1$ when reflected along $y = 0$. Label it $A_2B_2C_2$.

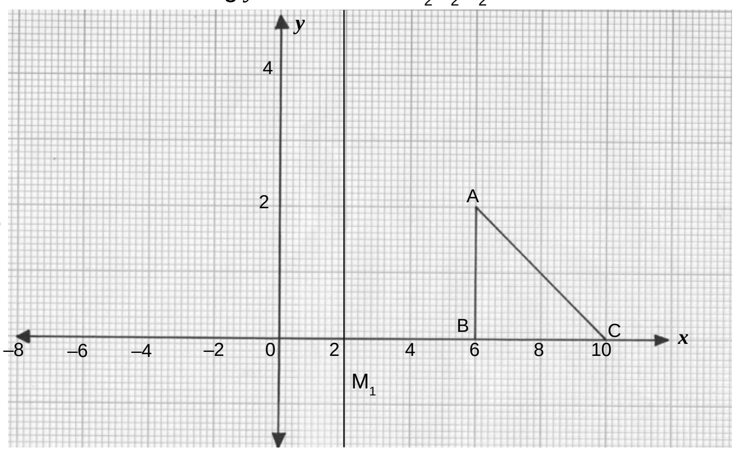


Fig. 20.8

6. a) Draw the x and y axis, for both value of x and y, ranging from –9 to 9 on the cartesian plane.

b) Draw and label triangle XYZ, given that:
X(5; 6) Y(1; 7) Z(1; 5)
Draw the lines $y = 1$ and $y = 0{,}5x$

c) Draw the image of $\triangle$XYZ after reflection in:
(i) the y-axis and label it T_1.
(ii) the line $y = 1$ and label it T_2.
(iii) the line $y = 0{,}5x$ and label it T_3.

d) Write down the matrix of the image of XYZ, in each case i.e. c(i) – (iii)

7. Copy Fig. 20.9 on squared or graph paper where P is the point (3; 5). Choose the correct scale on your diagram, mark the following points and state their co-ordinates:

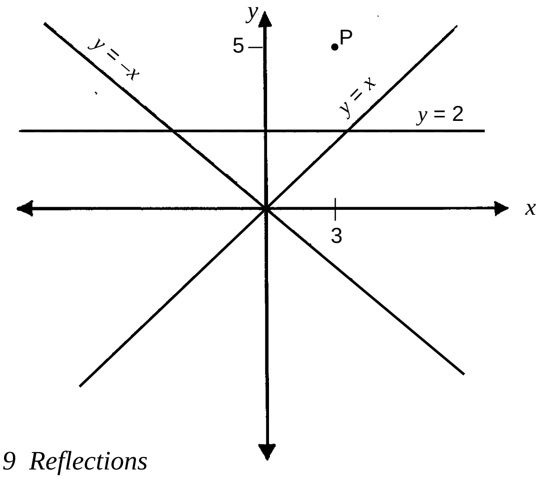


Fig. 20.9 Reflections

a) M, the reflection of P in the y-axis.

b) R, the reflection of P in the line $y = 2$.

c) Q, the reflection of P in the line $y = x$.

d) V, the reflection of P in the line $y = -x$.

8. Using axes numbered from 0 to 9 mark the following points A(4; 7) A_1(2; 5) B(6; 7) B_1(2; 3) C(7; 5) and C_1(4; 2). A_1 B_1 and C_1 are the images of A, B and C under reflection in a certain mirror line.

a) Draw this mirror line.

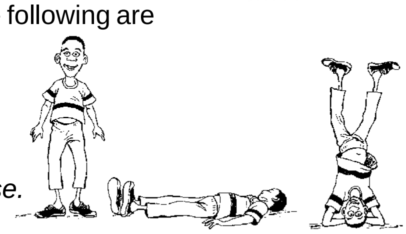
b) Deduce the equation of the mirror line.

B. ROTATION

A rotation is a transformation which involves the turning of a shape or figure, point or line either clockwise or anti-clockwise about a defined point called the centre of rotation.

Any rotation can only be described fully provided the following are given:

1. *a statement that there is rotation.*
2. *the centre of rotation (point of reference (x:y).*
3. *the angle to be rotated through.*
4. *direction of rotation (clockwise or anti-clockwise.*
5. *matrix can also be used when it is given.*



Tip

In this section, a clockwise rotation is ***positive*** *and an anticlockwise rotation is* ***negative****.*

Using the Geometric Method

Consider the following example

1. Rotate triangle ABC clockwise on the cartesian plane, about the origin, through 90^0.

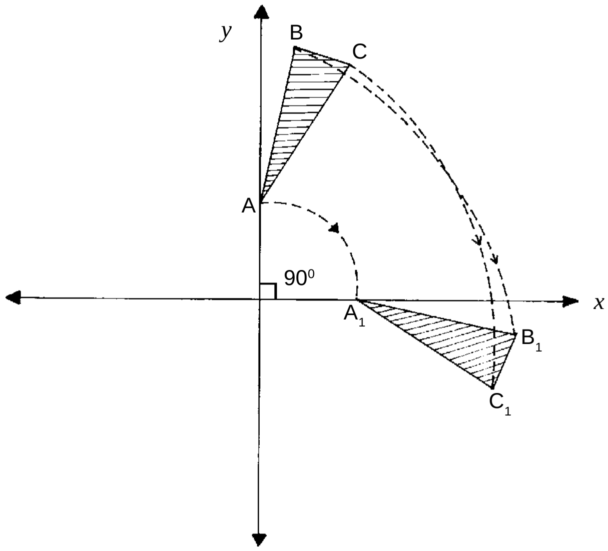
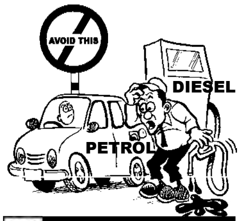


Fig. 20.10



Common Errors

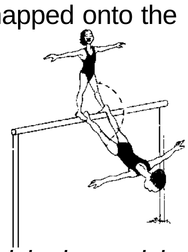
Rotating in the wrong direction ie anti-clockwise instead of clockwise or vice-versa.

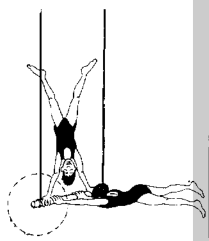
Points A, B and C move along an arc of a circle centre (0; 0) as illustrated in Figure 20.10.

Mathematically we say the object, i.e. $\triangle ABC$, is mapped onto the image $A_1B_1C_1$.

Thus, for this example, we say

A is mapped onto A_1.
B is mapped onto B_1.
C is mapped onto C_1.





In some cases you are given the original object and the image it is mapped onto. You are then required to find the angle of rotation, the centre of rotation and the direction of rotation.

Consider the following example

1. Consider triangle ABC, mapped onto the image $A_1B_1C_1$ as illustrated in Figure 20.11. Deduce the centre of rotation, the angle of rotation as well as its direction.

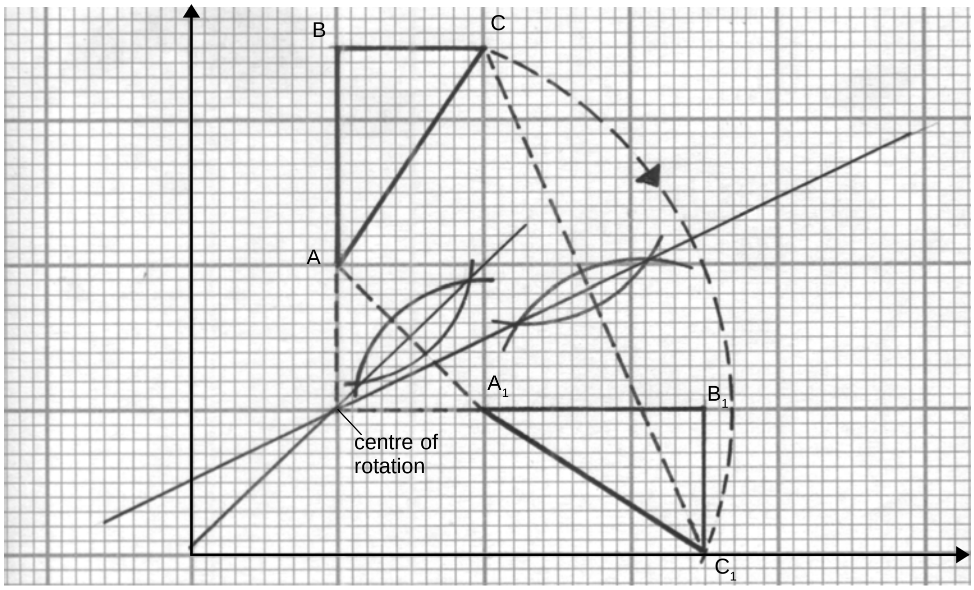
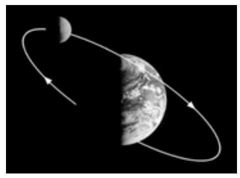


Fig. 20.11

Solution

1. **Facts** about Rotation in Fig 20.11

(i) The centre of rotation maybe found by drawing the perpendicular bisector of the line joining any two corresponding points e.g. A and A_1 or C and C_1. The centre of rotation is at the *intersection* of the two perpendicular bisectors. The centre of rotation is an invariant point under any rotation.



Hint

The angle, direction and centre of rotation can also be found using tracing paper and a sharp pencil placed where you think the centre of rotation is.

(ii) The angle of rotation may be found by:

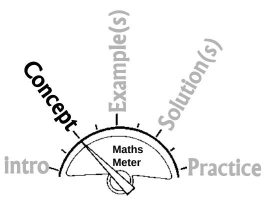
- Drawing lines from corresponding vertices to the centre of rotation.
- Measuring the angle formed by the two lines using a protractor (in this case = 90^0)

(iii) The direction of rotation is deduced by inspection or may be given.

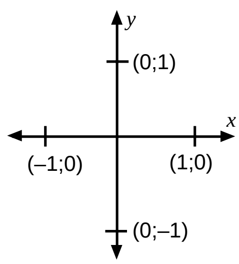
Matrix Method

The following matrices are used in the rotational transformation. Sometimes it is necessary to memorize them.

Table 20.2



Transformation	**Matrix used to multiply the matrix of the object**
Rotation of 90^0 centre origin (0; 0) anticlockwise	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation of 90^0 centre origin (0; 0) clockwise	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation through 180^0 about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



Consider the following example

2. Rotate $\triangle ABC$, with A(1; 6), B(1; 3) and C(3; 6) about the following angles, with the origin (0:0) as the centre of rotation:

a) 90^0 clockwise.
b) 90^0 anticlockwise.
c) 180^0.

Solution

2. a) 90^0 clockwise

matrix of rotation, matrix of triangle, matrix of image

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 6 & 3 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A_1 & B_1 & C_1 \end{array} \\ \begin{pmatrix} 6 & 3 & 6 \\ -1 & -1 & -3 \end{pmatrix} \end{array}$$

Therefore $A_1(6;-1)$ $B_1(3;-1)$ and $C_1(6;-3)$

Using the matrix method (Refer to Table 20.2)

b) 90^0 anticlockwise

matrix of rotation, matrix of triangle, matrix of image

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 6 & 3 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A_2 & B_2 & C_2 \end{array} \\ \begin{pmatrix} -6 & -3 & -6 \\ 1 & 1 & 3 \end{pmatrix} \end{array}$$

Therefore $A_2(-6;1)$, $B_2(-3;1)$ and $C_2(-6;3)$.

c) 180^0

matrix of rotation, matrix of triangle, matrix of image

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 6 & 3 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A_3 & B_3 & C_3 \end{array} \\ \begin{pmatrix} -1 & -1 & -3 \\ -6 & -3 & -6 \end{pmatrix} \end{array}$$

Therefore $A_3(-1;-6)$, $B_3(-1;-3)$ and $C_3(-3;-6)$.

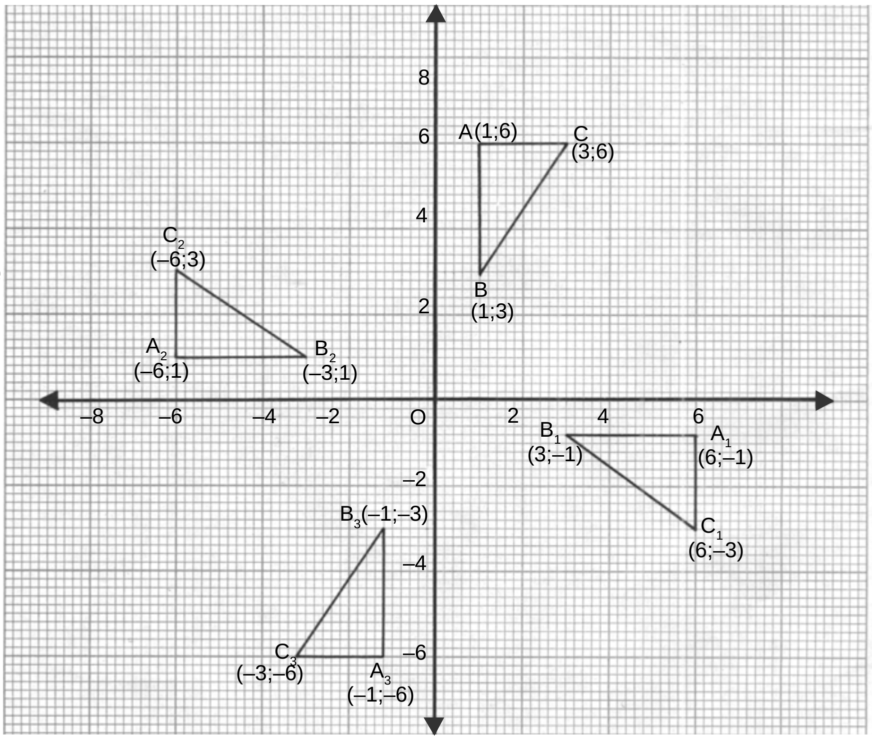


Fig 20.12

In questions 1 to 4, copy the diagrams in Fig. 20.13 on to a graph paper. By using the geometrical method, draw the image under the rotation . The angles of rotation are given. Take 0 as the centre of rotation in each case.

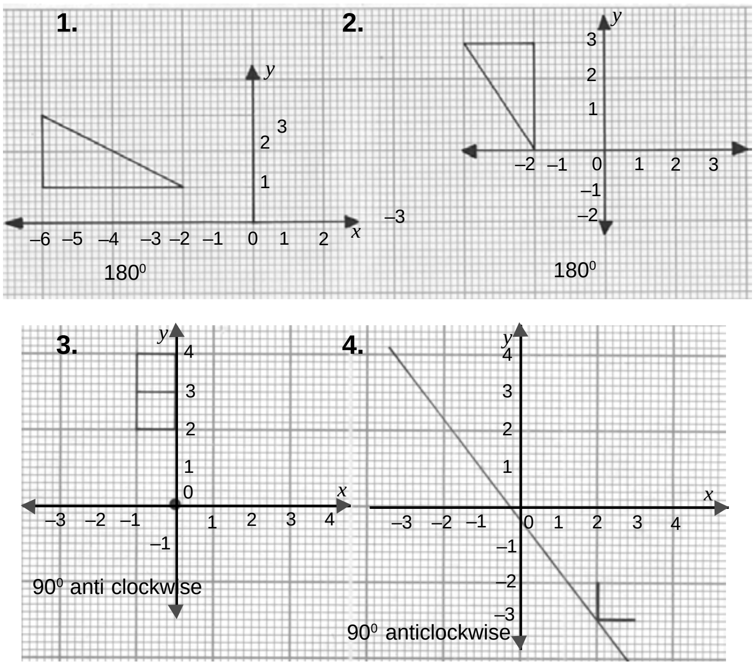


Fig 20.13

5. With the aid of a sketch, find the co-ordinates of the image (4:2) under an anticlockwise rotation of 90^0, about:

a) the origin (0:0).

b) the point (2:2).

c) the point (–3:–3).

6. In Fig 20.14 find the angle, the direction, and the centre of the rotation, given that MNV is mapped onto $M_1N_1V_1$.

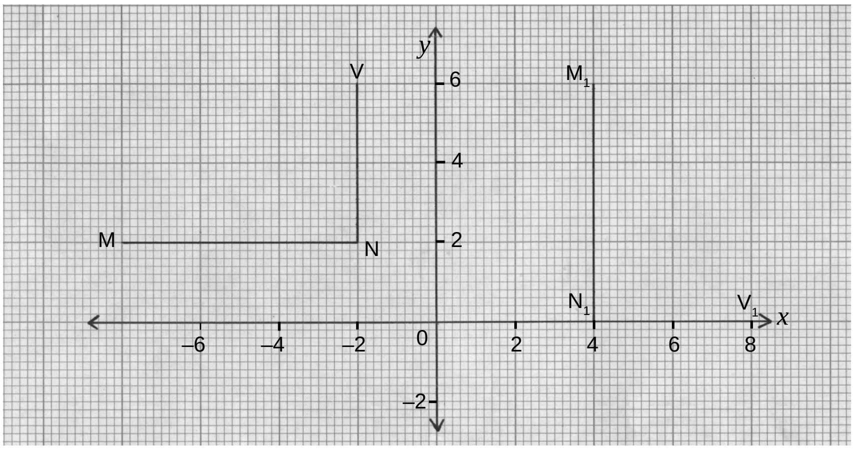


Fig. 20.14

7. On squared paper or graph paper, draw and label △ M with vertices at A (4; 4) B(1; 4) (1; 2) and △ N with vertices at A_1(3; –1) B_1(3; 4) C_1(5; 4)

Describe fully the transformation that maps △M onto △N.

8. Draw on squared or graph paper △R with vertices at A (2; 4), B(2; 8) C(4; 8) and draw the images of △R under the following rotations.

a) 90^0 anticlockwise centre (0:0) label it △ M.

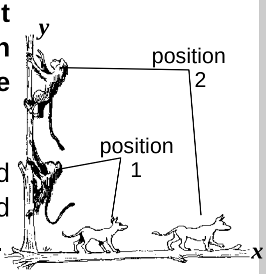
b) 180^0 centre (0:0) label △ N.

c) 90^0 clockwise centre (0:0) label △ V.

C. TRANSLATION

In everyday life “translation” in languages implies “transference” of a piece of spoken or written text from one language into another *without changing its sense*.

A mathematical translation involves the transfer or movement of an object (figure) or shape along a straight line, for a certain distance to a new position, without changing its appearance in any way.



On a cartesian plane, the translation of a shape can be described by the number of units moved along the x-axis (x) and that moved along the y-axis (y), i.e. as $\begin{pmatrix} x \\ y \end{pmatrix}$. This is called a **column vector**.

Translation Vector

As already mentioned, translation is denoted by a **column vector** in the form $\begin{pmatrix} a \\ b \end{pmatrix}$,

where a is the movement made by the shape in the x direction and b is the movement in the y direction.

Positive a means that the shape is to move to the right along the x axis and negative a means the shape will move to the left along the x axis.

Positive b means the shape is to move upwards along the y axis and negative b means the shape will move downwards along the y axis.

The key step in translation is to identify the points $(x;y)$ of vertices of the given shape, then rearrange these points into a vertical form as $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ instead of $(x;y)$. The point in such a form is said to be a column vector.

Tip

Remember a position vector is one in which the tail of that vector is the origin (0:0)

A last and most important thing to understand is the meaning of a **position vector**. *If the original point of an object is written as a position vector*, and the translation is represented by a column vector, then the image point is given by:

Image point	=	**Original point**	+	**Column vector (translation)**

Tip

Translation is the only transformation which can be described by a column vector.

Fig 20.15 shows this more clearly.

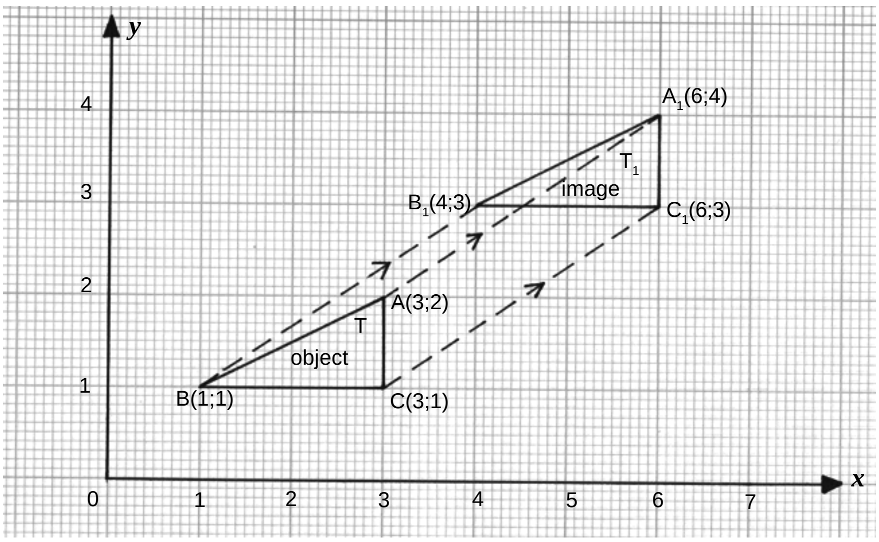
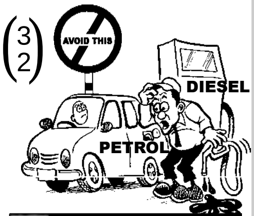


Fig. 20.15 Translation

Hint

In this translation, the x-coordinate of each point is changed by +3 and the y coordinate by +2. Thus, every point of the object moves 3 units to the right and 2 units upwards. Hence, point B(1;1) become $B_1(4;3)$. Deduce points A_1 and C_1 from the diagram. Note that, if corresponding vertices are joined (between object and image) the lines are parallel.

The translation in Fig 20.15 can be described by the column vector $\begin{pmatrix}3\\2\end{pmatrix}$



Common Error

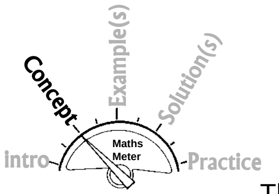
Most students fail to understand the link between the column vector (which is the translation description not a matrix) and the object matrix. The column vector must be added to each column of the matrix separately to effect the transformation. It is wrong to write:

image figure $= \begin{pmatrix}3 & 1 & 3\\2 & 1 & 1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$

because such matrices don't add and more so $\begin{pmatrix}3\\2\end{pmatrix}$ is a column vector not a matrix.

The original triangle, ABC, has vertices:
A(3; 2), B(1; 1), C(3; 1). Translate △ ABC by the translation $\begin{pmatrix}3\\2\end{pmatrix}$.
Using above formula you proceed as follows:

image point = original point + translation



$$\overrightarrow{OA_1} = \overrightarrow{OA} + \begin{pmatrix}3\\2\end{pmatrix}$$

$$\overrightarrow{OA_1} = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$$

$$\overrightarrow{OA_1} = \begin{pmatrix}6\\4\end{pmatrix}$$

Therefore $A_1 = (6:4)$

Similarly $\overrightarrow{OB_1} = \overrightarrow{OB} + \begin{pmatrix}3\\2\end{pmatrix}$

$$\overrightarrow{OB_1} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$$

$$\overrightarrow{OB_1} = \begin{pmatrix}4\\3\end{pmatrix}$$

Therefore B_1 (4:3)

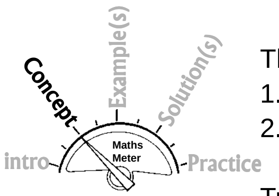
Also $\overrightarrow{OC_1} = \overrightarrow{OC} + \begin{pmatrix}3\\2\end{pmatrix}$

$$\overrightarrow{OC_1} = \begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$$

$$\overrightarrow{OC_1} = \begin{pmatrix}6\\3\end{pmatrix}$$

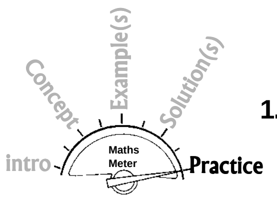
Therefore C_1(6;3)

Then draw △ $A_1B_1C_1$



The full description of a translation should include:
1. *naming that the transformation is a translation.*
2. *stating the translation vector.*

Translation is referred to as the "unique transformation" because all other transformations can be described by matrices but translation is described by a vector.



1. In this question draw x and y axes for $-9 \leq x \leq 9$ and $-9 \leq y \leq 9$.
  a) Draw the triangle ABC with vertices:
  A(–5;–2), B(–5;2) and C(–2;–2).

b) Draw the images of ABC which result from the translations described by the following column vectors:

(i) $\begin{pmatrix} 1 \\ -8 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ (iii) $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$

c) Write down the new coordinates of points A_1, B_1 and C_1 in each case.

2. Using axes numbered $0 < x < 8$ and $0 < y < 8$ mark the points (2;2) (4;2) and (4;3) and join them to form a triangle. Label the triangle M.

(i) Draw the image of M which results from the translation $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and label it N.

(ii) Draw the image N which results from the translation $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and label it V.

3. Copy the diagrams in Figure 20.16 and write down the column vector for each of the following translations.

a) A to F
b) D to B
c) B to E
d) F to C
e) C to B

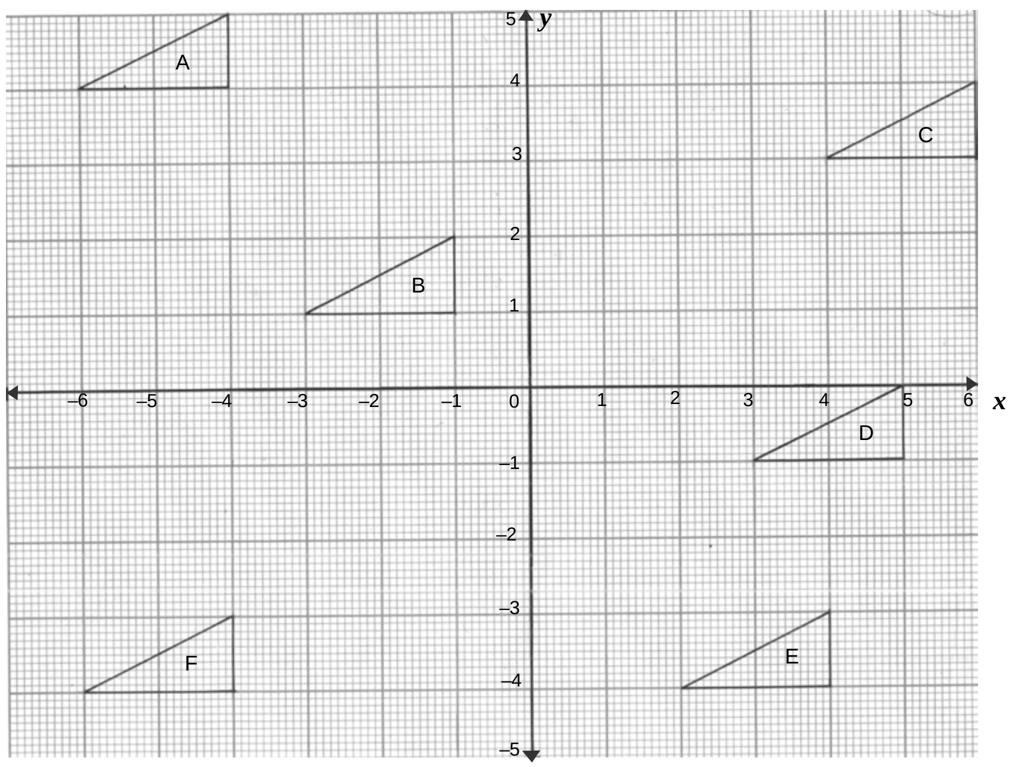
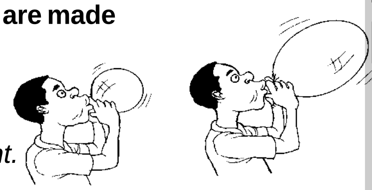


Fig. 20.16

D. ENLARGEMENT

An enlargement is a transformation in which shapes are made bigger or smaller proportionately.



A full description of enlargement should include:
1. *a statement that the transformation is enlargement.*
2. *stating the centre of the enlargement.*
3. *stating the enlargement factor (sometimes called scale factor).*

Geometrical Method

As already mentioned, geometrically, a full enlargement is described by two things:

a) the centre of enlargement.

b) the scale factor.

Consider the following example

1. A triangle ABC, (Fig 20.17), must go under enlargement using centre (0; 0) and

a) scale factor 2 to give $A_1B_1C_1$.

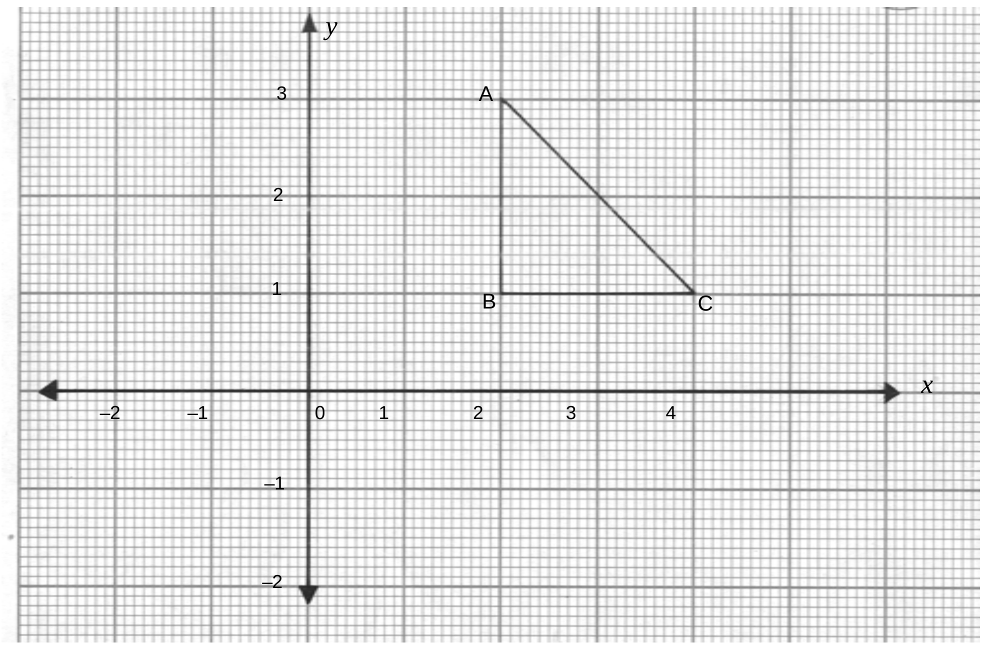


Fig. 20.17(a) Before Enlargement

b) scale factor $\frac{1}{2}$ to give $A_2B_2C_2$.

c) what happens to the enlargement image when the scale factor is negative?

Hint

Enlargement is not an isometry and the centre of enlargement is invariant under-enlargement.

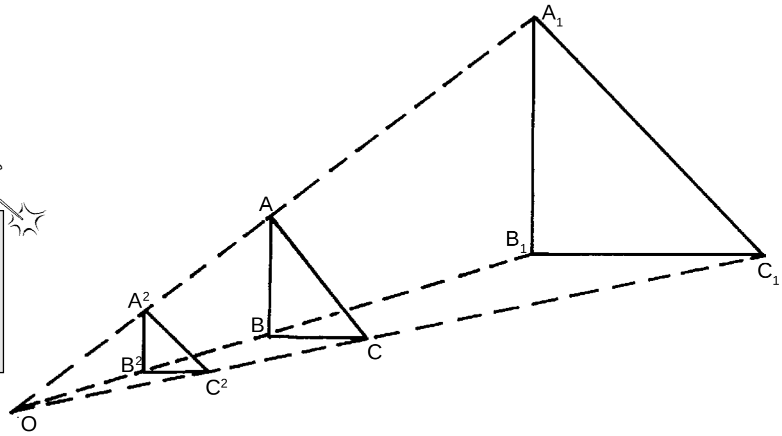


Fig 20.17(b) After enlargement

Hint

1. If the enlargement factor k is 0<k<1 the image shape will be between the centre of enlargement and the original shape. This means the image shape is ***smaller*** *than the original shape. Check $A_2B_2C_2$ in Fig 20.17(b).*

2. If the enlargement factor (k) where k>1 i.e. (a positive whole number) the original shape will be between the centre of an enlargement and the image shape. The image will be ***larger*** *than the original shape. ($A_1B_1C_1$ in Fig 20.17(b)).*

3. If the enlargement factor k is –1<k<0 the original shape and image shape are on opposite sides of the centre of enlargement and the image is ***inverted*** *and* ***smaller*** *than the original shape. ($A_3B_3C_3$ in Fig 20.17(c)).*

4. If the enlargement factor k where k<–1 the original shape and image shape are to the opposite sides of the centre of enlargement. Image shape is ***inverted*** *and* ***larger*** *than the original shape.*

Solution

1. a) Since O is the centre of enlargement and 2 is the scale factor, it follows that (Fig 21.17 (b))

$OA_1 = 2OA$
$OB_1 = 2OB$
$OC_1 = 2OC$

Thus, every point, of the image, is 2 times as far from O as the corresponding point of the object.

b) Similarly for image $A_2B_2C_2$ where the scale factor is $\frac{1}{2}$ it follows that

$OA_2 = \frac{1}{2}OA \qquad OB_2 = \frac{1}{2}OB \qquad OC_2 = \frac{1}{2}OC.$

c) When the scale factor is **negative** the enlarged image will be on the opposite side of the centre of enlargement and is always inverted, for example, an enlargement with a scale of factor = $-\frac{1}{2}$ (Fig 20.17(c)).

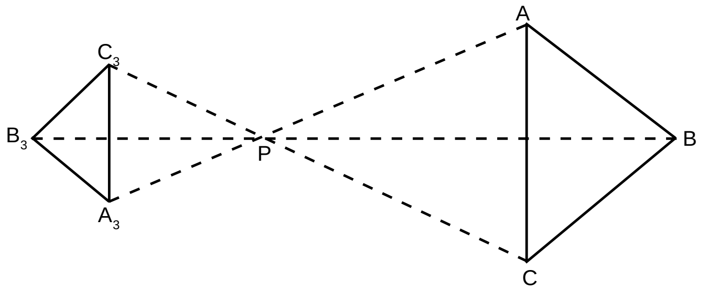


Fig 20.17(c) Enlargement

Common Error

Students forget to invert the image once the scale factor is given as negative especially when labelling the image.

Another very important skill is how to find the centre of the enlargement when its not given. It is the point where lines, joining the corresponding points intersect.

Consider another example

2. Consider the enlargement in (Fig 20.18) with a scale factor $-1\frac{1}{2}$. Find the centre of enlargement by geometrical means.

Hint

Always remember once the scale factor of enlargement is negative, this implies that the image and object are on opposite sides of the centre of enlargement and the image is inverted.

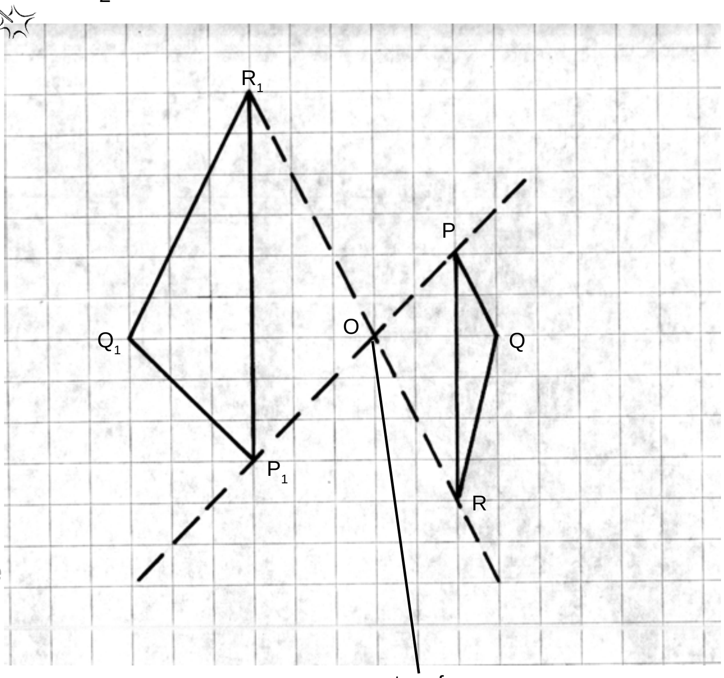


Fig 20.18 Enlargement

Solution

2. The following steps should be followed to find the centre of enlargement:

(i) Draw straight lines through the corresponding object and image vertices.

(ii) Where the straight lines intersect is the centre of enlargement.
Observe Fig 20.18.

Use of the matrix of enlargement.

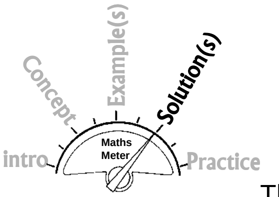
The matrix of enlargement, with the **origin as the centre of enlargement**, can be stated as $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$,where k is the enlargement factor.

Consider the following example

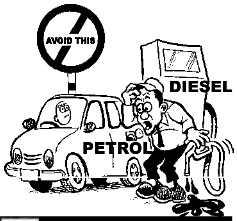
3. Enlarge $\triangle$ABC, with A(2;3), B(2;1) and C(4;1) by enlargement factor 2, with the origin as the centre of enlargement.

Solution

Using the **Matrix method**



$$\begin{array}{c} \text{Matrix of} \\ \text{rotation} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{array} \times \begin{array}{c} \text{Matrix of triangle} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 2 & 4 \\ 3 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \text{Matrix of image} \\ \begin{array}{ccc} A_1 & B_1 & C_1 \end{array} \\ \begin{pmatrix} 4 & 4 & 8 \\ 6 & 2 & 2 \end{pmatrix} \end{array}$$



Common Error

Students fail to realise that enlargement does not always mean the image must be **larger** than object.

Therefore $A_1(4:6)$, $B_1(4:2)$ and $C_1(8:2)$. See Fig 20.19.

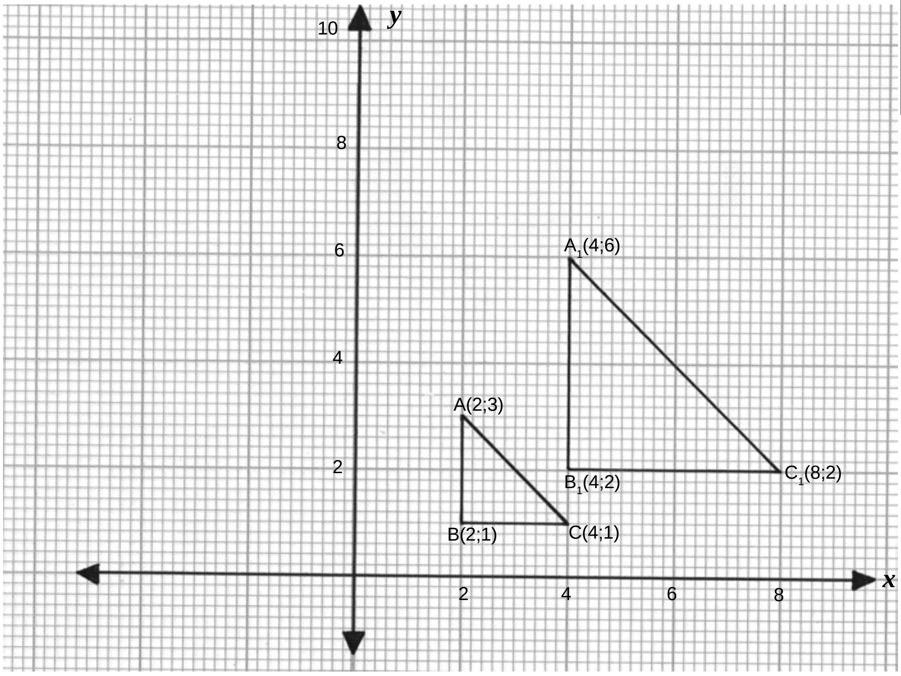
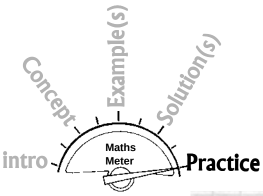


Fig 20.19



In questions 1 to 4, copy the given diagram and draw an enlargement, using the centre P and the scale factor given.

1. **2.** **3.** **4.**

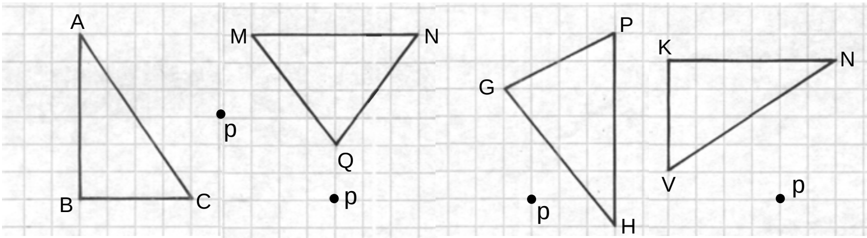


Fig 20.20

1.	2.	3.	4.
scale factor –3	scale factor $1\frac{1}{2}$	scale factor 2	scale factor –2

In questions 5 to 7 on the cartesian plane, enlarge the object, using the given centre of enlargement and scale the factor indicated.

	Object	Centre	Scale factor
5.	(3;5) (5;3) (6;6)	(2;3)	+2
6.	(2;2) (4;2) (4;3)	(5;4)	–2
7.	(8;1) (12;2) (12;6)	(6;5)	$-\frac{1}{2}$

8. Find the image and the centre of enlargement and the scale factor for:

a) Object A(3;2) B(6;2) C(4;4)
Image A_1(3;2) B_1(10;2) C_1(6;8)

b) Object A(0;5) B(3;5) C(2;–1)
Image A_1(10;4) B_1(6;4) C_1(7;10)

9. Draw the following two triangles, on the cartesian plane:

T_1 (–5;–1), (–5:7), (–8;–1)

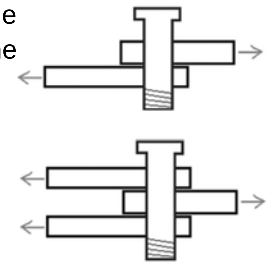
T_2 (–3;1), (–3;3), (–4;3)

Now describe fully the following transformation
$T_2 \rightarrow T_1$.

E. SHEAR

Geometry of shear transformation

It is a mathematical fact that parallelograms or triangles on the same base and between the same parallel lines have the same area. (Fig 20.21 a and b).



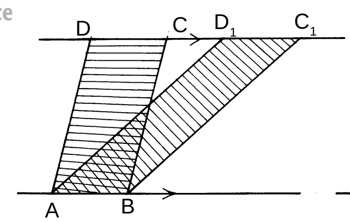


Fig 20.21(a)

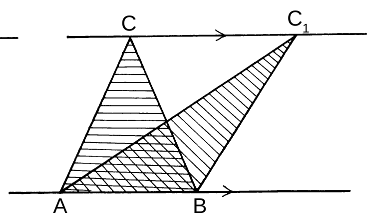


Fig. 20.21(b)

From Fig 20.21 the following points should be noted:

(i) Area of parallelogram ABCD = Area of parallelogram ABC_1D_1.

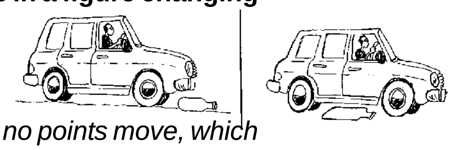
(ii) Area of $\triangle$ ABC = Area of $\triangle ABC_1$.

(iii) The bottom line AB did not move in either case and it is called an **invariant line.**

(iv) Points that are on the invariant line do not move and all other points/vertices, that are not on the invariant line, move **parallel** to the invariant line.

Supposing the shapes in Fig 20.21 are made of a soft elastic rubber, you could change their shapes without changing their areas by stretching them. That is an illustration of a transformation called **shear**.

A shear is a transformation which results in a figure changing shape but its area remains unchanged.



The following is used to describe a shear.

1. *The invariant line i.e. the line on which no points move, which is AB in (Fig 20.21)*
2. *The shear factor is calculated by:*

Hint

Note that in Fig 21.20 if DC had moved to the left, the scale factor would have been negative.

$$\text{Shear factor} = \frac{\text{distance moved by a point}}{\text{distance of the point from invariant line}}$$

In Fig 20.22 the shear of $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$, using a shear factor of 3 and ***x*-axis invariant**.

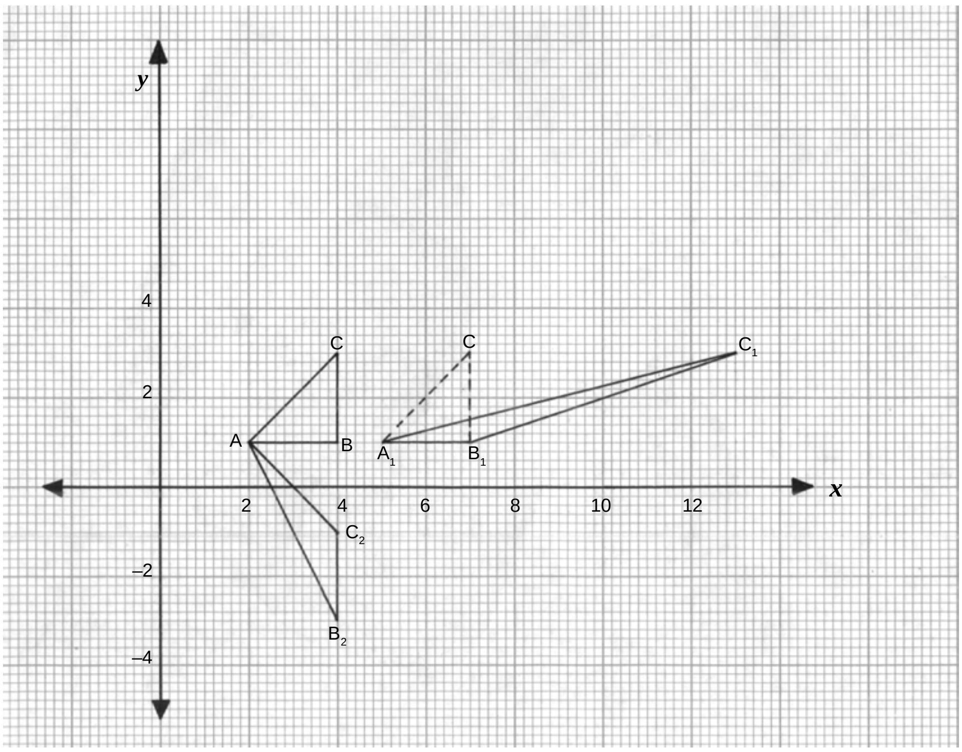
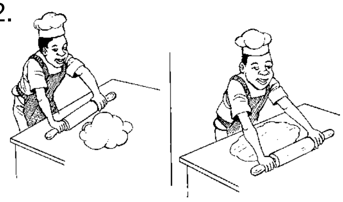


Fig. 20.22

$\triangle AB_2C_2$ indicates that A is on the invariant line and the movement of B and C implies that the invariant line is parallel to the y-axis and passes through A. It follows that the invariant line is $x = 2$.

Note Shear factor for $A_1B_1C_1$ using point

$$C_1 = \frac{\text{distance moved by C}}{\text{distance of C from invariant line } (x = 5)} = \frac{6}{2} = 3$$



Also Shear factor for AB_2C_2 using point

$$B_2 = \frac{\text{distance moved by B}}{\text{distance of B from invariant line } (x = 2)} = \frac{4}{2} = 2$$

As already noted, a full description of a shear should include the following:

1. *statement that the transformation is shear.*
2. *statement of the shear factor.*
3. *stating the invariant line (write down the equation of the invariant line)*
4. *stating of the matrix of the shear (where need be).*

Use of shear matrices

The general matrix is $\begin{pmatrix} \mathbf{1} & k \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ where k is the shear factor and x-axis invariant or $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ k & \mathbf{1} \end{pmatrix}$ where y-axis is invariant.

Consider the following example:

1. Shear rectangle OABC, with O being the origin (0;0), A(0;3) B(2;3) and C(2;0), by shear factor 2 with:

a) x axis as invariant line.

b) y axis as invariant line.

Solution

1. a) Matrix of shear, Matrix of rectangle, Matrix of image

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{array}{c} \begin{array}{cccc} O & A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O & A_1 & B_1 & C \end{array} \\ \begin{pmatrix} 0 & 6 & 8 & 2 \\ 0 & 3 & 3 & 0 \end{pmatrix} \end{array}$$

Therefore image is at O(0; 0), A_1(6; 3), B_1(8; 3) and C(2; 0) (Fig 20.23).

b) Matrix of shear, Matrix of rectangle, Matrix of image

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \times \begin{array}{c} \begin{array}{cccc} O & A & B & C \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O & A & B_2 & C_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 7 & 4 \end{pmatrix} \end{array}$$

Therefore image is at O(0;0), A(0;3), B_2(2;7) and C_2(2;4) (Fig 20.23).

Hint

In a shear note that the concept of "same base and between the same parallel lines" = same area is maintained..

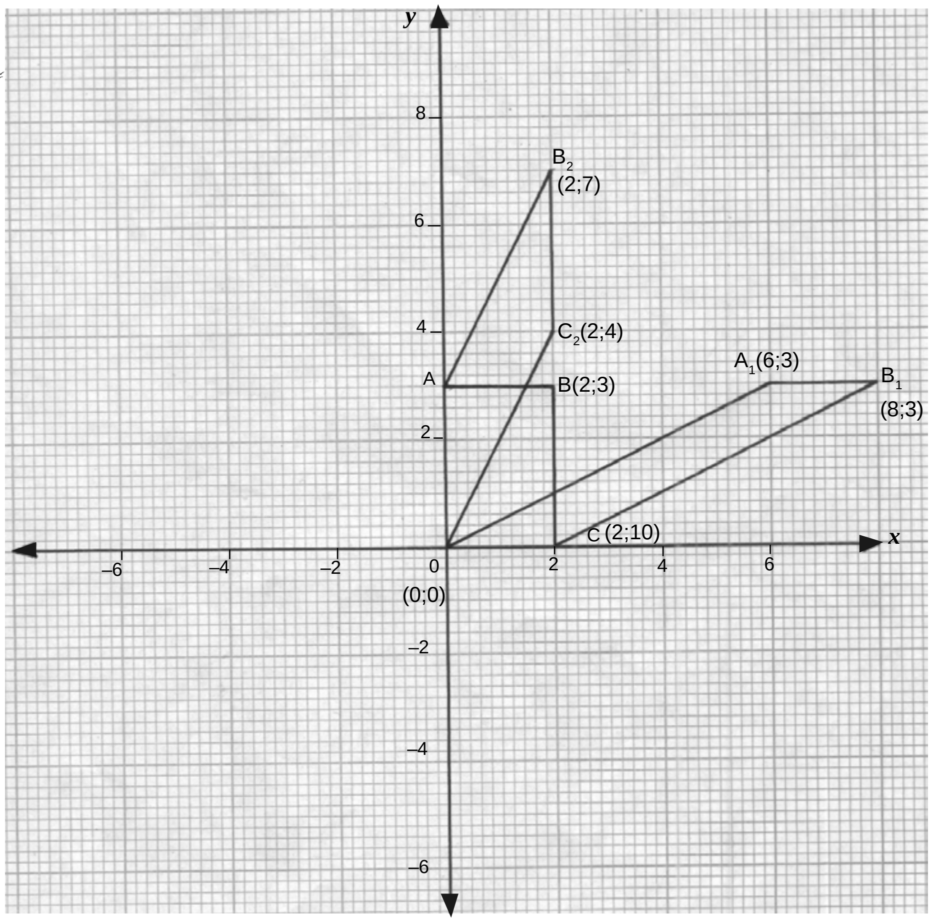


Fig. 20.23

Facts to note on shear transformation.

1. There is only one invariant line.
2. The area remains the same.
3. $$\text{Shear factor} = \frac{\text{distance moved by a point}}{\text{distance of the point from invariant line}}$$
4. Straight lines and parallel lines are unaltered in a shear.

In this section use either the geometric method or matrix method or use both to gain experience.

1. a) Draw rectangle O(0; 0) A(2; 0) B(2; 1) C(0; 1).

b) The shear H transforms rectangle OABC into OA_1B_1C where A_1 is (2; 2). Draw the image B_1.

c) Find the shear factor.

2. Using Fig 20.24.

a) describe, fully, the transformation where the parallelogram OMN_1V_1 is the image of rectangle OMNV.

b) copy the object OMNV and draw its shear image, with the y-axis as the invariant line, so that the image of M is at (2;–3)

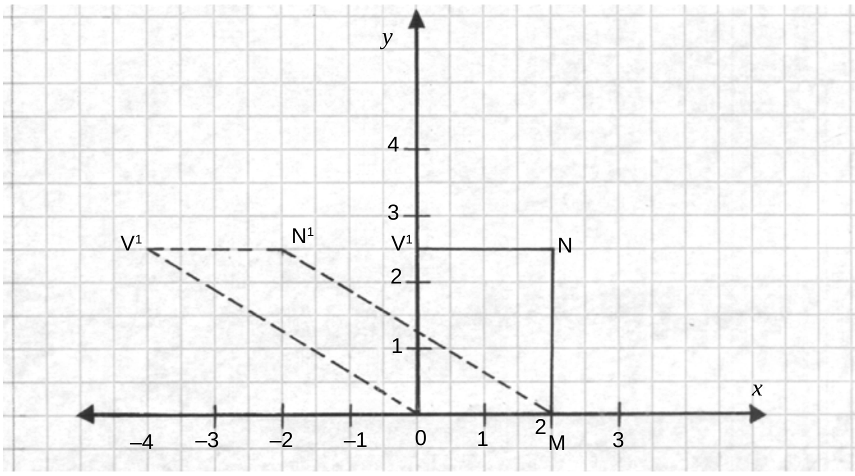


Fig. 20.24. Shear

3. a) Draw the figure indicated by (0;0), (0;2), (–1;1), (–1;–1).

b) Given that the transformation of this shape is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$ show geometrically, or otherwise, that the transformation is a shear and, hence, deduce the equation of the invariant line.

4. Fig. 20.25 illustrates a typical shear where ABCD is transformed to $A_1B_1C_1D_1$. The x-axis is the invariant line. Given that (0; 1) is mapped to (3; 1), deduce the following:

a) find the shear factor

b) find the area of ABCD and compare it with that of $A_1B_1C_1D_1$.

c) describe this transformation.

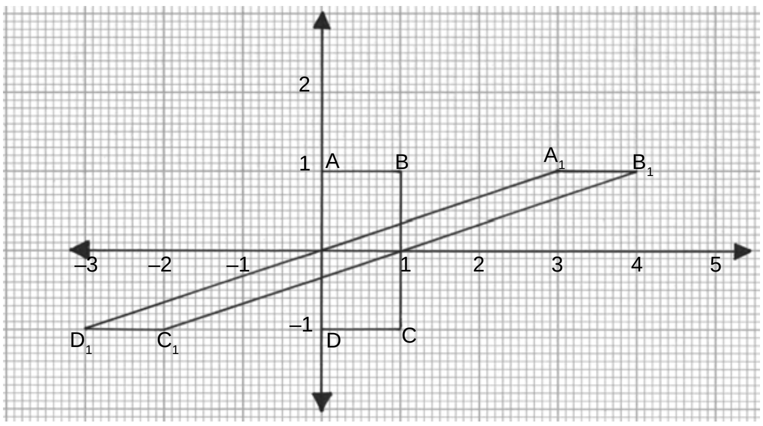
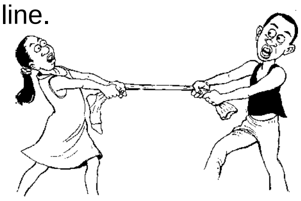


Fig. 20.25

F. STRETCH

A stretch is a transformation where distances are increased or decreased in one direction only.

Note that, in a stretch transformation, the point/vertices that lie on the invariant line do not move and points/vertices, which do **not** lie on the invariant line move perpendicularly to the invariant line.



Geometry of stretch transformation

A full description of stretch should include:

1. *statement that the transformation is a one way or two way stretch.*

2. *statement of the invariant line(s).*

3. *stating of the stretch factor(s)*

Consider the following example:

1. In Figure 20.26, rectangle OABC experiences two transformations which are both stretches. Describe them and, hence, deduce the formula required to calculate the area of the image.

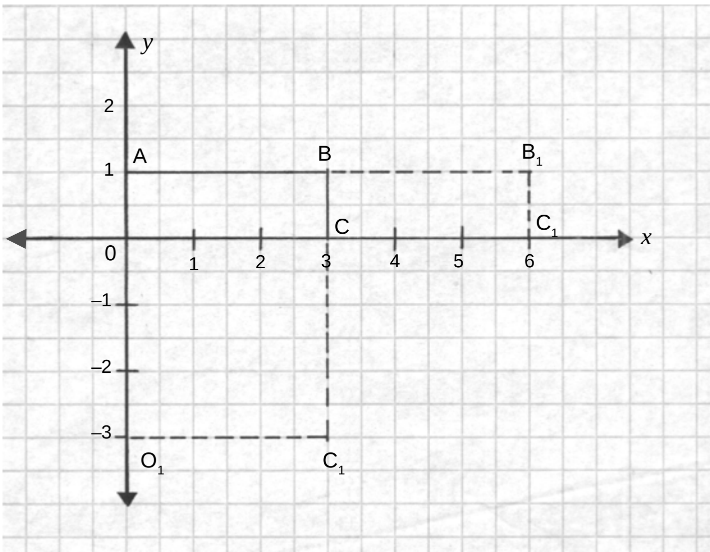


Fig. 20.26 Stretch

Solution

1. a) A stretch occurs when OABC is transformed to OAB_1C_1.
   (i) The stretch is along the positive x-axis.
   (ii) OA is the invariant line.

(iii) Stretch factor = $\dfrac{\text{distance of } B_1 \text{ from invariant line}}{\text{distance of B from invariant line}}$

$$= \frac{AB_1}{AB} = \frac{6}{3} = 2$$

b) A stretch also occurs when OABC is transformed to ABC^1O^1.
   (i) The stretch is along the negative y-axis
   (ii) AB is the invariant line
   (iii) The stretch factor $= \dfrac{BC_1}{BC} = \dfrac{-4}{1} = -4$

c) Area of image $\triangle$ = (Area of original figure) $\times$ stretch factor.

Matrix method of Stretch Transformation

The stretch matrix, in which the x axis ($y = 0$), is the invariant line is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, and when the y axis ($x = 0$) is the invariant line, it is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where k is the stretch factor.

Hint

Stretch is not isometry as it has an invariant line perpendicular to the direction of stretch.

A two way stretch has the matrix $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ when both the x-axis and y-axis are invariant lines. If $k_1 = k_2$ then the image becomes an enlargement with a scale factor k_1 and the centre of enlargement is the origin. (0; 0)

As already mentioned, the following is required to fully describe a stretch:

1. *the direction of the stretch.*

2. *the invariant line i.e. the base line from which distances are measured. This line does not necessarily have to be part of the object.*

3. *the magnitude of the stretch i.e. the ratio of corresponding lengths.*

1. Find and draw the **image** of the object described by the points: A(0;0), B(0; 2), C(2; 2) and D(2; 0)
under the transformation matrix of $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$
Describe this transformation fully.

2. a) Draw $\triangle$ABC with vertices A(6; 2), B(7; 3) and C(8; 2).

b) Draw and label the image $A_1B_1C_1$ which is a stretch with y-axis invariant and a scale factor of 2.

3. a) Draw $\triangle XYZ$ which has vertices X(0; 3), Y(0; 0) and Z(4; 2).

b) Draw and label the image.
$X_1Y_1Z_1$ with x-axis invariant and a stretch factor of 3.

4. a) Draw the image of the rectangle with vertices.
A(0; 0), B(3; 0), C(3; 2) and D(0; 2)
under the shear represented by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 3{,}5 \end{pmatrix}$

b) Find the invariant stretch point.

G. MORE ON TRANSFORMATION AND MATRICES

Inverse transformations

In translation, if A has a column vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, the translation which has the opposite effect, has the column vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and it is written as A^{-1}.

Similarly, if a rotation (R) moves through 90^0 anticlockwise about (0,0), then R^{-1} is the 90^0 clockwise rotation about (0,0).

The inverse of a transformation is the transformation which takes the image back to the object.

1. For point (6;3) apply each of these matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and show the image of (6; 3), state the type of transformation.

2. a) On a graph paper draw the triangle (T) with vertices: (3;3), (7;3) and (7;5).

b) Draw the image (T_1) from (T) using the transformation factor represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c) Draw the image (T_2) from (T) using the transformation factor represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

d) Describe the single transformation which would translate (T_1) to (T_2).

3. a) Draw the object A(2; 2), B(4; 4) and C(6; 2) and its image $A_1B_1C_1$ using Matrix M = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

b) Repeat part (a) with M = $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

c) Repeat part (a) with M = $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Draw the square with vertices:
A(1;1), B(2;2), C(1;3) and D(0;2)

a) Draw the image of the square using the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

b) Show that the transformation is a stretch and find the equation of the invariant line or point.

5. Draw the x-axis and y-axis with values from –8 to +8.

a) Draw triangle with vertices A(3;3), B(7;3) and C(7;5).

b) Find the image of ABC using the transformation represented by the following matrices:

(i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(v) Describe each transformation fully.

H. DETERMINING THE TRANSFORMATION MATRIX

In some situations, you are given information about the transformation and then you are required to determine the transformation matrix. The following ***facts*** may be recalled.

(i) Generally a transformation matrix is in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(ii) $\begin{pmatrix} \text{Transformation} \\ \text{matrix} \end{pmatrix} \times \begin{pmatrix} \text{Column vector} \\ \text{of object} \\ \textbf{point} \end{pmatrix} = \begin{pmatrix} \text{Column vector} \\ \text{of image} \\ \textbf{point} \end{pmatrix}$

The second point/fact is used to formulate the respective equations.

Consider the following examples

1. Given that a certain transformation maps point (3; 0), onto point (3; 6) and point (3; 6) is transformed on to the point (3; 1). Determine the matrix representing the transformation.

Solution

1. Required matrix = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Using fact 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

This means the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ has image $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$\Rightarrow$ multiplying the matrices

$$\begin{pmatrix} 3a \\ 3c \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$3a = 3$
$a = 1$

$3c = 6$
$c = 2$

Also we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$3a + 6b = 3$
$3c + 6d = 1$

Substituting for $a = 1$, $c = 2$

$3 \times 1 + 6b = 3$
$6b = 0$
$b = 0$

$3 \times 2 + 6d = 1$
$6 + 6d = 1$
$6d = -5$
$d = \frac{-5}{6}$

The required matrix = $\begin{pmatrix} 1 & 0 \\ 2 & -\frac{5}{6} \end{pmatrix}$

2. In a reflection of a shape its point (1; 3) is transformed onto (6; –3) and its other point (1; 6) is transformed onto (6; –1). Determine the matrix which represents the reflection.

Solution

2. Proceeding as in the first example, equate the matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

Equations formed are

$a + 3b = 6$ .... equation 1

$c + 3d = -3$ .... equation 2

Similarly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Equations formed

$a + 6b = 6$ .... equation 3

$c + 6d = -1$ .... equation 4

Solving equations 1 and 3.

$$a + 3b = 6$$
$$a + 6b = 6$$

$$a = 6$$
$$b = 0$$

Also solving equations 2 and 4

$$c + 3d = -3$$
$$c + 6d = -1$$

$$-3d = -3 + 1$$
$$-3d = -2$$
$$3d = 2$$
$$\therefore \quad d = \tfrac{2}{3}$$

$$c + 3 \times \frac{2}{3} = -3$$
$$c + 2 = -3$$
$$c = -5$$

$\therefore$ The required matrix is $\begin{pmatrix} 6 & 0 \\ -5 & \frac{2}{3} \end{pmatrix}$

1. In the following questions, determine the matrix which represents the transformation given, the following:

a) A(3; 3) $A_1(3; -3)$
C(–6;–6) $C_1(6; -6)$

b) M(4; 6) M_1(3; 2)
K(3; 1) K_1(–3; 4)

c) P(1; 0) P_1(–1; –1)
V(2; 3) V_1(4; 4)

d) R(2; 2) R_1(2; 4)
Q(3; 0) Q_1(0; 4)

e) N(2; 1) N_1(–2; –3)
X(4; 1) X_1(4; 0)

2. Verify all the transformation matrices in the table given in the summary of this chapter!

I. COMBINED TRANSFORMATION

When problems in the examination are set, examiners may combine two or more types of transformations in one problem. You need to recall the concepts related to the type of transformation being examined.

1. Define the following terms, mathematically, using diagrams where necessary:

a) translation.
b) enlargement.
c) rotation.
d) stretch.
e) shear.
f) reflection.

2. Using axes numbered from –8 to 8 on the same graph, draw the triangle with vertices A(1; 3), B(3; 3) and C(1; 2),

a) draw the image of triangle ABC transformed in an anticlockwise rotation of 90^0 about (3; 3) and label it P.

b) show the image of P transformed in an anticlockwise rotation of 90^0 about (1; 3) and label M.

c) State the transformation for which M is the image of P.

3. Draw triangle ABC with vertices A(1; 2), B(2; 2) and C(2; 4) and its image under reflection, with vertices A_1(5; 2), B_1(4; 2) and C_1(4; 4)

a) What is the equation of the mirror line?

b) Find the image of ABC under a reflection, in the line $x = 0{,}5$.

c) Find the transformation vector which maps this image onto $A_1B_1C_1$.

4. a) Draw the image of a rectangle (0; 0) (2; 1) (2; 0) (0; 1) transformed to the shear represented by the matrix.

$$\begin{pmatrix} 1 & -1 \\ 0 & 2{,}5 \end{pmatrix}.$$

b) Find the equation of the invariant line.

5. Draw the image of the square with vertices A(1; 1), B(2; 1), C(2; 2) and D(1; 2) under a transformation represented by the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

Describe this transformation fully.

6. Answer the whole of this question on a sheet of graph paper.

a) Draw x-axis for $-8 \leqslant x \leqslant 8$ and y-axis for $-8 \leqslant y \leqslant 8$. Draw triangle ABC with vertices A(2;1), B(3; 3) and C(4; 3) and label it T_1.

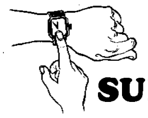
b) Draw the enlargement of triangle T_1, centre (0 0), scale factor 2. Label it T_2.

c) Draw the rotation of triangle T_1 through 90^0 anticlockwise about (0; 0), label it T_3.

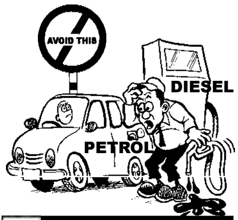
d) Draw the reflection of triangle T_1 in the line $y = -1$, label it T_4.

e) Draw the translation of triangle T_2 by the vector $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$, label it T_5.

f) Describe the transformation which is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.



SUMMARY



Common Error

Confusing the definition of transformation and translation.

1. A transformation is a change in position or magnitude, or both, for a given figure. It can either be isometry or non-isometric.

2. A reflection is a transformation in which a shape is translated to a new position, as an image of itself, with respect to a defined mirror line.

3. A rotation is a transformation which involves the turning of a shape either clockwise or anti-clockwise, about a defined point.

4. Translation is transformation which involves the transfer or movement of an object or shape, along a straight line for a certain distance, to a new position, without changing its appearance in any way.

- Translation is the "odd one out" because it is the only transformation which can be described by a vector.

5. An enlargement is a transformation in which a shape is made bigger or **smaller**, proportionately.

a) Enlargement factor (Scale factor) $= \dfrac{\text{Distance of Image from Centre}}{\text{Distance of Object from Centre}}$

b) Scale factor is the ratio of corresponding sides in similarity.

6. A shear is a transformation which results in an object changing shape but having its area unchanged.

a) Shear Factor $= \dfrac{\text{Distance moved by a point}}{\text{Distance of point } \textbf{from} \text{ the invariant line}}$

b) In shear, the movement is always parallel to the invariant line.

7. A stretch is a transformation where distances are increased or **decreased** in one direction only.

Stretch Factor $= \dfrac{\text{Distance of Image } \textbf{from} \text{ the invariant line}}{\text{Distance of Object } \textbf{from} \text{ the invariant line}}$

Note: The word '**from**' should always be written as it gives direction, and the sign of the factor.

Table 20.3 gives a summary of the matrix and the geometric transformation produced. Study the table thoroughly and verify each fact graphically.

Table 20.3

Transformation	Matrix
Translation	$\begin{pmatrix} a \\ b \end{pmatrix}$
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection on the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection on the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Rotation of 90^0 centre origin anticlockwise.	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation of 90^0 around the origin clockwise.	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation through 180^0 about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Enlargement centre, the origin,with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Shear with x-axis invariant, scale factor k	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$
Shear with y-axis invariant, and shear factor k	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$
One way stretch parallel to x-axis (y-axis invariant) with a scale factor k.	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
One way stretch parallel to y-axis (x-axis invariant) with a scale factor k.	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Two way stretch, with scale factors k_1 and k_2	$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$

EXAM PRACTICE 20

The following example may help you to master questions from this chapter.

Answer the whole of this question on a single sheet of graph paper.

1. Using a scale of 2cm to represent 1 unit on each axis, draw axes for values of x and y in the ranges $-3 \leqslant x \leqslant 5$ and $-6 \leqslant y \leqslant 4$ respectively.

a) Triangle XYZ has vertices at X(1;2), Y(3;2) and Z(2;4) respectively.

b) Triangle $X_1Y_1Z_1$ represents the image of triangle XYZ transformed by reflection on the y-axis. Draw and label triangle $X_1Y_1Z_1$.

c) Triangle $X_2Y_2Z_2$ is the image of triangle XYZ transformed by an enlargement, scale factor of –2 and centre (2, 0). Draw and label triangle $X_2Y_2Z_2$.

d) Triangle $X_3Y_3Z_3$ is the image of triangle XYZ transformed by a **single** transformation. Describe **fully** this transformation.

Solution

1. a to c refer to the graph (Fig 20.27).

d) Rotation 90^0 anticlockwise (or 270^0 clockwise) around centre of rotation (0;0).

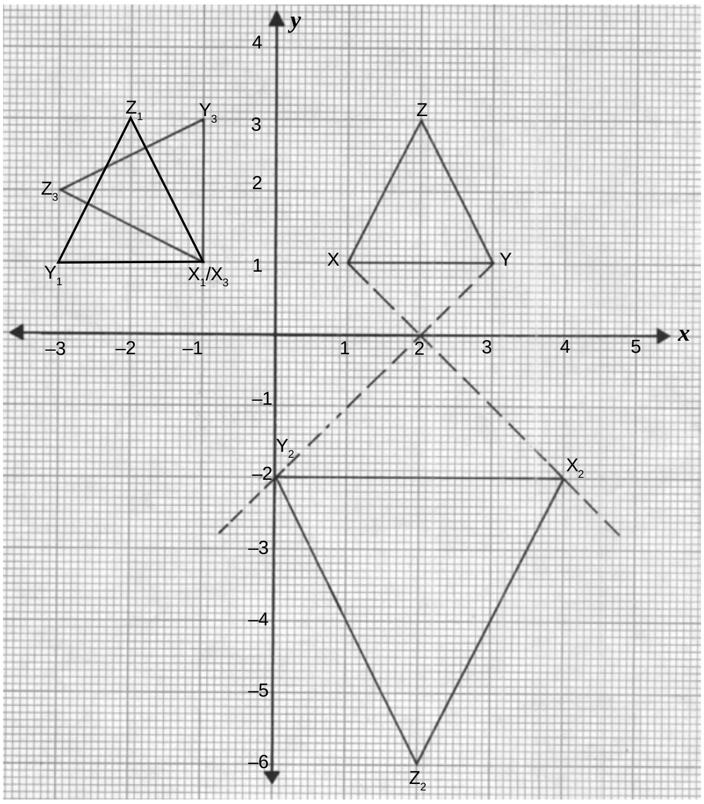


Fig. 20.27

Now do the following:

1. Allowing a range of 0 to +12 on the x-axis, and a range of –3 to +7 on the y-axis, draw, on graph paper, the triangle whose vertices are A(1,1), B(3,1) and C(1,2).

a) Draw the image of triangle ABC under an enlargement with the origin as the centre and, having a scale factor 3. Label $\triangle A_1B_1C_1$.

b) State the coordinates of the images of A_1 B_1 and C_1 under this enlargement.

c) Draw the image of triangle ABC under enlargement, centre point (0;2) and having a scale factor 4.

d) State the coordinates of the image A_2 B_2 and C_2 resulting from this enlargement.

2. *Answer the whole of this question on a sheet of graph paper.*

a) Using a scale of 2cm, to represent 2 units on each axis, draw axes for values of x and y in the ranges $-10 \leqslant x \leqslant 8$ $0 \leqslant y \leqslant 14$, respectively. Draw triangle V, whose vertices are at the points (2; 2), (5; 2) and (3; 4). Label the triangle V.

b) (i) W is the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.
Calculate the co-ordinates of the points onto which points (2;2), (5;2) and (3;4) are mapped by the matrix W.

(ii) Draw the image of triangle V resulting from the transformation represented by W and label the triangle V_1.

(iii) Triangle V_1 is mapped onto triangle V_2, by an anticlockwise rotation through 90^0, about the point (1;1). Draw the triangle and label it V_2.

3. *Answer the whole of this question on a sheet of graph paper*

a) Using a scale of 2cm to represent 2 units on each axis, draw x- and y-axes for $-6 \leqslant x \leqslant 10$ and $0 \leqslant y \leqslant 16$ respectively.
Draw and label the triangle whose vertices are X(2;5), Y(3; 5) and Z(3; 2).

b) The enlargement E has the origin as its centre and transformed $\triangle XYZ$ onto $\triangle X_1Y_1Z_1$. Given that X_1 is the point (6; 15),
(i) draw and label the triangle $X_1Y_1Z_1$.
(ii) write down the scale factor of E.

c) The transformation (R), an anticlockwise rotation of 90^0 about the origin, maps $\triangle XYZ$ to $\triangle X_2Y_2Z_2$. Draw and label $X_2Y_2Z_2$ and find the matrix representing R.

4. A point (A) has co-ordinates (0; –1) and point B is at (2; 1,5). B_1, is the image of B under rotation centre C(–1,5; 2), through 90^0 anticlockwise.

a) Plot A, B and B_1 and C on graph paper.

b) Locate A_1 the image of A after a rotation 90^0 anticlockwise centre (–1,5; 2).

5. ABCD is a rectangle in which A,B,C,D has co-ordinates of (1; 2), (6;2), (6;4) and (1;4) respectively. That transformation (T) is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and transformation U is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In transformation (T), the image points of A,B,C,D are $A_1B_1C_1D_1$, respectively. In transformation (U) the image points of $A_1B_1C_1D_1$ are $A_2B_2C_2D_2$, respectively.

a) Determine the co-ordinates of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$.

b) Show the rectangle ABCD and the image figures $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ in a diagram.

c) Describe the transformation T and U fully.

6. Use the letters A, B, C, D, E to answer the following questions concerning Fig 20.28.

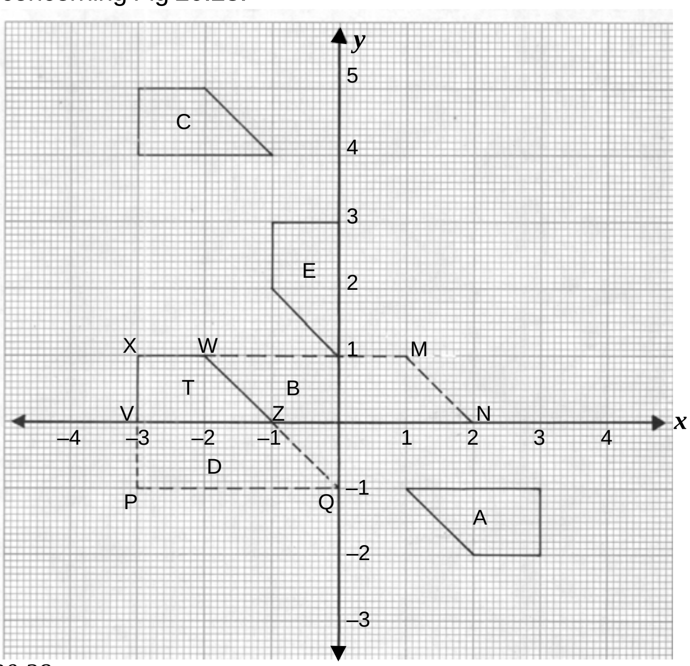


Fig. 20.28

a) Which trapezium is (T) mapped onto by a translation? Write down the translation vector.

b) Which trapezium is (T) mapped onto by a rotation? Write the co-ordinates of the centre of rotation.

c) Which trapezium is (T) mapped onto by a reflection? Write down the equation of the mirror line.

d) Which trapezium is T mapped onto by a stretch with the
   (i) x-axis invariant
   (ii) y-axis invariant
   (iii) Write down the stretch factor of the stretches, respectively.

7. a) On graph paper, using a scale 1cm to represent 1 unit on both axes for $-9 \leqslant x \leqslant 9$ and $-10 \leqslant y \leqslant 10$:
   (i) plot the point (2,1). Label this as point P.
   (ii) draw the displacement vectors:

$$\overline{PQ} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \text{ and } \overline{QR} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Note: *Indicate the direction of vectors by using arrows. Label the points Q and R. Join R to the point P with a straight line and label the triangle as shape A.*

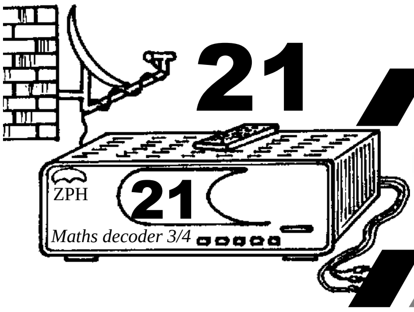
   (iii) Give a 2×3 matrix representation for shape A.

b) If $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

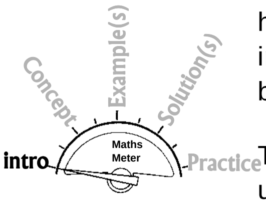
and $B = T \times A + X$,
   (i) calculate matrix B
   (ii) plot, draw and label shape B on the same sheet of graph paper as shape A.

c) If $T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $C = T \times B$
   (i) calculate matrix C
   (ii) plot, draw and label shape C on the same sheet of graph paper as shape A.

d) Describe fully, the single transformation that:
   (i) moves shape A to shape B.
   (ii) moves shape B to shape C.
   (iii) moves shape A to shape C.

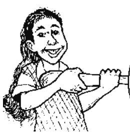


21 Gradient and Travel Graphs



The study of the motion of objects in Mathematics and Science has always been interesting. There are many types of motions in everyday life. But most of them are beyond the scope of this book.

The Mathematics syllabus at 'O' Level requires us to study uniformly accelerated motion where equations of motion, can be applied.



Syllabus Expectations

By the end of this chapter, students should be able to:

1. identify lines with positive or negative gradient.
2. calculate gradient by measurement or use of given points.
3. interpret and obtain the equation of a straight line in the form $y = mx + c$.
4. identify parallel lines using the concept of gradient.
5. estimate the gradients of curves at a given point.
6. draw and interpret displacement against time graphs.
7. draw and interpret velocity and acceleration as well as area under a v–t graph.
8. use cartesian coordinates to interpret and infer from graphs.
9. draw graphs from given data.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- identify and/or draw a right-angled triangle where needed.
- identify and/or read coordinates of given points and plot points on a Cartesian plane.
- draw, linear, quadratic, cubic and inverse function graphs.
- find gradient of a curve or line.
- manipulate problems involving direct and inverse variation.
- draw a linear graph, given the necessary data.
- estimate enclosed area using graph paper.

A. CONCEPT OF GRADIENT

Kinematics is primarily dependent on a concept called **gradient**. It is therefore, important that this concept is dealt with before the actual work on kinematics is covered.

Consider the six lines below.

a) b) c) d) e) f)

Line (a) is **horizontal** i.e. it has no slope, meaning it has a gradient equal to zero.

Lines (b) and (c) are **positively inclined** to the horizontal meaning they have a positive gradient.

Lines (d) and (e) are **negatively inclined** to the horizontal meaning they have a negative gradient.

Line (f) is **vertical** i.e. too steep to be a slope. Such lines are said to have an undefined gradient.

Gradient is a numerical measure of the slope of a given straight line.
If lines (b) and (c) are considered more closely, it is seen that their slope is different.

This part of the chapter discusses methods of calculating the gradient of given lines.

B. GRADIENT BY MEASUREMENT

Consider the diagram below.

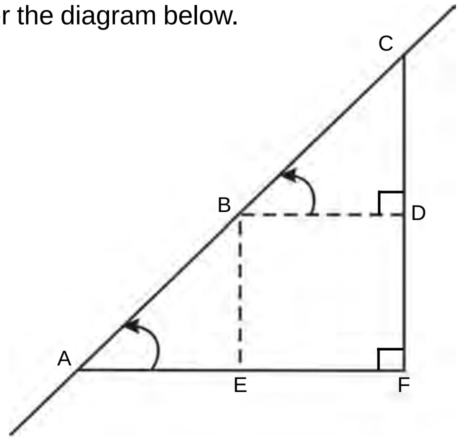


Fig 21.1

The line ABC is uniformly inclined to the horizontal AEF or BD, so ABC is a straight line.

The gradient of ABC can be measured using either of triangles ABE, BCD or ACF. These three triangles are actually, similar.

So the ratio $\frac{BE}{EA} = \frac{CD}{DB} = \frac{CF}{FA}$ gives the gradient of the line ABC.

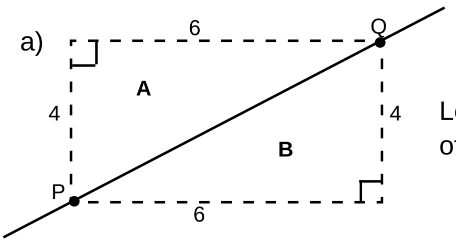
By definition

Gradient = $\frac{\text{change in } y}{\text{change in } x}$ or $\frac{y\text{-movement}}{x\text{-movement}}$

Note that this change or movement, can be positive or negative resulting in a positive or negative gradient.

The following examples will illustrate this better.

1. a)



Let us find the gradient of line PQ.

Fig 21.2

Solution

1. The slope should be measured from a point where you can read off the y-axis direction first (vertical).
Thus using triangle A, measure from P.

Hence, gradient PQ = $\frac{+4}{+6}$

$= \frac{2}{3}$

using triangle B, measurement is from Q.

Thus, gradient PQ $= \frac{-4}{-6}$

$= \frac{2}{3}$

2.

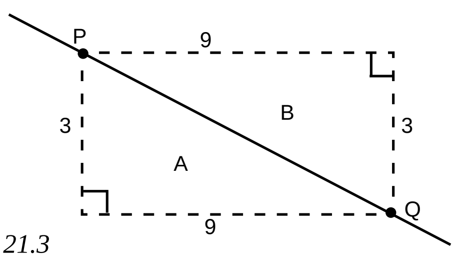


Fig 21.3

Hint

Any graph/line slanting towards the left or having its highest point on the left has a negative gradient.

Solution

2. From triangle A, (start from P).

Gradient PQ $= \frac{-3}{+9}$

$= \frac{-1}{3}$

From triangle B (start from Q)

$= \frac{+3}{-9}$

$= \frac{-1}{3}$

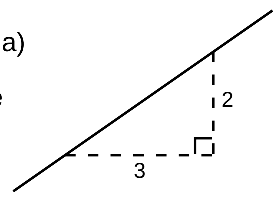
Note that you will be given one right-angled triangle on the line.

- measurement should always start from a point on the line which enables measurement in the y-direction vertically to form the numerator.
- Taking the right angle turn is then in the x-direction horizontal to the other point on the line, to form the denominator.

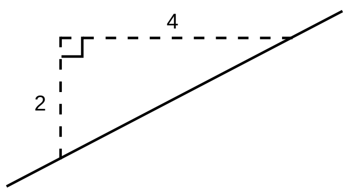
PRACTICE 21A

1. Use the dimensions on the triangles to calculate the gradient of each line.

a)



b)



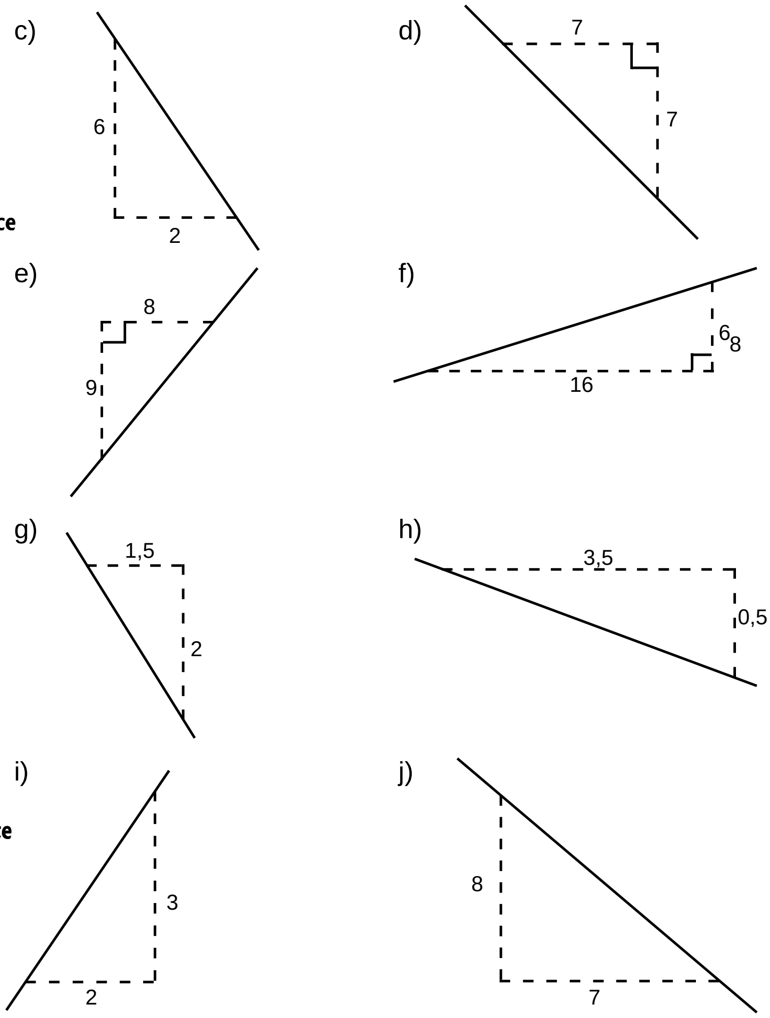


Fig 21.4

2. Copy the graph below and on each line, draw a convenient right angled triangle, as above, and find the gradient of the line.

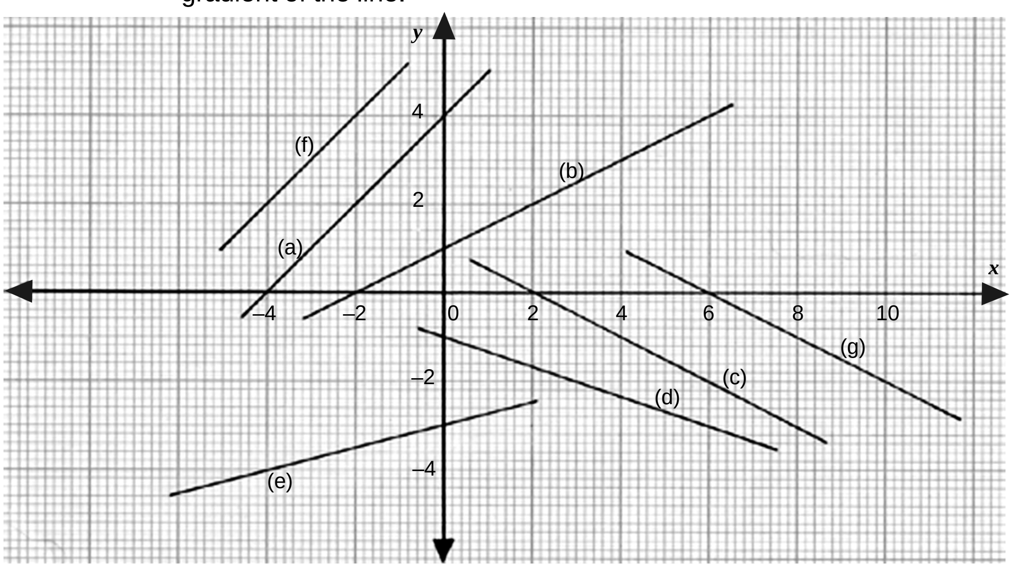


Fig 21.5

Note: *Do not forget to use the scale of the graph when measuring the sides of the triangle.*

Note from Fig 21.5 is the relationship between lines (a) and (f) and lines (c) and (g).

Did you note that gradient of (a) = gradient of (f) and gradient of (c) = gradient of (g)?

This includes that these pairs of lines are parallel to each other.

Parallel lines have the same gradient.

C. FINDING GRADIENTS FROM COORDINATES.

Refer to Fig 21.5.

Line (b) passes through points (–2;0) and (4;3). (You may choose any two points on the line). Using these two points we find the differences between the x points and the y points respectively.

change in $y = 3 - 0$ and change in $x = 4-(-2)$

$= 3$ $\qquad= 6$

Hint

Remember coordinates are written with the x position first.

$\therefore$ Gradient of line (b) $= \dfrac{3}{6}$

$= \dfrac{1}{2}$

The gradient of line (c) using points (4; –1) and (6; –2)

is $\dfrac{-1-(-2)}{4-6} = \dfrac{1}{-2}$

$= \dfrac{-1}{2}$

The above example shows that given any two points on a line, it is possible to calculate the gradient of the line without necessarily drawing the line itself. In general, given $(x_1; y_1)$ and $(x_2; y_2)$

$$\text{Gradient} = \frac{y_1 - y_2}{x_1 - x_2}$$

Consider the following example :

1. Find the gradient of the line passing through the following points:

a) (2;1) and (7; 3) b) (6;0) and (–2;4)

c) $(\frac{1}{2}; \frac{2}{3})$ and $(2\frac{1}{2}; 1)$

Hint

Swop the order of the points if you see it is easier to find the difference that way.

Solution

1. a) Using (7; 3) as $(x_1; y_1)$:

Gradient $= \dfrac{3-1}{7-2}$

$= \dfrac{2}{5}$

b) Using (6;0) as $(x_1;y_1)$:

$$\text{Gradient} = \frac{0-4}{6-(-2)}$$

$$= \frac{-4}{8}$$

$$= \frac{-1}{2}$$

c) Using $(2\frac{1}{2}\,;\,1)$ as $(x_1;y_1)$:

$$\text{Gradient} = \frac{1-\frac{2}{3}}{2\frac{1}{2}-\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{2}$$

$$= \frac{1}{6}$$

This way of finding the gradient from two given points can be applied to find an unknown.

2. The points (2; –2), (8; 2) and (14; k) lie on the same straight line. Find k.

Solution

2. Using (2; –2) and (8; 2) = Using (8, 2) and 14, k)

$$\frac{2-(-2)}{8-2} = \frac{k-2}{14-8}$$

$$\frac{4}{6} = \frac{k-2}{6}$$

$$\therefore\quad k-2 = 4$$

$$k = 6$$

1. Find the gradient of each line passing through the given points.

a) (11; 7), (6; 3)

b) (–1; –4), (4; 7)

c) (0; 0), (–3; 5)

d) (4; 2), (–1; –6)

e) (0; 5), (–5; 0)

f) (–3; 3), (10; 8)

g) (0,3; 2,7), (2,3; 6,7) h) (–3,6; –1,5), (–0, 3; –4, 7)

i) $\left(\frac{2}{3}; \frac{1}{4}\right)$, $\left(\frac{-1}{3}; \frac{2}{3}\right)$ j) $\left(2\frac{2}{3}; -3\frac{1}{3}\right)$, $\left(3\frac{7}{10}; -2\frac{3}{5}\right)$

2. The gradient of the straight line joining the points (5; 7) and (3; y) is 6. Find y.

3. The straight line passing through the points (x; –2) and (–4; 8) has a gradient of $2\frac{1}{2}$. Find the value of x.

4. The points (3; p), (2; 4) and (1; 1) lie on the same straight line. Find p.

5. If points (3; –5), (4: y) and (–1; –3) form a straight line, find the value of y.

6. The points (0; 4), (2; p) and (2p; 9) form a straight line. Find the values of p.

D. THE EQUATION OF A STRAIGHT LINE

Consider the following activity:

1. a) Use squared paper or graph paper to draw the graph of: (i) $y = 4x + 2$. (ii) $2x + 3y - 6 = 0$.

b) By taking measurements or using points on the graph, find the gradient of each line.

Solution

1.

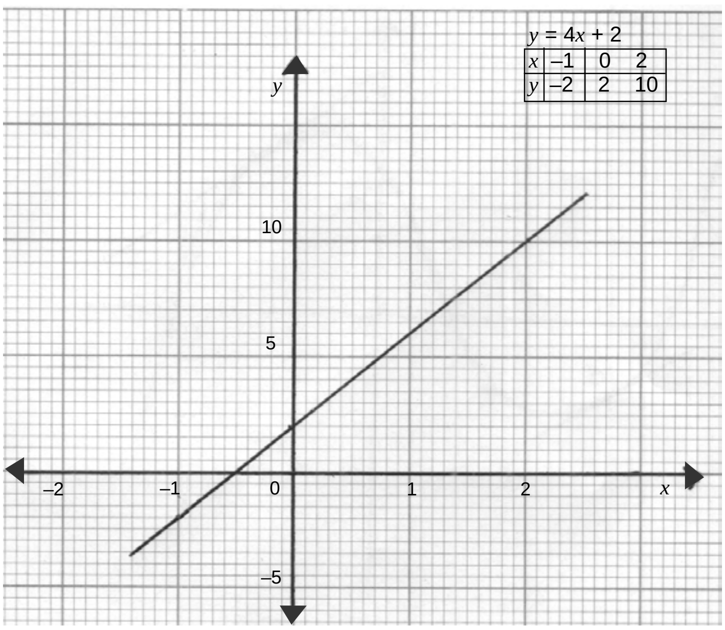


Fig. 21.6

b)(i) Using points (–1; –2) and (2; 10)

$$\text{Gradient} = \frac{10 - (-2)}{2 - (-1)}$$

$$= \frac{12}{3}$$

$$= 4$$

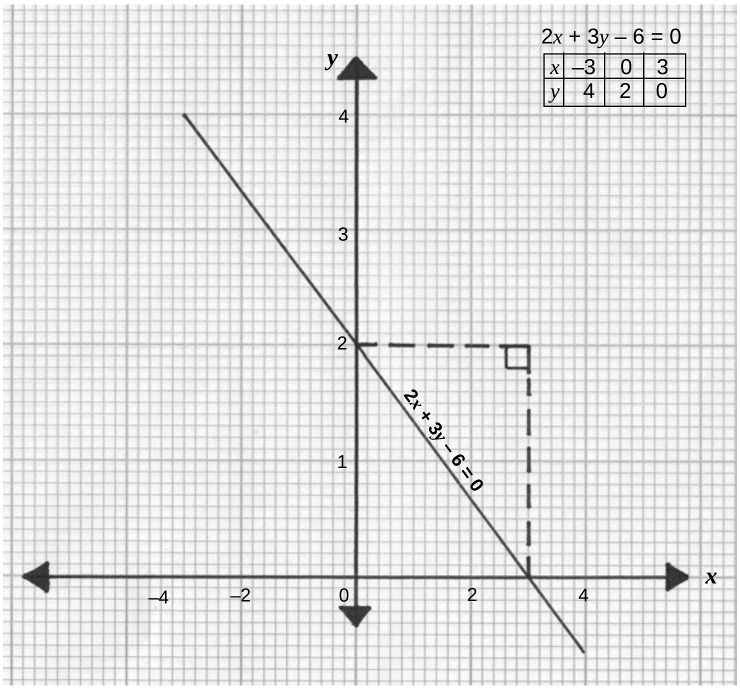


Fig 21.7

b)(ii) By taking measurements

Gradient $= \frac{+2}{-3}$

$= \frac{-2}{3}$

The two previous activities help us to notice relationships in the equation of the straight line:

From $y = 4x + 2$

The gradient $= 4$ i.e. co-efficient of x, when y is the subject.
y-intercept = 2 i.e. the constant in the equation when y is the subject.
Note that, getting the correct answer depends on y being made the subject of the equation first.

Remember the y-intercept is where the graph cuts the y-axis.

a(ii) From $2x + 3y - 6 = 0$

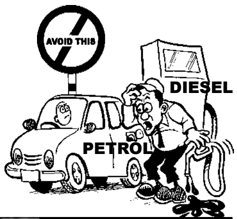
Make y the subject first.

$3y = 6 - 2x$

$y = 2 - \frac{2}{3}x$

Gradient $= \frac{-2}{3}$

y-intercept = 2



Common Error

Picking the co-efficient of x as the gradient before making y the subject e.g in $2x + 3y - 6 = 0$ Gradient = 3 is wrong.

The equation of a straight line can be expressed as:

y = mx + c

where m is the gradient of the line and c the y-intercept.

PRACTICE 21C

Find, by inspection, *(do not draw the graph):*

a) the gradient.

b) the y-intercept of the following equations.

1.	$y = 4x - 2$	**2.**	$y = 2 - 4x$
3.	$y = -x + 3$	**4.**	$x + y = 0$
5.	$4x + 2y = 4$	**6.**	$3x = y - 6$
7.	$2y = 3x - 4$	**8.**	$4x - 3y - 5 = 0$
9.	$5x + 3y + 10 = 0$	**10.**	$8x + 5y - 4 = 0$

E. FORMING THE EQUATION OF A STRAIGHT LINE

Given a point on the line and its gradient.

Consider the example below:

1. A straight line, of gradient 7, passes through point (–3; 6). Find the equation of the line.

Solution

1. **Method 1** (Using $y = mx + c$)

$m = 7$

substitute $m = 7$, $x = -3$ and $y = 6$ from (–3; 6) into

$y = mx + c$ (to find c).

$6 = 7(-3) + c$

$c = 27$

$\therefore$ The equation is $y = 7x + 27$

Method 2 (Using gradients)

Introduce a second unknown point $(x;y)$.

Find gradient between (–3; 6) and $(x; y)$

i.e. $\dfrac{y-6}{x-(-3)} = \dfrac{y-6}{x+3}$

This gradient must equal 7 as it is for the same line.

$$\therefore \quad \frac{y-6}{x+3} = 7$$

$$y - 6 = 7x + 21$$

$$y = 7x + 27$$

F. GIVEN TWO POINTS ON THE LINE

Consider the example below:

1. A straight line passes through points (1; 4) and (4; 7). Find its equation.

Solution

Method 1 (Using $y = mx + c$)

Find m from the two points.

$$m = \frac{7-4}{4-1} = \frac{3}{3}$$

$$= 1$$

Find c by substituting this value of m and any of the two points. e.g. (1; 4)

$4 = 1(1) + c$

$c = 3$

The equation is $y = x + 3$

Method 2 (Using gradients)

Find the gradient using the two points.

$$\text{Gradient} = \frac{7-4}{4-1} = \frac{3}{3}$$

$$= 1$$

Introduce a third point $(x;y)$ and use it with one of the two points, e.g. (1; 4) to find the gradient.

$$\text{Gradient} = \frac{y-4}{x-1}$$

Equate the two gradients and simplify.

$$\frac{y-4}{x-1} = 1$$

$$y - 4 = x - 1$$

$$y = x + 3$$

1. Find the equation of the straight line passing through the given point and having the given gradient. Give your answer in the form (i) $y = mx + c$.

(ii) $ax + by = c$, where a, b and c are integers.

Table 21.1

	Point	Gradient
a)	(1; 3)	5
b)	(1; 3)	$-\frac{2}{7}$
c)	(–2; –6)	3
d)	(0; –4)	–2,3
e)	(–4; 1)	–7

2. Find the equation of the straight line passing through the given points. Give your answer in the form:
(i) $y = mx + c$,
(ii) $ax - by + c = 0$, where a, b and c are integers.

a) (4; 2), (1; 6)
b) (7; –1), (3; 9)
c) (0, –1), (7; –2)
d) (0; 0), (–3; 5)
e) (–9; –13), (–2; 1)

3. Line p passes through the point (8; –3).
Line q passes through the point (–5; 12).
Find the equations of the two straight lines if they both pass through (1; 4).

4. Show the equation of a straight line, which passes through the points (0; $2a$), and (2; a), is $2y + 4x = 4a$.

5. a) Write down an expression for the gradient of a straight line joining the points (5; 2) and (3; k).

b) Find the value of k if the gradient is –3.

c) Another line is parallel to the line in (a) and passes through $(-5; -\frac{1}{2})$, find the equation of this line.

6. Two points $(-\frac{1}{2}; \frac{1}{2})$ and $(\frac{3}{4}; \frac{1}{4})$ are on the same straight line. Find the equation of the line, giving your answer in the form $ax + by = c$, where a, b and c are integers.

7. Two straight lines p and r are parallel. If p passes through (0; 5) and r has a gradient of $-\frac{2}{9}$ and passes through (5; 0), find the equations of the two lines, in their simplest form.

G. GRADIENT OF A CURVE

The gradient of a curve changes from one point to another. This means one can only find the gradient of a curve at a particular point. To get this, a tangent to the curve is drawn at the point and the gradient of the tangent is what is found.

In Fig 21.8 tangents have been drawn at points A, B, and C and D of the curve.

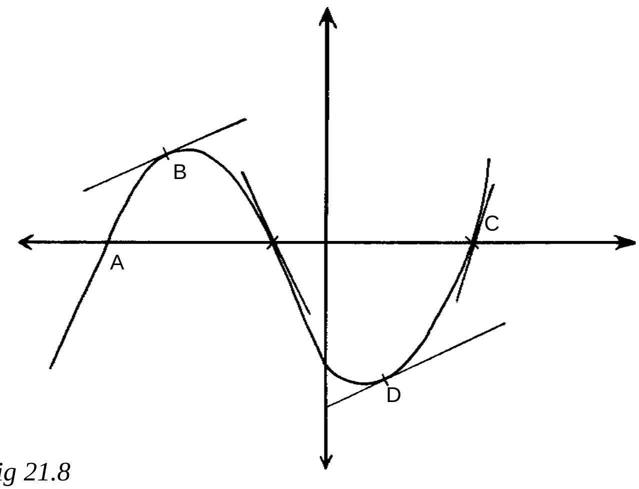


Fig 21.8

Consider the following example:

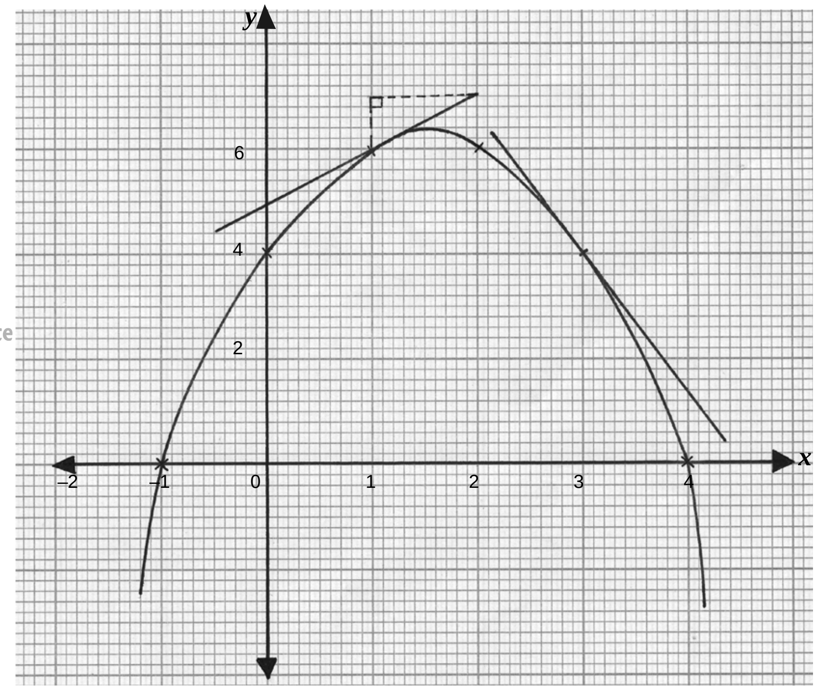


Fig. 21.9

1. Fig 21.9 is the graph of the quadratic equation $y = 4 + 3x - x^2$. Use the graph to find the gradient of the curve at

a) $x = 1$

b) $x = 3$

Solution

1. a) By measurement: when $x = 1$

- Draw a tangent, at the point on the graph where $x = 1$, (i.e. A).
- Draw a right angled triangle on the tangent, as illustrated. (Any triangle will do as long as it is right-angled).
- Use the y and x sides of the triangle to find the gradient.

i.e. Gradient $= \dfrac{+1}{+1}$

$= 1$

b) By picking points on the tangent (3; 4) and (4; 1,4). When $x = 3$

Gradient $= \dfrac{4 - 1{,}4}{3 - 4}$

$= \dfrac{2{,}6}{-1}$

$= -2{,}6$

Either approach can be used to find the gradient of a curve at a given point.

1. Fig 21.10 shows a cubic curve.

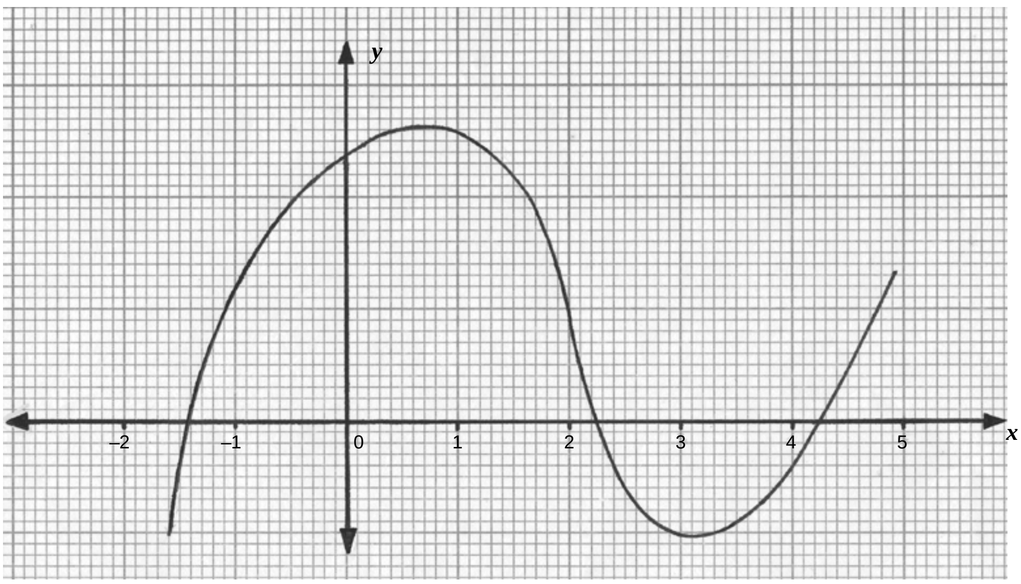


Fig 21.10

Use it to find the gradient of this curve at:

a) $x = -1$.

b) $x = 1$.

c) $x = 3$.

2. a) Use appropriate scales to draw the graph of $y = 6x - x^2$ for values of x from –2 to 6.

b) Find the approximate gradients of the curve at the points $x = -1$, 0, 2 and 5.

3. a) Draw the graph of $y = \frac{1}{x^2}$, for values of x from $\frac{1}{4}$ to 2, using a scale of 2cm to 5 units, vertically.

b) Use the graph to estimate the gradient of the curve at $x = \frac{1}{2}$.

4. a) Copy and complete the table of values for the equation $y = 2x^2 - 5x - 3$.

x	–2	–1,5	–1	–0,5	0	0,5	1	1,5	2	2,5	3	3,5	4
y	15	9			3	5				3			9

b) Draw the graph of $y = 2x^2 - 5x - 3$, using suitable scales.

c) Use the graph to find the approximate gradient of the curve at (i) $x = -0,5$.
(ii) $x = 1,3$.

H. THE MEANING OF GRADIENT

In the chapter on variation it was shown that, when two or more quantities have a relationship there must be a constant value included in the governing law. The constant may be a fractional or integral number or, even another quantity.

Consider the following derivatives

(i) For a triangle its area (A) and its height (h) are related as follows, if its base is kept constant.

$A \propto h$

$\Rightarrow A = \underbrace{k}_{\frac{1}{2}b}\, h$

$\Rightarrow A = \frac{1}{2}bh$

hence $\boxed{\mathbf{A = \frac{1}{2}bh}}$

(ii) For a rectangle, if you keep, the width constant then, its area (A) is directly proportional to its length (l).

$$A \propto l$$

$$A = k\, l$$

hence $\boxed{A = W \times L}$

Tip

In this section of the chapter we are concerned with the physical meaning of these constants and also study how graphs may be used to deduce these constants using the gradient. In particular we are going to look into.
(i) The physical meaning of the gradient.
(ii) The physical meaning of a graph in respect to its shape.
(iii) The physical meaning of the area under a given graph.

(iii) For a circle, the circumference (C) of a circle is directly proportional to its diameter (D).

$$C \propto D$$

$$C = \underset{\pi}{\underset{\downarrow}{k}}\, D$$

$$C = \pi D$$

$$\mathbf{C = \frac{22}{7} D}$$

Consider the following example:

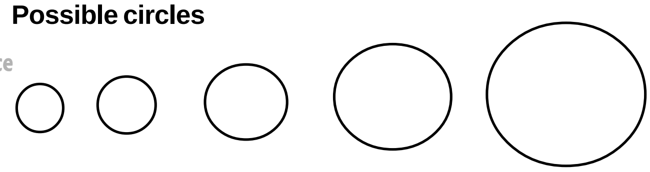
1. Draw five circles of different radii and determine the gradient of the graph of circumference against diameter.

Solution

Step 1

1. Draw the circles using a pair of compasses.

Possible circles



Step 2

Using a piece of string and a ruler, measure the circumference and diameter of each circle and record them in a table of values.

Possible values

Table 21.2

Circumference	11	13	22	28	37
Diameter/cm	3,5	4	7	9	12

Step 3

Plot the graph of circumference (y-axis) against diameter (x-axis).

Step 4

Find the numerical value of the gradient of your graph.

Gradient will always give you a

Value $= \pi = \frac{22}{7} = 3{,}142$

Possible graph

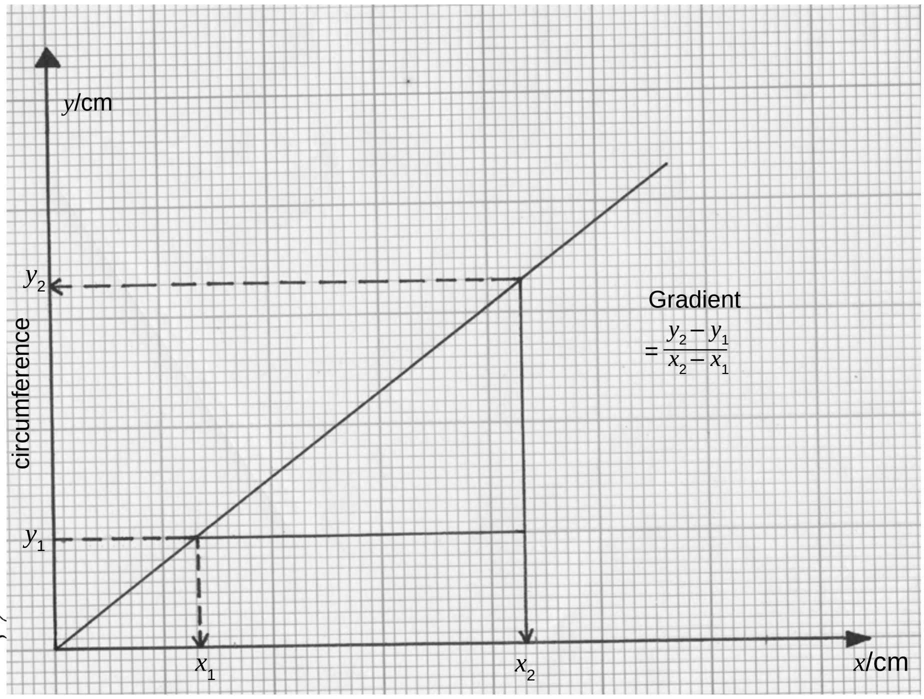


Fig 21.11

Tip

The gradient of circumference in relation to the diameter of a circle gives us a constant called pi, $\pi = \frac{22}{7}$

In practical situations (everyday life) by plotting values of defined (real) quantities, the gradient will give us a value of a real (defined) quantity.

The physical quantities we are going to deal with are:

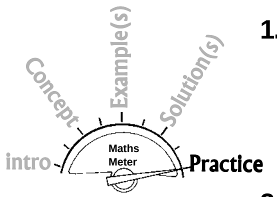
Distance – any length covered by a moving object.

Displacement – Distance or length covered by a moving object in a specified direction, It is the shortest distance between two points. It is a vector quantity.

Velocity – Displacement per unit time. It is also a vector which takes the direction of displacement.

Acceleration – The rate of change of velocity or velocity per unit time. (Also a vector).

Time – Always an independent variable (x-axis) and normally measured in seconds.



1. For the circles you drew in example 1, measure the radii and calculate the areas using, $A = \pi r^2$. Construct a table of values of area and the radius squared. Plot the graph of A against r^2 and so find the gradient.

2. The values of mass of water and its corresponding volume are given in table 21.3.

Hint

The gradient resulting from a relationship of a mass – volume gives us density.

Table 21.3

Mass/g	4	5	9	11	14
Volume/cm^3	4	5	9	11	14

Plot the graph and find its gradient.

3. Plot a graph of displacement in relation to time,1 using the values in table 21.4. Find the gradient.

Hint

The gradient resulting from displacement – Time graph gives us velocity.

Table 21.4

Displacement/m	4	6	8	10	12
Time/s	1	2	3	4	5

4. Plot a graph of velocity in relation to time, using the table 21.5. Find the gradient. What is the physical meaning of the gradient?

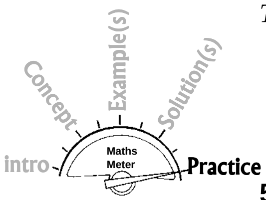


Table 21.5

Velocity	0	1	2	3	4	5
Time	0	2	4	6	8	10

5. Plot a graph of displacement in relation to time, using the values in table 21.6. Comment on the physical meaning of the gradient.

Table 21.6

Displacement	4	2	0	–2	–4
Time	1	2	3	4	5

Hint

All the characteristics of a gradient are strongly linked to the shape of the graph (curve). With practice all these characteristics maybe deduced by inspection.

Note that besides the possibility of having a physical meaning, the gradient of any graph will always have the following characteristics:

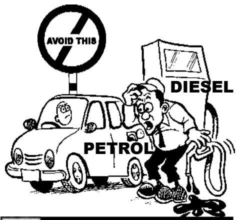
a) a sign (+ or –).
b) being constant in value or changing in value.
c) if changing, it is either increasing or decreasing.
d) if constant, it can also be zero in value.
e) it may be undefined in which case it is infinity.

I. THE BASIC EIGHT TYPES OF GRAPHS

Table 21.7 gives a summary of all the 8 kinds of graphs found in Mathematics. All other graphs are combinations of these eight basic graphs. Study them thoroughly as you will need to recognise them in future. (+ve) and (–ve) denote positive and negative respectively.

Hint

a) A displacement – Time (s – t) graph will give us the object's velocity.
b) The gradient of a velocity – time (v–t) graph will give us the object's acceleration.
c) A negative acceleration implies the object is slowing down (reducing velocity). Negative acceleration is also known as retardation or deceleration.
d) Remember, displacement, velocity and acceleration are vector quantities.

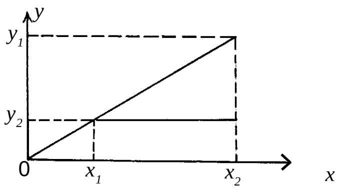


Common Error

Once you write retardation or deceleration, then there is no need to write the negative sign. Thus:
Retardation/deceleration = 3m/s^2 – **correct**
Retardation/deceleration = –3m/s^2 – **wrong**
Acceleration = –3m/s^2 – **correct**.

Table 21.7

1. Shape of Graph



Notes on Gradient

- Gradient is positive (+ve).
- Gradient is not changing, its constant.

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

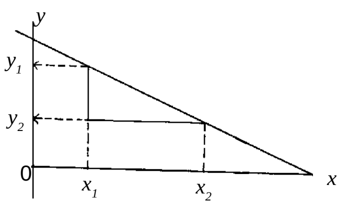
Physical meaning of motion if it is *s–t* graph

- Gradient gives us velocity.
- Object is moving with a constant velocity.
- Velocity is positive implying object is moving forward and not reversing i.e. moving in the opposite direction.

Physical meaning of motion if it is *v–t* graph

- Gradient gives us acceleration.
- Object is moving at a constant acceleration.
- The acceleration is positive. implying the object is gaining velocity (speed) not reducing it.

2. Shape of Graph



Notes on Gradient

- Gradient is negative (–ve)
- Gradient is not changing, it is deceleration/retardation.

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

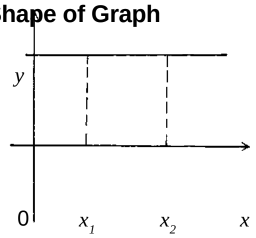
Physical meaning of motion if it is *s–t* graph

- Gradient gives us velocity.
- Object is moving at constant velocity.
- Velocity is negative implying the object is in reverse i.e. the vector direction is reversed.

Physical meaning of motion if it is *v–t* graph

- Gradient gives us negative acceleration i.e. constant.
- Acceleration is negative. implying that object is reducing (retarding) its velocity (speed).
- An example would be smooth braking of a vehicle or bicycle.

3. Shape of Graph



The gradient of horizontal lines is zero.

Physical meaning of motion if it is *s–t* graph

- Gradient = velocity.
- Velocity = zero, implies the object is not moving, it is stationary.

Intro Concept Example(s) Solution(s) Practice
Maths Meter

Table 21.7 continued

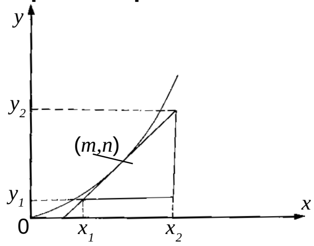
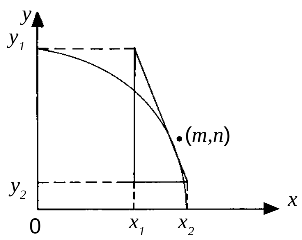
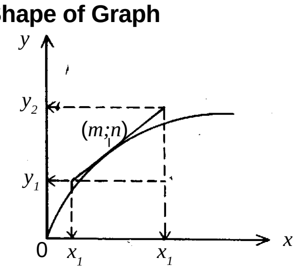
Notes on Gradient ⅄ Gradient is neither negative (–ve) nor positive (+ve) it is zero since $y_2 - y_1 = 0$. ⅄ Gradient is constant i.e equal to zero and not changing. Gradient = $\frac{0}{x_2 - x_1} = 0$	**Physical meaning of motion if it is *v–t* graph** Gradient = acceleration ⅄ Zero acceleration does not mean the object is stationary if it is a *v–t* graph. ⅄ It implies the object is moving at a constant velocity (speed), it is neither decreasing nor increasing its velocity.
4. Shape of Graph  	**Physical meaning of motion if it is *s–t* graph** ⅄ Gradient = velocity. ⅄ Object is moving gaining velocity, its not constant. ⅄ The velocity is positive (+ve) showing the object is moving forward.
Notes on Gradient ⅄ Gradient is positive (+ve). ⅄ Gradient is not constant, it is continuously changing i.e increasing with time. ⅄ Tangent at any point gives gradient at the point. At point $(m;n)$ Gradient = $\frac{y_2 - y_1}{x_2 - x_1}$	**Physical meaning of motion if it is *v–t* graph** ⅄ Gradient = acceleration. ⅄ Object is gaining acceleration continuously (it is not constant). ⅄ Acceleration is positive (+ve) showing object is gaining velocity.
5. Shape of Graph  	**Physical meaning of motion if it is *s–t* graph** ⅄ Gradient = velocity. ⅄ Object is moving with decreasing velocity, is not constant. ⅄ The velocity is negative (–ve) showing object is moving in opposite direction.
Notes on Gradient ⅄ Gradient is negative (–ve). ⅄ Gradient is not constant its continuously changing i.e increasing negatively e.g 0, –5, –8. i.e. it is actually decreasing ⅄ Tangent at any point gives gradient at the point. At point $(m;n)$ Gradient = $\frac{y_2 - y_1}{x_2 - x_1}$	**Physical meaning of motion if it is *v–t* graph** Gradient = acceleration ⅄ Object is retarding. ⅄ Acceleration is negative (–ve) implying the velocity is being reduced.
6. Shape of Graph  	**Physical meaning of motion if it is *s–t* graph** ⅄ Gradient = velocity. ⅄ Object is moving with decreasing velocity i.e. it is slowing down. ⅄ The velocity is negative (–ve) showing object is moving forward.

Table 21.7 *continued*

Notes on Gradient

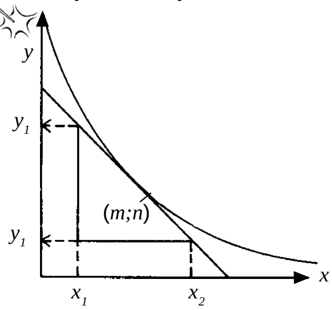
- Gradient is positive (+ve).
- Gradient is not constant but continuously changing i.e decreasing with time.
- tangent at any point gives gradient at that point.
  At $(m;n)$ Gradient = $\frac{y_2 - y_1}{x_2 - x_1}$

- Gradient = acceleration.
- Object is decreasing in acceleration.
- Acceleration is positive (+ve).
- An example would be when one is overtaking a vehicle (high acceleration then reduced thereafter).

Hint

Such a graph is called an exponential curve and never touches the x-axis.

7. Shape of Graph



Physical meaning of motion if it is *s–t* graph

- Gradient = velocity.
- The object is reversing since velocity is negative.
- The reverse velocity starts very high and then reduces with time.

Notes on Gradient

- Gradient is negative (–ve). It starts being hugely negative and then reduces.
- Gradient at point $(m;n)$ is

Gradient = $\frac{y_2 - y_1}{x_2 - x_1}$

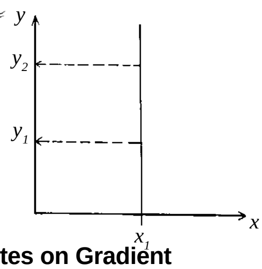
Physical meaning of motion if it is *v–t* graph

- Gradient = acceleration.
- Acceleration is negative implying there is deceleration or retardation.

Hint

If a number is divided by zero the answer is undefined.

8. Shape of Graph



Physical meaning of motion if it is *s–t* graph

- Velocity is undefined – it is an ideal situation and in everyday life you cannot have this kind of velocity not even a launched rocket could attain it!

Notes on Gradient

- The gradient of vertical lines is infinite i.e. undefined.

Gradient = $\frac{y_2 - y_1}{0}$

A number divided by zero is so huge that it cannot be written, hence, it is undefined.

e.g. $\frac{1}{0{,}000001} = 1000000$

$\frac{1}{0{,}00000000001} = 1\ 000\ 000\ 000$

J. COMBINING GRAPHS

An object may have a motion described by two or more of the above **eight** graphs combined.
At this stage, we are not concerned with the quantities on the y-axis or x-axis but with the gradient.

Consider the following examples:

1. a) Start with graph 1, add graph 3 to its end and then graph 2.

2. b) Describe the behaviour of the gradient at each stage.

Solution

1. a)

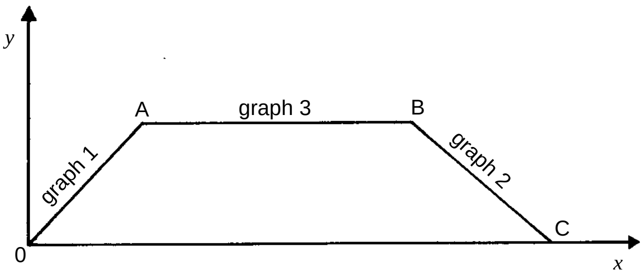


Fig. 21.12

b) For OA – Gradient is positive and constant.

For AB – Gradient = 0.

For BC – Gradient is negative and constant.

2. a) Starting with the graph at point (0,4) on the y-axis, combine the following graphs in their respective order: graph 6, then graph 3 and finally graph 5.

b) Describe the behaviour of the gradient in each section.

Solution

2. a)

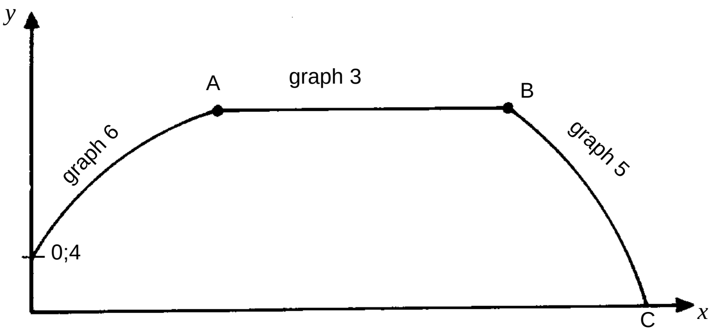


Fig. 21.13

2. b) Graph 6 – Gradient is increasing and is positive (+ve).
Graph 3 – Gradient is constant and = zero.
Graph 5 – Gradient is decreasing.

Combine the following graphs in their respective order and describe the gradient of each section.

a) $4 \longrightarrow 3 \longrightarrow 5$

b) $1 \longrightarrow 2$

c) $3 \longrightarrow 1 \longrightarrow 2$

d) $5 \longrightarrow 3 \longrightarrow 2$

e) $1 \longrightarrow 3 \longrightarrow 1$

f) $3 \longrightarrow 5$

g) $3 \longrightarrow 6 \longrightarrow 2$

K. THE AREA UNDER A VELOCITY-TIME (V – T) GRAPH

The area under a graph may or may not have physical meaning.

The area under a velocity-time graph gives us the distance or displacement covered by the object.

However the area under a displacement-time ***(s – t)*** graph or an acceleration-time ***(a – t)*** graph is meaningless.

Where area is of a particular interest, it may be found by using one of the formulae for areas of shapes formed by the graph, or by estimation, using graph paper.

Consider the following example:

1.

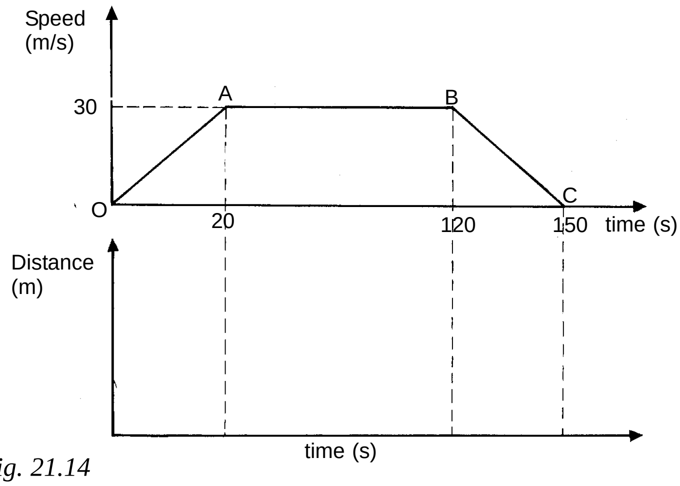


Fig. 21.14

Fig 21.14 shows the speed – time graph of a journey covered by a moving object.

a) Calculate
   (i) the retardation (deceleration) during the last 30 seconds.

   (ii) the total distance travelled.

b) On the axes provided in Fig 21.14, complete a sketch of the distance – time graph for the same journey.

Solution

1. a) (i) Find retardation by finding gradient of BC

$$\text{Retardation} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{V_2 - V_1}{T_2 - T_1}$$

$$= \frac{(0 - 30)\text{ m/s}}{(150 - 120)\text{s}}$$

$$= \frac{30}{30}$$

Retardation $= 1\text{m/s}^2$ or Acceleration $= -1\text{m/s}^2$

(ii) Total distance = area under graph

= area of trapezium OABC

$= \frac{1}{2}(100 + 150) \times 30$

Total distance = 3 750m

Hint

Note that the values on the y-axis i.e 300, 3300 and 3750 are cummulative areas under the speed-time graph which are equal to distance travelled.

b)

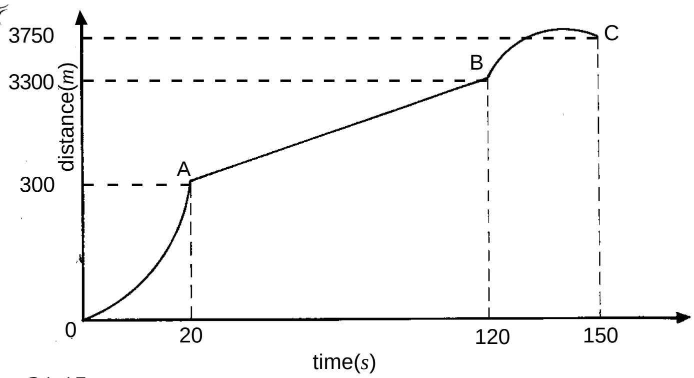
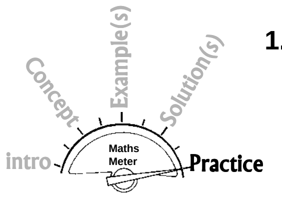


Fig. 21.15



1.

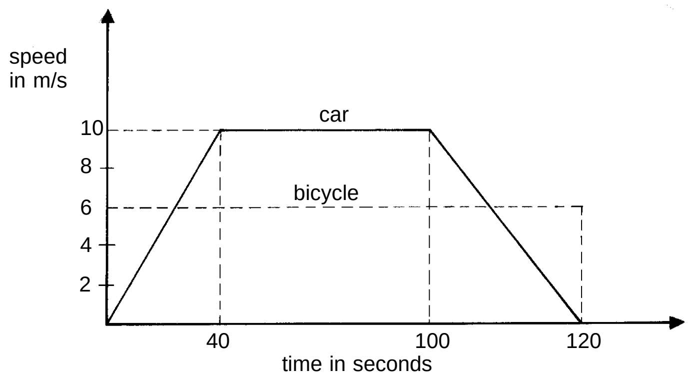


Fig 21.16

Fig 21.16 shows the speed-time graphs of a cyclist and a car during a period of 120 seconds.

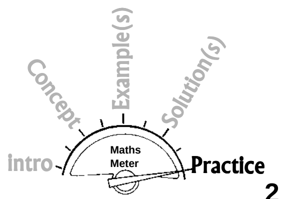
Calculate:

a) the distance travelled by the car in the first 100 seconds.

b) the distance travelled by the cyclist during the 120 seconds.

c) the acceleration of the car during the last 20 seconds.

d) the difference between the distances covered by the cyclist and the car, in the 120 seconds they travelled.



2. A car starts from a resting position and moves for 20 seconds, with a uniform acceleration until it attains a velocity of 12m/s. It then moves, at this constant velocity, for the next ten seconds, before starting to accelerate uniformly at twice the initial acceleration, for 10 seconds, then emergency brakes are applied which bring it to a stop, after 8 seconds.

a) Draw the velocity time graph of this motion.

b) Calculate the acceleration of the car between the 30th and 40th seconds of its journey.

c) Calculate the retardation (deceleration) of the car during the last part of its journey.

d) Use your graph to work out the total distance of the journey.

3. Fig 21.17 shows the distance – time graph of an object moving from point A to point E. Use the graph to answer the following questions.

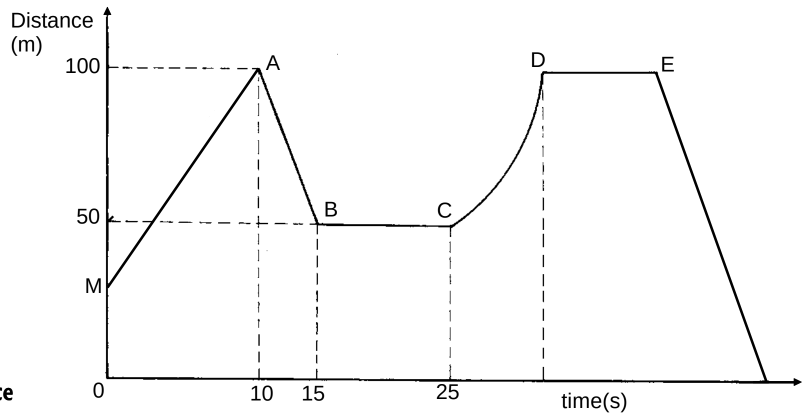


Fig. 21.17

a) Between which points is the object moving with zero acceleration?

b) Between which points is the object moving with constant positive velocity?

c) Between which points is the object at rest?

d) Between which points is the object showing continuous change in velocity?

4. Fig 21.18 represents a speed-time graph of a moving object during a period of 35 seconds.

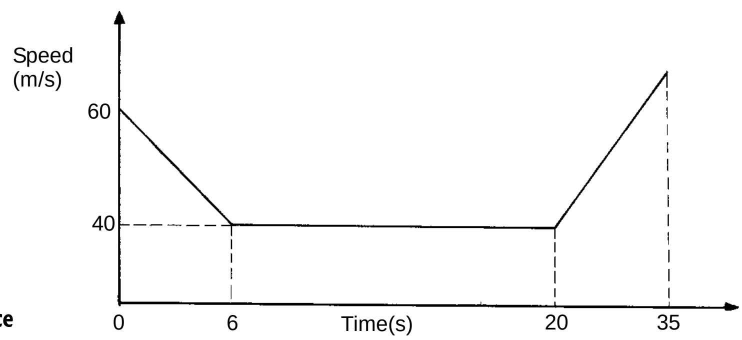


Fig. 21.18

a) Calculate the speed of the object when $t = 3$ seconds.

b) Calculate the total distance travelled during the first 20 seconds.

c) The acceleration between the 20th seconds and 35th second is $2m/s^2$. Calculate the speed of the object when $t = 35$ seconds.

5.

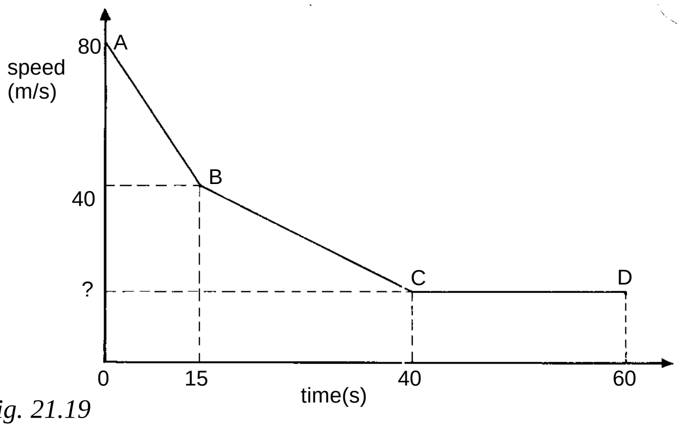


Fig. 21.19

Fig 21.19 shows the speed-time graph of a bus which uniformly decelerated from point A to point B and then from point B to point C before attaining a constant velocity of 20m/s.

Calculate:

a) the retardation during the first 15 seconds.

b) the acceleration between C and D.

c) the total distance travelled in the last 45 seconds of the journey.

6.

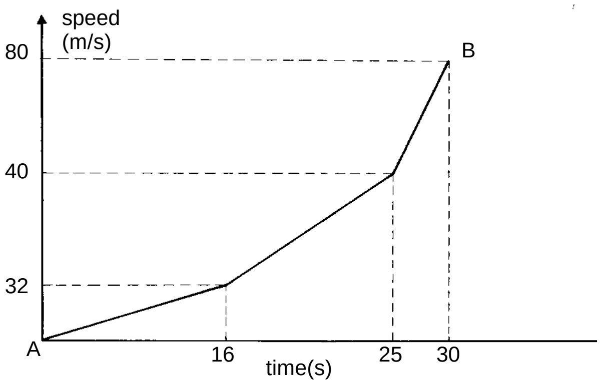


Fig. 21.20

Fig 21.20 is a speed-time graph of an object moving from point A to point B.

Work out:

a) the time it takes for the object to reach the speed of 20m/s.

b) the value of *time* (t) for which speed (v) = 35m/s.

c) the distance travelled in the first 18 seconds.

7. Sketch the speed-time graph from the distance-time graph given. The axes have already been drawn for you.

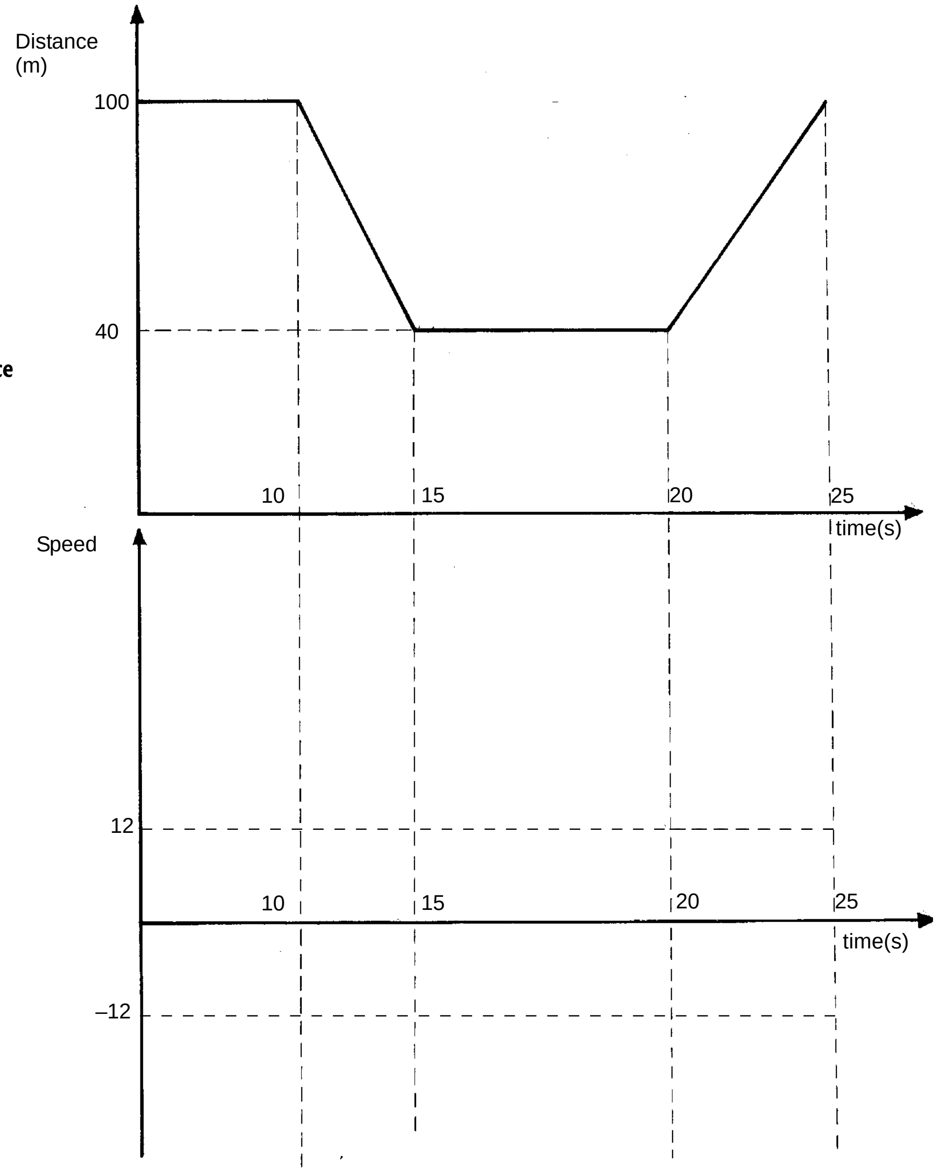


Fig. 21.21

8. Sketch the acceleration-time graph indicated below the given velocity-time graph.

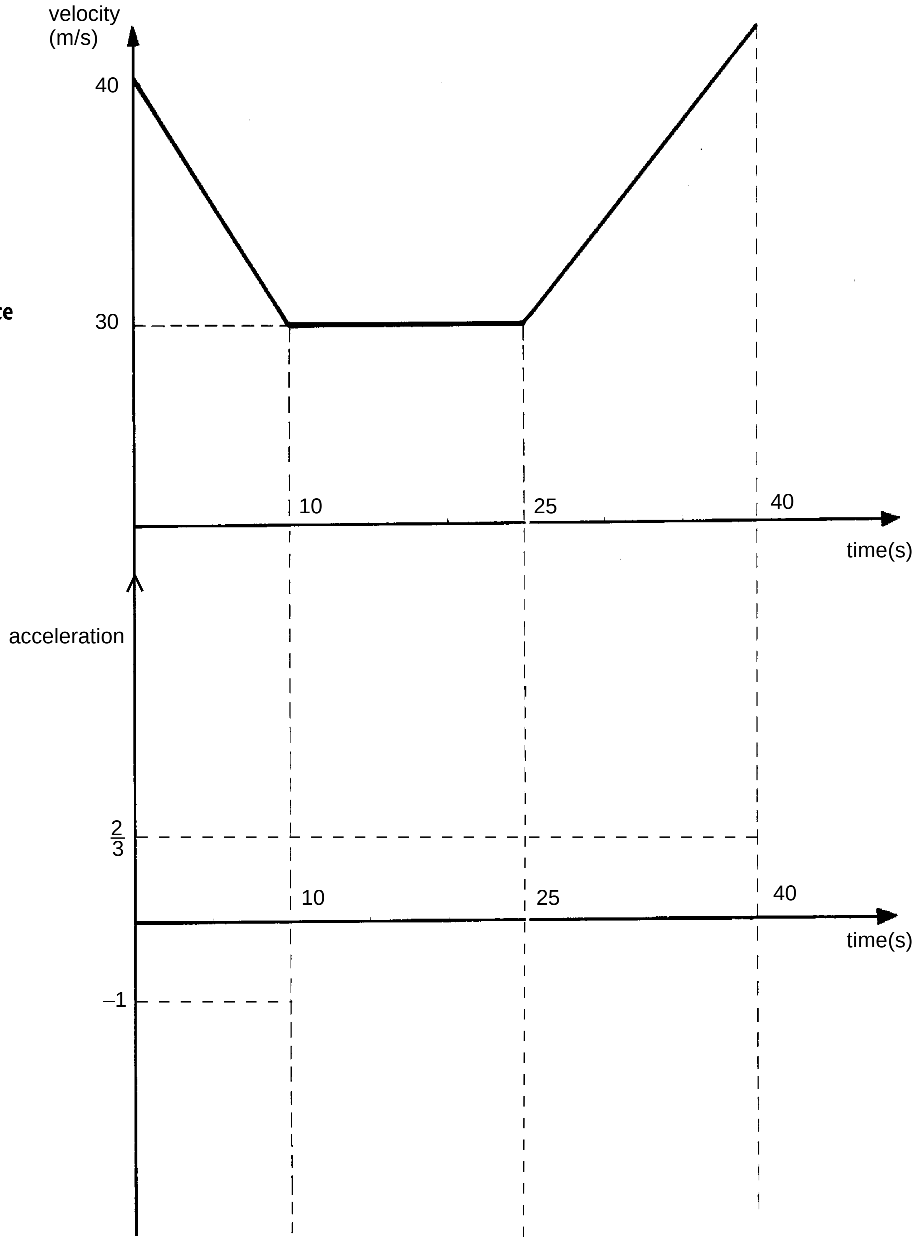
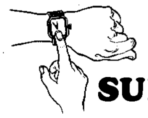


Fig. 21.22



SUMMARY

1. Gradient $= \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\text{increase in } y\text{-axis}}{\text{increase in } x\text{-axis}}$

2. Two points can be used to find gradient in the form
$\dfrac{y_1 - y_2}{x_1 - x_2}$ or $\dfrac{y_2 - y_1}{x_2 - x_1}$
where $(x_1;y_1)$ is the first point and $(x_2;y_2)$ is the second point.

3. Parallel lines have the same gradient.

4. Horizontal lines have a gradient of zero.

5. The equation of any straight line can be expressed in the form $y = mx + c,$ where m is the gradient and c the y-intercept.

6. The gradient of a curve is the gradient of the tangent, at the given point on the curve.

7. Kinematics is the study of objects moving in a straight line (rectilinear motion) with uniform acceleration.

8. The gradient of a distance against time graph gives us speed (velocity).

9. The gradient of a velocity/speed against time graph gives us acceleration.

10. The area under a velocity-time graph gives us distance.

11. Another name for deceleration is retardation.

12. Graphs slanting towards the right have a positive gradient while those slanting towards the left have a negative gradient.

EXAM PRACTICE 21

Hint

Never leave answer as $\frac{5}{-13}$

Consider the example below:

1. Two points, (–5; 2) and (8; –3), lie on a straight line q and point (–13; 10) lies on another straight line t.

a) Find the gradient of line q.

b) Given that line q is parallel to line t, find the equation of line t, in its simplest form.

Solution

1. a) Gradient of line $q = \dfrac{2-(-3)}{-5-8}$

$$= \frac{5}{-13}$$

$$= \frac{-5}{13}$$

b) Since line q and t are parallel, it means their gradients are the same.

$\therefore$ Gradient of line $t = \dfrac{-5}{13}$

in $y = mx + c$

$m = \dfrac{-5}{13}$ and point (–13; 10)

$10 = \dfrac{-5}{13}(-13) + c$

$c = 5$

$y = \dfrac{-5}{13}x + 5$ Multiply through by 13

$13y = -5x + 65$

Hint

Simplest form means free of fractions.

Let us look at another example:

2.

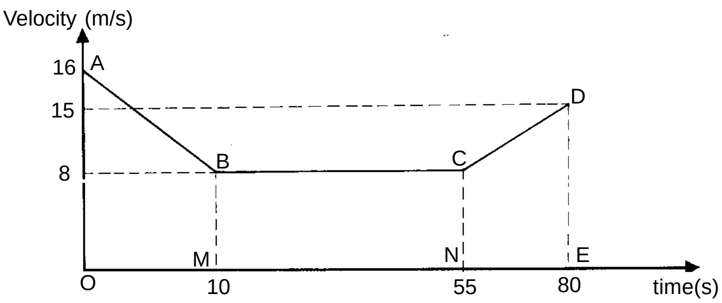


Fig. 21.23

The graph A–B–C–D describes the motion of a train.

a) Describe what is happening to the train's acceleration along:
(i) AB (ii) BC (iii) CD

b) Calculate the retardation of the train during the first 10 seconds.

c) Show that the total distance travelled by the train is 780m.

Solution

2. a) (i) AB $\rightarrow$ Object is moving with constant deceleration (since gradient is constant and negative).

(ii) BC $\rightarrow$ Object is moving with zero acceleration (since gradient = 0).

(iii) CD $\rightarrow$ Object is moving with constant acceleration (since gradient is constant and positive).

Hint

The negative (–ve) is not written once you write retardation.

b) Acceleration = Gradient of graph AB

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 16}{10 - 0} = \frac{-8}{10} = -0{,}8\text{m/s}^2$$

$\therefore$ Retardation = 0,8m/s^2

c) Total distance equals the area under graph ABCD.

Total area = Area of trapezium OABM + Area of rectangle MBCN + Area of trapezium NCDE

$$\text{Total area} = \tfrac{1}{2}(8 + 16) \times 10 + (45 \times 8) + \tfrac{1}{2}(8 + 16) \times 25$$

$$= \tfrac{1}{2} \times 24 \times 10 + 360 + \tfrac{1}{2} \times 24 \times 25$$

$$= 120 + 360 + 300$$

$$= 780\text{m}$$

Now do the following:

1. (–3; 0), (0; 3) and (x; 12) are three points on the same straight line. Find the value of x.

2. Fig 21.4 illustrates the speed-time graph of an object, which has travelled from one point to another. For both axes, the graph has been drawn using a scale of 2cm: 2 unit.

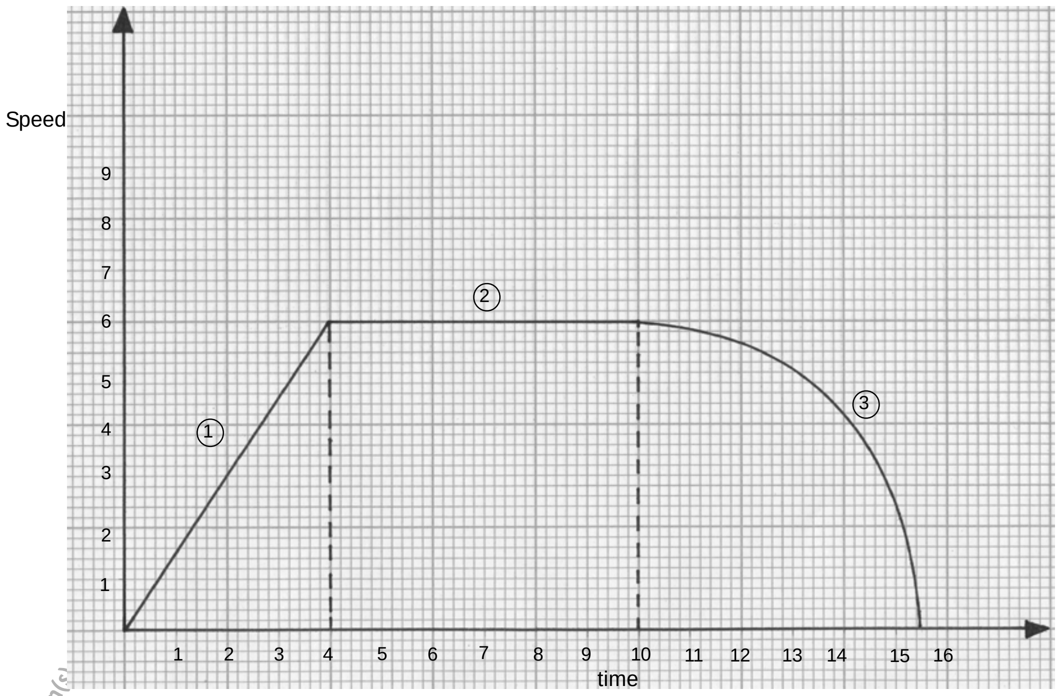


Fig. 21.24

a) Describe the gradient at stages 1, 2 and 3.

b) Estimate the distance travelled by the object for the complete journey.

3.

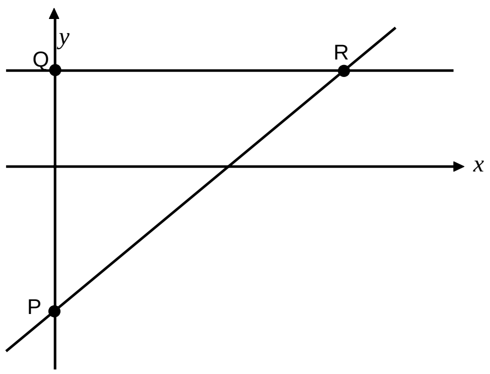


Fig. 21.25

The diagram shows points P(0; –6), q(0; 6) and R, the intersection of the sloping line though P and the horizontal line through Q.

a) Write down the equation of the line:
(i) PQ (ii) QR

b) Given that the gradient of the line PR is 3, find the equation of the line PR, in the form $ax + by = c$, where a, b and c are integers.

c) Calculate the coordinates of point R.

d) Find the area of triangle PRQ.

4. The diagram below (Fig 21.26) is the speed-time graph of a moving object during a period of 20 seconds.

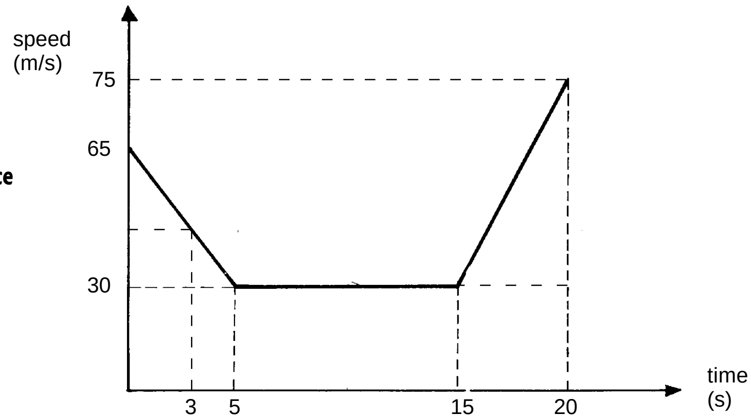


Fig. 21.6

a) Calculate the speed of the object when $t = 3$ seconds.

b) Calculate the distance travelled in the first 15 seconds.

c) Find the acceleration of the object during the last 5 seconds.

5. *Answer the whole of this question on a sheet of graph paper.* A solid object is thrown into the air. Its height (h) in metres after t seconds is given by the formula.

$h = 40t - 8t^2 + 80$.

Below is the table of values for
$h = 40t - 8t^2 + 80$.

Table 21.8

Time (t/s)	0	1	2	3	4	5	6	7	8
height (h/m)	80	112	128	m	112	80	n	–32	p

a) Find the value of m, n and p.

b) Draw the graph of

$h = 40t - 8t^2 + 80$ for $0 \leqslant t \leqslant 8$.

c) Use your graph to find:
(i) the maximum height reached by the solid object.
(ii) the velocity of the object when $t = 2$ seconds.
(iii) the *times* when the object is at a height of 100m.

6.

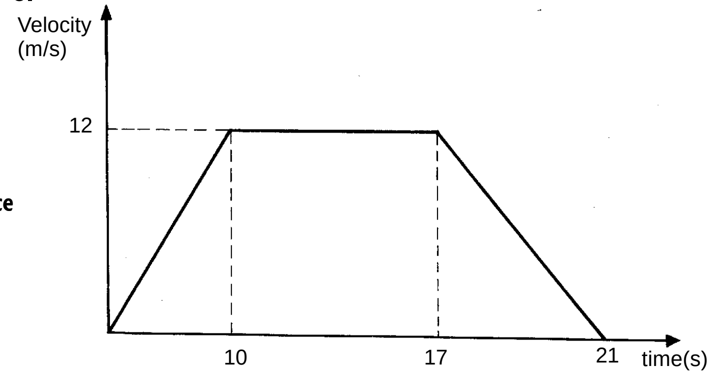


Fig. 21.27

Fig 21.27 shows the velocity–time graph for a bus which starts its journey from a resting position.

a) Find the acceleration of the bus during the first 10 seconds.

b) Find the distance travelled by the bus in the first 17 seconds of its motion.

c) On the diagram below, sketch an acceleration–time graph for the bus for the whole 21 seconds of its motion.

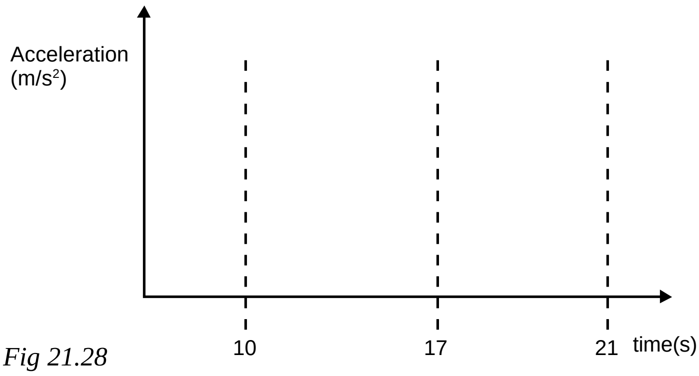


Fig 21.28

7. The travel graph illustrates the movements of two motorists A and B, who set out in opposite directions.

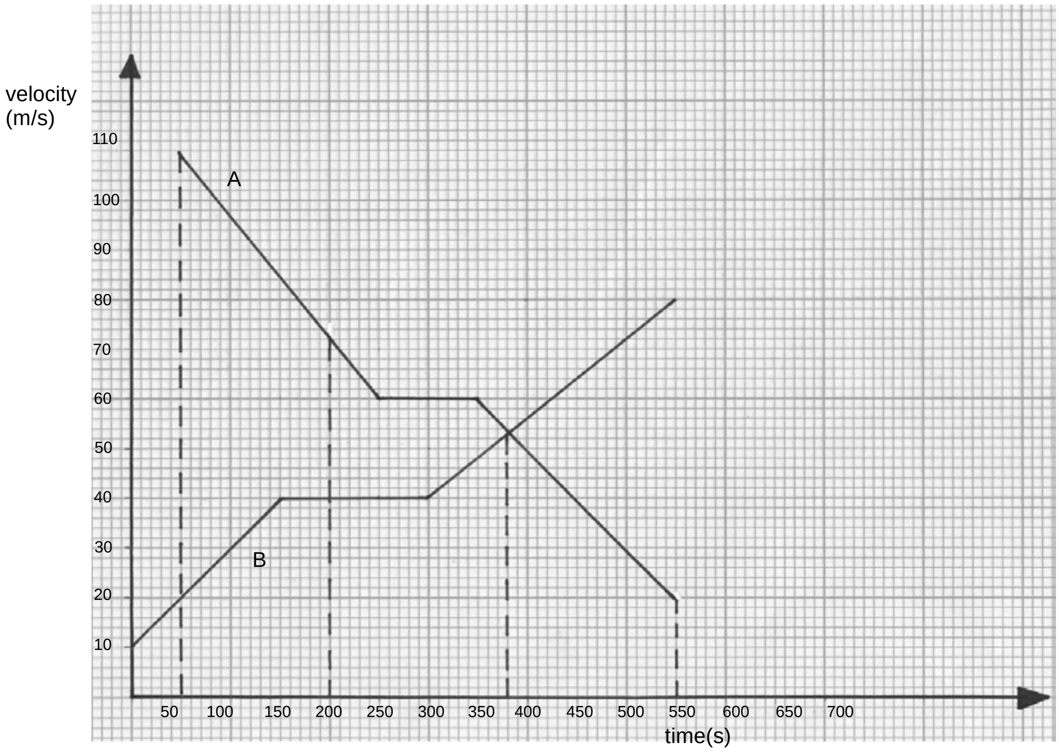
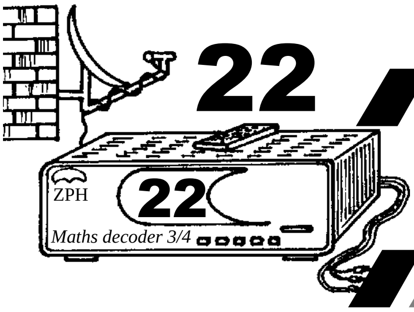


Fig. 21.29

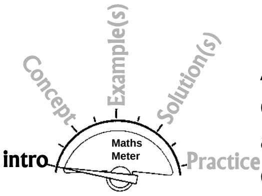
Find:

a) the approximate time for which their velocity is the same.

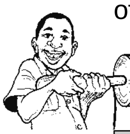
b) the distance they each travelled after they had met.



22 Construction and Loci



A mathematical set has the appropriate tools to be used in constructing angles, lines, and various plane shapes. Constructions are done with a pair of compasses, a pencil and ruler. Any straight edge may be used as a ruler. Sometimes the use of a protractor or other instruments is needed.



Syllabus Expectations

By the end of this chapter, students should be able to:

Using a ruler and compasses only

1. construct an angle bisector, perpendicular bisector, angles of 30^0, 45^0, 60^0 and 90^0.
2. construct a perpendicular, from a given point to a given line (altitude)
3. construct triangles, parallelograms and simple regular *n*-sided polygons (protractors may be used where necessary).
4. produce scale drawings using an appropriate/given scale.
5. construct and use the locus (in two-dimensions) of a point equidistant from
   - a fixed point
   - two given points
   - a given straight line
   - two intersecting straight lines.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that pupils are able to:

- use a pair of compasses, a protractor and a ruler to make constructions.
- define basic terms in circle geometry such as arc, chord and segment.
- apply the characteristics of plane shapes especially that of a triangle and a parallelogram.

The following basic constructions are critical and should be practised and understood thoroughly.

1. Construction of a line segment of given length.
2. Perpendicular bisector of a line.
3. Bisecting a given angle.
4. Construction of 60^0, 30^0, 90^0, 45^0, 135^0 and 120^0 angles using a ruler and a pair of compasses *only*.
5. Construction of a triangle.
6. Construction of a parallelogram.
7. Construction of a circumscribed circle for a triangle.
8. Construction of an inscribed circle for a triangle.
9. Dropping/ constructing a perpendicular from the vertex of a triangle.
10. Construction of parallel lines.

A. CONSTRUCTIONS

Concept 1

Construction of a straight line OB of a given length = 7,5. (Fig 22.1)

Hint

Draw a point (O) on a line, open compasses to given length, place point of compass on point and sweep through line for point B.

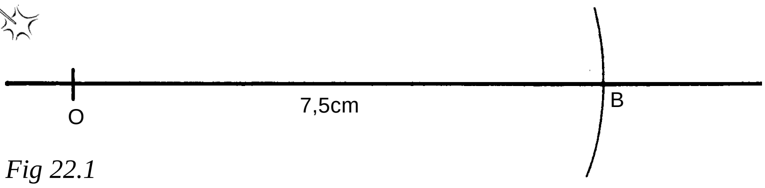


Fig 22.1

Concept 2

Perpendicular bisector of a line (Fig 22.2(a) and (b)).

Hint

In all constructions the arcs and lines should be clearly shown. A line to a point should be drawn to extend through the point.

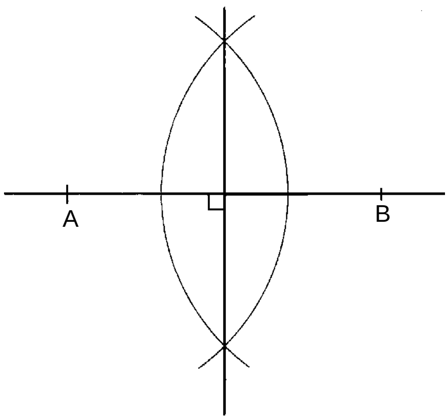
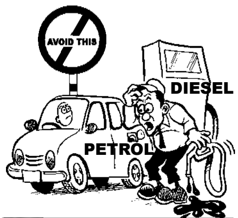


Fig 22.2(a)



Common Error

Arcs not intersecting but just touching.

Sometimes the arcs are just shortened as in Fig 22.2 (b).

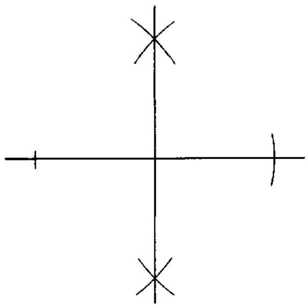


Fig 22.2(b)

Concept 3
Bisecting a given angle. (Fig 22.3)

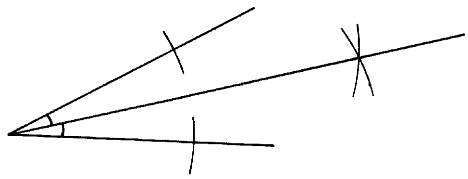


Fig 22.3

Concept 4
Construction 60^0, 30^0, 90^0, 45^0, $135^{0,}$ and 120^0 using a ruler and compasses only. Fig 22.4(a – f)

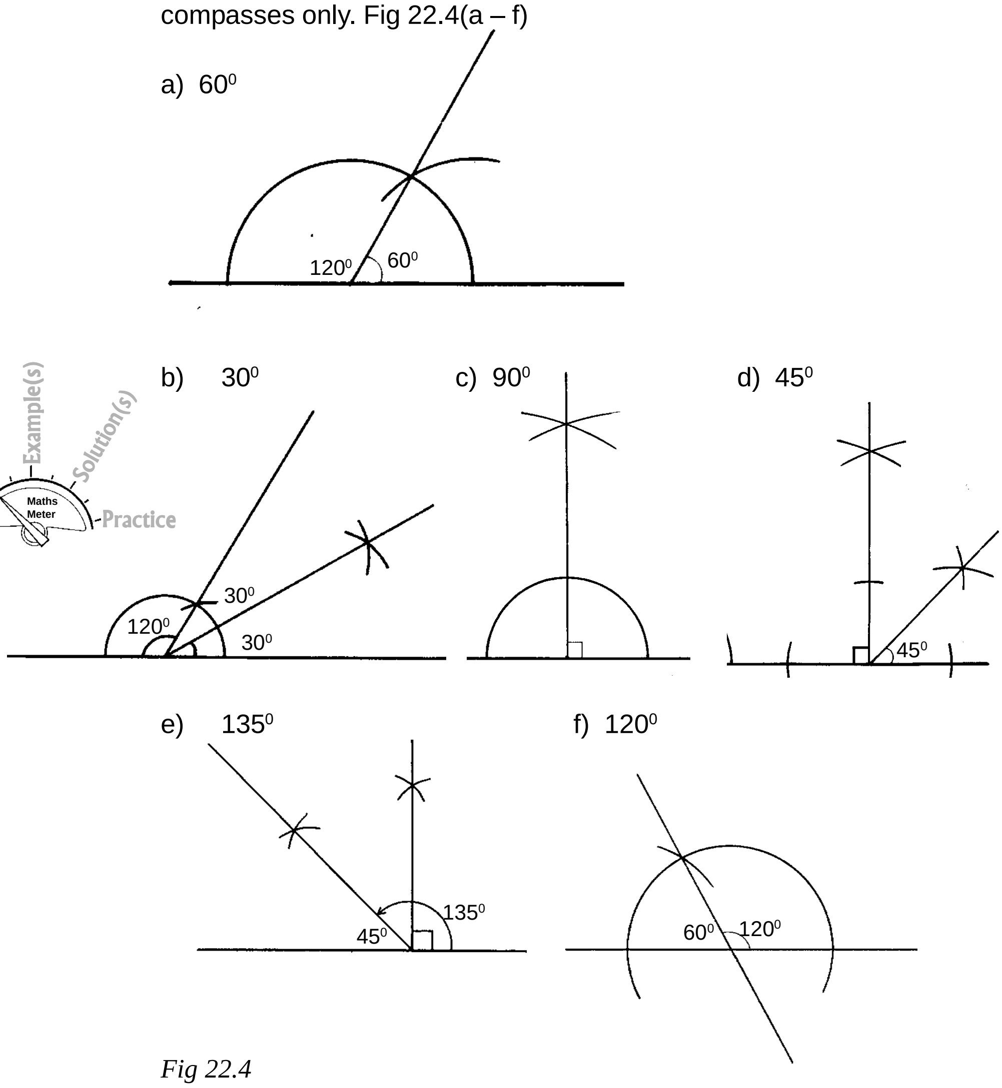


Fig 22.4

Concept 5
Constructing a triangle using a ruler and compasses and a protractor.

Hint

Always start by making a freehand sketch and label it when constructing a shape.

Construct a triangle ABC with AB = 9cm BC = 5cm CA = 8cm
Begin with a sketch (Fig 22.5a)

Tip

A triangle can only be constructed if given the following:
a) lengths of all the three sides(SSS)
b) lengths of two sides and an included angle (SAS)
c) two angles and the length of a side (AAS) and in this case the third angle can be found.

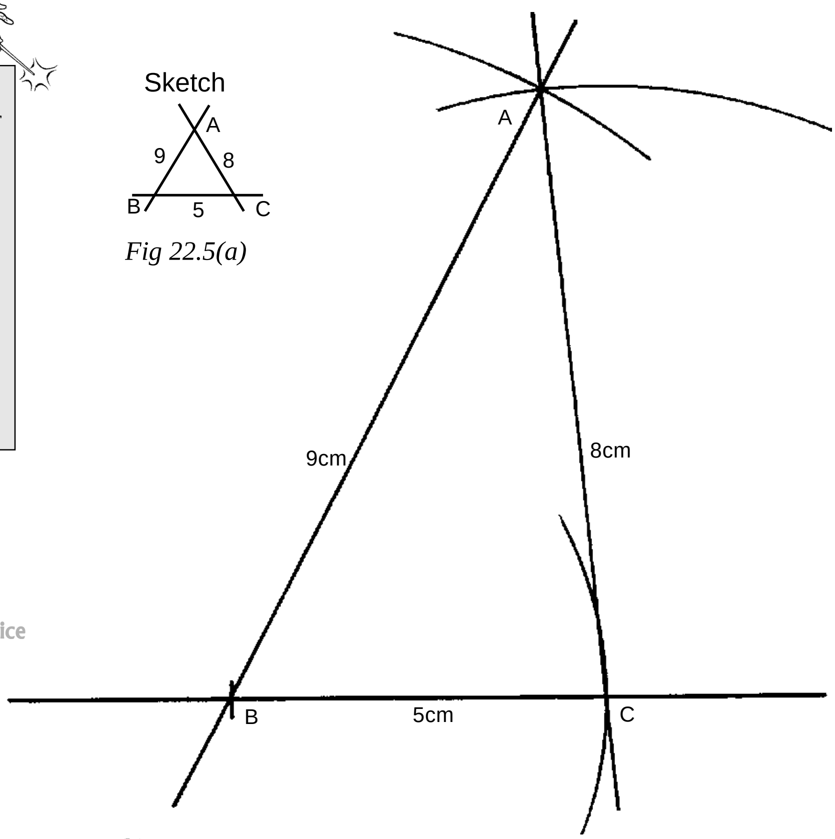


Fig 22.5(a)

Fig 22.5(b)

Hint

The construction of parallel lines can also be achieved using a ruler and set square only. Discuss with your teacher or friends how this can be done.

Concept 6
Construction of parallel lines.

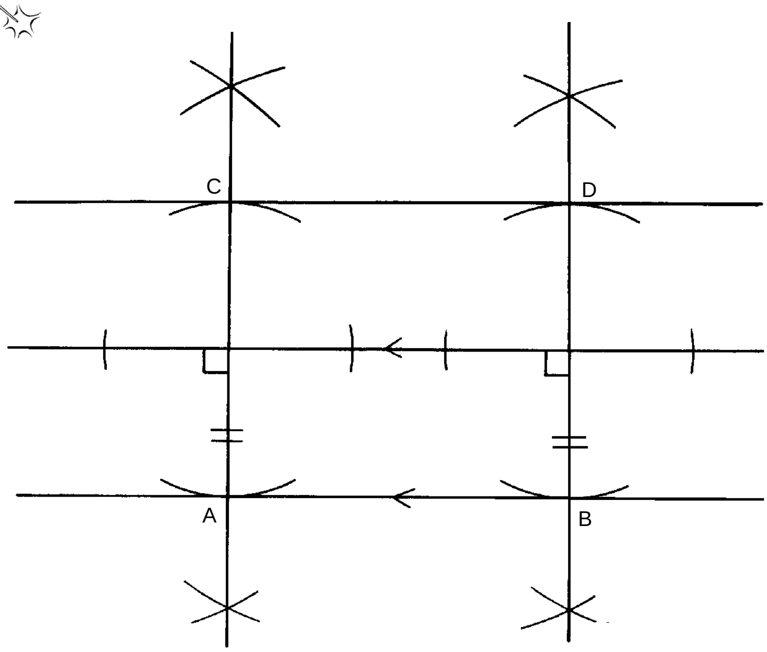


Fig 22.6

AB is parallel to CD.

Concept 7

Construction of a parallelogram ABCD/parallel lines.

e.g. construct a parallelogram ABCD given the following:

AB = DC = 8cm
AD = BC = 4cm
$D\hat{A}B = 120^0$

Hint

In a sketch the measurements do not have to be accurate as it is a mere guide.

Begin with a sketch (Fig 22.6a)

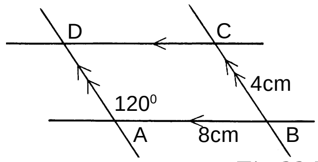


Fig 22.7(a)

Fig 22.6(b) illustrates the required parallelogram ABCD.

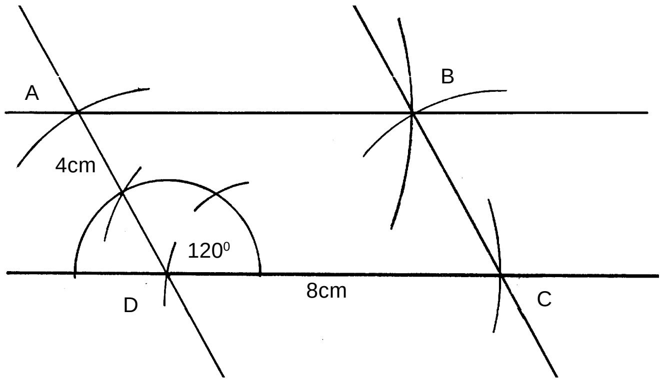


Fig 22.7(b)

(Not to scale)

Concept 8

Circumscribed circle. This is a circle which passes through all the vertices of a triangle, its centre found by bisecting any two sides. (Fig 22.8)

Hint

Bisect any two sides of a given triangle. The point where the two bisectors intersect is the centre of the circle.

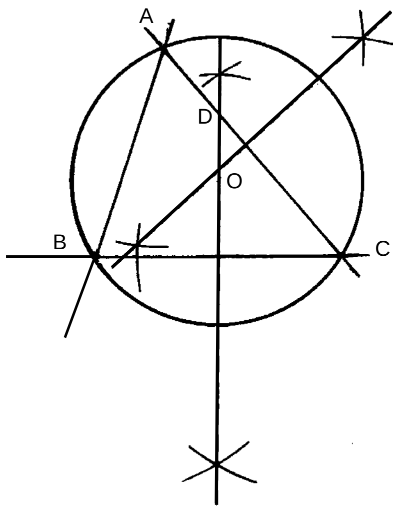


Fig 22.8

Concept 9

Inscribed circle. This is a circle which touches each side of the triangle and it is enclosed in the triangle. (Fig 22.9). Its centre is found by bisecting any two angles.

Hint

You bisect any two angles of the given triangle to get centre of the circle where the two bisectors meet

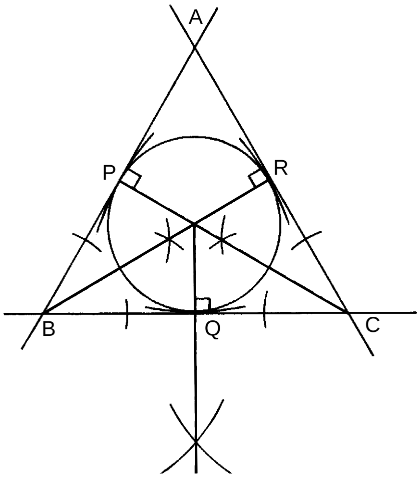


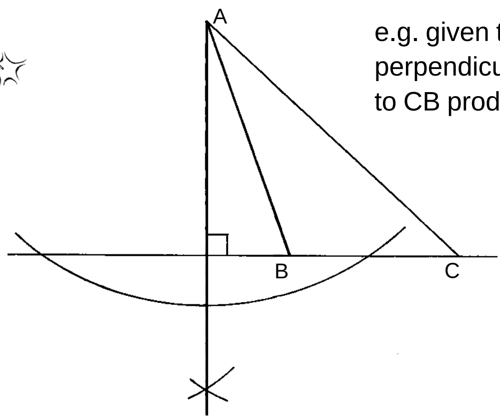
Fig 22.9

Concept 10

Constructing a perpendicular from a given point to a given line. (Dropping a perpendicular from the vertex). (Fig 22.10).

Hint

Line CB is deliberately extended in this case.



e.g. given triangle ABC. Drop a perpendicular line from vertex A to CB produced.

Fig 22.10

PRACTICE 22A

1. Using a protractor, draw the following angles and bisect them using a ruler and a pair of compasses:

a) 40^0 b) 130^0 c) 75^0 d) 140^0 e) 100^0

2. Using a ruler and compasses only, construct the following angles:

a) 60^0 b) 45^0 c) 150^0 d) 135^0 e) 120^0

3. Using any instruments, construct two intersecting straight lines which are both 8cm long with 120^0 as one of the angles between them. Bisect the acute angle between them.

4. Construct triangle ABC where $A\hat{B}C = 90^0$, AB = 7cm and BC = 9cm. Construct an inscribed circle to triangle ABC.

5. Draw an equilateral triangle with sides of length 8cm, using a ruler and compasses only. Construct the circumscribed circle.

6. Construct a parallelogram ABCD with one of its internal angle $A\hat{D}C = 120^0$ and AD = 9cm and CD = 5cm. Construct a circle with a radius of 4cm and point A as its centre .

B. LOCUS

The plural for locus is ***loci***.

Locus of a point

The locus of a point is the path traced by a moving point as it obeys a given law.

Some standard (common) loci are defined in table 22.1. The illustrations follow below this table.

Hint

Students must appreciate that the loci in 3-dimesnion is beyond the scope of this book otherwise the answer for loci in 3 dimension (space) for 1st concept would be a sphere!

Table 22.1 **Common loci**

	The locus of a point (points)	**Locus defined in 2 dimensions (plane)**
1.	which is/are at a given distance (r) from a given point (O).	A circle, centre O and radius r
2.	which is/are at a given distance (r) from a given straight line.	A pair of lines parallel to the given line *l*. One on either side of the given line.
3.	which are equidistant from two given fixed points A and B.	The perpendicular bisector of the line AB.
4.	which are equidistant from two given intersecting straight lines.	A pair of lines bisecting the angles between the given lines
5.	at which a given segment of a straight line (AB) subtends a given angle	A pair of equal arcs of equal circles on opposite sides of the line segment AB.
6.	at which a given line segment subtends an angle of 90^0.	A circle with the given line segment as its diameter.

Illustration of the common loci in table 22.1.

In the diagrams that follow, the loci is shown by *dotted lines*.

1. The locus of a point which moves such that it is a fixed distance (r) from a given fixed point (A), is the circumference of a circle drawn with the fixed point as its centre and the given distance (r) as its radius. (Fig 22.11)

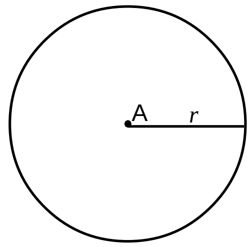


Fig 22.11

Hint

Parallel lines maybe constructed using a ruler and compasses only or with a set square. (Revisit concept)

2. The locus of a point which moves such that it is a fixed distance (h) equidistant from a given straight line (l) is a pair of straight lines parallel to (l) which are on either side of l. (Fig 22.12)

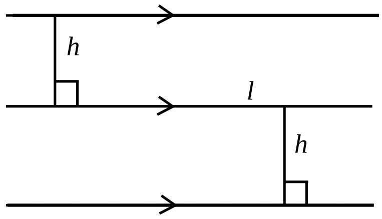


Fig 22.12

3. The locus of a point, which is equidistant from two given *fixed points* A and B is a perpendicular bisector of the line joining the two points. (Fig 22.13)

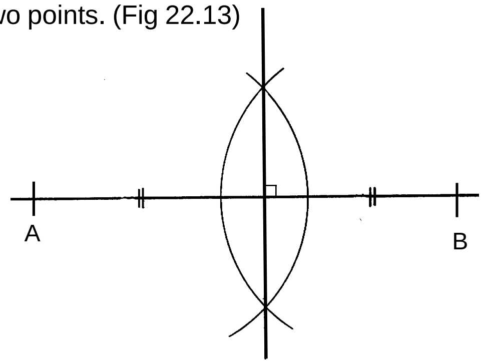


Fig 22.13

4. The locus of a point, which is equidistant from two given intersecting straight lines l and m, is a pair of bisectors of the angles between them (l and m). (Fig 22.14)

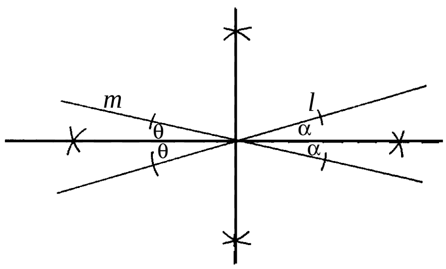


Fig 22.14

5. The locus of a variable point such that a given segment of a straight line AB subtends a constant angle θ is a pair of congruent arcs on opposite sides of AB. (Fig 22.15)

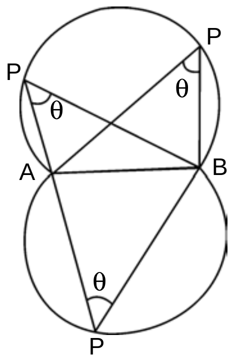


Fig 22.15

6. The locus of a point which a given segment of a straight line AB subtends an angle 90^0 is a circumference of a circle drawn with AB as its diameter. (Fig 22.16)

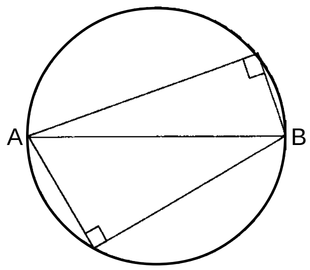


Fig 22.16

Consider the following example

1. Using a ruler and compasses only,

a) construct on a single diagram:

(i) a quadrilateral ABCD with AD parallel to BC; AB = 8cm; BC = 10cm; angle $A\hat{B}C = 90^0$ and angle $B\hat{C}D = 60^0$.

b) (i) Construct the locus of points equidistant from A and B.

(ii) Construct the locus of points equidistant from AD and DC.

(iii) Point P lies inside the quadrilateral ABCD such that it is equidistant from AD and CD, and equidistant from A and B. Label point P.

c) (i) Construct a circle which passes through vertices A, B and C.

(ii) Give the measurement of the radius of the circle.

d) Region T is such that it is nearer B than C, nearer B than A, and lies inside the quadrilateral ABCD.
Shade region T.

e) What is the name of the shape quadrilateral ABCD?

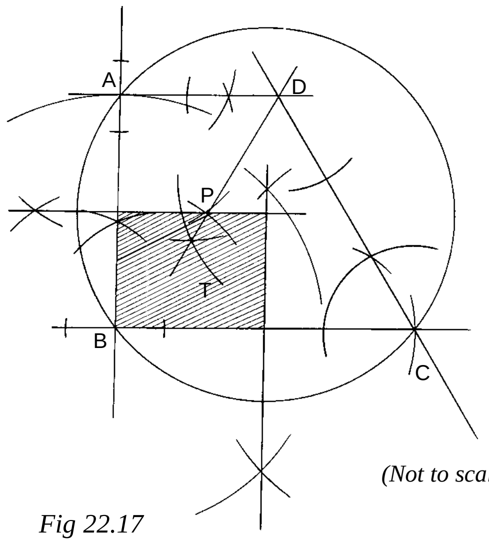
f) Find the area of the shaded region.

Hint

A triangle may be constructed provided you are given the following.
a) SSS – all lengths of sides given
b) SAS – lengths of two sides given
c) SAA – two angles and a length of a side given.

Solution

1. a(i) b(i)–(iii) c(i) d) (shown on the diagrams)



(Not to scale)

Fig 22.17

c)(ii) Radius = 5,4cm

e) Trapezium

f) 20cm^2

PRACTICE 22B

In the following question make a sketch before you make your construction (use a ruler and compasses only).

1. Draw triangle ABC with
AB = 6cm, BC = 8cm and AC = 10cm.

2. Construct triangle XYZ with
XZ = 6cm and $X\hat{Z}Y = 60^0$ and YZ = 7cm. Draw the inscribed circle.

3. Using a ruler and compasses only construct the following angles:
a) 15^0
b) 150^0
c) 75^0
d) 135^0
e) 165^0

4. A church is to be constructed at a point central to three villages X,Y and Z. The distances between the villages are:
XY = 8km, XZ = 7km and YZ = 8,8 km.
a) Draw a diagram to scale showing the three villages.

b) Locate the position of the church by constructing points equidistant from points X and Y as well as from X and Z. Mark this point C.

c) Measure the distances CX, CY and CZ.

d) Draw a circle which would pass through all the three villages.

5. a) Draw triangle XYZ with XY = 14cm. $Z\hat{X}Y = 40^0$ and ZX = 12cm. Measure and write down the length of YZ.

b) On the same diagram,
   (i) draw the locus of points within the triangle XYZ which are 3cm from XZ.
   (ii) construct the locus of points equidistant from XY and YZ.

c) A point lies inside the triangle XYZ. The position of P is such that it is more than 3cm from XZ but its distance from YZ is less than its distance from XZ.
Indicate clearly, by shading, the region in which P must lie.

6. a) Construct a triangle ABC, in which AB = 8,5cm, BC = 7cm and $A\hat{B}C = 90^0$.

b) Measure and write down the length of AC.

c) Draw the locus of points which are 6cm from point C.

d) Draw the locus of points which are 3cm from the line BC and on the same side of BC as A.

e) Mark the two points, P inside the triangle and Q outside the triangle, which are 6cm from C and 3cm from BC respectively.

7. Two pegs, peg A and Peg B, are 10 metres apart and a goat is tied to each peg. The goat tied to peg A is tethered with a 7 metre rope while the other goat is tied to a 6 metre rope. Shade the locus of the space in which both goats can graze.

8. Draw a quadrilateral, ABCD, such that AB = 12cm, $D\hat{A}B = 60^0$, AD = 6cm, CD = 7cm and BC = 5cm. Construct a point X on BC, such that $B\hat{X}A = C\hat{X}D$.
Measure and record, $A\hat{X}D$.

9. a) Construct a trapezium ABCD, with AB parallel to DC, $A\hat{D}C = 60^0$, AD = 5cm, DC = 5cm and AB = 8cm

(i) Construct a perpendicular from point B to line DC produced and measure its length.

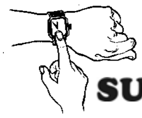
(ii) Calculate the area of the trapezium ABCD.

10. a) Construct a triangle ABC with BC = 6cm, $A\hat{B}C = 60^0$ and AC = 7cm.

b) Construct the locus of points which are equidistant from A and C.

c) Construct the locus of points which are equidistant from AB and AC.

d) Construct an inscribed circle for triangle ABC.



SUMMARY

1. Using a ruler and compasses only, the following fundamental angles can easily be constructed:

60^0 90^0 120^0

The other angles can then be found by either bisecting or constructing multiples or sub-multiples of these angles e.g.

15^0, 30^0, 45^0, 135^0, 150^0

2. An inscribed circle is a circle which is drawn inside a shape but touches all sides. For *any triangle* the centre for its inscribed circle is where the bisectors of all angles of the triangle meet.

3. A circumscribed circle is a circle which is drawn outside the shape passing through all its corners (vertices). For *any triangle*, the centre for its circumscribed circle is where the bisector of all sides of the triangle meet (or simply bisectors of any two sides).

4. A locus is a line or curve along which it is possible for a point to be located according to a given circumstance e.g. 6cm away from a line or a point.

5. The locus of a point which is equidistant from a given fixed point is the circumference of a circle drawn with the fixed point as centre and the given distance as radius.

6. The locus of a point which varies, such that it is equidistant from two given fixed points, is the perpendicular bisector of a line joining them.

7. The locus of a point which is equidistant from two given intersecting straight lines is a pair of bisectors of the angles between them.

8. The locus of a point which moves such that it is at a given distance from a given straight line is a pair of lines parallel to it and sandwiching it.

EXAM PRACTICE 22

Consider the example below

Intro Concept Example(s) Solution(s) Practice Maths Meter

1. a) Construct on a single diagram:
 (i) a parallelogram XZMY with XY = 10cm, XZ = 8cm and the angle $Z\hat{X}Y = 60^0$,
 (ii) the locus of points equidistant from XY and XZ,
 (iii) the perpendicular bisector of YM.

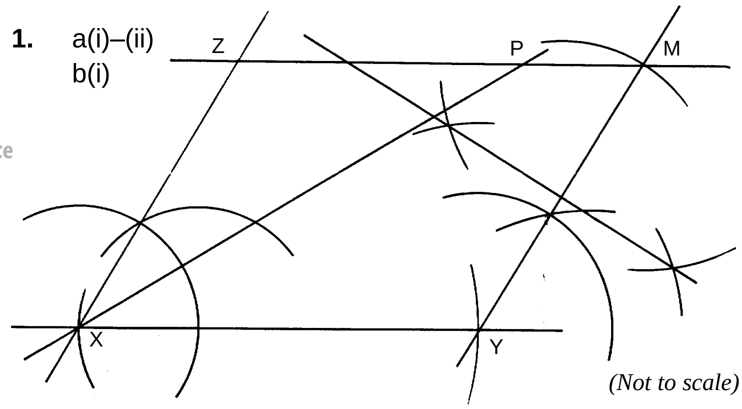
 b) (i) Mark the point P that lies on ZM and is equidistant from XY and XZ.
 (ii) Measure and write down the length of PM.

 c) Describe the locus that the perpendicular bisector YM represents.

Solution

Intro Concept Example(s) Solution(s) Practice Maths Meter

1. a(i)–(ii)
b(i)



(Not to scale)

 b) (ii) ZM = 3,6cm

 c) Points that are equidistant from Y and M.

Now do the following:

In all questions, use a ruler and compasses only. All construction lines and arcs must be clearly shown.

Intro Concept Example(s) Solution(s) Practice Maths Meter

1. Construct, on a single diagram:
 a) a triangle ABC, in which AC = 9,0cm, $B\hat{A}C = 45^0$ and AC = 6,8cm,
 b) the locus of points equidistant from A and B.
 c) the locus of points equidistant from B and C.
 d) the circle passing through A, B and C.
 e) Measure and write down the radius of the circle in (d).

2. a) Construct on a single diagram:

   (i) triangle XYZ, in which YZ = 10cm, ZX = 7,5cm and XY = 8,3cm.

   (ii) the locus of a point P, such that YP = PZ.

   (iii) the locus of a point R, such that XR = 5cm.

b) Given that R is on XZ, measure and write down the length of YR.

c) Shade the region inside the triangle XYZ, in which a point Q lies, given that $YQ < QZ$ and $XQ < 5$cm.

3. A playground is in the form of a quadrilateral ABCD. AB = 80m, BC = 100m, CD = 140m, $A\hat{B}C = 90^0$ and $B\hat{C}D = 120^0$.

a) (i) Using a scale of 2cm to represent 20m, construct an accurate scale drawing of the playground.

   (ii) Measure and write down $B\hat{A}D$.

b) A tree in the playground is equidistant from AB and BC and also equidistant from A and B.

   (i) Using clear and relevant constructions, show, using T, the position of this tree.

   (ii) Find the actual distance of the tree (T) from the corner A.

4. On a single diagram construct:

a) triangle XYZ, in which XY = XZ = 10cm and $Y\hat{X}Z = 75^0$.

b) the locus of points which are equidistant from XY and XZ.

c) triangle YZN, on the same side of YZ as X, which is equal in area to triangle XYZ.

5. On a single diagram construct:

a) a line OQ, 10cm long.

b) a circle with centre O, and radius 4cm.

c) the locus of points which are equidistant from O and Q.

d) the circle whose diameter is OQ to cut the circle with centre O at R and M.

e) Describe fully the locus OQ, given that Q is an external point for two tangents, of the circle with centre O.

6. a) Construct a regular hexagon of sides 5cm.

b) Measure the largest diagonal of the hexagon and hence draw the circle which circumscribes the hexagon.

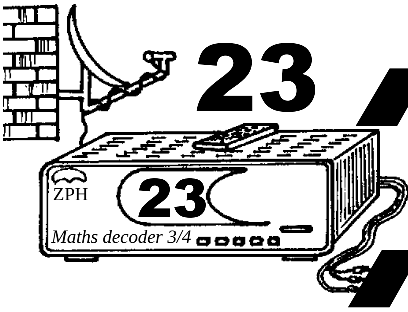
c) Construct triangle ABC, in which AB = 7,3cm, AC = 8cm and angle $B\hat{A}C = 30^0$. On the opposite side of AC from B construct a triangle ACX, such that AX = CX = 9,5cm. If a point P is on the same side of BX as C and its distance from CX is the same as the distance C from BX, sketch the locus of P.

7. a) Construct triangle ABC when
AB =10cm, BC = 17cm and $A\hat{B}C = 60^0$.

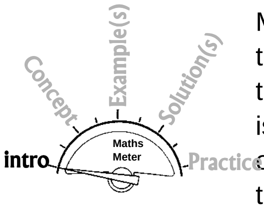
b) Construct the inscribed circle of the triangle.
Measure and write down its radius (*r*).

c) Construct the locus of points which are 3cm from C.

d) Shade all the space which is more than 3cm from C but within the inscribed circle.



23 The Non-Right-angled Triangle



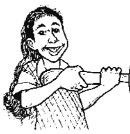
Most triangles are naturally non-right-angled. The Pythagoras theorem has limitations in that it only applies to the right-angled triangle. The same applies to the trigonometric ratios. The challenge is that we still need to find the sides (lengths) and internal angles of non-right-angled triangles. In chapter 12 we studied how to get the angles and sides of a right angled triangle. Remember:

Trigonometry is a branch of mathematics which deals with all the computations involving triangles.

When dealing with **non-right-angled** triangles two important formulas may be used. These are

a) The sine rule/formula

b) Cosine rule/formula



Syllabus Expectations

By the end of this chapter, students should be able to:

1. apply the sine rule to solve non-right-angled triangles.
2. apply the cosine rule to solve non-right-angled triangles.
3. solve three dimensional problems involving the angle between a line and a plane.
4. use the formula Area = $\frac{1}{2}$ abSinC to calculate the area of a triangle.
5. define the bearing of a point and work out angles of elevation and depression for a given problem.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- define all types of triangles and their properties.
- construct angles and triangles.

- construct parallel lines and perpendicular lines.
- use a protractor to measure the size of an angle.
- understand the concept of direction.

A. TRIGONOMETRICAL RATIO FOR OBTUSE ANGLES

Consider Fig 23.1. It illustrates another way to measure angles. Angles measured in this way are always measured in an anticlockwise direction (rotation).

The starting point is the positive, x-axis Fig 23.1 and then move the initial length x as radius of a circle.(Fig 23.2). This length becomes h while the x-value is continuously reduced.

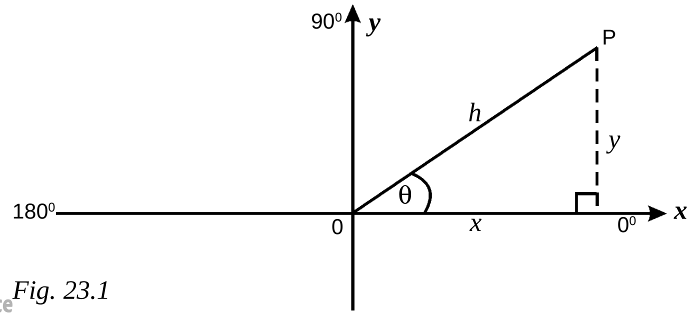


Fig. 23.1

$$\text{Sin } \theta = \frac{y\text{-coordinate of P}}{\text{radius OP}} = \frac{y}{h} \quad \text{(where } \theta \text{ is acute)}$$

$$\text{Cos } \theta = \frac{x\text{-coordinate}}{\text{radius OP}} = \frac{x}{h}$$

$$\text{Tan } \theta = \frac{y\text{-coordinate of P}}{x\text{-coordinate}} = \frac{y}{x}$$

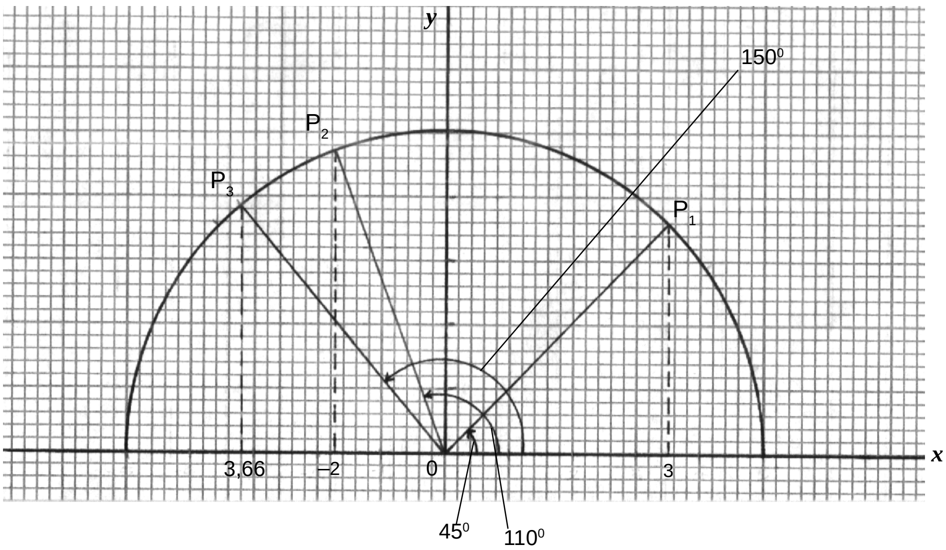


Fig. 23.2

Consider the following example:

Use Fig 23.2

1. a) For the acute angle 45^0, find its:

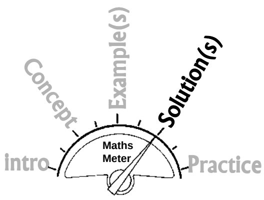
(i) Tan (ii) Cos (iii) Sin

b) For the obtuse angle 110^0, find its:

(i) Sin (ii) Cos

Solutions

From *Fig 23.2*



1. a) For acute angle, use radius OP, in 1st quadrant

$$\text{Tan } 45^0 = \frac{3}{3} = 1$$

$$\text{Cos } 45^0 = \frac{3}{4,24} = 0,7075$$

$$\text{Sin } 45^0 = \frac{3}{4,24} = 0,7075$$

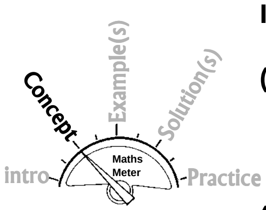
b) For obtuse angle, use radius OP_2 in the 2nd quadrant

$$\text{Sin } 110^0 = \frac{3,5}{4,24} = +\ 0,8255$$

$$\text{Cos } 110^0 = \frac{-2}{4,25} = -\ 0,4706$$

Try to find the sine and cosine of 150^0 using the given measurements in the diagram.

Important points to remember



(i) An angle more than 0^0 and less than 90^0 is called an **acute angle**. An angle more than 90^0 but less than 180^0 is called an **obtuse** angle.

(ii) Two angles which add up to 90^0 are called **complementary angles**.

(iii) Two angles which add up to 180^0 are called **supplementary angles.**

(iv) The sine of an obtuse angle is equal to the sine of its supplement. (i.e. the acute angle)

sin A = sin (180 – A)

(v) The Cosine of the obtuse angle is equal to the negative of the cosine of its supplement. (i.e. the acute angle)

cos A = –cos (180 – A)

(vi) Supposing θ is an obtuse angle bigger than 90^0 but less than 180^0) i.e. ($90^0 < \theta < 180^0$),
then, trigonometric ratios are found by using the supplement of the angle.

$$\sin\theta = \sin(180^0 - \theta)$$
$$\cos\theta = -\cos(180^0 - \theta)$$
$$\tan\theta = -\tan(180^0 - \theta)$$

Consider the following example:

1. Use tables or a calculator to find the value of:
a) sin 120^0 b) cos 120^0 c) tan 120^0

Solutions

1. a) *For Sine*

sin 120^0 = sin ($180^0 - 120^0$)

sin 120^0 = sin 60^0

sin 120^0 = 0,8660

b) *For Cosine*

cos 120^0 = – cos ($180^0 - 120^0$)

cos 120^0 = – Cos 60^0

cos 120^0 = – 0,5

c) *For Tangent*

tan 120^0 = – tan ($180^0 - 120^0$)

tan 120^0 = – tan (60^0)

tan 120^0 = – 1,7321

1. Use tables or a calculator to find the Sine of:

a) 100^0 b) 116^0 c) $154^0 46'$
d) $139^0 50'$ e) $164^0 30'$

2. Use tables or a calculator to find the Cosine of:

a) 100^0 b) 150^0 c) $168{,}8^0$
d) $157^0 16'$ e) $149^0 55'$

3. Use tables or a calculator to find the Tangent of:

a) 96^0 b) 110^0 c) 154^0
d) $162^0 33'$ e) $154{,}4^0$

4. Express the following trigonometrical ratios as ratios of acute angles and use a calculator or tables to find the values:

a) $\cos 146^0$ b) $\sin 160^0$
c) $\cos 172^0$ d) $\sin 111^0$
e) $\cos 157^0$ f) $\cos 126^0\ 14'$
g) $\sin 163^0\ 21'$ h) $\cos 176^0\ 39'$
i) $\cos 144^0\ 44'$

5. Use tables or a calculator to find angle θ given that $0^0 < \theta < 180$

a) $\cos\theta = -0{,}0871$ d) $\sin\theta = 0{,}8660$
b) $\cos\theta = -0{,}9431$ e) $\cos\theta = -0{,}4641$
c) $\sin\theta = 0{,}5664$ f) $\cos\theta = -0{,}8660$

B. THE SINE RULE (FORMULA)

Consider non-right angled triangle ABC with the side opposite the vertex (A) being a units, opposite (B) being b units long and opposite (C) being c units. (Fig 23.3). The perpendicular height is h.

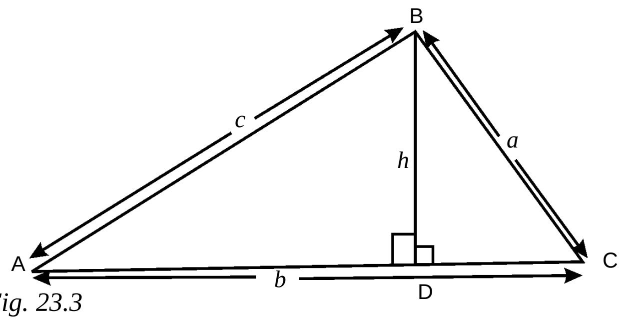


Fig. 23.3

Using the sine rule we can write:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

OR

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof of the sine formula

In $\triangle$ABD, $\dfrac{h}{c}$ = Sin A

h = c Sin A ....(i)

In $\triangle$ACD, $\dfrac{h}{a}$ = Sin A

h = a Sin C...........(ii)

From equations (i) and (ii)

c Sin A = a Sin C

$$\frac{c \text{ Sin A}}{\text{Sin C}} = a$$

$$\frac{c}{\text{Sin C}} = \frac{a}{\text{Sin A}}$$

By dropping a perpendicular from C to AB it can also be shown that:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore \boxed{\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}}$$

Tip

This formula can be used for a non-righted triangle when:
(i) two sides of the triangle and at least one angle opposite one of the sides is given.
(ii) two or three angles and one side are given.
Remember once you are given 2 angles of a triangle, the third one can easily be deducted.

Consider the examples below:

1. Find b in Fig 23.4(a).

2. Find $M\hat{N}K$ in Fig 23.4(b).

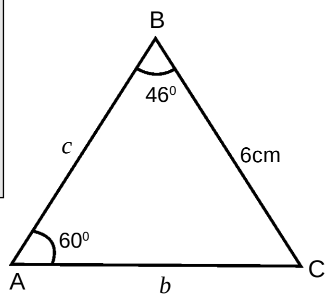


Fig 23.4(a)

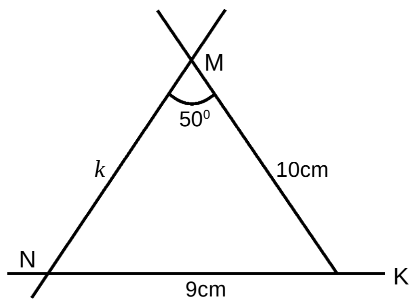
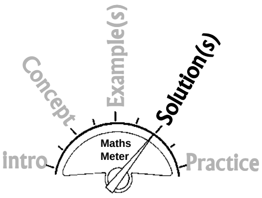


Fig 23.4(b)

Solutions



Tip

Value of b can be worked out using a calculator or log tables i.e. logarithm of sine.

1. $$\frac{b}{\text{Sin B}} = \frac{a}{\text{Sin B}}$$

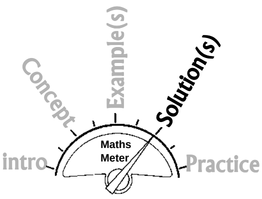
$$\frac{b}{\text{Sin } 46^0} = \frac{6}{\text{Sin } 60^0}$$

$$b = \frac{6}{\text{Sin } 60^0} \times \text{Sin } 46^0$$

$$b = \frac{6 \text{ Sin } 46^0}{\text{Sin } 60^0}$$

Using log tables

No.	Log
6 Sin 46^0	0,7782 $\bar{1}$,8569
Numerator Sin 60^0	0,6351 $\bar{1}$,9375
4,9839	0,6976



$$\therefore\ b = 4,9839$$

Using the calculator

$$b = \frac{6 \times 0,7193398}{0,8660}$$

$$b = 4,983878524$$

$$b = 4,9839$$

Hint

Sine is positive for the acute angle and its supplement.

2. $$\frac{\text{Sin } M\hat{N}K}{10} = \frac{\text{Sin } 50^0}{9}$$

$$\text{Sin } M\hat{N}K = \frac{10 \times \text{ Sin } 50^0}{9}$$

$$\text{Sin } M\hat{N}K = \frac{10 \times 0,7660}{9}$$

$$\text{Sin } M\hat{N}K = 0,8511$$

$$\therefore\ M\hat{N}K = 58,3^0 \text{ or } 121,7^0$$

The ambiguous case

Using Fig 23.5

1. Find $A\hat{C}B$

2. Find $A\hat{B}C$

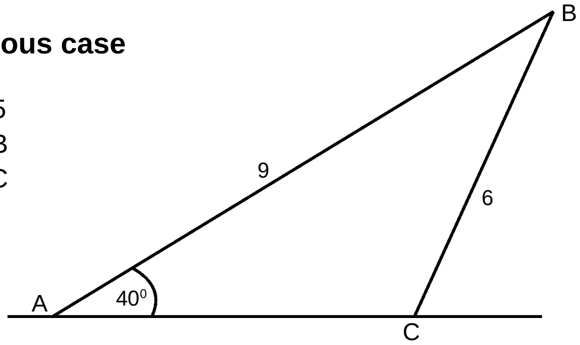


Fig 23.5

Solutions

1.

$$\frac{\text{Sin } C}{c} = \frac{\text{Sin } A}{a}$$

$$\frac{\text{Sin } A\hat{C}B}{9} = \frac{\text{Sin } 40^0}{6}$$

$$\text{Sin } A\hat{C}B = \frac{9 \times \text{Sin } 40^0}{6}$$

$$= \frac{\overset{3}{\cancel{9}} \times 0{,}6428}{\underset{2}{\cancel{6}}}$$

$$\text{Sin } A\hat{C}B = 0{,}9642$$

$$A\hat{C}B = 74^0\ 37'$$

Since ACB can also be an obtuse angle, from Sin A = Sin (180^0 – A), we get two possible answers hence the ambiguity (Fig 23.6).

$$\therefore \quad A\hat{C}B = 74^0\ 37' \text{ or } 105^0\ 23'$$

Hint

Sin θ = Sin (180 – θ)

∴ Another answer is $180^0\ 0' - 74^0\ 37'$ $= 105^0\ 23'$

2. If $A\hat{C}B = 74^0\ 37'$

then $A\hat{B}C = 180^0 - (74^0\ 37' + 40^0)$

$= 180^0 - 114^0\ 37'$

$= 65^0\ 23'$

If $A\hat{C}B = 105^0\ 23'$

then $A\hat{B}C = 180^0 - (105^0\ 23' + 40^0)$

$= 180^0 - 145^0\ 23'$

$A\hat{B}C = 34^0\ 37'$

Hint

The two possible triangles are as illustrated in (Fig 23.6). The two triangles are ABC_1 and ABC_2.

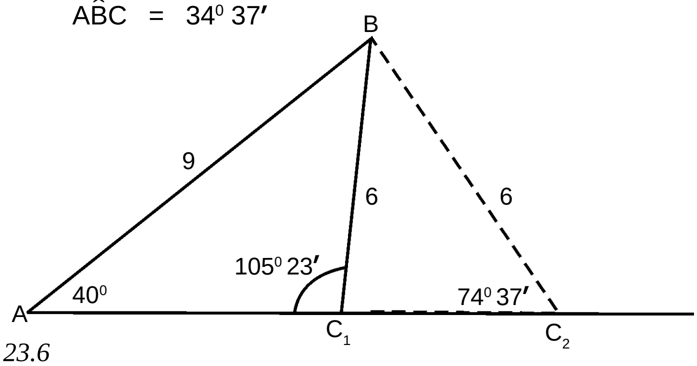
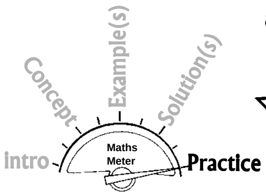


Fig 23.6

Hint

Where obtuse angles are given use their supplements.

1. In Fig 23.7 find the marked angle θ or the unknown side x.



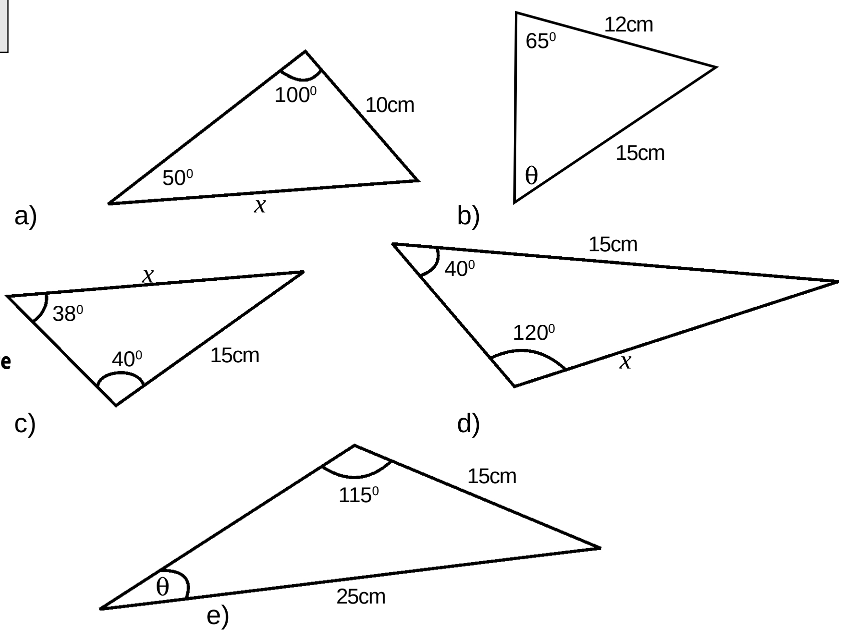


Fig. 23.7

Hint

Watch out for the ambiguous case.

2. Find all the lettered sides and angles marked θ in the following triangles.

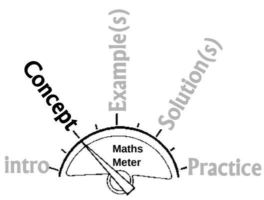
a) $\hat{B} = 54^0\ 38'$: $a = 73$m $b = 62$m

b) $\hat{A} = 89^0$ $\hat{B} = 63^0\ 14'$ $b = 15$m

c) $\hat{C} = 72^0\ 18'$ $b = 18{,}6$cm $c = 21{,}5$cm

C. THE COSINE RULE/FORMULA

Consider the two triangles in Fig 23.8.



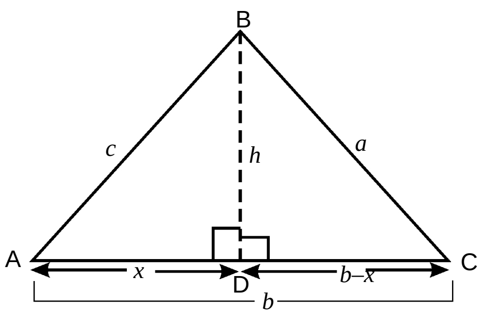


Fig. 23.8(a)

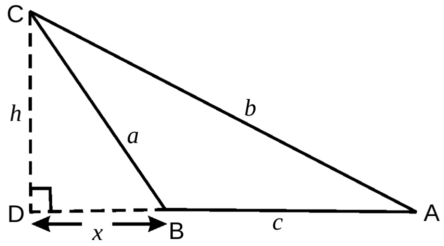


Fig. 23.8(b)

From Fig. 23.8(a)
Using Pythagoras Theorem for △ ABD

$c^2 = x^2 + h^2$

for △BDC $a^2 = h^2 + (b - x)^2$

$a^2 = h^2 + b^2 - 2bx + x^2$

But $h^2 + x^2 = c^2$

$\therefore\ a^2 = c^2 - 2bx + b^2$

$a^2 = b^2 + c^2 - 2bx$

For △ ABD Cos A $= \dfrac{x}{c}$

$x = c\text{Cos A}$

Hence substituting for x:

$$\boxed{\therefore\ \boldsymbol{a^2 = b^2 + c^2 - 2bc\ \textbf{Cos A}}}$$

In Fig 23.8(b)

AD $= c + x$

$b^2 = (c + x)^2 + h^2$

$\Rightarrow\ b^2 = c^2 + 2cx + x^2 + h^2$

From Pythagoras Theorem $a^2 = h^2 + x^2$

$\therefore\ b^2 = c^2 + a^2 + 2cx$

Angle ABC is obtuse, so the value of its cosine is the negative of the cosine of its supplement, C$\hat{\text{B}}$D.

Cos B = – Cos A$\hat{\text{B}}$C $= \dfrac{-x}{a}$

Cos B $= \dfrac{-x}{a}$

$x = -a\text{Cos B}$

Substituting for x

$$\boxed{b^2 = c^2 + a^2 - 2ca \text{ Cos B}}$$

Similarly

$$\boxed{c^2 = a^2 + b^2 - 2ab \text{ Cos C}}$$

Note, the three formulae are versions of one and the same thing. In the above format of the cosine rule is used to calculate sides since a side is the subject of each formula.
Also it can be deduced that

$$\text{Cos C} = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\text{Cos B} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc}$$

The cosine rule is used in this format to calculate the angles since the angle functions are on their own side.

Hint

The Cosine rule is used to solve a non-right-angled triangle when:
a) the values of 2 sides and the included angle are given (SAS)
b) the values of all three sides only are given (SSS)
– It is more complicated compared to the sine rule and where a choice is possible, use the sine rule unless otherwise stated.

Consider the following examples:

1. Find b in Fig 23.9(a)

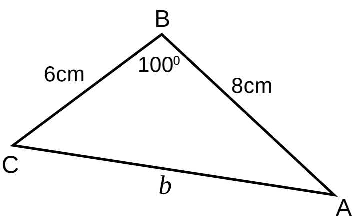


Fig. 23.9(a)

2. Find $\hat{B}$ in Fig 23.9(b)

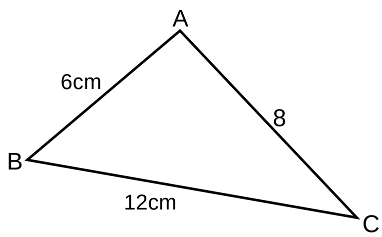


Fig 23.9(b)

Solutions

Hint

Cos100⁰ = – cos80⁰

1. $b^2 = a^2 + c^2 - 2ac \text{ Cos B}$

$b^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \text{ Cos } 100^0$

$b^2 = 36 + 64 - 96 \text{ Cos } 100^0$

$b^2 = 100 - 96 \times (-\text{Cos } 80^0)$

$b^2 = 100 - (96 \times -0{,}1736)$

$b^2 = 100 + 16{,}6656$

$b^2 = 116{,}6656$

$b = \sqrt{116{,}6656}$

$b = \sqrt{1{,}166656 \times 100}$

$b = 10{,}80\text{cm}$

2. $\text{Cos }\hat{B} = \dfrac{a^2 + c^2 - b^2}{2ab}$

$\text{Cos }\hat{B} = \dfrac{12^2 + 6^2 - 10^2}{2 \times 12 \times 10}$

$\text{Cos }\hat{B} = \dfrac{144 + 36 - 100}{240}$

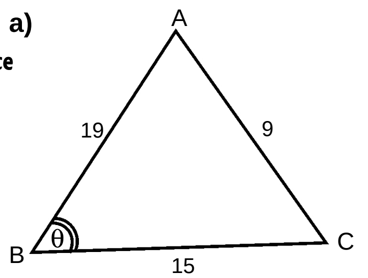
$\text{Cos }\hat{B} = \dfrac{180 - 100}{240}$

$\text{Cos }\hat{B} = \dfrac{80}{240} = \dfrac{8}{24} = 0{,}3333$

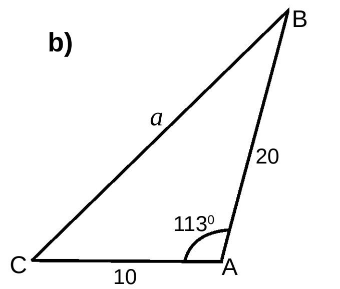
$\hat{B} = 70^0\,32' \text{ or } 70{,}5^0$

1. Find the marked angle θ or the lettered side in Fig 23.10 below. All measurements are in cm.

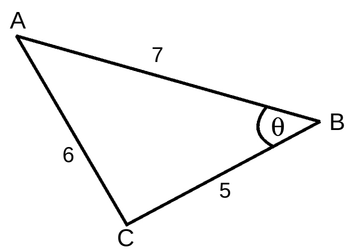
a)



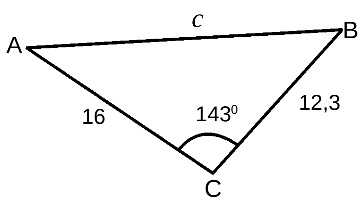
b)



c)



d)



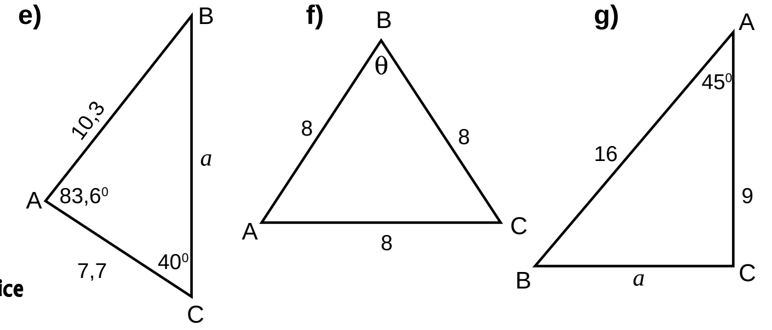


Fig. 23.10

D. THE AREA OF A TRIANGLE

The final operation concerned with the triangle that we will cover is its area. Consider triangle ABC (Fig 23.11.)

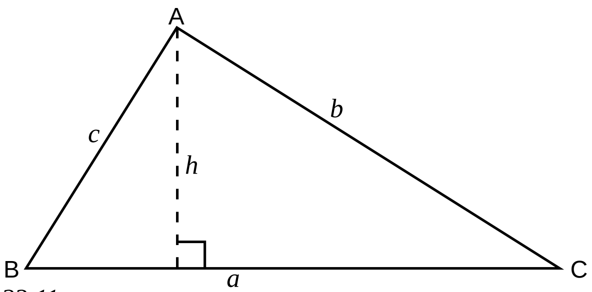


Fig. 23.11

Three possible formulae may be used to calculate the area of a triangle. These formulae are:

1. **Area** = $\frac{1}{2}$ **base × height** Given perpendicular height

2. **Area** = $\frac{1}{2}\,bc\text{Sin}A = \frac{1}{2}\,ab\text{Sin}C = \frac{1}{2}\,ac\text{Sin}B$ given (SAS) in a non-right-angled triangle.

3. **Area** = $\sqrt{s(s-a)(s-b)(s-c)}$ given (SSS) in a non-right-angled triangle where $s = \frac{1}{2}(a + b + c)$

Hint

The first two formulas are popular in examinations. The third one, called Hero's formula, has just been added to increase your knowledge.

Consider the example below:

1. Find the area of triangle ABC (Fig. 23.12.)

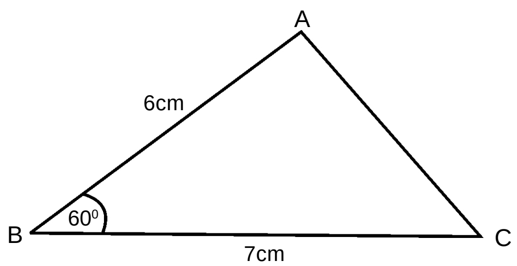


Fig. 23.12

Solution

1. Area $= \frac{1}{2}\, ac\text{Sin}B$

Area $= \frac{1}{2} \times 6 \times 7 \text{ Sin } 60^0$

$= 21 \times 0{,}8660$

Area $= 18{,}1860$

$= 18{,}19\text{cm}^2$ (2d.p)

Find the areas of all the triangles in “Practice 23C” Fig 23.10.

E. BEARING, ANGLE OF DEPRESSION AND ANGLE OF ELEVATION

Bearing helps to define the direction of a point from another given point. It is used in navigation and many other situations.

The first step in any bearing problem is to locate or draw the “North” line passing through the reference point. This line is always perpendicular to an “imaginary” horizontal axis. The point where the “imaginary” line and the “north” line meet is called reference point (0;0).

The second step is to draw a straight line from the reference point to the point to be located in (Fig 23.13). O is the reference point and P is the point to be located. The bearing of P from O is 150^0 or $S30^0E$. In fact, bearing can be quoted in two forms i.e. the ***three-figure bearing*** and the ***compass bearing***.

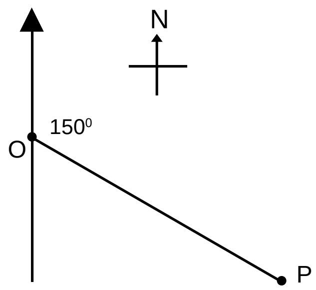
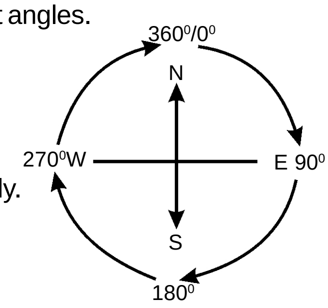


Fig. 23.13

a) **Three figure bearing**

1. Is always measured from a line that is pointing North in a clockwise direction.
2. Always use 3 digits. Use prefix zero for 1 or 2 digit angles. e.g. 080^0.



b) **Compass bearing**

1. Is measured from North or South.
2. Both directions, clockwise or anticlockwise apply. e.g. S40^0E.

Hint

Compass bearing is always N or S ... acute angle ... then E or W.

Quoting bearing of a point

A bearing may be quoted either in three figure format or compass format. Can you figure out the following, diagrammatically

060^0 = N60^0E
300^0 = N60^0W
200^0 = S20^0W

Note that, North, East, South and West are described by the word (due) and their direction (Fig 23.14).

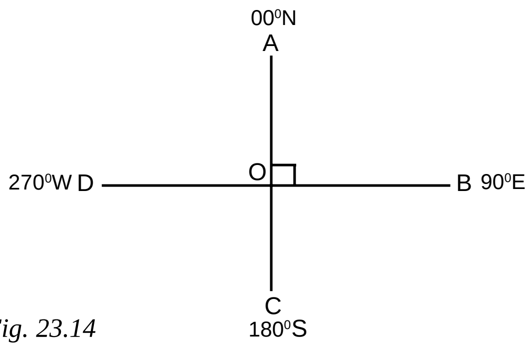


Fig. 23.14

(i) The bearing of A from O is 00^0 or due North.
(ii) The bearing of B from O is 090^0 or due East.
(iii) The bearing of C from O is 180^0 or due South.
(iv) The bearing of D from O is 270^0 or due West.

Consider the following examples:

1. From Fig 23.15, given that N stands for North, write down the bearing of A, B, C and D from O, both as three figure bearing and compass bearing.

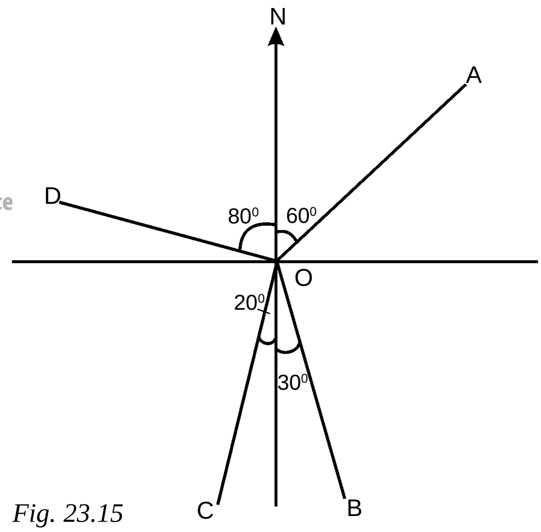


Fig. 23.15

Solution

1. Bearing of
A from O is 060^0 or $N60^0E$
B from O is 150^0 or $S30^0E$
C from O is 200^0 or $S20^0W$
D from O is 280^0 or $N80^0W$

2. In the diagrams (Fig 23.16(a) and (b)) find the bearing of A from B both as compass bearing and three figure bearing.

(i)

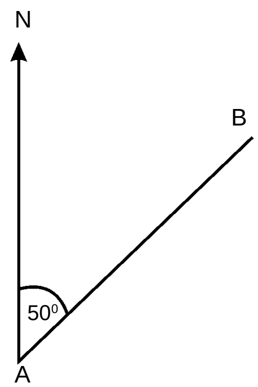


Fig. 23.16(a)

(ii)

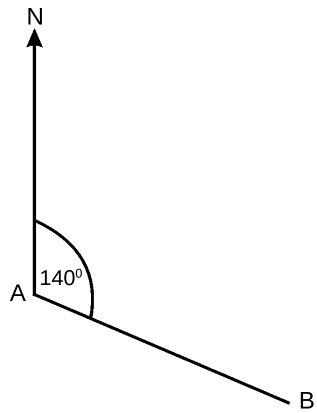


Fig. 23.16(b)

Solution

2. Draw a line **parallel** to the North line passing through B. Then, measure the reflex angle formed between the north line and the line joining the two points to find the three figure bearing. For the compass bearing, observe the direction of the line going to A. If it goes upward the first compass direction is N, if downward it is S. The second compass direction is taken from whether it points to the left or right of the North line. If it points to the left, W is the direction, to the right, E is the direction.

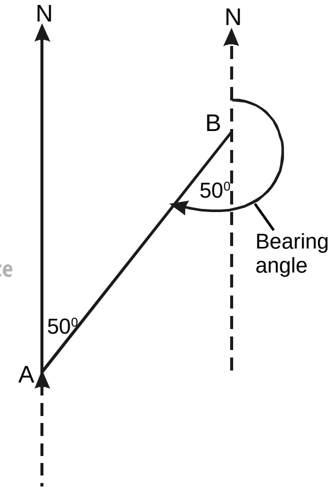


Fig. 23.17(a)

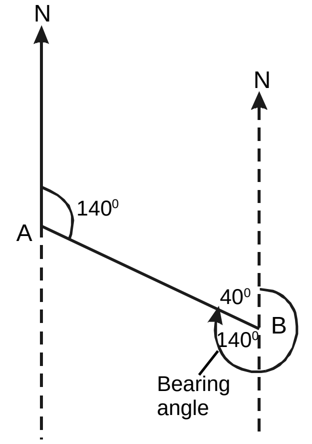


Fig. 23.17(b)

From the diagrams in Fig 23.17.
For case (i) Fig 23.17(a) the bearing of A from B is $180^0 + 50^0 = 230^0$ or $S50^0W$
For case (ii) Fig 23.17(b) the bearing of A from
$B = 180^0 + 140^0$
$= 320^0$ or $N40^0W$

Hint

In making reference to an object in this situation, it is important to treat the object as a point or very small particle, e.g. in our example, the bird should be treated as a small particle.

Angle of elevation or depression

Consider a bird on the top of a flagpole (Fig 23.18) . The angle of elevation of the bird (B) from A is the angle which AB makes with the horizontal AC.

Similarly the angle of depression the bird makes from B is the angle BC makes with the horizontal, BD.

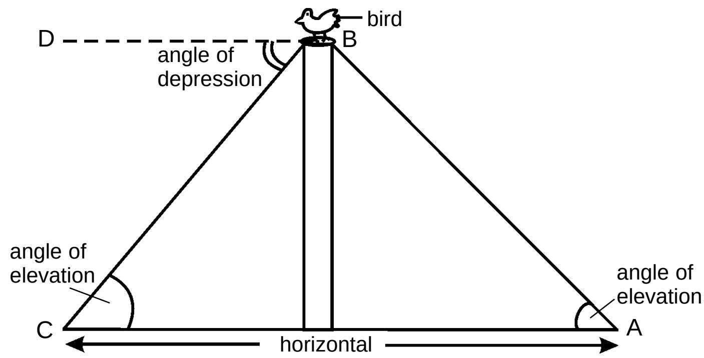


Fig. 23.18

Consider the following example:

1. Tinashe is 1,2m tall and is 10m from the wall of a building and the angle of elevation from where he is to the roof top is 60^0 find:

a) the height of the building.

b) the angle of depression, if Tinashe is 6 metres from the building.

Solution

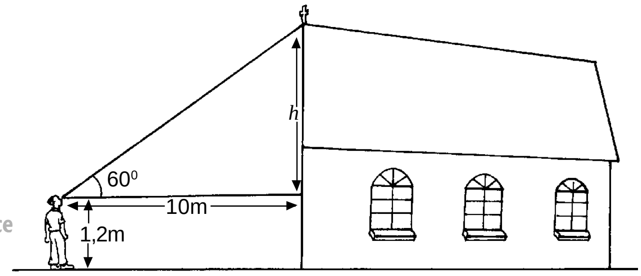


Fig 23.19a)

1. a) $\dfrac{h}{10} = \tan 60^0$

$h = 10 \tan 60^0$

$h = 10 \times 1{,}7321$

$h = 17{,}32$

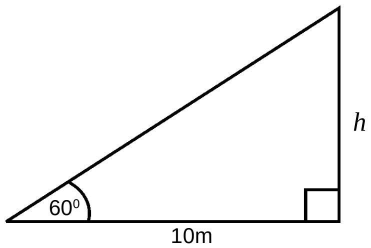


Fig23.19b)

$\therefore$ Height of building = 17,32 + 1,2
= 18,52m

b) $\tan \theta = \dfrac{17{,}32}{6}$

$\theta = \tan^{-1} (2{,}8866)$

$\theta = 70{,}89'$

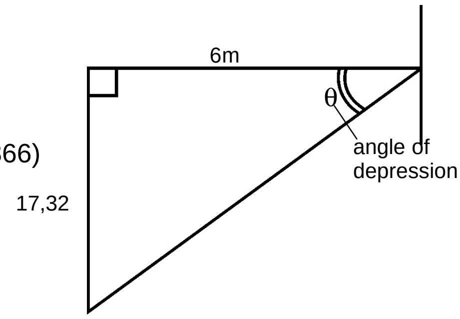


Fig. 23.20

1. Three towns, XYZ, form a triangular shape when connected as in (Fig 23.21).

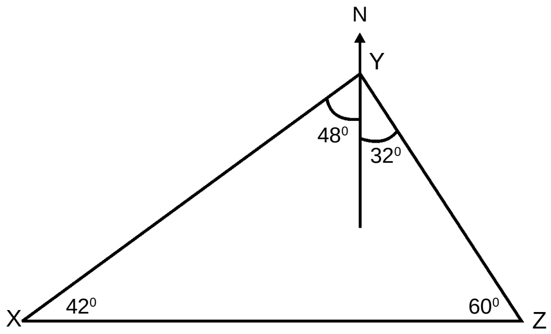


Fig. 23.21

a) Calculate the 3 figure bearing of:
   (i) X from Y (ii) Y from X

b) (i) Z from Y (ii) Y from Z

c) (i) X from Z (ii) Z from X

2. Find the bearing of:

a) A from B using the compass bearing.

b) B from A using the three figure bearing. in the following diagrams (Fig 23.22)

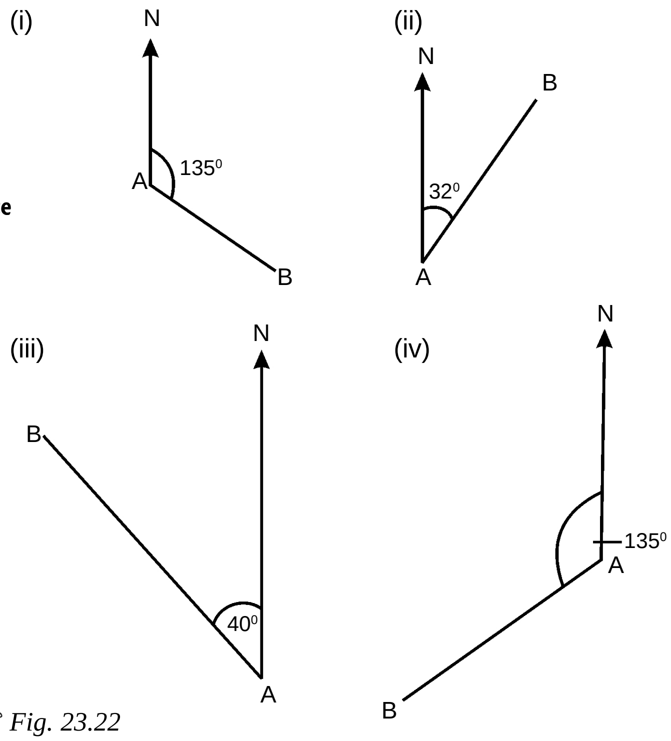
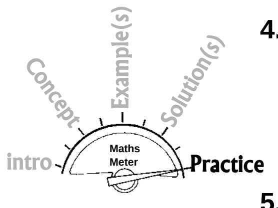


Fig. 23.22

Hint

The ducks and the base of the cliff are in a straight line.

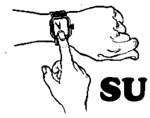
3. From the top of a cliff, 80 metres high, the angles of depression of two ducks in the dam are 15^0 and 20^0. Calculate the distance of each duck from the cliff and hence the distance between the ducks.



4. An observer with his eye 1,5 metres above ground level, is at a distance of 32 metres from the base of a tower. If the angle of elevation of the tower-top, from the observer is 40^0, calculate the height of the tower.

5. A soldier, lying on flat ground, observes that the angle of elevation to the top of a tree-top is 20^0 when she is lying some distance from the foot of the tree. If the height of the tree is 13 metres, how far is the soldier from the foot of the tree?

6. Dumisani leaves school A and travels 18km on a bearing N60^0W to school B. He then changes direction and travels for 12km on a bearing N40^0E to school C. Find the distance AC.



SUMMARY

1. For θ bigger than 90^0, but less than 180^0 i.e. $90^0 < \theta < 180^0$

$\text{Sin }\theta = \text{Sin }(180^0 - \theta)$

$\text{Cos }\theta = -\text{Cos }(180^0 - \theta)$

$\text{Tan }\theta = -\text{Tan }(180^0 - \theta)$

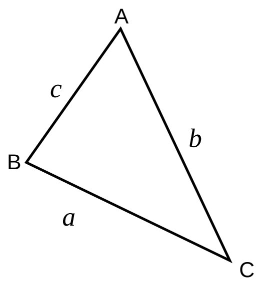
2. Sine Rule:

$$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$$

Ideal for finding sides

$$\frac{\text{Sin A}}{a} = \frac{\text{Sin B}}{b} = \frac{\text{Sin C}}{c}$$

Ideal for finding angles



3. $a^2 = b^2 + c^2 - 2bc \text{ Cos A}$

$b^2 = a^2 + c^2 - 2ab \text{ Cos B}$

$c^2 = a^2 + b^2 - 2ac \text{ Cos C}$

Ideal for finding sides.

or

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{Cos B} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Cos C} = \frac{a^2 + b^2 - c^2}{2ab}$$

Ideal for finding angles.

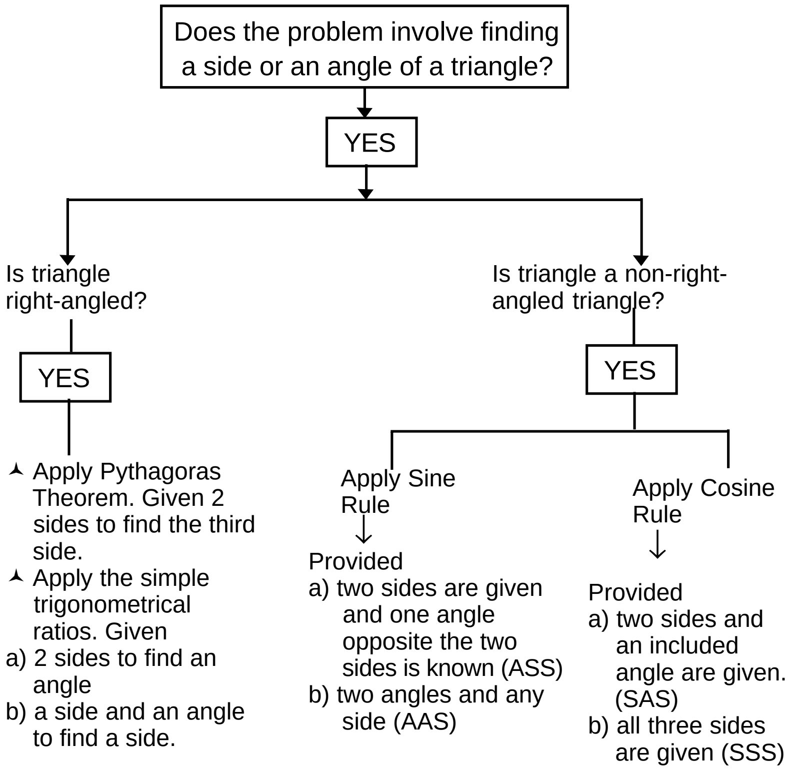
4. Area of a Triangle

$$\text{Area} = \tfrac{1}{2}\text{base} \times \text{height}$$

$$= \tfrac{1}{2}\, bc \text{ Sin A} = \tfrac{1}{2}\, ab \text{ Sin C} = \tfrac{1}{2}\, ac \text{ Sin B}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \tfrac{1}{2}(a+b+c)$$

5. The thinking process for solving problems involving triangles, in an exam, may be aided by using the flow chart below.



EXAM PRACTICE 23

The following example may help you master questions from this chapter.

1. Two shopping centres, B and C, are 12 km and 6km from shopping centre A respectively. C is due East of A.

a) Given that $B\hat{A}C = 150^0$,
calculate:
(i) the bearing of B from A.
(ii) the bearing of A from B.

b) A road is to be built linking B and C. Find the length of the road.

2. Calculate the area bound by triangle ABC and leave your answer in km^2.

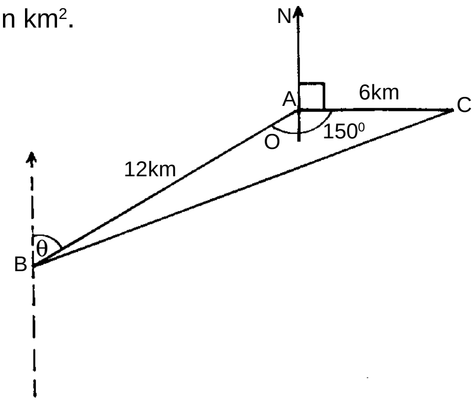


Fig. 23.23

Solutions

1. a) (i) Using the three-figure bearing, the bearing of B from A is given by $90^0 + 150^0$
$= 240^0$
Hence Bearing of B from A is 240^0.

(ii) $N\hat{A}B = 360^0 - 240^0$
$= 120^0$
$\therefore \theta = 180^0 - 120^0$
$= 60^0$
The bearing of A from B is 060^0.

b) Using the cosine rule

$BC^2 = 12^2 + 6^2 - 2(12)(6)\text{Cos } 150^0$
$= 144 + 36 - 2(-\text{Cos } 30^0)$
$= 144 + 36 + 2\text{Cos}30^0$
$= 180 + 2 \times 0{,}8660$
$BC = \sqrt{181{,}732}$
$BC = 13{,}5$ (3 sig fig)

Hence the length of the road is 13,5km.

2. Area of $\triangle$ ABC $= \frac{1}{2}\, ab\text{Sin}\theta$

$= \frac{1}{2} \times (12)(6)\text{ Sin } 150^0$

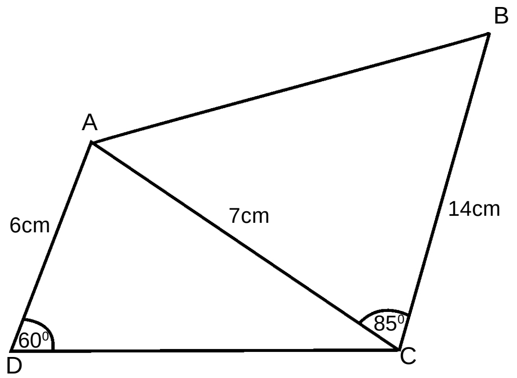
$= \frac{1}{2} \times (12)(6)\text{ Sin } 30^0$

$= 18\text{km}^2$

$\therefore$ Area of $\triangle$ABC = 18km^2

Now do the following:

1.

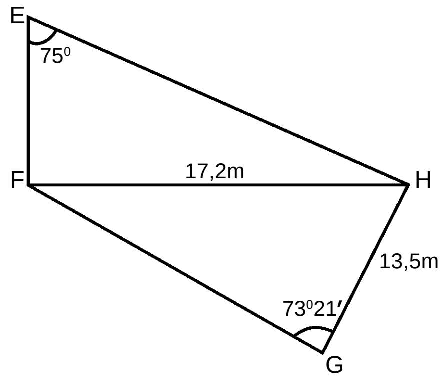


In the above diagram, calculate:

a) $D\hat{C}A$.

b) Calculate the length AB.

2.



In the diagram E, F, G and H are four corners of a sports field on level ground. E is due north of F, H is due east of F, $F\hat{E}H = 75^0$, FH = 17,2m, HG = 13,5m and $F\hat{G}H = 73^0\ 21'$.

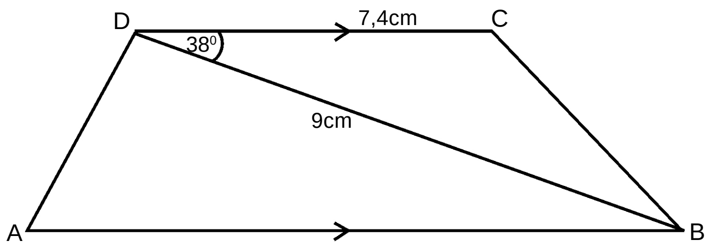
a) Calculate:

(i) the distance EF.

(ii) $G\hat{F}H$.

(iii) the bearing of G from F.

3.



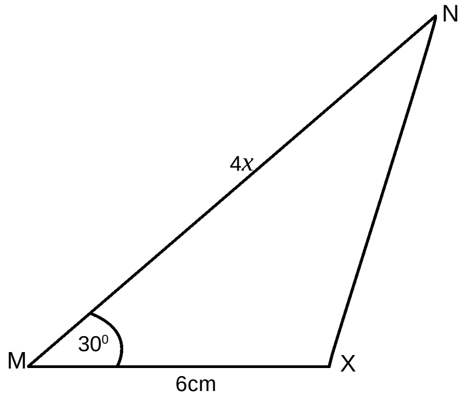
In the diagram, ABCD is a trapezium, in which DC = 7,4cm, BD = 9cm, AB = 2DC and $B\hat{D}C = 38^0$.

Find:

a) $A\hat{B}D$.

b) the area of triangle ABD.

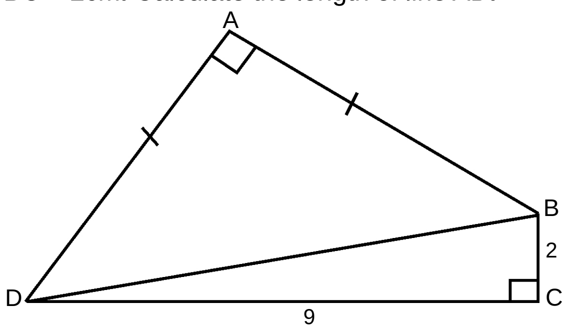
4.



The diagram shows triangle MNX with MN = $4x$ cm, MX = 6cm and NMX = 30^0. Given that the area of the triangle is 15cm^2, find the value of x.

[sin 30^0 =0,5; cos 30^0 = 0,87; tan 30^0 = 0,58]

5. In the diagram, $D\hat{A}B = D\hat{C}B = 90^0$, AB = AD, DC = 9cm and BC = 2cm. Calculate the length of line AD.

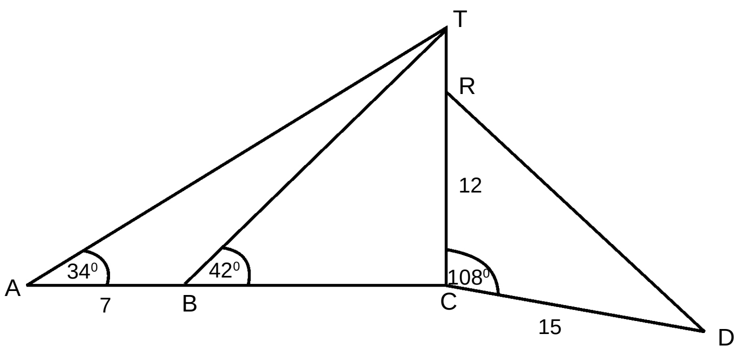


6. A rhombus has sides of length 10cm and one of its angles is 80^0. Calculate

a) the length of the longer diagonal of the rhombus.

b) the area of the rhombus.

7.



A, B and C lie in a straight line on level ground. T is the top of a vertical flagpole, TC.

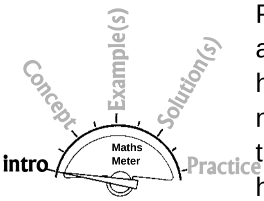
a) Joyce wants to find the height of the flagpole. She measures the angle of elevation to the top of the flagpole from A and finds that it is 34^0.
   She then walks 7m, to B, and finds that the angle of elevation is now 42^0.

   Calculate (i) $A\hat{T}B$.
   (ii) the length BT.
   (iii) the height, TC, of the flagpole.

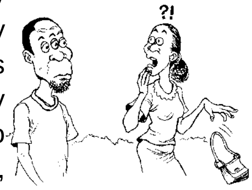
b) On the opposite side of the flagpole, the ground slopes down to D such that $T\hat{C}D = 108^0$.
   A rope is stretched from D, which is 15m from C, to a point R, where CR = 12m.
   Calculate the length of the rope DR.

24 Probability

Boy or girl

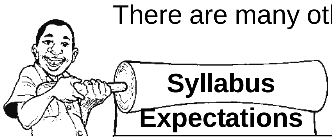


Probability is a numerical measure of the chances or likelihood of an event happening or not happening. Some events will certainly happen e.g. death. On the other hand some events will certainly not happen, e.g. meeting a person with three eyes! In some cases the chances of an event happening are high (i.e.. closer to certainly happen) whilst in other cases the chances are low (i.e.. closer to "certainly not happen"). This means, probability, as a measure, lies between the two extremes 'will certainly happen' and 'will certainly not happen'.



Three eyed man

Probability is used by, **physicists** in studying the various gas and heat laws and in the theory of atomic physics, **biologists** use it in genetics and the theory of natural selection and **managers** in government and industry in economic decision making processes. There are many other uses.



By the end of this chapter, students should be able to:

1. use the following probability terms correctly: random, event, trial, sample space, equally likely, mutually exclusive events and independent events.
2. distinguish between experimental and theoretical probability.
3. solve simple problems involving:
   (i) single events.
   (ii) more than one event.
4. use probability diagrams, like outcome tables and/or tree diagrams, to make calculations.

Pear in an orange tree

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- multiply and add fractions (common or decimals).

- reduce common fractions to their lowest terms or to their simplest form.
- carry out operations on sets of numbers.
- use statistical tables.

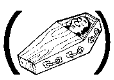
A. BASIC PROBABILITY TERMS

Supposing a bag contains 5 red balls and 7 blue balls; all identical except for colour. Before a ball is picked from the bag, the bag is shaken to mix the balls thoroughly. Mixing the balls creates a situation where each ball is **equally likely** to be picked. (All balls have an equal chance to be picked). In a fair game, a ball is picked **randomly**. The act of picking the ball is called an **experiment**. The pick made is called **an outcome or an event**. The bag represents the **probability space** meaning the possible outcomes. (In this case, the 12 balls of which 5 are red and 7 are blue).

Theoretical probability uses known facts about the probability space. For example, the probability of picking a blue ball from the bag above is $\frac{7}{12} = \frac{\text{Number of blue balls}}{\text{Total balls in the bag}}$

Experimental probability depends on the outcome of experimenting. Here the experiment has to be done several times recording the successes e.g. picking a blue ball. The successes are then compared to the total number of times the experiment has been done. For an example, supposing the experiment (i.e. picking a ball randomly from the bag above) is done 50 times and supposing the blue ball appeared 28 times. The probability of picking a blue ball from this bag will be $\frac{28}{50} = \frac{\text{Number of successes}}{\text{Total possible outcomes}}$

The Theoretical and the Experimental probabilities above show that the numerical measure is actually a fraction! Notice here that simple or common fractions lie between 0 and 1. 0 stands for the probability of an event which **certainly will not happen** whilst **1** stands for the probability of an event which will **certainly happen**.

Thus **P(**  **) = 0** $0 \leqslant$ **Probability** $\leqslant 1$ **P(**  **) = 1**

oranges in banana tree *death*

The size of the fraction indicates whether the chances are high or low.

i.e.

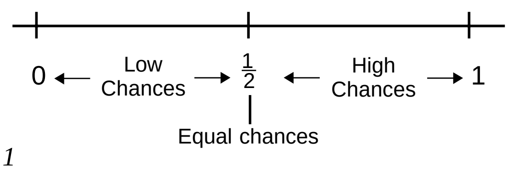


Fig. 24.1

Events are normally denoted by capital letters and capital **P** usually stands for the probability.
e.g. P(A) reads 'the probability of event A happening'.
P(Red) reads 'the probability of picking a red ball'.
It is also important to note that opposing events have complementary probabilities
e.g. If P(Red) $= \frac{5}{12}$, then P(Not Red) $= 1 - \frac{5}{12}$

$$= \frac{7}{12}$$

$\therefore$ P(A) + P($\bar{A}$) = 1. ($\bar{A}$ meaning 'Not A')


Dice

The following probability games are some which are used in formulating probability questions.

(i) Die throwing (plural of die is dice)
The number of sides of the die will be specified.

(ii) Coin tossing
A coin has two sides, a *Head* and a *Tail*.

(iii) A pack of playing cards has 52 cards
26 Black + 26 Red
(13 *Spades* + 13 *clubs*) (13 *Hearts* + 13 *Diamonds*)
It is important that you have a thorough knowledge of the contents of the pack of cards.
Whilst these are the most common games used, other games exist and their appropriate rules should be followed.


Playing cards

Consider the examples below:

1. A die is thrown and the outcome noted. Find the probability that the outcome is:
a) a prime number b) a factor of 18

Solution

1. a) Primes 2, 3, 5
$\therefore$ P(prime) $= \frac{3}{6}$

$= \frac{1}{2}$

b) F18 = 1, 2, 3, 6
$\therefore$ P(F18) $= \frac{4}{6}$

$= \frac{2}{3}$

2. A card is drawn or picked at random from a pack of 52 cards
Find the probability that the card is:
a) a Queen b) a black Ace c) not a Heart

Solution

2. a) There are 4 Queens in the pack
$\therefore$ P(Queen) $= \frac{4}{52}$

$= \frac{1}{13}$

b) There are 2 black Aces in the pack

$\therefore$ P(a black Ace) $= \frac{2}{52}$

$= \frac{1}{26}$

c) There are 13 Hearts in the pack

$\therefore$ P(Not a Heart) $= 1 - \frac{13}{52}$

$= \frac{39}{52} \quad = \frac{3}{4}$

3. There are 42 students in a class. Given that the probability of picking a girl from the class is $\frac{4}{7}$, find the number of boys.

Solution

3. **Method 1**

P(Girl) $= \frac{4}{7}$ means $\frac{4}{7}$ of the class are girls

$\therefore$ Number of Boys $= 42 - \frac{4}{7} \times 42$

$= 42 - 24$

$= 18$

Method 2

P(Girls) $= \frac{4}{7}$

P(Boys) $= 1 - \frac{4}{7}$

$= \frac{3}{7}$

$\therefore$ Number of Boys $= \frac{3}{7}$ of 42

$= 18$

1. A die is thrown and the outcome noted. What is the probability of getting:

a) a 4? b) an odd number?

c) a prime number?

2. A letter is picked at random from the alphabet. Find the probability that:

a) it is not a vowel.

b) it is one of the letters in the following words

(i) Mathematics.

(ii) Statistics.

(iii) Probability.

3. a) In a raffle, 600 tickets are sold. The probability that Sibongile wins the first prize is $\frac{1}{25}$. How many tickets did Sibongile buy?

b) John bought 45 tickets for this raffle. What is the probability of John winning the first price?

4. A bag contains 3 red balls, 5 black balls and 4 yellow balls all identical except for colour. One ball is picked at random from this bag. What is the probability of picking:

a) a yellow ball? b) no red ball?

c) a black or a yellow ball?

5. What is the probability that the next person you meet was born:

a) in January, June or July?

b) on 1, 10 or 30 August?

c) on 30 February?

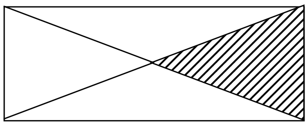
6. 

Fig 24.2 shows a rectangle measuring 12cm by 8cm.

Fig 24.2

A pin is thrown into this rectangle randomly. What is the probability that it lands in the shaded area?

7. A letter is chosen at random from the letters of the word MATHEMATICIA. What is the probability that the letter is

a) A? b) a vowel? c) not M? d) not E?

8. A group of 30 students were asked if they watched ZBC or SABC news on a certain day and time.
If 2 watched neither, 6 watched both and 15 watched SABC news, what is the probability of picking a student who watched

a) SABC news only? b) ZBC news?

9. In a car park in Gwanda, there is a $\frac{5}{8}$ probability of that a car picked at random, is Japanese made. If there are 192 cars in the car park, how many of them are not made in Japan?

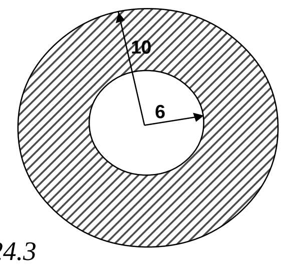
10. 

Fig 24.3 shows a circular disc of radius 10cm with a hole of radius 6cm through the centre. A gambler tries to throw a small stone through the hole. What is the probability of the stone:

Fig 24.3

a) going through the hole?

b) landing in the shaded region?

B. COMBINED PROBABILITIES

This section will discuss mutually exclusive as well as independent events.
Mutually exclusive events cannot happen together.
This means that one excludes the other.

Consider the following situations:

1. A = {the first 5 letters of the alphabet}
B = {vowels}
C = {the last 5 letters of the alphabet}

Playing with the ball *Reading a book*

Notice

- Events A and B are not mutually exclusive as they have common elements between them.
- A and C are mutually exclusive as they have no common elements.
- B and C are also mutually exclusive.

Suppose we want the probability of *either A or C* happening.
A or C implies all the elements of A combine with all the elements of C to form the successful outcomes.

$$\therefore\ P(A \text{ or } C) = P(a, b, c, d, e, v, w, x, y, z)$$
$$= \frac{10}{26}$$
$$= \frac{5}{13}$$

But instead of combining all the elements of the two events, separate probabilities can be used to produce the same result.

$$P(A \text{ or } C) = P(A) + P(C)$$
$$= \frac{5}{26} + \frac{5}{26}$$
$$= \frac{10}{26}$$
$$= \frac{5}{13}$$

This idea gives us the ***Addition law*** of probabilities.

If events Q and R are mutually exclusive then the probability of either Q or R happening is the sum of their probabilities.

$$P(Q \text{ or } R) = P(Q) + P(R)$$

Notice:
Mutually exclusive events are more likely to happen since the successes space is widened by the 'either or' set up, hence addition of the probabilities – to make the result bigger.

However, in case the events are not mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$ coming from the set truth that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

2. Independent events have no effect on the other. For example picking a vowel from the alphabet and throwing a die. What is happening in the alphabet has no effect on what is happening in the die throwing!

If A = {Pick a vowel from the alphabet} and
B = {Getting an even number from a die}.

$P(A) = \frac{5}{26}$, $\quad P(B) = \frac{3}{6}$

$\therefore$ P(A <u>and</u> B) $= P(A) \times P(B)$

$= \frac{5}{26} \times \frac{3}{6}$

$= \frac{5}{52}$

This gives us the ***Product law*** of probabilities.

If two events are independent, then the probability of both happening is the product of their respective probabilities.

P(A and B) = P(A) $\times$ P(B)

Notice: Independent events are less likely to happen since the success space is made less likely by the *'and'* set up. The Product law – makes the result smaller.

Hint

Catch words to note!

(i) **either** A **or** B implies **addition** law.
(ii) A **and** B, A **followed by** B, A **then** B **both** A and B imply **product** law.

Consider the following example:

1. A pack of cards is randomly shared.
What is the probability of one getting:
a) a spade or a 2 of hearts?
b) an 8 of clubs and an 8 of diamonds?

Solution

1. a) P(spade or a 2 of hearts) = P(spade) + P(2 of hearts)

$= \frac{13}{52} + \frac{1}{52}$

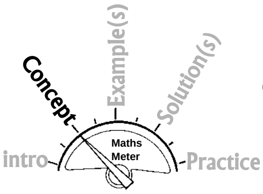
$= \frac{14}{52}$

$= \frac{7}{26}$

b) P(8 of spade and an 8 of diamonds) $= \frac{1}{52} \times \frac{1}{52}$

$= \frac{1}{2704}$

The experiemnt in (b) can be carried out in two ways:
Pick the first card, note it, *replace* it, then pick the second card! (shown by the calculation above).



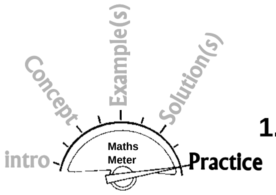
If the first card is not replaced, then P(8 of spades *then* an 8 of diamonds) $= \frac{1}{52} \times \frac{1}{51}$

$= \frac{1}{2652}$

Note that:

After the first pick, the total cards changes from 52 to 51, hence $\frac{1}{51}$ for the second pick.

Watch out for the '*with or without replacement' statements*.



1. From a pack of 52 playing cards, what is the probability of picking:

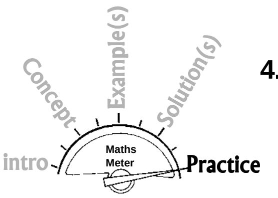
a) an ace? b) either a club or a red ace

2. If two cards are picked from a 52 pack of playing cards, with replacement, find the probability that:

a) the first card is black and the second also black.

b) a 4 of spades followed by a 2 of diamonds are picked.

3. If the situation in question 2 was 'without replacement', recalculate the probabilities asked.



4. A coin is tossed and a die is thrown. This is done simultaneously. What is the probability of getting:

a) a six and a head.

b) a perfect square and a tail.

5. A bag contains 3 red discs, 4 black discs and 5 yellow discs, all identical except for colour. Find the probability of picking:

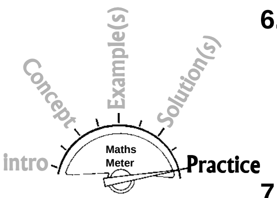
a) a yellow or a black disc.

b) neither a red nor a yellow disc.

c) a disc.

d) a pink disc.

Use the bag in question 5, for questions 6 and 7.



6. Two discs are picked one after the other, with replacement. Find the probability that:

a) the first disc is black and the second red.

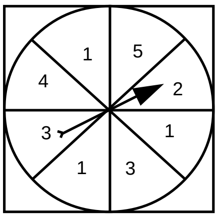
b) both the first and the second one are yellow.

7. Two discs are picked, one after the other, without replacement. Find the probability that:

a) the first disc is black and the second red.

b) the first and second discs are yellow.

c) the first is red and the second is yellow.

8. 

The pointer on the number wheel, in Fig 24.4, is spun and, when it stops spinning, it could be pointing at any one of the numbers marked.

Fig 24.4

Find the probability of the arrow pointing at:

a) a 3 or a 5. b) a 1 or a 3.

c) a 1 or a 4. d) a 5 or a 4.

9. If, the arrow in Fig 24.4 is spun twice, find the probability that it points at:

a) a 2 followed by a 1.
b) a 1 followed by a 1.
c) a 5 then a 3.
d) a 4 in both cases.

10. The probability of an event, A, happening is $\frac{1}{4}$ and the probability of an event, B, happening is $\frac{3}{5}$. Given that A and B are independent, calculate the probability that:

a) both events will happen.
b) B will happen but A will not.

11. A fridge contains x Fanta cans and y Cola cans. If a can is taken from the fridge at random, the probability that it is a Fanta can is $\frac{2}{7}$.

a) Write down an equation connecting x and y.
b) If there were 3 more Fanta cans in the fridge, the probability of taking a Fanta can would be $\frac{1}{3}$.
   (i) Form another equation connecting x and y.
   (ii) Find the value of x and the value of y.

C. OUTCOME TABLES AND TREE DIAGRAMS

Sometimes diagrams simplify or clarify given situations. Suppose 2 six sided dice are thrown and the results noted. Without a diagram it is not easy to see how many possible combinations there are. Below is a typical **outcome table**.

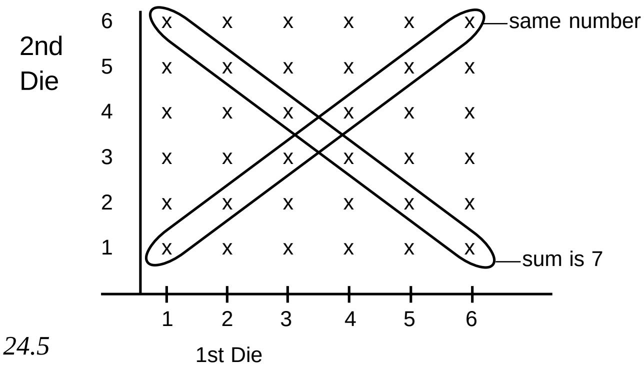


Fig 24.5

Notice the possible combinations are clearly 36.
Thus the favourable outcomes are simply picked from the total 36.

Consider the following examples:

1. Find the probability that when two dice are thrown
a) both show the same number.
b) the sum of the faces shown is 7.

Solution

1. a) From the diagram, mark combinations of same numbers.

$$\therefore \ P(\text{Same number}) = \frac{6}{36} = \frac{1}{6}$$

b) From the diagram, mark combinations which give a sum of 7.

$$P(\text{Sum is 7}) = \frac{6}{36} = \frac{1}{6}$$

Tree diagrams are also used to solve probability problems. A tree diagram is made up of branches, each of which represents a probability. Each branch splits into further branches.

2. If a bag contains 5 red and 7 blue identical cards, find the probability, without replacement, of picking 2 cards which are
a) red followed by blue.
b) of the same colour.
In this situation a Tree diagram is most convenient to use. See fig 24.6

Solution

2.

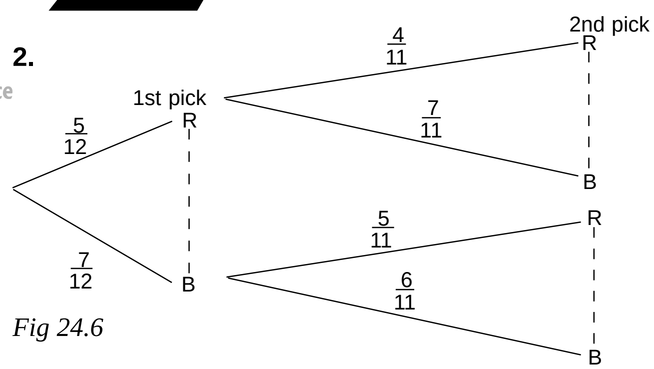


Fig 24.6

Notice that:

- the tree starts with 2 branches since there are 2 colours in the question. It then follows that if there were 3 colours, the tree would start with 3 branches.
- each branch stands for the probability of getting the colour at the end of that branch, i.e. red (R) blue (B).

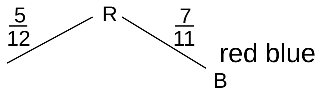
A closer study of the branches will reveal that:

- each pair of branches is complementary i.e. the 2 probabilities add up to 1.
- the denominator at each pick is the same and depends on the 'with or without' condition.

To illustrate: if the first pick was red, 11 cards remain (since it is a 'without' replacement situation) of which 4 are now red and 7 blue.
That is why P(R) $= \frac{4}{11}$ and P(B) $= \frac{7}{11}$ in the branches which follow the first red. The same explanation goes for blue as the first pick.

Also if there was to be a third pick, 4 pairs of branches would have to be added, using denominator 10.
Let us use the diagram to answer the questions asked.

Diagram



a) P(RB)

$= \frac{5}{12} \times \frac{7}{11}$

$= \frac{35}{132}$

Now you should understand why the probabilities are multiplied

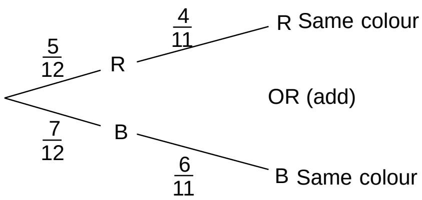
b) P(RR or BB)

$= \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11}$

$= \frac{20}{132} + \frac{42}{132}$

$= \frac{62}{132}$

$= \frac{31}{66}$

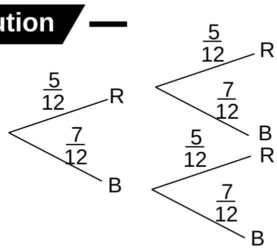


3. If a bag contains 5 red and 7 blue identical, find the probability, with replacement, of picking 2 cards which are

a) red followed by blue.
b) of the same colour.

Using the same bag as in example (2) on page 154 but this time with replacement, the tree diagram would look like

Solution



a) P(RB) $= \frac{5}{12} \times \frac{7}{12}$

$= \frac{35}{144}$

b) P(RR OR BB) $= \frac{5}{12} \times \frac{5}{12} + \frac{7}{12} \times \frac{7}{12}$

$= \frac{25}{144} + \frac{49}{144}$

$= \frac{74}{144}$

$= \frac{37}{72}$

Hint

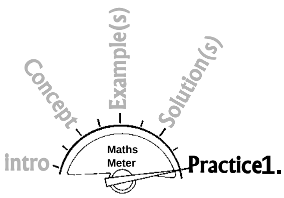
Some problems do not say whether it is a 'with or without' replacement situation. However the item under discussion should give a guide.

e.g. (1) Tossing a coin 3 times

➤ is a **with** replacement situation since the coin has to be used again or you cannot continue, the result is always, head or tail.

(2) If people are involved (e.g. choosing prefects), this is a **without** replacement situation since once one is chosen, they can no longer be part of the number under consideration again. It is the rest who are to compete next.

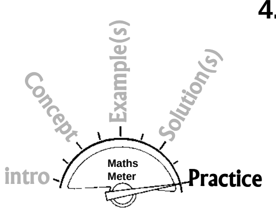
Watch out *and make a logical decision.*



1. A coin is tossed and a die is thrown and the results noted. Use an outcome table to calculate the probability of getting:
a) a head and a 5.
b) a tail and a factor of 6.

2. Two coins are tossed together,
a) show all the possible outcomes on an outcome table.
b) use the table to find the probability that the two coins show different faces.

3. Two dice are thrown. Find the probability that:
a) the sum of the two numbers shown is a perfect square.
b) the products of the two numbers shown is bigger than 10.
c) the two numbers differ by 2.



4. There are 20 prefects at a certain school. Of these 12 are boys and 8 are girls. A sub committee (the entertainment committee) of 3 members is to be created from the group of prefects.
What is the probability that the committee is made of:
a) 1 boy, 2 girls?
b) 2 boys, 1 girl?
c) all members of the same sex?

5. From a pack of seeds, the probability of a seed germinating is $\frac{3}{5}$. If three seeds are picked at random, find the probability that:
a) they all do not germinate.
b) they all germinate.
c) only 2 germinate.

6. The table below shows the distribution of the marks scored by 200 students in their final examination.

Marks	0 to 19	20 to 39	40 to 59	60 to 79	80 to 99
f	11	30	55	72	32

a) A students is selected randomly from the 200. What is the probability that the student scored 59 or less?

b) Another student is selected randomly from the rest. What is the probability that the student scored marks of 80 to 99.

c) Two student are selected from the 200, one after the other. What is the probability that both students scored marks between 20 and 39.

7. A coin is tossed 3 times. Draw tree diagrams to find the probability of getting:

a) a Head followed by a Head then a Tail

b) the same side and three times.

c) at least two tails.

8. A bag contains 3 black, 4 white and 5 red balls all identical except for colour.

A ball is picked at random from the bag, its colour noted and it is put away. Another ball is then picked from the bag. What is the probability that:

a) the first ball is red and the second white?

b) the two balls are of the same colour?

c) at least one of the balls is black?

9. Two spinners, the first numbered 1 to 5 and the other second numbered 11 to 15, are spun together.

a) Show, on an outcome table, all the possible combinations resulting from the double spin.

b) Using the table or otherwise, find the probability that

(i) the total of the scores will be more than 15 but less than or equal to 18.

(ii) the product of the scores will be less than 30.

(iii) the first shows a prime number whilst the second shows is an odd number.

10. The Tree diagram below in Fig 24.8 shows the probabilities of picking counters from a box containing red (R) and white (W) counters.

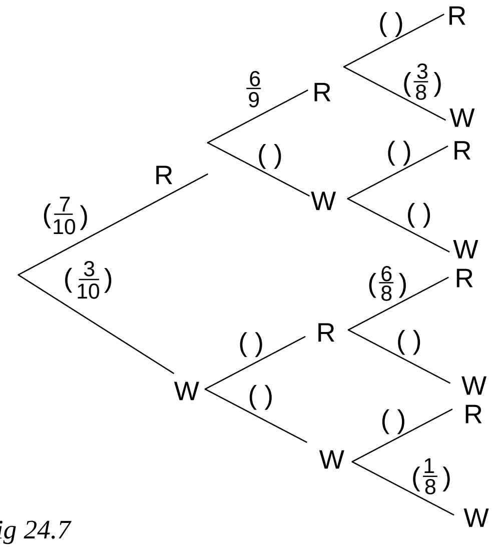
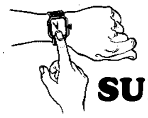


Fig 24.7

a) Say whether the situation is a 'with or without' replacement.
b) Copy and complete the tree diagram.
c) How many counters are in the bag at the beginning?
d) Using the diagram or otherwise, find the probability of picking:
   (i) at most two red counters.
   (ii) at least two white counters.
   (iii) three counters of the same colour.



SUMMARY

1. $0 \leqslant$ **Probability** $\leqslant$ **1**

   $\swarrow$ **Will certainly not happen** $\searrow$ **Will certainly happen**

2. $P(A) + P(\bar{A}) = 1$ The events A and $\bar{A}$ are complements!

3. Addition law – from ‘either or’ e.g. P(A) + P(B) = P(A or B)

4. Product law – from ‘and, followed by then’ etc.
   P(A and B) = P(A) $\times$ P(B)

5. Outcome tables and Tree diagrams can be used.

6. Probabilities on branches of tree diagrams are always complementary.

7. ‘With replacement’ maintains the total i.e. denominator.

8. ‘Without replacement’ changes the total i.e. denominator at every following stage or pick.

9. Independent events have no effect on each other. They are linked to the product law of probabilities.

10. Mutually exclusive events cannot happen together. They are linked to the addition law of probabilities.

11. Theoretical probability uses known facts about the probability space.

12. Experimental probability depends on the outcome of experimenting.

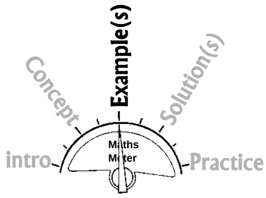
EXAM PRACTICE 24

The following examples may help you master skills taught in this chapter. Study them carefully. Questions usually state the nature of the answer required.

e.g. give the answer as a common fraction in its lowest terms.
or give the answer as a decimal.

Candidates often lose vital marks due to not paying attention to this instruction. ***Watch out***!

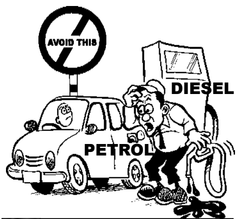
Consider the following examples



1. A bag contains mint, toffee and chocolate sweets, all identical except for their flavour. The probability of picking a mint is $\frac{3}{5}$ and that of picking a chocolate is $\frac{1}{4}$.

a) A sweet is picked at random from the bag. What is the probability of picking either a mint or a toffee. Give your answer as a decimal.

b) If two sweets are picked from the bag, one after the other, with replacement, calculate the probability, as a percentage, that one is a mint and the other one is a chocolate.



Common Error

1) Candidates have a tendency to simply use what is given, hence $\frac{3}{5} + \frac{1}{4}$ is calculated.

Solution

Hint

The probability of picking a toffee is not given but can be found.

1. a) P(a Toffee) $= 1 - (\frac{3}{5} + \frac{1}{4})$

$$= 1 - \frac{12+5}{20} = 1 - \frac{17}{20} = \frac{3}{20}$$

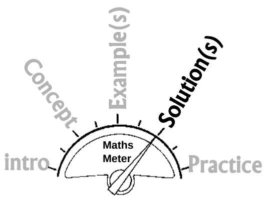
$\therefore$ P(a Mint or a Toffee) $= \frac{3}{5} + \frac{3}{20}$

$$= \frac{12+3}{20}$$

$$= \frac{15}{20}$$

$$= \frac{3}{4}$$

$$= 0{,}75$$



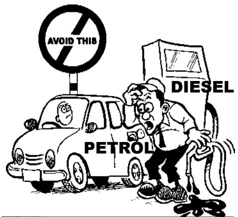
b) $\therefore$ P(MC) or (CM) $= \frac{3}{5} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{5}$

$$= \frac{3}{20} + \frac{3}{20}$$

$$= \frac{6}{20}$$

$$= \frac{6}{20} \times \frac{100}{1}\%$$

$$= 30\%$$



Common Error

2) The answer is usually left as $\frac{3}{4}$, disregarding the requirements of the question.

1) Only one way

$\frac{3}{5} \times \frac{1}{4}$

is considered.

2) Disregarding the instructions.

2. There are x red and y yellow discs in a bag.
A disc is drawn at random from the bag.
a) Give, in terms of x and y, an expression for the probability of drawing a yellow disc.
b) Also given that the probability in (a) is $\frac{3}{7}$, express y in terms of x.

Solution

2. a) $\frac{y}{x+y}$

b)
$$\frac{y}{x+y} = \frac{3}{7}$$
$$7y = 3x + 3y$$
$$4y = 3x$$
$$\therefore y = \frac{3}{4}x$$

Now do the following:

1. In a school choir, there are 18 girls. Given that the probability of picking a girl from the group was $\frac{6}{11}$:
a) find the number of boys in the choir.
b) find the number of girls who joined the choir given that 12 more students joined the choir, making the probability of choosing a girl $\frac{3}{5}$.

2. The probability that Zanele is early for school is $\frac{3}{8}$ and that Dumisani is late for school is $\frac{2}{5}$. Giving each answer as a common fraction in its lowest terms, find the probability that on a given day:
a) Zanele is late for school.
b) they are both early for school
c) either Zanele or Dumisani is late for school.

3. The table below shows the frequency distribution of the number of counters children had at a certain creche.

No. of counters	0	1	2	3	4
No. of children	3	4	8	2	3

Pupils are selected at random from the group. Giving your answer as a common fraction, calculate the probability of selecting.
a) a pupil with 2 counters.
b) two pupils, one after the other, with 1 counter each.

4. A bag contains 7 mints and 3 toffees, all identical except for their flavour. A girl takes 3 sweets out of the bag randomly, one after the other, without replacement. Find the probability that:
a) all three sweets taken out are toffees.
b) at least one, of the three sweets, is a mint.

5. A woman and her son are both HIV positive. The woman estimates that the probability that she will be alive in the next 10 years is 0,6 and that her son will be alive in that time is 0,4.
What is the probability that:
a) both will be alive in 10 years?
b) at least one of them will die within 10 years?

6. Two six-sided dice are thrown at the same time.
What is the probability that:
a) the sum of the two faces showing is (i) odd?
(ii) perfect?
b) both faces show even numbers?

7. A bag contains 7 red balls and 5 black balls, all identical except for colour. 3 balls are picked at random, one after the other, without replacement.
a) Copy and complete the tree diagram below.

Note R $\Rightarrow$ Red
B $\Rightarrow$ Black

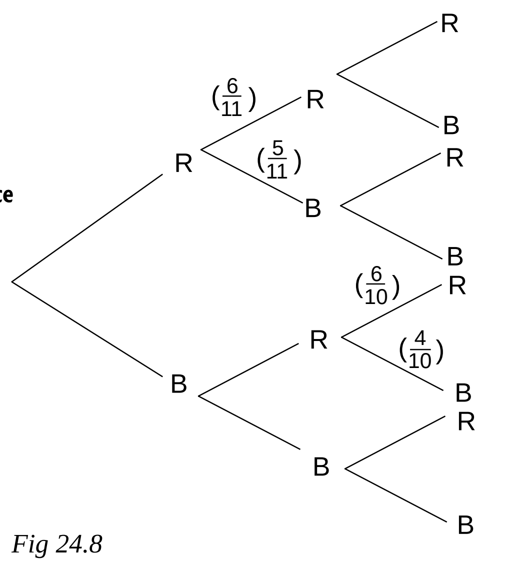
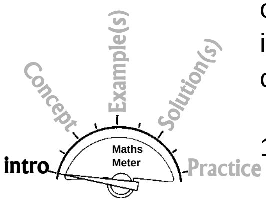


Fig 24.8

b) Find the probability that two, of the 3 balls picked, are black.

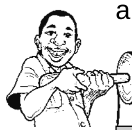
25 Statistics



Statistics is that branch of Mathematics which deals with the collection, presentation, analysis and interpretation of numerical information (data). Below are examples of organisations busy collecting and processing statistical data on a day to day basis.

1. The Police : road carnage during festive period. 'Don't' drink and drive, lest you become a statistic'.
2. Hospital Personnel: recording births and deaths at that place and time. A baby born at a certain institution becomes a statistic.
3. Relief Agencies : there has been a drought in the country. Information is collected on numbers of people who need relief aid and what sort of aid they need.

This chapter is going to deal mainly with the presentation, analysis and intepretation of data that has already been collected.



Syllabus Expectations

By the end of this chapter, students should be able to:

1. read, interpret, draw and make simple inferences from bar charts, pie charts, histograms, frequency tables and frequency polygons.
2. calculate the mean, median and mode from given data and distinguish the purposes for which each is used.
3. use an assumed mean to calculate the mean.
4. read and interpret data presented in class intervals.
5. draw and use the cumulative frequency curve/ogive.

ASSUMED KNOWLEDGE

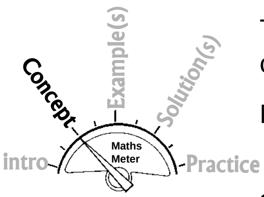
In order to tackle the work in this chapter, it is assumed that students are able to:

- arrange numbers in order of size

- apply the four arithmetic operations, $(+, -, \times, \div)$, to simple numbers, directed numbers and expressions.
- draw axes correctly and apply scale to graduate them and plot points on the Cartesian plane.
- work out probability questions.
- reduce fractions to their lowest terms.

A. UNGROUPED DATA

Averages are measures of central tendency. This section will deal with three types of averages, the mean, the median and the mode.



The Arithmetic Mean (Mean)
This is what is generally referred to as the average and is calculated as:

$$\text{Mean} = \frac{\text{Sum of terms in data}}{\text{Total number of terms in data}}$$

1. Supposing you are given the following data 3, 1, 3, 0, 5, 3, 4, 3

$$\text{Mean} = \frac{3+1+3+0+5+3+4+3}{8}$$

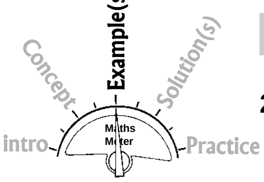
$$= \frac{22}{8}$$

$$= 2\tfrac{3}{4} \text{ or } 2{,}75$$

Notice that the repeated 3 is not taken as a single item but as four separate items.

The Mode
This is the most frequent term in the data (frequency is the number of times a term appears in the list)
In the above data, the mode is 3.

The Median
This is the middle term **when the terms are arranged in order of size**, ascending or descending.



Consider the examples below:

2. Given 3, 1, 2, 5, 3, find the median.
a) Arrange in order of size first: 1, 2, $\boxed{3}$, 3, 5
middle term

$\therefore$ The Median = 3

Common Errors
Failing to arrange data in order of magnitude first. Hence 2 is given as the median in this case.

b) Given 7, 4, 7, 4, 5, 4, find the median.
Arrange first! 4, 4, $\boxed{4, 5}$ 7, 7
middle two terms

$$\therefore \text{ The median} = \frac{4+5}{2} = 4\tfrac{1}{2}$$

This approach for finding the averages, still applies if the terms are fractional, as given below.

3. $\frac{3}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{1}{3}$ and $\frac{3}{5}$

Mode is clearly = $\frac{3}{5}$

Median. Order first!
i.e. $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{3}{5}$, $\frac{3}{5}$

Median = $\frac{3}{5}$

$$\text{Mean} = \left(\frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{3}{5} + \frac{3}{5}\right) \div 5$$

$$= \left(\frac{10 + 15 + 18 + 18 + 18}{30}\right) \div 5$$

$$= \frac{79}{150}$$

The three examples show that:
(i) the median can be picked straight from the data (when the total number of terms is odd).
(ii) median is the average of the two middle terms (when the total number of terms is even).

Any one, of the three averages, can represent or stand for the entire data, hence they are referred to as **measures of central tendency.**

Frequency Table

Sometimes the terms are not listed individually, as in the examples above, but are given in a frequency table as shown below.
Table 25.1 shows the number of sisters each student in Form 4C had.

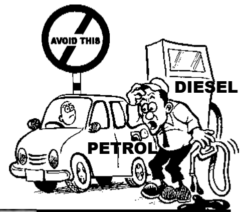
Table 25.1

Number of sisters	0	1	2	3	4
Number of students (frequency)	9	6	7	5	3

In this case, the number of students with the given number of sisters is the frequency of that term, hence the **frequency table**. For example, the frequency of 2 is 7 (2 appears 7 times) and that for 1 is 6 (1 appears 6 times) etc.

From the table

- The frequency row always gives the total participants in the exercise, (in this case 30 students = 9 + 6 + 7 + 5 +3).
- The mode is clearly 0 since it is the term with the highest frequency.
- The median is equally easy to find. The advantage of a frequency table is that the data is already arranged in order of size (from 0 to 4). Now using the frequencies from either side.



Common Errors

The frequency (9 in this case) is given as the mode instead of the term.

No. of sisters	0	1	2	3	4
Number of students (frequency)	9	6	7	5	3

9 + 6 = 15 → ← 15 = 7 + 5 + 3

Add frequencies from 9 to the right and from 3 to the left. These additions show that the median is between the last 1 and the first 2, hence $\frac{1+2}{2} = 1{,}5$

Common Error

Median = 2 since it is the middle term of the list 0, 1, 2, 3, 4.
This is wrong since there are 9 zeros in the table instead of 1 zero and so forth.

This agreed with the result, if the data was listed individually.

15 terms

0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,2,2,2,2,2,2,2,3,3,3,3,3,4,4,4

15 terms   middle two

***Note that** there is no need to list terms individually like this.*

When the total frequency is odd, the sharing will leave a term in the middle. That one will be the median.

For an example

No.	2	3	4	5	6
f	2	3	1	5	2

2 + 3 + 1 = 6 → 5 left out ← 6 = 4 + 2

Common Error

Using some and not all the terms $\frac{0+1+2+3+4}{5}$

There are more terms than 5.

So the median = 5

the mean $= \frac{9 \times 0 + 6 \times 1 + 7 \times 2 + 5 \times 3 + 4 \times 3}{30}$

$= \frac{47}{30}$

$= 1\frac{17}{30}$

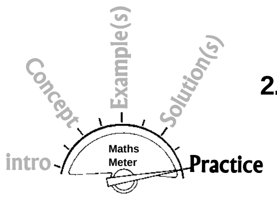
The multiplication in the numerator is to find the total of each term.

Hint

When total frequency is odd – Median is the number in the middle of the data. When the total is even – median is the average of the middle two numbers.

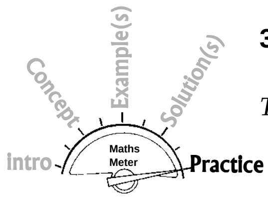
1. Say whether the average in the given statement is mean, mode or median.

a) I earn a living mainly by farming activities.
b) Most Form 3A students put on size 7 shoes.
c) The average rainfall this month was 20ml.
d) I wish to be in the top half of the class this term
e) Her average mark in Mathematics this term was 36%.
f) Tendai was voted class monitor for Form 3A.



2. Calculate the mean, median and mode in each of the following,

a) 2, 3, 3, 4, 4, 4, 5
b) 5, 7, 8, 9, 9, 10
c) 2, 1, 3, 5, 2, 6, 2, 4, 2
d) 11, 20, 18, 15, 13, 20, 19, 15, 17, 18, 20, 12
e) 5, –3, –1, 3, –3, 6
f) –0,9; 0,3; 1,2; –2,7; 1,2; 0,5; 1,2; 0,6; –1,2.
g) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{1}{3}$
h) 0,3; 0,25, 1,76; 3,2; 0,3
i) $\frac{2}{7}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{7}$
j) 3,23; 1,06; 3,23; 2,53



3. From the frequency tables below, find the mean, median and mode, in each case.

Table 25.2

a)

Age (yrs)	11	12	13	14	15
Frequency	2	9	6	4	4

Table 25.3

b)

Distance (km)	1	2	3	4	5
Frequency	3	5	10	1	1

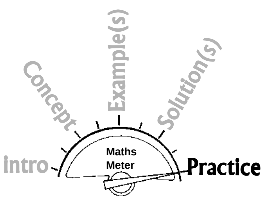
Table 25.4

c)

No. of crayons	0	1	2	3	4	5
No. of pupils with	5	13	8	4	6	4

4. Below are the results, in percentages of five students who took an entrance test, in three subjects.

Table 25.5



	English	Maths	G. Paper
Brighton	69	52	17
Tendekai	53	80	49
Sipho	60	26	73
Billy	43	62	66
Tandiwe	37	44	58

Find a) the mean mark of each subject.
b) the median mark of each student.

B. THE ASSUMED MEAN

This is sometimes called the **Working Mean** and can be used to calculate the mean.

Let us go back to data given earlier on.

1. 3, 1, 3, 0, 5, 3, 4, 3

Note that the words assumed 'mean' implies a guess. One is expected to inspect the data and guess the mean before calculating it. Ideally, the guess should be between the smallest and the biggest term in the list. The guess can be one of the terms in the list or not. Suppose someone chooses 2 as the assumed mean for the list above. From this, **deviations**, are calculated by subtracting the assumed mean from each term to see how much it deviates (differs) from them.

Thus **Deviation = Term – Assumed Mean**

3 – 2, 1 – 2, 3 – 2, 0 – 2 etc

Deviations are 1, –1, 1, –2, 3, 1, 2, 1

Total deviations = 6 (add all deviations)

Mean deviation $= \frac{6}{8} \left(\frac{\text{Total deviations}}{\text{No. of terms}}\right)$

$\therefore$ **The Actual Mean = Assumed Mean + Mean deviation**

$= 2 + \frac{6}{8}$

$= 2\frac{3}{4}$

If someone had picked 3 to be the Assumed Mean,

Total Deviations = –2

Mean deviation $= -\frac{2}{8}$

Actual Mean $= 3 + (-\frac{2}{8})$

$= 2\frac{6}{8}$

$= 2\frac{3}{4}$

2. When the terms are numerous, as in question 2d of Practice 25A on page 167, the layout may be modified.
Suppose the assumed mean in question 2d is 16.

	16
–5	
	4
	2
–1	
–2	
	4
	3
–1	1
	2
	4
–4	
–14	20
	6

Note, the deviations are separated, i.e. negatives from positives.

Each column is added.
The two totals are combined and the result put on the relevant side (positive side if positive, negative side if negative).

Mean deviation $= \frac{6}{12}$

$= \frac{1}{2}$

$\therefore$ Actual Mean $= 16 + \frac{1}{2}$

$= 16\frac{1}{2}$

Tip

When the total deviation happens to be zero then the assumed mean is indeed the actual mean.

Using 2 as the assumed mean for data in table 25.1 on page 165.

Total deviation $= (-2 \times 9) + (-1 \times 6) + (0 \times 7) + (1 \times 5) + (2 \times 3)$

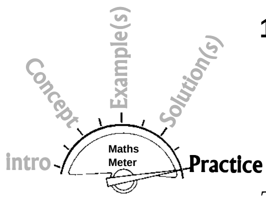
$= -18 - 6 + 0 + 5 + 6$

$= -13$

Mean deviation $= \frac{-13}{30}$

$\therefore$ Actual Mean $= 2 + (-\frac{13}{30})$

$= 1\frac{17}{30}$



1. Use the given assumed mean to calculate the mean of the following data.

a) 126, 127, 131, 133, 134 Assumed Mean 130

b) 9, 2, 3, 9, 10, 11, 8, 3, 6, 6, 8, 1, 4, 4, 3, 9, 3. Assumed Mean 6

Table 25.6

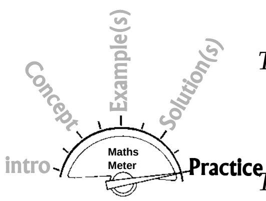
c) Assumed Mean 13

Age (yrs)	11	12	13	14	15
Frequency	2	9	6	4	4

Table 25.7

d) Assumed Mean 3

No of crayons	0	1	2	3	4	5
No. of pupils with	5	13	8	4	6	4



2. Choose an assumed mean and use it to calculate the mean of the following:

a) 5, 9, 3, 4, 3, 8, 7, 3, 8, 4

b) 78, 82, 86, 89, 74, 81, 74, 76, 80, 88, 90, 74, 77, 85, 81

Table 25.8

c)

Distance (km)	1	2	3	4	5
Frequency	3	5	10	1	1

Table 25.9

d)

No. of seeds	0	1	2	3	4
No. of pods with	5	6	8	11	5

C. REPRESENTING UNGROUPED DATA

This part of the chapter will deal with some of the basic statistical diagrams used to represent data in diagram form.

The Bar Chart

Consider the example below:

1. 50 students were asked how they got to school on a particular day. The table below shows the results of the survey.

Table 25.10

Transport	Bus	Walking	Car	Bicycle
No. of Pupils	10	20	8	12

The Bar Chart of the above data would look like Fig 25.1 below.

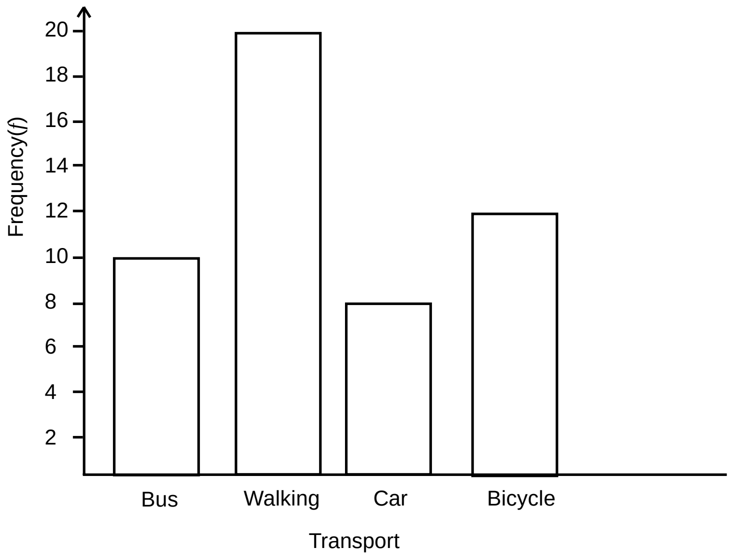


Fig. 25.1

OR

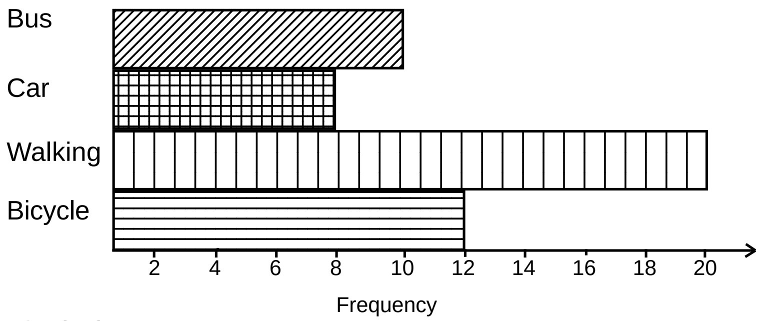


Fig. 25.2

OR

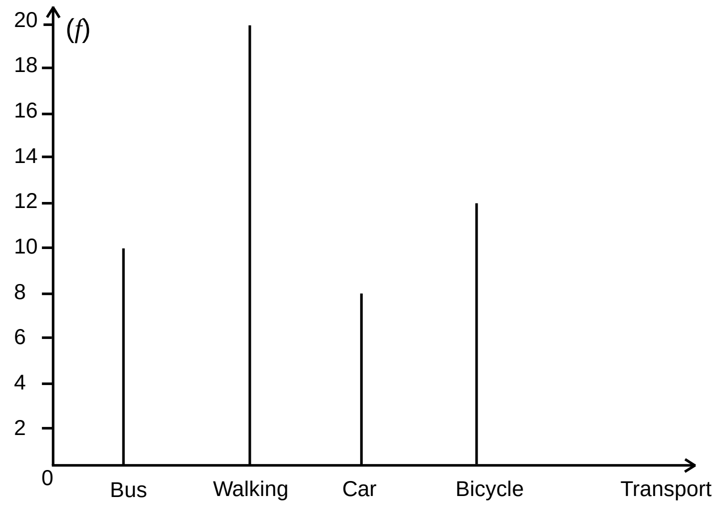


Fig. 25.3

Features of a Bar Chart:

- Only the frequency axis is scaled.
- The bars are of uniform thickness and evenly spread.
- The thickness of the bars is insignificant.
- The diagram can be vertical or horizontal.
- The bars can be coloured differently or shaded differently.
- The height/length of bar is what is significant (catches the eye).

The Pie Chart

The same type of information which can be represented on a bar chart can be represented on a pie chart.
A pie chart is a circle divided proportionally into slices or sectors, each representing a particular part of the given data.
From Table 25.10 each means of travel will be a sector of a pie chart.
The size of each sector is proportional to the respective frequency.
Angles are needed to cut the circle into the respective sectors.
Below is a table showing how the angles are obtained.

2. *Table 25.11*

Transport Means	Frequency	Angle at Centre
Bus	10	$\frac{10}{50} \times 360 = 72^0$
Walking	20	$\frac{20}{50} \times 360 = 144^0$
Car	8	$\frac{8}{50} \times 360 = 57{,}6^0$
Bicycle	12	$\frac{12}{50} \times 360 = 86{,}4^0$
Total	= 50	Total $= 360^0$

In summary, **angle of a Sector** $= \dfrac{\textbf{frequency}}{\textbf{Total frequency}} \times 360^0$

When the angles have been calculated and checked that they add up to 360^0, the diagram can be drawn.

i.e. A circle of convenient radius (preferably more than 5cm) is drawn. The four angles are then measured starting from a drawn radius.

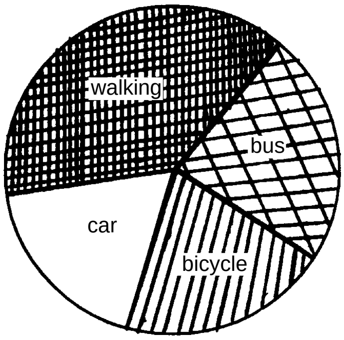


Fig. 25.4

Features of a Pie Chart:

- The radius of the circle is insignificant.
- Each sector is labelled using the data it is representing.
- The angles at the centre may not be indicated.
- The sectors can be coloured differently.

Sometimes students are asked to answer questions which require them to interpret charts.

Consider the examples below:

3.

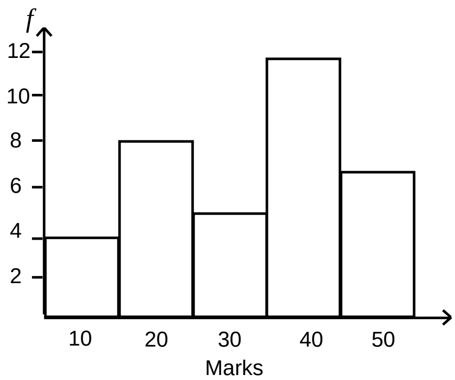


Fig 25.5

Fig 25.5 is a bar chart showing the distribution of marks obtained by students in a certain test.

a) How many students were in the survey?

b) Which marks were attained by the same number of students?

c) What is the modal mark?

d) Find the mean mark of the test.

Solution

3. a)

Mark	No. of students
10	4
20	8
30	5
40	12
50	7
Total =	36

b) None No bars have the same height.

c) Modal mark = 40 (Has the highest bar)

d) $$\text{Mean} = \frac{10\times 4 + 20\times 8 + 30\times 5 + 40\times 12 + 50\times 7}{36}$$

$$= \frac{1180}{36} = 32\frac{7}{9}$$

4.

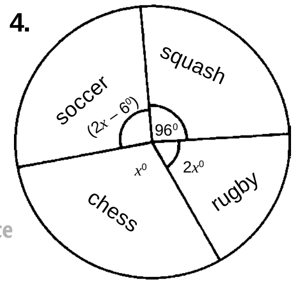


Fig 25.5 shows the popularity of the various games offered by a Sports Academy. Every student takes part in at least one of the games in the diagram.

Fig. 25.6

a) Find the value of x
b) If 180 students joined rugby, find the total academy enrolment.
c) How many students take part in soccer?

Solution

Tip

Angles at a point add up to 360^0.

4. a) $x + 2x + 2x - 6^0 + 96^0 = 360^0$

$5x = 270^0$

$x = 54^0$

b) Angle for rugby $= 2\times 54$

$= 108^0$

i.e.

108^0	180
360^0	? more

$\therefore$ Total enrolment $= \frac{360^0}{108^0} \times \frac{180^0}{1}$

$= 600$ students

b) Angle for soccer $= 2(54^0) - 6^0$

$= 108^0 - 6^0$

$= 102^0$

No. for soccer $= \frac{102}{360} \times \frac{600}{1}$ or $\frac{102^0}{108^0} \times 180$

$=$ 170 students

Other diagrams worth mentioning are the Pictogram and the Line graph.

The Pictogram

This diagram uses symbols of the term in question to represent numbers.

Consider the example below:

Way of Travelling	No. of Users
Walking	18
Car	15
Bus	9

The pictogram for this data could look like this:

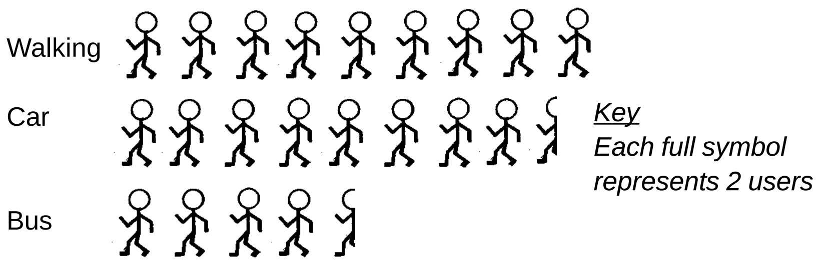


Fig 25.7

Line 1 ***Notice*** here that, without the key, the diagram would be meaningless. From the key, the nine person symbols represents 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 18 users.

Line 2 The eighth symbol is not complete meaning it represents a number less than the 2, that is 1. Thus the whole picture is 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 = 15.

Line 3 2 + 2 + 2 + 2 + 1 = 9.

The Line Graph

The temperature chart of a patient in hospital, is a good example.

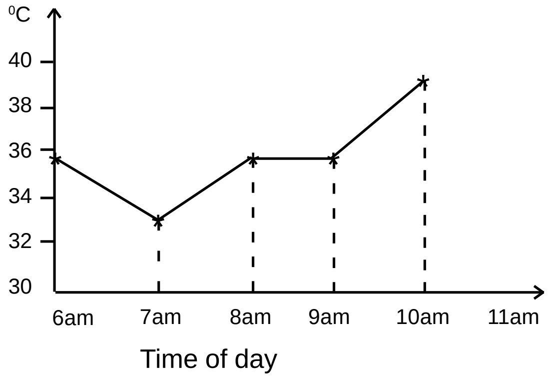


Fig. 25.8

For a Line graph, it is the points being joined (normally by a straight line segment) which have meaning. The line segments between points do not tell us what happened in between. One cannot predict the temperature for 11am in Fig 25.8.

1. The following table (25.12) shows what drinks customers ordered at lunch, at a certain shop.

Table 25.12

Drink Type	No. of customers
Coffee	4
Mahewu	7
Tea	3
Soft drink	6

a) Show this information on a (i) Bar Chart (ii) Pie Chart

b) State the modal drink.

2. Grades obtained by a group of students in a test are shown below

Table 25.13

Grade	A	B	C	F
Frequency	6	18	12	4

a) How many students took the test?

b) Show this information on a (i) Bar Chart (ii) Pie Chart

3. A dice was thrown 30 times and the results are shown on the bar chart below.

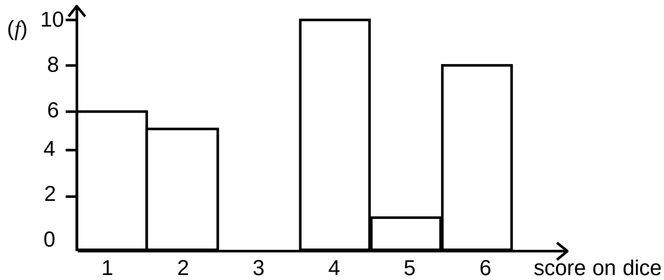


Fig. 25.9

a) How many times did a three show?
b) If a Pie Chart, for the same data, was to be drawn, find the angle for the sector representing a score of 6.
c) State the modal score.
d) Find the: (i) median score.
(ii) the mean score.
e) Express the score for a 2 as a percentage of all scores.

4. Fig 25.10 is a Pie Chart showing the distribution of grades, obtained by 50 students, in a test.

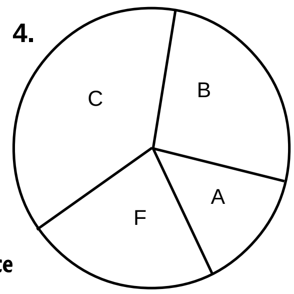


Fig 25.10

a) Sector A has $28,80^0$ at the centre. How many students scored grade A?
b) If 36% of the students scored grade C, find the angle at the centre for sector C.
c) Given that $\frac{2}{5}$ of the students failed the test (grade F), find the angle at the centre for grade B.

D. GROUPED DATA

Data collection can be on a much larger scale than has been handled so far. It then becomes tedious and senseless to think of individual terms.

For example, results of a national examination are usually graded A, B, C, D, E, and U for 'O' Level. The grades accommodate marks from a certain mark to another mark. (e.g. D denotes 40–44.

The '40–44' is called a *class interval* of 40 to 44, where 40 is the lower class boundary and 44 the upper class boundary.

The *width* of this class is 5 if both boundaries are included.

Common Errors

Class width when given 40 – 44
= 44 – 40
= 4

Let us see how mode and mean are found when data is grouped.

Consider the example below:

1. Table 25.14 shows the distribution of marks of 20 students.

Table 25.14

Class	1–5	6–10	11–15	16–20	21–25
Frequency	2	4	7	5	2

Notice that in table 25.14 there are students who got marks between 6 and 10. It does not specify exactly which marks!

The **modal class** in this case is **11 – 15**. It is not accurate to talk of a mode in this case.

How about the mean of the distribution?

Method (1) Find the class centre of each class and use that as the fair class representative.

$$\textbf{Class centre} = \frac{\textbf{Lower boundary + upper boundary}}{\textbf{2}}$$

$$\text{e.g class centre for } (6 - 10) = \frac{6 + 10}{2} = 8$$

$$\therefore \text{ mean} = \frac{\underline{3} \times 2 + \underline{8} \times 4 + \underline{13} \times 7 + \underline{18} \times 5 + \underline{23} \times 2}{20}$$

$$= \frac{265}{20}$$

$$= 13\tfrac{1}{4} \text{ or } 13{,}25$$

Method (2) Choose an assumed mean and proceed as follows.
Assumed Mean = 15

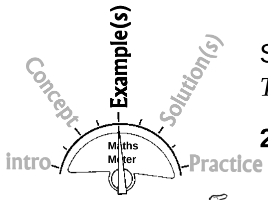
Table 25.15

Class	Class Centre	Frequency (f)	Deviations(d)	$f \times d$
1 – 5	3	2	3 – 5 = –12	2×(–12) = –24
6 – 10	8	4	8 – 15 = –7	4×(–7) = –28
11 – 15	13	7	13 –15 = –2	7×(–2) =–14
16 – 20	18	5	18 –15 = +3	5×(3) =+15
21 – 25	23	2	23 –15 = +8	2×(+8) =<u>+16</u>
				Total = –35

Mean deviation $= -\frac{35}{20}$ $\qquad \therefore$ Mean $= 15 + (-1\tfrac{3}{4})$

$= -1\tfrac{3}{4}$ $\qquad = 13\tfrac{1}{4}$

In both methods, the class centre has been used to represent the class. This means the final mean is only *an estimate* since the class centre is not necessarily the correct term.



Suppose the table is as follows

Table 25.16

2.

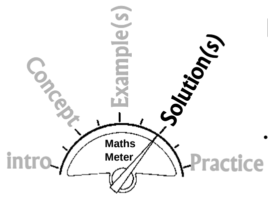
Class	**1 – 10**	**11 – 20**	**21 – 30**	**31 - 40**	**41 – 50**
f	2	4	4	9	1

Hint

When the class centre involves fractions, choose an assumed mean which involves that fraction so that it is cleared when finding the deviations.

Table 25.17 Assumed Mean 20,5

Class	**Class/Centre**	**Freq (f)**	**dev (d)**	**f x d**
1 – 10	5,5	2	–15	–30
11 – 20	15,5	4	–5	–20
21– 30	25,5	4	+5	+20
31 – 40	35,5	9	+15	+135
41 – 50	45,5	1	+25	+25
	Check these	20	Check these	Total = 130



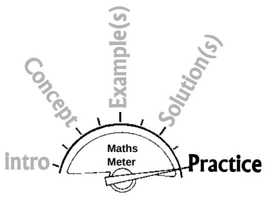
Mean deviation = $\frac{130}{20}$

$= 6\frac{1}{2}$

$\therefore$ Mean $= 20\frac{1}{2} + 6\frac{1}{2}$

$= 27$

1. Use **method(1)** on page 177 to calculate the mean of the given distribution.



a)

Class	1 – 5	6 – 10	11 – 15	16 – 20	21 – 25
Frequency	2	3	5	1	1

b)

Class	19–20	20–21	21–22	22–23	23–24	24–25
f	2	5	8	18	12	5

c)

Class	0 – 9	10–19	20–29	30–39	40–49	50–59
f	8	12	33	11	9	7

2. Choose a suitable assumed mean and calculate the mean of each distribution, in 1 above.

E. GROUPED DATA DIAGRAMS

1. Consider the information in Table 25.18 below.

Table 25.18

Class	1–5	6–10	11–15	16–20	21–25
f	2	4	7	5	2

As has been previously said, this data is not specific. Bar Charts and Pie Charts are no longer suitable. There is need for different diagrams which represent this form of data meaningfully.

The Histogram

This diagram is often confused with the Bar Chart. Table 25.19 show four main differences between the two.

Table 25.19

BAR CHART	**HISTOGRAM**
* Represents **simple ungrouped data**	Represents complex (grouped) data
* **One** axis is scaled (the frequency axis)	Both axes must be scaled.
* Height of bar is proportional to the the **frequency.**	**Area** of bar is proportional to the **frequency**
* A frequency axis is used	A frequency density axis is used

Something extra is required to use area in a histogram diagram. This introduces the **frequency density**, to take the place of frequency, as data on an axis.

$$\textbf{Frequency density} = \frac{\textbf{frequency}}{\textbf{class width}}$$

Let us develop the data in Table 25.18 above so we can draw a histogram from it.

Table 25.20

a)

CLASS	1–5	6–10	11–15	16–20	21–25
Frequency	2	4	7	5	2
Class width	5	5	5	5	5
Frequency density	0,4	0,8	1,4	1	0,4

The bars are drawn using the frequency density.

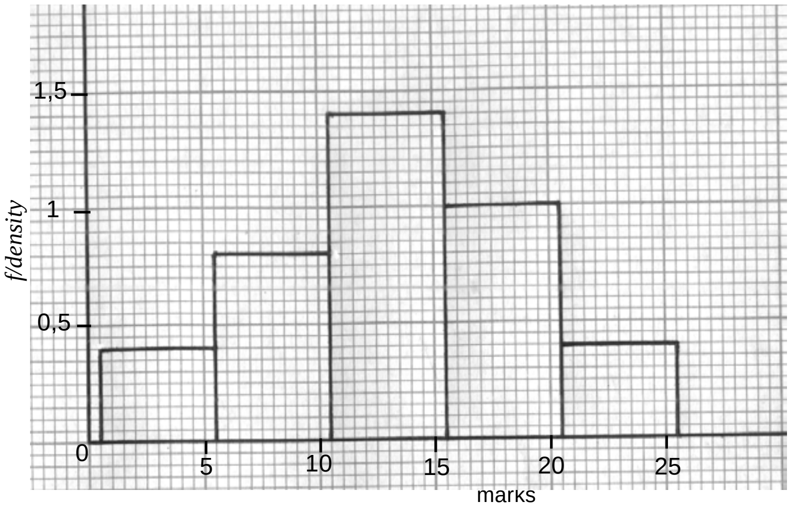


Fig 25.11

Notice that

- all the classes have the same width hence the bars have the same thickness. Classes which differ in width, will have bars which differ in thickness.

- the data is not continuous, hence there will be gaps between bars, to remedy this each class boundary is moved by $\frac{1}{2}$ i.e. $\frac{1}{2}$ to 5 $\frac{1}{2}$ for first bar 5 $\frac{1}{2}$ to 10 $\frac{1}{2}$ for second bar and so forth.

- in the example we use the bars cover the classes (there are no gaps between the classes)

2. Table 25.21, below, shows the distribution of average heights (cm) of soya bean plants grown by 50 resettled farmers.

Table 25.21

Height (cm)	$5<h\leqslant 9$	$9<h\leqslant 15$	$15<h\leqslant 20$	$20<h\leqslant 30$	$30<h\leqslant 45$
Number of farmers	8	8	10	15	9
Class width	4	6	5	10	15
Frequency/density	2	1,3	2	1,5	0,6

The histogram from this data would look like the one in Fig 25.12.

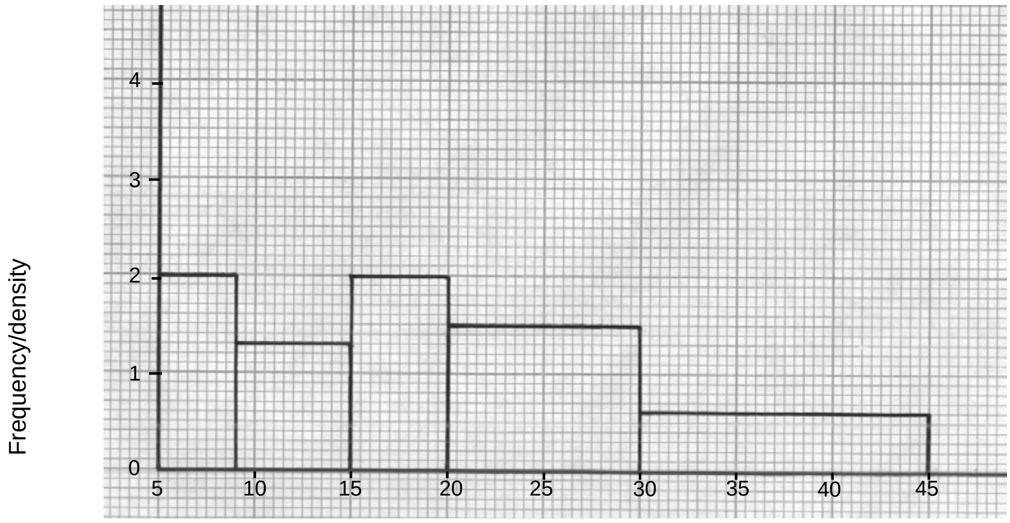


Fig 25.12 height of plants (cm)

Note that

- The data, i.e. heights, starts at more than 5. This means the heights axis does not have to start at zero! It should start from where the data starts.
- The classes seem to be overlapping. The bars simply meet at that number.
- Same frequency does not necessarily mean same height of bars. (Check the first two bars)
- **Frequency = Area of bar**

The Frequency Polygon

Here, points (**class centre; frequency density**) are plotted and joined with straight line segments. This means both axes need to be scaled.

3. The following is a frequency polygon drawn from Table 25.22 below.

Table 25.22

Class	1 – 5	6 –10	11 – 15	16–20	21 – 25
frequency	2	4	7	5	2
Class centre	3	8	13	18	23
density	0,4	0,8	1,4	1	0,4

Hint

The frequency polygon for data, where the class widths are the same throughout, may be drawn using points (class Centre; frequency). Otherwise frequency density is essential.

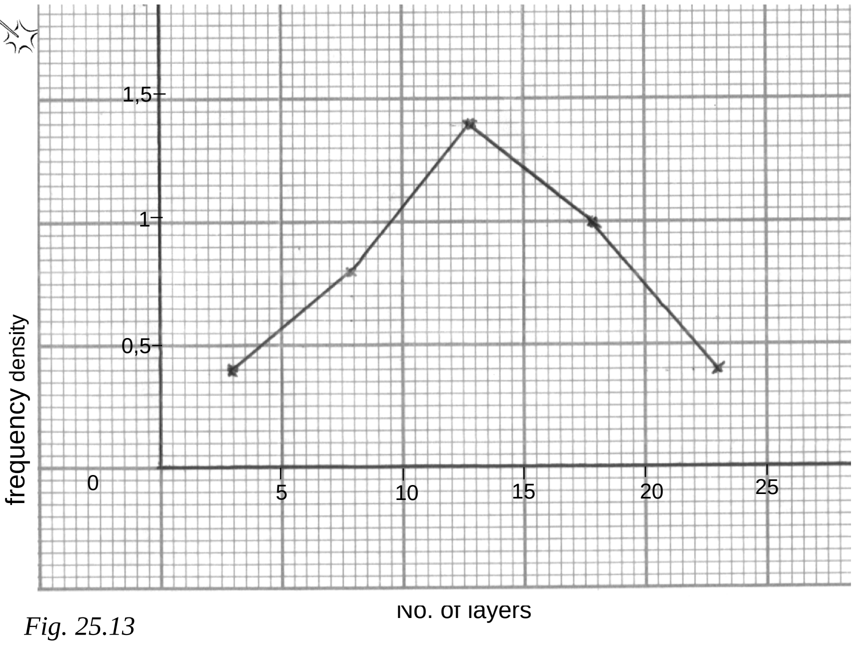
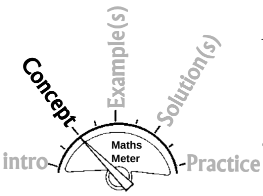


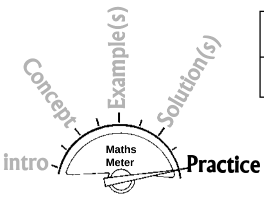
Fig. 25.13



In this case, points ((3;0,4); (8;0,8), (13;14); (18;1) and (23;4), are plotted (normally on graph paper) and joined using a ruler. Some authorities go on to join the first point to the origin and the last point to the last number on the horizontal axis.

1. The age distribution of 50 mature-entry students at a certain university is shown below.

Table 25.23



Age(y year)	$30<y\leqslant 35$	$35<y\leqslant 40$	$40<y\leqslant 45$	$45<y\leqslant 50$	$50<y\leqslant 55$
No.of students	4	10	22	12	2

Show this distribution on:

a) a histogram,

b) a frequency polygon on the same axes as the histogram.

2. Use the table 25.24 below and suitable scales to:

a) draw the histogram of the distribution.

b) draw the frequency polygon on the same axes as the histogram in (a).

Table 25.24

Class	1–5	6–10	11–15	16–20	21–25	26–30
f	2	18	24	20	12	4

3. From table, 25.25 below, draw a histogram for the given data.

Table 25.25

Class	$10 < x \leqslant 15$	$16 < x \leqslant 25$	$26 < x \leqslant 30$	$31 < x \leqslant 40$	$41 < x \leqslant 60$	$61 < x \leqslant 100$
f	10	27	35	10	10	8

4. The scores of 70 teams in a Rugby tournament are shown in Table 25.26, below.

Table 25.26

No. of scores	$10 < x \leqslant 15$	$16 < x \leqslant 25$	$26 < x \leqslant 30$	$31 < x \leqslant 40$	$41 < x \leqslant 60$	$61 < x \leqslant 100$
No. of Teams	3	10	10	25	16	6

Show this distribution on a histogram

5. The marks scored by 200 candidates from a certain school are shown in table 25.27, below.

Table 25.27

Marks	$10 < x \leqslant 25$	$25 < x \leqslant 40$	$40 < x \leqslant 50$	$50 < x \leqslant 60$	$60 < x \leqslant 80$	$80 < x \leqslant 100$
Freq (f)	45	30	55	30	28	12

Use the data in the table to draw: a) a Histogram.
b) a frequency polygon.

F. THE CUMULATIVE FREQUENCY CURVE (OGIVE)

This is also used to represent grouped data in diagram form. Consider the table 25.28 which shows the height of 50 tomato seedlings'.

1. *Table 25.28(a)*

Height *mm*	$10 < h \leqslant 15$	$15 < h \leqslant 20$	$20 < h \leqslant 30$	$30 < h \leqslant 40$	$40 < h \leqslant 50$	$50 < h \leqslant 60$
No. of plants	2	3	13	22	8	2

From this data another table which illustrates cumulative frequency more clearly, can be developed.

Table 25.28(b)

Height (*mm*)	$h \leqslant 10$	$h \leqslant 15$	$h \leqslant 20$	$h \leqslant 30$	$h \leqslant 40$	$h \leqslant 50$	$h \leqslant 60$
Cumulative Frequency	0	2	5	18	40	48	50

For explanation refer to table 25.28(a).
There are no seedlings which are less than or equal to 10mm. Those less than or equal to 40mm are 2+3+13+22 = 40.
Note that it is this addition process which produces the cumulative frequencies.
The last frequency in the table should equal the total frequencies in the table.

The second table, 25.28(b), is the one used to draw the cumulative frequency curve, using points (10;0), (15;2), (20;5), (30;18) etc from there.

The Ogive

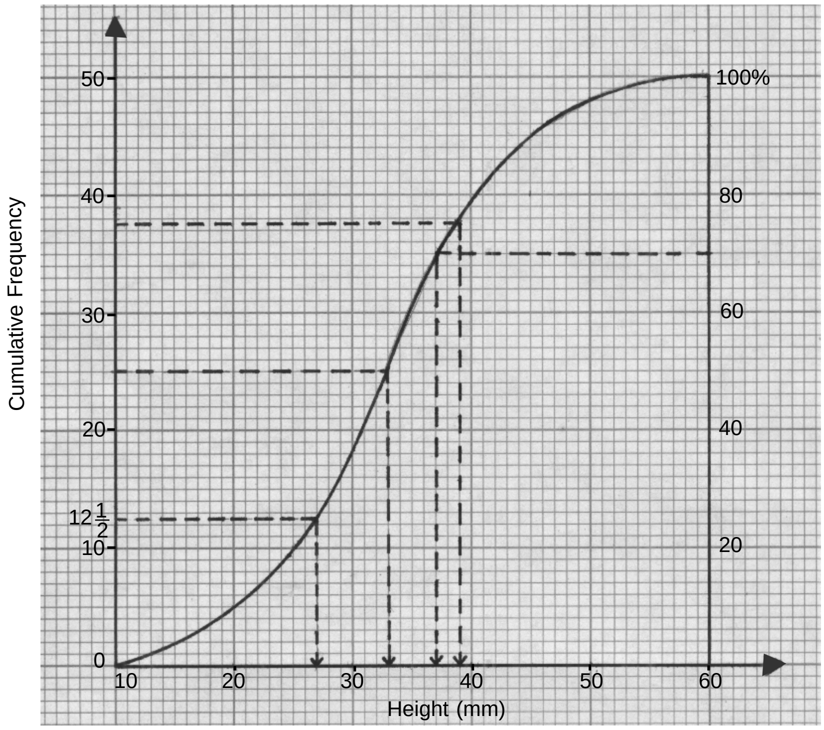


Fig. 25.14

The curve usually takes the shape of an expanded or stretched S.
This curve is used to make very useful estimations in statistics.

a) the median i.e. the height of the plant, at the middle of the arrangement i.e. 25th plant = 33mm
$\therefore$ median height = 33mm

b) the interquartile range (quartile comes from the word quarter.)

(i) Lower quartile $\longrightarrow$ $\frac{1}{4}$ of distribution i.e. at $12\frac{1}{2}$

(ii) Upper quartile $\longrightarrow$ $\frac{3}{4}$ of distribution i.e. at $37\frac{1}{2}$

$\therefore$ **Interquartile range = Upper Quartile (UQ) – Lower Quartile (LQ)**
= 39 – 27
= 12

c) Semi-interquatile range = $\frac{UQ - LQ}{2}$ i.e. $\frac{1}{2}$of Interquartile Range

$= 12 \div 2$

$= 6$

d) Percentiles (from percent) are distributions divided into 100 parts. These are easier to get if a percentages axis is drawn at the end of the curve, as illustrated.
The 70th percentile in this case is 37mm.
Percentiles and Quartiles are **measures of dispersion**. Measures of dispersion give an idea of how the data is spread.

Statistics questions are usually tested with a probability component included.
To illustrate this, from the previous example,

e) a seedling is chosen at random. Calculate the probability that the plant's height is greater than 35mm but less than or equal to 45mm.

Solution

Number of plants in the range $35 < h \leqslant 45 = 45 - 30$ (see curve)

$= 15$

$\therefore$ Probability $= \frac{15}{50}$

$= \frac{3}{10}$

f) If three plants are chosen at random, what is the probability that they are all less than or equal to 25mm?

Solution

From the curve, 10 plants are less than or equal to 25mm high.

$\therefore$ P (less than or equal to 25) $= \frac{10}{50} \times \frac{9}{49} \times \frac{8}{48}$

$= \frac{3}{490}$

Common Errors

$\frac{10}{50} \times \frac{10}{50} \times \frac{10}{50}$
This is wrong!

From knowledge of probability one should appreciate the change in the fractions at the second and third plant.

2. The diagram, in Fig 25.15, is a cumulative frequency graph showing the weights lifted by 60 junior body builders.

Use the graph to estimate:

a) the median weight.
b) the semi-interquartile range.
c) the 83rd percentile.
d) the probability, of picking at random, someone who can lift more than 90kg.

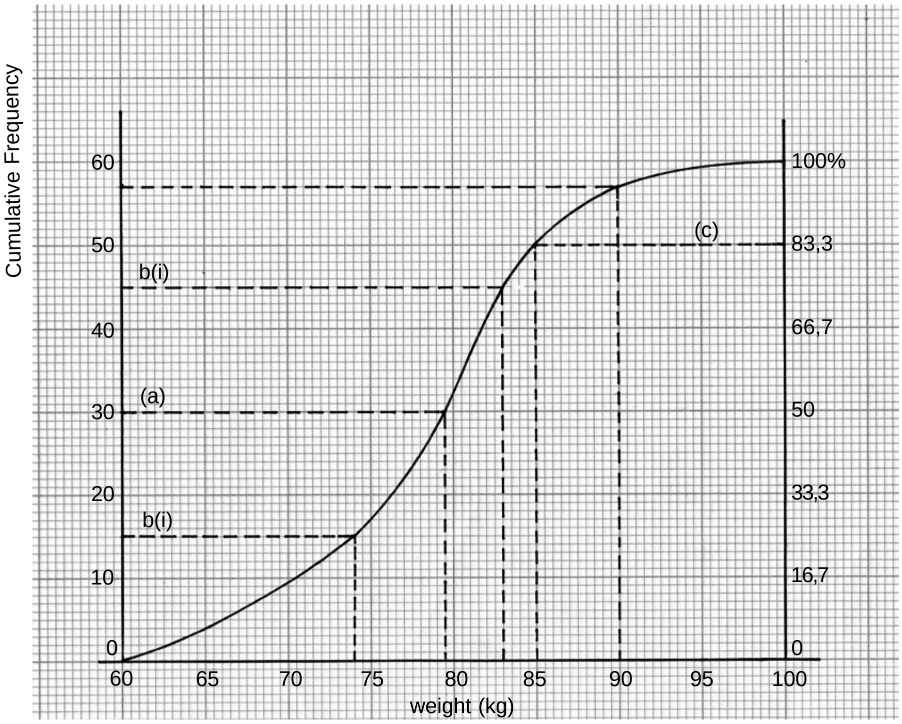


Fig 25.15

a) The 30th lifter, lifts 79,5kg.
$\therefore$ Median weight = 79,5kg.

b) Semi-interquartile Range $= \dfrac{UQ - LQ}{2}$

$= \dfrac{83 - 74}{2}$

$= \dfrac{9}{2}$

$= 4,5$

c) 83rd Percentile is almost at 83,3% $\approx$ 85kg

d) Number who can lift less than 90kg = 57
$\therefore$ Number who lift more than 90kg = 60 – 57
= 3

$\therefore$ P(Lifts 90kg) $= \dfrac{3}{60}$

$= \dfrac{1}{20}$

PRACTICE 25F

1. The distribution of maize yield of 200 farmers, at a certain settlement, is shown in the table 25.29 below.

Table 25.29

No. of bags (x)	$0<x\leqslant 10$	$10<x\leqslant 20$	$20<x\leqslant 30$	$30<x\leqslant 40$	$40<x\leqslant 50$	$40<x\leqslant 50$
No.of farmers	18	27	65	48	29	15

Table 25.30

No. of bags (x)	$x\leqslant 0$	$x\leqslant 10$	$x\leqslant 20$	$x\leqslant 30$	$x\leqslant 40$	$x\leqslant 50$	$x\leqslant 60$
No.of farmers	0	p	45	q	156	185	200

a) Find the value of p and q.

b) Using suitable scales on the axes, draw a smooth ogive for the data.

c) Use the graph to estimate:
   (i) the median yield.
   (ii) the interquartile range of the yield.

d) If two farmers are chosen at random, find the probability that their yields are more than 25 bags but less than, or equal to, 45 bags.

2. The percentages scored by 100 pupils, in a certain test are shown below:

Table 25.31

% (x)	$0<x\leqslant 10$	$10<x\leqslant 20$	$20<x\leqslant 30$	$30<x\leqslant 40$	$40<x\leqslant 50$	$50<x\leqslant 60$
Frequency	5	21	24	32	q	3
F.density	0,5	p	2,4	3,2	1,5	r

a) How many pupils scored a percentage less than, or equal to 30%?

b) Find the values of p, q and r in the table.

c) Using a scale of 2cm to represent 10%, on the horizontal axis, and 2cm to represent 0,5 units, on the vertical axis, draw a histogram which illustrates this data.

d) Copy and complete the table below.

Table 25.32

% (x)	$x\leqslant 0$	$x\leqslant 10$	$x\leqslant 20$	$x\leqslant 30$	$x\leqslant 40$	$x\leqslant 50$	$x\leqslant 60$
Cum frequency	0	5	26		82		100

e) Using a scale of 2cm to represent 10%, on the horizontal axis, and 2cm to represent 10 pupils, on the vertical axis, draw a smooth ogive, which illustrates this data.

f) Use your graph to estimate:
(i) the semi-interquartile range.
(iii) the cut-off point if the top 12% of the students are in grade A.

3. The water consumption of 30 households, at a Mission Centre for a certain month, are shown below.

Table 25.33

Volume (vcm^3)	$0 < v \leqslant 5$	$5 < v \leqslant 10$	$10 < v \leqslant 15$	$15 < v \leqslant 20$	$20 < v \leqslant 25$	$25 < v \leqslant 30$
Number of Consumers	3	7	9	6	3	2

a) Copy and complete the cumulative frequency table below.

Table 25.34

Volume (v/m^3)	$v \leqslant 0$	$v \leqslant 5$	$v \leqslant 10$	$v \leqslant 15$	$v \leqslant 20$	$v \leqslant 25$	$v \leqslant 30$
Cum. frequency	0	3	10			28	30

b) Using a scale of 2cm to represent $5m^3$, horizontally and 2cm to represent 5 households, vertically, draw a smooth cumulative frequency curve for the above data.

c) Using the graph, estimate:
(i) the median volume.
(ii) the number of households consuming volumes more than $17\frac{1}{2}$ cm^3 but less than, or equal to $27m^3$.

d) Calculate the average consumption, of the last 6 households.

4.

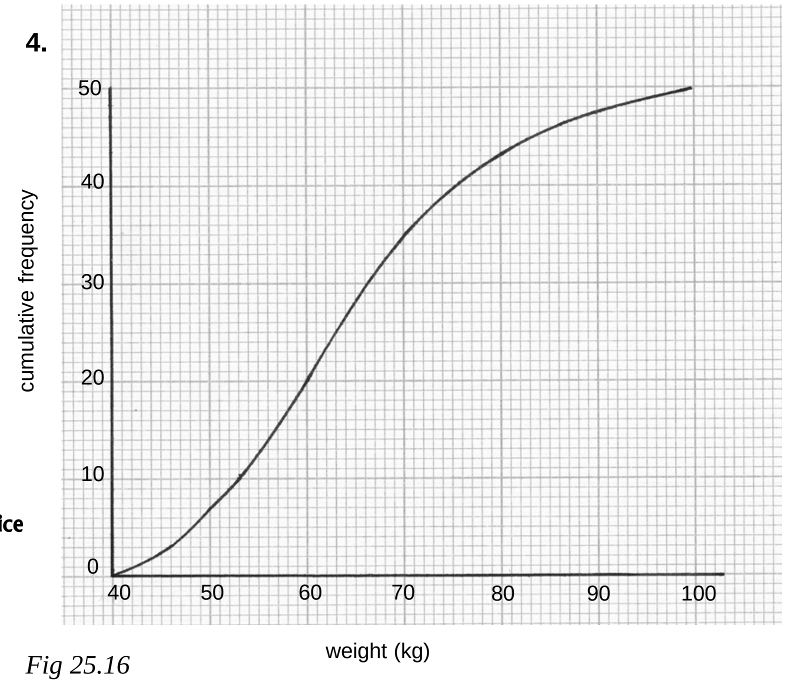


Fig 25.16

The diagram, Fig 25.16, is an Ogive, showing the distribution of the weights of 50 pigs, from a piggery project.

a) Use the graph to estimate:
   (i) the median weight.
   (ii) the number of pigs weighing more than 60kg but less than or equal to 85 kg.

If a pig weighs more than 85kg, it is rated as grade A.

b) Find the percentage of the pigs in grade A.

c) A pig is picked at random from the 50. Find the probability, as a common fraction in its lowest terms, that the pig weighs less than or equal to 55kg.

5. The heights of 80 maize plants, at a research station, are given in the table below.

Table 25.35

Height x cm	$0<x\leqslant5$	$5<x\leqslant10$	$10<x\leqslant15$	$15<x\leqslant20$	$20<x\leqslant25$	$25<x\leqslant30$	$30<x\leqslant35$	$35<x\leqslant40$
No.of plants	5	12	16	r	14	7	4	2
Cumulative frequency	5	p	q	52	66	74	78	80

a) Find the value of p, q and r.

b) Using suitable scales, draw a smooth Ogive for the given data.

c) Use the graph to estimate,
   (i) the median height of the plants.
   (ii) the semi-interquartile range.
   (iii) the 75th percentile.

d) Two plants are picked at random from the group. Find the probability that one plant is, at most, 12cm tall and the other one is more than 32cm tall.

SUMMARY

1. Statistical information is often called **data**.

2. Ungrouped data is compiled by listing terms or by using frequency tables.

3. **Averages**

- Mode – most frequent item
- Median – the middle term (Arrange in order of size first)
- Mean – $\frac{\text{Sum of terms}}{\text{Total number of terms}}$

or use assumed mean (Deviation = term – assumed mean)

4. Ungrouped/simple data is best represented on a bar chart, pie chart, pictogram or line graph.

5. More complex/grouped data is best represented by:

a) frequency polygon – plot (class-centre; frequency density) and the points are joined using a ruler.

b) Histogram – find frequency density = $\frac{\text{frequency}}{\text{class width}}$

c) Cumulative frequency curve (Ogive) – compile a cumulative frequency row and use this as the vertical axis and plot against the upper limit of each class.

6. Use the cumulative frequency curve to estimate:

Median – Mid-cumulative frequency axis.

Quartiles – Quarters

Percentiles– Percentages (draw a percentages axis at the end of the curve)

EXAM PRACTICE 25

Hint

Note, when the instruction 'Answer the whole of this question on a single sheet of graph paper' accompanies a question, it means answers to all of the questions under that question number should be on the same sheet of graph paper.

Study the following examples carefully

1. The terms in the distribution –3, –2, 0, 4, p, q, 9, 12, 15, 16 are arranged in ascending order of size. Given that for this distribution, the mode is 4 and the median is 6.

a) find the value of (i) p and (ii) q.

b) calculate the mean of the distribution.

Solution

1. a) (i) For 4 to be the mode, it must appear more frequently than the rest of the numbers.

$\therefore \; p = 4$

(ii) $$\text{Median} = \frac{p+q}{2}$$

$$\text{Thus } \frac{4+q}{2} = 6$$

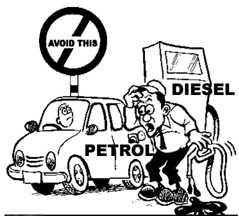
$4 + q = 12$

$\therefore \quad q = 8$

b) $$\text{Mean} = \frac{-3+-2+0+4+4+8+9+12+15+16}{10}$$

$$= \frac{63}{10}$$

$$= 6{,}3$$



Common Errors

1) Failing to simplify –3 –2.
2) Ignoring the zero and then dividing by 9 instead of 10.
3) Continuing to use p and q in the mean and ending up with an expression.

Answer the whole of this question on a single sheet of graph paper.

2. The marks, out of 50, for fifty students, are distributed as shown in table 25.36, below.

Table 25.36

Marks	0 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency	5	13	22	8	2
Cumulative Frequency	5	18	p	q	50

a) Calculate the values of p and q.

b) Using a scale of 2cm to represent 10 marks, on the horizontal axis, and 2cm to represent 10 students, on the vertical axis, draw a smooth Ogive, to represent the above data.

c) Use the graph to estimate
   (i) the median mark.
   (ii) the number of students who scored more than 35 marks.
   (iii) the 25th percentile.

d) Two students were picked randomly from the group, one after the other. Find the probability that they both had at least 37 marks. Give your answer as a fraction in its lowest terms.

Solution

2.

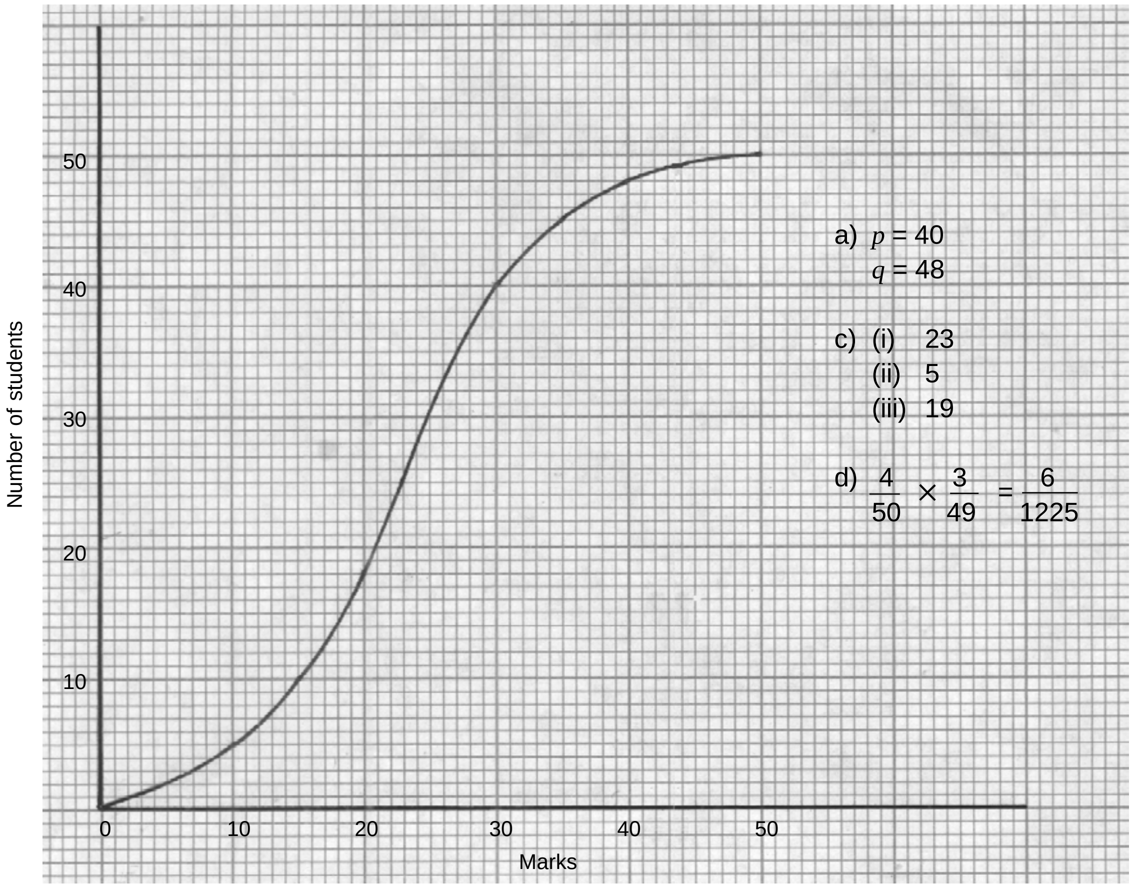


Fig 25.17

Now do the following:

1. Answer the whole of this question on a sheet of graph paper The table below gives the age distribution of 60 athletes in a certain team.

Table 25.37

Age (*y*yrs)	$12 < y \leqslant 14$	$14 < y \leqslant 16$	$16 < y \leqslant 18$	$18 < y \leqslant 20$	$20 < y \leqslant 24$	$24 < y \leqslant 27$	$27 < y \leqslant 30$
f	5	11	16	12	11	3	2

a) Copy and complete the cumulative frequency table below.

Table 25.38

Age (*y*yrs)	12	14	16	18	20	24	27	30
No. with this age or less	0	5	16			55	58	60

b) Using a horizontal scale of 2cm to represent 2 years and a vertical scale of 2cm to represent 10 athletes, draw a smooth Ogive for the distribution.

c) Use the graph to estimate

(i) the median age.

(ii) the number of athletes in the age group $15 < y \leqslant 19$.

d) If two athletes are chosen at random, find the probability, as a common fraction in its lowest terms, that both are more than 17 years but less than or equal to 25 years of age.

2.

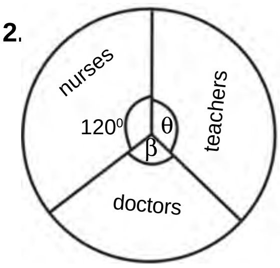


Fig. 25.18

Table 25.39

Profession	Angle Sector	Number of Professional
Doctors	β	*n*
Nurses	120^0	540
Teachers	θ	720

Fig 25.19 and table 25.39 show the same information relating to professionals in a certain town. Calculate: a) θ

b) β

b) *n*

3. a) Answer the whole of this question on a single sheet of graph paper. The table below shows the ages of 80 vehicles which passed through a police road block, on a certain day.

Table 25.40

Ages (y yrs)	$0 < y \leqslant 2$	$2 < y \leqslant 5$	$5 < y \leqslant 10$	$10 < y \leqslant 15$	$15 < y \leqslant 25$
f	16	p	31	14	8
f-density	8	7	q	2,8	r

(i) Find the values of p, q and r.

(ii) Using a horizontal scale of 2cm to represent 5 years, and a vertical scale of 2cm to represent 2 units, draw a **histogram** for this distribution.

b) Copy and complete the following table, using an assumed mean of 10,5.

Table 25.41

Ages (y yrs)	Frequency	Deviation	$f \times d$
1	16	–9,5	–152
3,5	p		
7,5	31	–3	–93
12,5	14		
20	8		
	Total 80		Total =

Hint

You already have the value of p!

Calculate the mean of the distribution.

4. a) The table below shows the marks obtained by 50 students in a certain class.

Table 25.42

Marks	5	6	7	8	9	10
Frequency	7	9	15	11	5	3

Find: a) the modal mark.

b) the median mark.

c) the mean mark.

5.

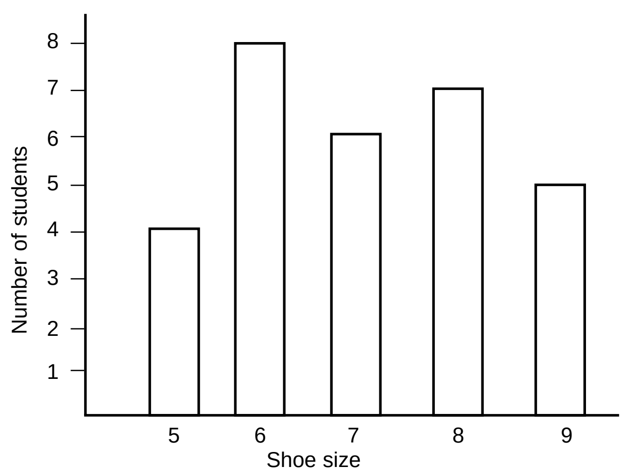


Fig. 20.19

In a survey, some Form 6 students were asked what shoe size they wore. The bar chart Fig 25.19 illustrates the results of the survey.

a) State the mode of the distribution.

b) How many students were involved in the survey?

c) Find the median shoe size.

6. Fig 25.20 is a line graph showing the lowest temperatures recorded, at a certain station over 7 days.

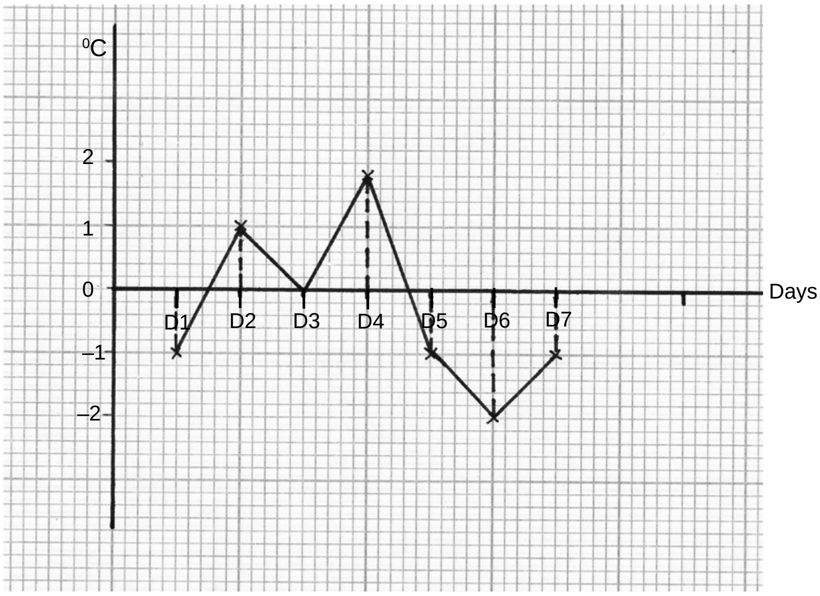


Fig. 25.20

From the graph, find:

a) the modal temperature for the week.

b) the median temperature for the week.

c) the mean temperature for the week.

7. a) A resettled farmer has 7 cows, 4 heifers, 5 steers and 4 oxen.

(i) Construct an accurate pie chart representing this data.

(ii) Two animals are selected at random.
Calculate the probability that one is a cow and the other one a steer. Leave your answer as a fraction, in its lowest terms.

b)

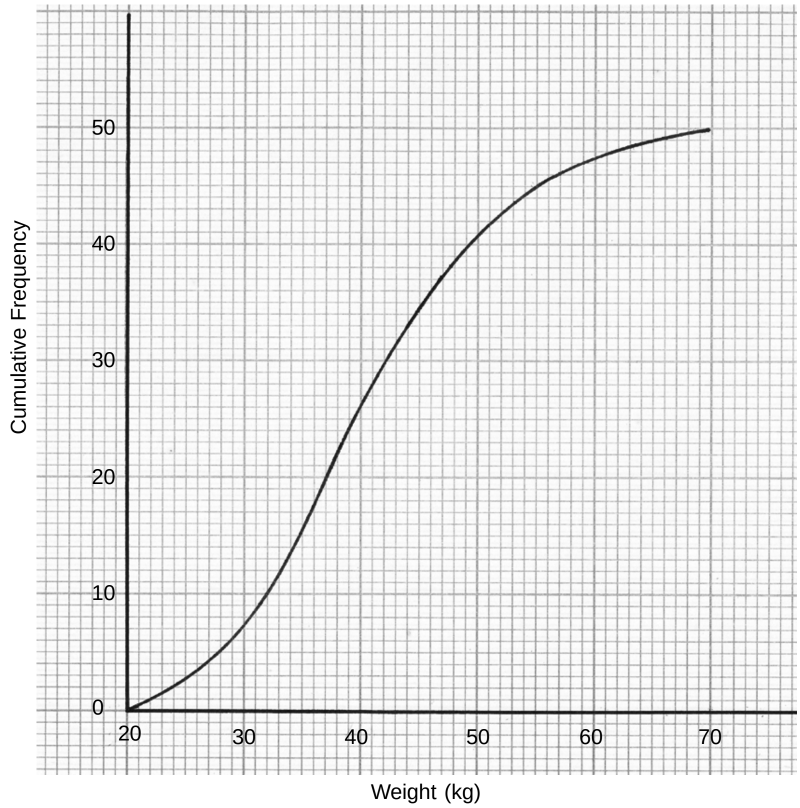


Fig 25.21

Fig 25.19 is an Ogive showing the distribution of weights of patients attended at a polyclinic, on a certain day.

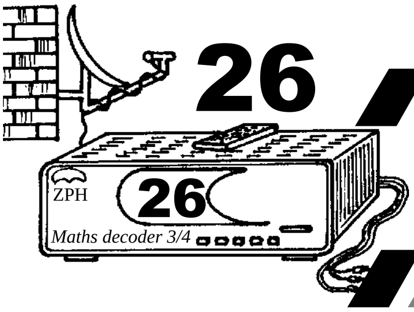
Use the diagram to estimate:

(i) the median weight.

(ii) the 68th percentile.

(iii) the number of patients weighing more than 30kg but less than or equal to 45kg.

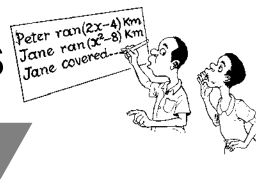
26

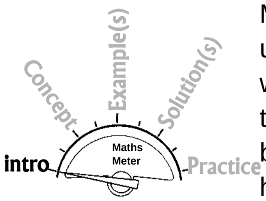


Product (×) The difference (–) Sum (+) Times (×)

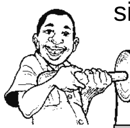
Words to numbers

Together (+) Consecutive (x, $x + 1$) Combined (+)

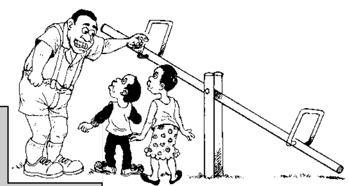




Most algebraic topics in Mathematics textbooks end with what is usually termed 'Word Problems'. Here, a situation is narrated in word form. The objective is for the learner to be able to translate the words in a situation into algebraic expressions and, where need be, create an equation or equations whose solution or solutions help to answer the question about the situation. This chapter endeavours to give hints and tips which will help the learner arrive at with correct numerical or algebraic answers to a situation. The focus is on the formation of linear expressions and equations, linear simultaneous equations and quadratic equations.



Syllabus Expectations



By the end of this chapter, students should be able to:

1. transform words into respective arithmetic processes.
2. form algebraic expressions from word stories.
3. form and solve simple linear and fractional algebraic equations from word stories.
4. form and solve simple linear simultaneous equations (by graphs, substitution, elimination and matrices).
5. form and solve quadratic equations by factorisation and by use of formulae.

ASSUMED KNOWLEDGE

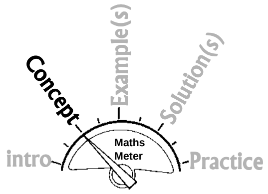
In order to tackle work in this chapter, it is assumed that students are able to:

- represent numbers with letters.
- solve linear, simultaneous and quadratic equations.
- identify properties of plane shapes and the angles inside them.

Below are examples of translations of number stories into algebraic expressions. Study them carefully before you attempt the problems that follow.

(i) 2 less than x $\Rightarrow$ $x - 2$ not $2 - x$

(ii) 5 more than y $\Rightarrow$ $y + 5$



Terms that translate into operations:

(i) Double and squaring.
Double means times 2 e.g. x doubled means $2x$ not x^2:
Raised to power two means square
e.g. x squared $\Rightarrow$ x^2 not $2x$

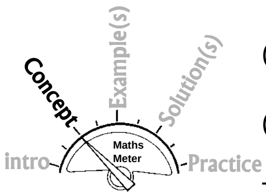
(ii) Sum, altogether, combined $\Rightarrow$ Addition

(iii) The difference $\Rightarrow$ subtraction.

(iv) Product $\Rightarrow$ multiplication.

(v) Consecutive $\Rightarrow$ one after the other.

(vi) One and half times $\Rightarrow$ multiply.



The following steps will help students to arrive at the desired interpretation.

1. Read the problem carefully (preferably more than once) noting the mathematical terms in the story.

2. Choose a letter or letters to stand for what is not known in the question.

Hint

Always watch the minus sign
$- \times - = +$
$- \times + = -$
$- \times - \times - = -$
$(-x)^4 = +x^4$
$-(-x) = x$
$-(3x - 2a) = -3x + 2a$

3. Translate each part of the problem into algebra using the letter/s in (**2**) and carry out the operations between the algebraic symbols.

4. Look for the words or phrases which direct you to the correct arithmetic process to be employed.

5. If an equation has been formed in (**4**), solve it.

6. Check if the solution is appropriate for the given situation.

A. LINEAR EXPRESSIONS AND/OR EQUATIONS

Consider the examples below:

Hint

a) The main mathematical term in the story is younger less or $\Rightarrow$ smaller than.

1. Chipo, who is $(2x - 7)$ years old, is younger than her cousin who is $(5x - 15)$ years old.

a) How much younger is Chipo than her cousin?

Hint

b) Sum ⇒ Add
The question implies an equation has to be formed.

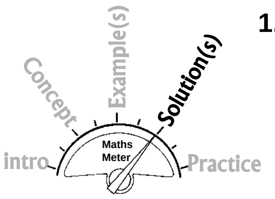
b) Given that the sum of their ages is 55 years, find the value of x.

c) Hence calculate how old Chipo's cousin is.

Common Errors

Either $2x - 7 - (5x - 15)$ or no use of brackets leading to $5x - 15 - 2x - 7$

Solution



1. a) $(5x - 15) - (2x - 7)$ and simplify

$= 5x - 15 - 2x + 7$

$= (3x - 8)$ years

b) Thus $(5x - 15) + (2x - 7) = 55$

$$5x - 15 + 2x - 7 = 55$$
$$7x - 22 = 55$$
$$7x = 77$$
$$\therefore x = 11$$

c) $5(11) - 15 = 55 - 15$

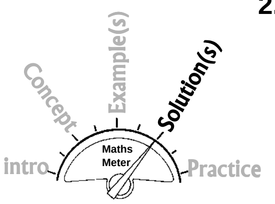
$\therefore$ Chipo's cousin is 40 years old

Hint

Establish the dimensions of the rectangle in terms of a chosen letter.

2. The length of a rectangle is 5cm less than twice its width. If its perimeter is 38cm, find the area of this rectangle.

Solution



2. Let the width be x cm

$\therefore$ Length $= (2x - 5)$ cm (from 5cm less than twice its width)

Now, use the concept of perimeter to find the actual dimensions.

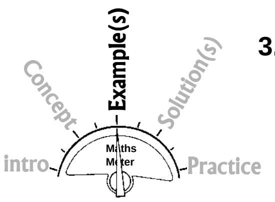
i.e. $2(2x - 5 + x) = 38$

$$3x - 5 = 19$$
$$3x = 24$$
$$x = 8$$

Hence, width $\leftrightarrow = 8$ cm

Length $= 2(8) - 5$
$= 16 - 5$
$= 11$

$\therefore$ Area $= 8 \times 11$
$= 88\text{cm}^2$



3. Four employees, at a company, are aged x years, $(2x + 3)$ years, $(3x - 1)$ years and $(x + 4)$ years old.

a) Write down an expression for the total ages of the four.

b) If the total ages of the four is 181 years find x.

c) Find the age of the oldest of the four.

Solution

3. a) $x + 2x + 3 + 3x - 1 + x + 4 = 7x + 6$

b) $7x + 6 = 181$

$7x = 175$

$x = 25$

c) The oldest $= 3(25) - 1$

$= 75 - 1$

$= 74$ years old

1. James has $(5x - 4)$ counters, whilst Jimmy has $(2 - x)$ more.

a) How many counters does Jimmy have?

b) How many counters do the two boys have altogether?

2. Farai got $(2x - 7)$ marks in a test whilst Job got $(3x + 2)$ marks. How many more marks did Job get compared to Farai?

3. A girl is $(2x + 3)$ years old and her mother is $(5x - 3)$ years old.

a) How much older, in terms of x, is the mother than the daughter?

b) Given that the sum of their ages is 56 years, find the value of x.

4. A shopkeeper ordered two brands of drinks, Tru-Apple and Tru-Guava. He ordered $(3x - 5)$ crates of Tru-Apple.

a) If this is a third of the Tru-Guava crates, how many crates, in terms of x, did he order altogether?

b) Given that he ordered 40 crates in all, find the value of x.

5. A boy weighs $(6x - 18)$kg and his elder brother weighs $(x + 5)$kg.

The total mass of the boy, his elder brother and their father is $(11x - 3)$kg. Find, in terms of x, the mass of their father.

6. Six of the angles of a nonagon are $(3x - 10)^0$ each and the other three are 75^0, 92^0 and 55^0.

a) Find, in terms of x, the simplified expression for the sum of those angles.

b) Find the value of x and give the size of the largest angle in this nonagon.

7. In a certain season, a farmer planted $(6x - 3)$ ha of maize, $(x + 1)$ ha of groundnuts and $(x - 6)$ ha of sunflowers. The rest, of the 20ha of arable land was left idle. How much land, in terms of x, was left idle?

Hint
Let playing time be x hours

8. On a certain day, a boy worked out that he used his time as follows:
Sleeping time was four times the playing time. Household chores took six hours less than the playing time.
How much of his day was spent for playing?

B. SIMULTANEOUS EQUATIONS

Consider the examples below:

Hint
Represent the two numbers with different letters i.e. Let the numbers be x and y.

1. The sum of two numbers is 22. Their difference is 8. Find the two numbers.

Solution

1. Let the numbers be x and y.

Thus $x + y = 22$ (Sum is 22)
$x - y = 8$ (Difference is 8)
and solve simultaneously
$2x = 30$
$x = 15$
$y = 7$

$\therefore$ The two numbers are 15 and 7.

2. The sum of the two digits of a two digit number is 12. The difference between the digits is 6. Given that the tens digit is bigger than the units digit, find the number.

Solution

2. Let x be the tens digit and y be the units digit
$x + y = 12$
$x - y = 6$
$2x = 18$
$x = 9$
$y = 12 - 9$
$= 3$
$\therefore$ The number is 93.

3.

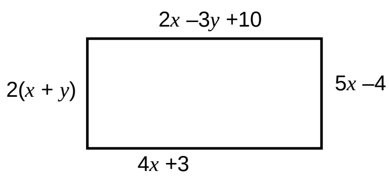


Fig. 26.1

Hint
Find actual length and width first. Simultaneous equations can be formed by equating sides.

Fig 26.1 shows a rectangle with the dimensions in cm. Find the actual area of the rectangle.

Solution

3. i.e. $4x + 3 = 2x - 3y + 10$

$\Rightarrow\ 2x + 3y = 7$

and $5x - 4 = 2x + 2y$

$\Rightarrow\ 3x - 2y = 4$

Solving simultaneously $2x + 3y = 7$

$3x - 2y = 4$

Using the Matrix method

$$\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{13} \begin{pmatrix} -2 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$= \frac{-1}{13} \begin{pmatrix} -26 \\ -13 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore\ x = 2$ and $y = 1$

Thus Actual Length $= 2(2) - 3(1) + 10$ or $4(2) + 3$

$= 11\text{cm}$

Actual Width $= 5(2) - 4$ or $2(2 + 1)$

$= 6\text{cm}$

$\therefore$ Actual Area $= 11 \times 6$

$= 66\text{cm}^2$

PRACTICE 26B

1. The sum of two numbers is 30. Their difference is 6. Find the two numbers.

2. The opposite interior angles of a cyclic quadrilateral are x and y. Their difference is 94. Find the value of x and the value of y.

Hint

Can also be done using one letter

3. The perimeter of a rectangle is 30cm. If the length is 5cm more than the width, find the length and the width of the rectangle.

4. Fig 26.2 shows a rectangle with the dimensions in cm.

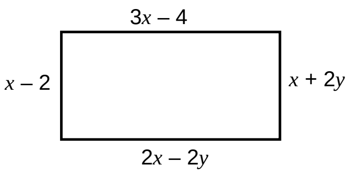


Fig. 26.2

Find:

a) the values of x and y.

b) the actual (i) perimeter.

(ii) area of the rectangle.

5. Two numbers add up to 14. If $\frac{2}{3}$ of the first number added to $\frac{1}{4}$ of the second number gives 6, find the two numbers.

6.

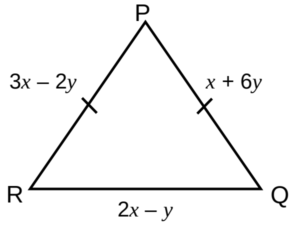


Fig. 26.3

PQR in Fig 26.3 is an isosceles triangle in which PQ = PR. If PQ = $(x + 6y)$cm, PR = $(3x – 2y)$cm and QR = $2x – y$, and the perimeter is $(x + 23)$cm, find:

a) the values of x and y.

b) the actual perimeter of the triangle.

7. Dumisani's age and Chipo's age add up to 48 years. Four years ago Dumisani was three times as old as Chipo. Find their present ages.

8. A boy cycles for x hours at 10km/h and y hours at 15km/h. Altogether he cycles 75km in 6 hours.
Find x and y.

9. If 2 is added to both the numerator and the denominator of a fraction, the fraction becomes $\frac{1}{2}$. However if 5 is added to both, the fraction becomes $\frac{4}{7}$.
What is the fraction?

10. The sum of the digits of a two digit whole number is 12. If the tens digit is 3 more than twice the units digit, find the number.

C. QUADRATIC EXPRESSIONS AND EQUATIONS

Study the examples below carefully:

1. The product of two negative consecutive even numbers is 48. Find the numbers.

Solution

Hint

Ensure that you answer the question asked!

1. Let the first number be x

The next = $(x + 2)$

Hence $x(x + 2) = 48$

$x^2 + 2x - 48 = 0$

$(x + 8)(x - 6) = 0$

$x = -8$ or 6

The numbers are -8 and

$-8 + 2 = -6$

or alternatively

Let the first number be x

The next $= (x - 2)$

Hence $x(x - 2) = 48$

$x^2 - 2x - 48 = 0$

$(x - 8)(x + 6) = 0$

$x = 8$ or -6

The numbers are -6 and

$-6 - 2 = -8$

Notice that, 6 and 8 are not correct since the question is about negatives.

2. A cyclist rode a distance of 30km at an average speed of $x$km/h.

a) Write down an expression in hours for the time he took for the journey.
Another cyclist rode the same distance at 2km/h more than the first cyclist's average speed.

b) Write down an expression for the time in hours this second cyclist took for the journey.

c) Given that, the time taken by the first cyclist is 30 minutes more than the time taken by the second cyclist, form an equation in x and show that it reduces to
$x^2 + 2x - 120 = 0$

d) Solve this equation and find the time taken by the second cyclist for the whole journey.

Solution

Hint

The letter to use is already introduced.

2. a) $\text{Time} = \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{30}{x}$ h

b) Speed $= (x + 2)$ km/h

$\therefore$ Time $= \dfrac{30}{x + 2}$ h

Hint

As fractions $\dfrac{30}{x} > \dfrac{30}{x + 2}$

c) $\therefore \dfrac{30}{x} - \dfrac{30}{x + 2} = \dfrac{1}{2}$

$60x + 120 - 60x = x^2 + 2x$

$x^2 + 2x - 120 = 0$

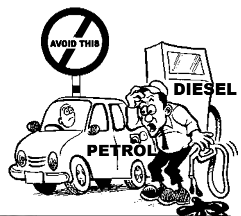
d) $x^2 + 2x - 120 = 0$

$(x - 10)(x + 12) = 0$

$x = 10$ or -12

$\therefore$ Second cyclist time $= \dfrac{30}{12}$

$= 2\frac{1}{2}$ h



Common Errors

$\dfrac{30}{x} - \dfrac{30}{x+2} = 30$

The RHS is in minutes whilst the LHS is in hours! Work with the same units.

3. If nine times a certain integer is subtracted from five times its square, the result is 2. Find the integer.

Solution

3. Let x be the integer
Nine times = $9x$
Five times its square = $5x^2$

$$\therefore \quad 5x^2 - 9x = 2$$
$$5x^2 - 9x - 2 = 0$$
$$(5x + 1)(x - 2) = 0$$
$$x = -\tfrac{1}{5} \text{ or } 2$$

The integer is 2.

Common Errors

1) $9x - 5x^2 = 2$

2) Five times its square is taken to mean $(5x)^2$

Hint

Note that 2 sets of answers are required here!

1. Find two numbers which differ by 5 and whose product is 14.

2. A student is 6 times younger than her teacher. Three years ago, the product of their ages was 36. Find their present ages.

3. Zandile is 4 years older than Thoko. Four years ago, the product of their ages was 165. Find their present ages.

4. The product of two consecutive even numbers is 24. Find the numbers.

5. Two square tiles have a total face area of 40cm^2. One tile is 4cm longer each way than the other. Find the dimensions of the two tiles.

6. The length of a room is 4 times longer than its width. Its area is 36m^2. Find the dimensions of the room.

7. A rectangular piece of cardboard measures 15cm by 10cm. Strips of equal width are cut off one side and one end. The area remaining is 50cm^2. Find the width of the strips removed.

8.

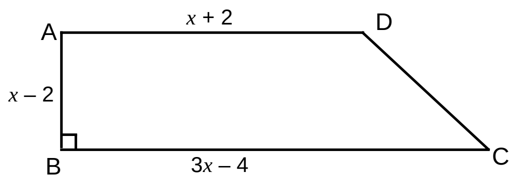


Fig. 26.4

Fig 26.4 shows trapezium ABCD, in which AD is parallel to BC and $\hat{B} = 90^0$.

a) Given that AD = (x +2) cm, AB = (x – 2)cm and BC = (3x – 4)cm, find, in terms of x, the expression for the area of the trapezium.

b) Given that this trapezium has an area of 14cm^2, form an equation, in x, and show that it reduces to $2x^2 - 5x - 12 = 0$.

c) Solve this equation and, hence, find BC.

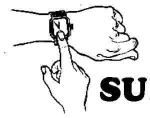
9. Mr. Gijima drove a distance of 200km in x hours.

a) Write down an expression in terms of x for his average speed for the journey.

b) Using the same route, Mr Gijima's return journey took one hour less.
Write down, in terms of x, the average speed for the return journey.

c) It was established that Mr Gijima's average speed for the return journey was 10km/h faster than the outward journey.
Form an equation, in x, and show that it reduces to $x^2 - x - 20 = 0$.

d) Solve this equation, and hence, write down the average speed for the return journey.



SUMMARY

1. Word problems all require accurate reading and thorough understanding of language. It is important to read the problem thoroughly paying particular attention to technical words like, twice, product, sum and extra.

2. When required to introduce a letter, make sure it stands for the key item in the question asked. In most cases, the letter stands for the answers of the question asked.
When asked to find area, the letter should stand for the dimensions which will be used to find the area.

3. When reading the story, take note of those words or phrases which have a mathematical meaning and use them to form appropriate expressions.

4. Where roots of a quadratic equation have been found, be cautious and check if both roots are relevant to the question. Usually only one of the roots is required.

EXAM PRACTICE 26

The following examples may help you to master questions from this chapter.

Hint

This type of problem is more straight forward. Most important features are given. You only need to remember how to find the area of a circle.

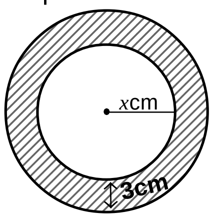
1. 

Fig 26.5

Fig 26.5 shows a washer with a hole of radius $x$cm through the centre. If the shaded area is 3cm wide,

a) find an expression in x and π, for the shaded area of the washer.

b) if the area of the shaded portion is 66cm^2, and π is $\frac{22}{7}$, find the value of x.

Solution

Hint

This is a difference of 2 squares.

1. a) Area of a circle $= \pi r^2$

$\therefore$ Shaded area $= \pi(x + 3)^2 - \pi x^2$

$= \pi\{(x + 3)^2 - x^2\}$

$= \pi(x + 3 + x)(x + 3 - x)$

$= \pi(2x + 3)3$

$= 3\pi(2x + 3)$ cm^2

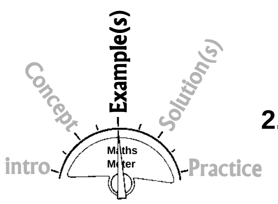
b) $3\pi(2x + 3) = 66 \times \frac{22}{7} \times \frac{1}{3}$

$2x + 3 = 66$

$2x + 3 = 7$

$2x = 4$

$x = 2$



2. The sum of the digits of a two digit number is 11. If the units digit is 1 more than four times the tens digit, find the two digit number.

Common Error

The error is to use only one letter for the number, when the story is about two digits.

Solution

Hint

Introduce 2 letters since the number in question has 2 digits.

2. i.e. Let x be the tens digit and y the units digit.

$\therefore$ $x + y = 11$ (from the first sentence)

$y - 1 = 4x$ (from the second sentence)

$4x - y = -1$

Hint

Let the girl be x years old now.
∴ The mother is 3x years old now

Solve $x + y = 11$ and $4x - y = -1$ simultaneously

$$\begin{aligned} x + y &= 11 \\ 4x - y &= -1 \\ 5x &= 10 \\ x &= 2 \\ y &= 9 \end{aligned}$$

∴ The number is 29

3. A girl is 4 times as young as her mother, 4 years ago, the product of their ages was 112. Find their present ages.

Solution

3. Let the girl be x years old now

∴ The mother is $4x$ years old

4 years ago: Girl's age = $(x - 4)$ years

Mother's age = $(4x - 4)$ years

Product means multiplying these ages to give 112,

i.e.
$$\begin{aligned} (x - 4)(4x - 4) &= 112 \\ 4x^2 - 20x + 16 &= 112 \\ 4x^2 - 20x - 96 &= 0 \\ x^2 - 5x - 24 &= 0 \\ (x + 3)(x - 8) &= 0 \\ x &= -3 \text{ or } 8 \end{aligned}$$

Note that the root – 3 is not appropriate to the above story.

∴ Present ages: Girl is 8 years old

Mother is 4×8 years = 32 years

Now do the following:

1. Farida has $(3 - 5x)$ oranges whilst Zep has $(x - 2)$ oranges less than Farida. Find in terms of x,

a) the number of oranges Zep has.

b) the total number of oranges they have between them.

2. The hypotenuse of a right-angled triangle is $(x + 4)$cm and the other two sides are each $(x - 3)$cm and 10 cm long.

a) Form an equation in x.

b) Solve this equation.

c) Find the length of the other side, correct to the nearest tenth.

3. A shopkeeper orders boxes of chocolate bars and boxes of milkshake powder.

She realises that 5 boxes of chocolate bars and 2 boxes of milkshake powder cost \$62 whilst 2 boxes of chocolate bars and 4 boxes of milkshake powder cost \$44.

a) Find the cost of each box of chocolate bars and the cost of each box of milkshake powder.

b) If each box of chocolate bars contains 25 bars, find the cost of each bar.

4. A rectangular backyard measures 20m by 12m. A path, of uniform width, runs along one side and one end. If the total area of the garden and the path is 384m^2, find the width of the path.

5. A certain girl ran a distance of 400m in $(x + 2)$ seconds.

a) Find an expression for the speed of the girl, in m/s.

b) Express this speed in km/h.

6. The bigger of two consecutive odd numbers is x.
If the sum of the two numbers is $5x + 7$, find the smaller of the numbers.

7.

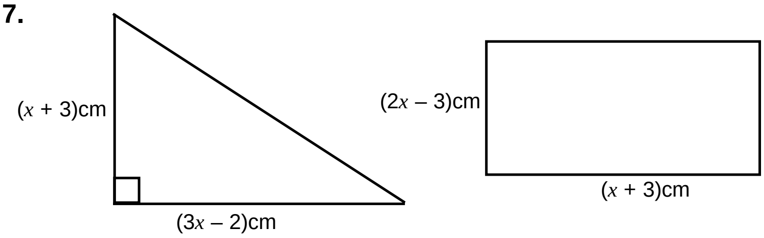
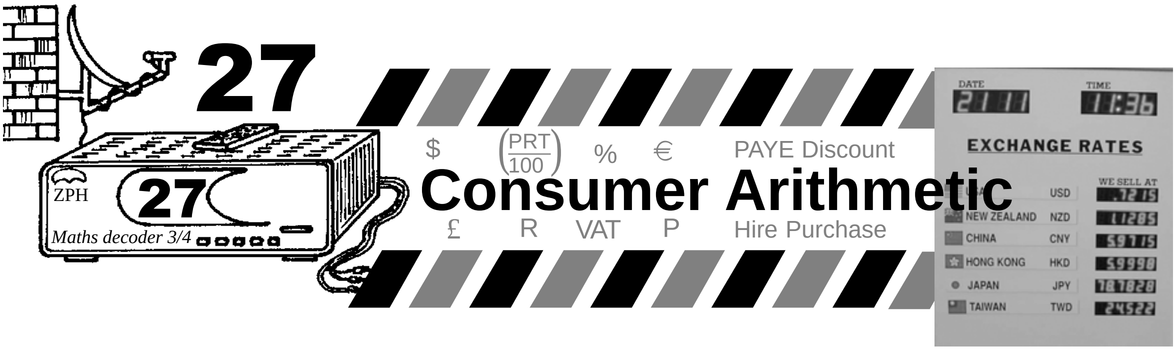


Fig 26.6

Fig 26.6 shows a right-angled triangle and a rectangle with the given measurements. Given that, the two have the same area,

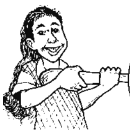
a) form an equation in x and show that it reduces to $x^2 - x - 12 = 0$

b) Solve this equation and, hence, give the dimensions of the rectangle.



27 Consumer Arithmetic

This chapter is all about money. It shows how people acquire and spend money and how they make various payments when buying goods and/or services.



Syllabus Expectations

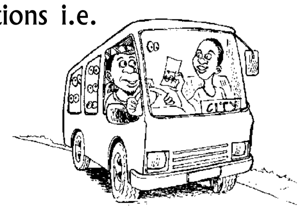
By the end of this chapter, students should be able to:

1. interpret data in the form of documents like water, electricity and/or phone bills.
2. read and understand documents like bank statements and mortgage statements.
3. solve problems related to budgets, rates, insurance premiums, wages, simple interest and/or compound interest, discount, commission, depreciation, sales or income tax, hire purchase and bank accounts.
4. read, interpret and use data presented in charts, tables, maps and graphs.
5. convert currencies from one form to another.
6. calculate simple interest and/or compound interest.

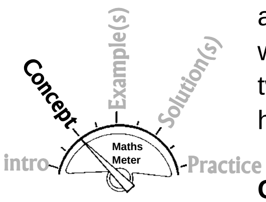
ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

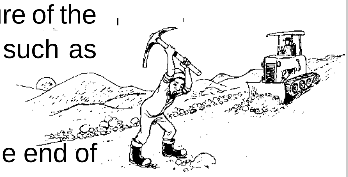
- calculate percentages given relevant values.
- work comfortably with the four arithmetic operations i.e. addition, subtraction, multiplication and division.
- convert currencies using simple figures.



A. EARNINGS



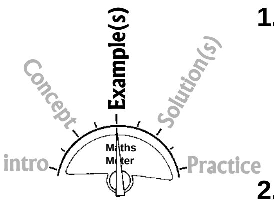
The saying, 'You reap what you sow', is strongly related to earnings at work places. Some earnings are fixed, like salaries and wages whilst others depend on production and some are a mixture of the two. Workers can be paid after varying periods of time such as hourly, daily, weekly, monthly or even yearly.



Gross income is everything the earner is entitled to at the end of the agreed time.
Take-home or **Net** income is usually less than the gross income since there may be deductions made from it. Deductions can be in the form of pension, medical aid, tax and other contributions.

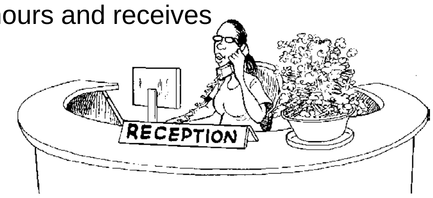
Overtime is normally paid at a higher rate than the usual working hours.
Workers, such as sales personnel, may be paid a fairly low basic wage plus a **commission**. The commission is usually a percentage of the value of goods or services sold.

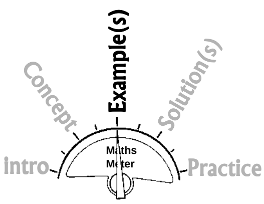
Consider the examples below:



1. Ms Mango sells fruit for a basic weekly wage of $23 plus a commission of 30c for every 100kg of fruit sold. In a certain week, she sold 6 tonnes of fruit. What was her total income for that week?

2. Thoko and Betty work in different factories. Thoko is paid $3,20 per hour for a basic working week of 35 hours and receives time-and-a-half for any overtime. Betty is paid $3,50 per hour for a basic working week of 39 hours and receives double-time for any overtime.
Find
a) Thoko's basic weekly wage.
b) Thoko's overtime rate per hour.
c) Betty's basic weekly wage.
d) Betty's overtime rate per hour.
e) Who receives the greater gross pay in a week if they both worked for 45 hours?





3. In addition to a basic monthly salary of $205, a sales lady received a commission of 2% for any sales above $500.
a) Calculate her gross salary for the month if she made sales of $1 550.
The following deductions were made:

Pension	**Medical Aid**	**PAYEE**
$3,90	$1,65	$4,50

b) Calculate her net salary for the month she made sales of $1 550.

Solutions

1. Total income $= \$23 + \frac{30}{1\cancel{00}} \times 60\cancel{00}$

$= \$23 + 18,00$

$= \$41$

2. **Thoko**

a) Basic weekly wage $= \$3,20 \times 35$

$= \$112,00$

Hint

Time-and-a-half means $1\frac{1}{2}$ times.

b) $\therefore$ Overtime rate $= \$\overset{1,60}{\cancel{3,20}} \times \frac{3}{\cancel{2}_1}$

$= \$4,80$ per hour

Betty

c) Basic weekly wage $= \$3,50 \times 39$

$= \$136,50$

d) Overtime rate $= \$3,50 \times 2$

$= \$7,00$ per hour

e) Thoko's overtime $= (45 - 35)h = 10h$

$\therefore$ Thoko's gross wage $= \$112,00 + \$4,80 \times 10$

$= \$112,00 + \$48 = \$160$

Betty's overtime $= (45 - 39)h$

$= 6h$

$\therefore$ Betty's gross wage $= \$136,50 + (\$7,00 \times 6)$

$= \$136,50 + \$42,00$

$= \$178,50$

$\therefore$ Betty received more

3. a) Basic salary $= \$205$

Commission received $= \frac{2}{100} \times 1050$ $(1550 - 500 = 1050)$

$= \$21,0$

$\therefore$ Gross salary $= \$(205 + 21)$

$= \$226$

b) Total deductions $= \$(3,90 + 1,65 + 4,50)$

$= \$10,05$

$\therefore$ Net salary $= \$(226 - 10,05)$

$= \$215,95$

1. Mr Pedro receives a wage of \$60 per week and receives a commission of 3%, on all sales over \$600. Find his income for a week he sold goods worth \$7 700.

2. The hourly rate at Dhlodho Auctions for a 43-hour week is \$4,30. Overtime is paid at time-and-a-half.

a) One week Ben worked for 51 hours. Calculate his gross pay.

b) In the same week Kim worked for 55 hours. Find her gross pay.

c) The following week Ben in (a) worked for 57 hours. How much more did he earn this week than last week.

3. Ms Khupe's time-sheet, for a week, is shown in table 27.1 below.

Table 27.1

Name: Jean Khupe
Works no.: 78
Week no.: 13

DAY	IN	OUT	IN	OUT
Monday	8.05	12.00	13.00	16.32
Tuesday	8.00	12.01	13.00	16.30
Wednesday	8.02	12.03	12.58	16.30
Thursday	7.58	12.00	13.20	16.31
Friday	8.00	11.59	13.00	15.30

Use this time-sheet to answer the following questions.

a) What is the clock-in time supposed to be?

b) How long is the lunch break?

c) How long should Ms Khupe work in the mornings?

d) On which day of this week did Ms Khupe have an extended lunch break?

e) How long is the basic working week if the time-sheet shows that Ms Khupe worked the required time on the Friday?

f) Calculate the gross wage for this week if the basic hourly rate is \$3,90.

4. Mr Midzi is paid a basic salary of \$122 per month plus a commission of $1\frac{1}{2}$ %, on sales over \$1 500. Find his gross salary, for a month in which he made sales of \$16 600.

Hint

Since there is no direct comparison between £ and (P), go through the Rand

5. Mrs Dube sells building materials.
Apart from a basic wage of \$56 per week, she is paid commission at the following rates

up to \$600	none
from \$601 to \$1 500	4%
above \$1 500	$2\frac{1}{2}$%

Calculate her income in a week when she made sales of \$3 220.

Hint

\$56 + 4% of \$1500 + 2 % of (\$3220 – \$1500)

6. Mr Longfields wants to raise money to buy 5 bags of fertilizer costing $34 each, for the coming season. He decides to earn it by washing cars at $1,20 per car.
   a) How many cars must he wash to raise the amount needed?
   b) How many days would it take him if he washes 22 cars each day?

7. The table 27.2 below shows the number of spark plugs produced by three workers each day, for a certain week.

Table 27.2

Name of worker	Mon	Tue	Wed	Thur	Fri
Tindo	37	38	36	42	40
Tom	42	44	–	48	43
Thelma	35	44	48	43	38

The rate of pay is 55c per plug for up to 25 plugs and 70c for each plug above 25 per day.
   a) How many plugs did each person produce in this week?
   b) Find the gross income for each worker for this week.
   c) Which day had the highest production?
   d) Find how much was paid out altogether to the three workers on Tuesday.

8. Table 27.3 below, shows details of the pay earned by four employees, of a certain parastatal, in a certain month.

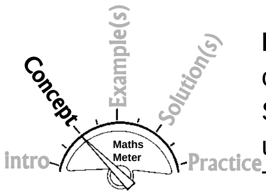
Table 27.3

Employee	**Gross Pay($)**	**Deductions**			**Net Pay($)**
		Medical Aid ($)	**Income Tax($)**	**Pension Fund($)**	
Chung	176,35	3,61	–	7,15	$156,89
Musa	204,00	7,60	11,89	9,11	–
Khama	$184,00	–	10,45	8,55	$137,50
Mafy	–	8,64	10,08	6,76	143,33

Find the missing item on each of the employee's pay slip.

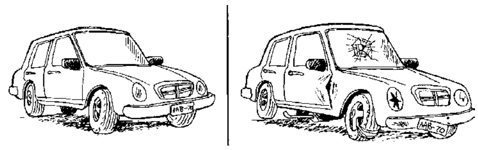
B. BUYING AND SELLING

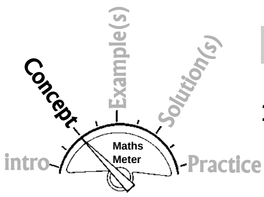
Purchasing goods for sale is a cost. After selling, one makes either a profit or a loss.
Thus
Selling Price (S.P) – Cost Price (C.P) = Profit (if **SP > CP**) However, it becomes a loss (if SP < CP), hence
Loss = Cost Price (CP) – Selling Price (SP)



Discount is sometimes given, especially, to customers who pay cash. This is an amount deducted from a marked price.
Sometimes the price of goods is so high that it is impossible to pay using cash. In such cases, a **Hire Purchase** arrangement is made. This is a system where one makes a down payment (deposit) and the remaining payments are spread over a period of time. Each payment for the rest of the amount is called **an instalment**. Note that paying on a **Hire Purchase** (HP) system is more expensive than paying cash because you have to pay interest. During the period one is paying the installments, the goods are on hire to the buyer, and a percentage is added to each payment.

Durable goods lose value with time, for example, a car's worth next year is lower than it is this year. This loss of value is called **depreciation**.



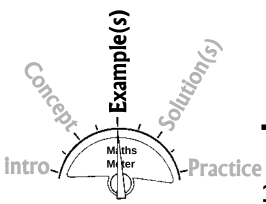


Consider the following examples

1. A car costs \$2 650. A 7% discount is given for cash. However the hire purchase price of this car is 15% deposit and 24 monthly payments of \$110. Calculate the difference between paying cash and paying by hire purchase.

2. When an article is sold for \$15,25 a profit of 25% is made. Find the cost price of the article.

3. Buildings on a farm are said to depreciate at the rate of 3c in the dollar, annually. A shed worth \$1 360 was built on the farm. Find its worth two years later.



Solutions

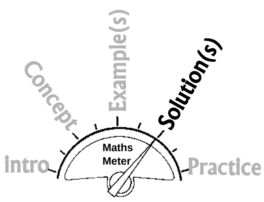
1. Cash price $= \$2\,650 - \frac{7}{100} \times 2\,650$

$= \$2\,650 - \$185{,}50$
$= \$2\,464{,}50$

or

Discount reduces the price to 93%

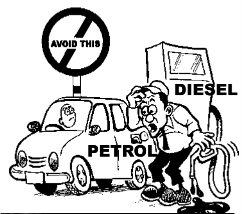
$\therefore$ Cash price $= \frac{93}{100} \times 2\,650$
$= \$2\,464{,}50$



The H.P price $=$ Deposit + Total Installments

$= \frac{15}{100} \times 2650 + 24 \times \$110$
$= \$397{,}50 + \$2\,640$
$= \$3\,037{,}50$

$\therefore$ Difference $= \$3\,037{,}50 - \$2\,464{,}50$
$= \$573{,}00$



Common Errors

The deposit is often forgotten and not included in the H.P price.

Notice that the difference between the H.P price and the marked price of the article gives the cost of hiring it. Thus in this case Cost of Hire = 3 037,50 – 2 650

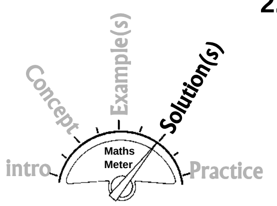
= $387,50

Common Errors

$\frac{25}{100} \times 1525$

is wrong!

Percentage cannot be found on Selling Price for it is not basic.



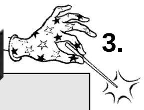
2. Selling Price = $15,25

Profit = 25%

Selling Price percentage = 125%

$\therefore$ Cost price $= \frac{100}{125} \times 1525$

= $12,20



Hint

3c in the dollar means 3%

3. Value of building now = $1 360

Value after one year $= \frac{97}{100} \times 1360$ or $1360 - \frac{3}{100} \times 1360$

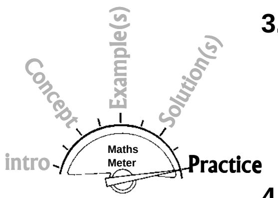
= $1319,20

Value after second year $= \frac{97}{100} \times 1319{,}20$

= $1 279,62

1. A discount of $22,00 is allowed for an article priced at $330,00. Find the percentage discount, to the nearest whole number.

2. When an article is sold for $12,90, a loss of $7\frac{1}{2}$% is made. Find the cost price of the article, to the nearest cent.



3. A car radio costs either $280,50 cash or 24 weekly payments of $13,50 with no down payment.

a) Find the cost if bought through hire purchase.

b) Find the cost of the hire of the radio for the 24 weeks.

4. Dick bought a gas lamp for $210 and sold it to Eva at a profit of 19%. Eva later sold the lamp to Adam at a loss of 17%. How much did Adam pay for the lamp? Give your answer to the nearest dollar.

5. Sipho paid $310,60 cash for a lounge suite after being allowed a discount of 15%. What was the marked price of the lounge suite to the nearest cent?

6. A farmer sells a cow to a dealer at a discount of 28%. The dealer then sells the cow at a profit of 12% to a butcher who paid \$336. Find the original price received by the farmer, to the nearest cent.

7. The value of a certain car depreciate by 25% annually. After one year, an owner of the car, decided to sell it to a dealer, at a discount of 10% of the current market value. If the owner managed to get \$2 790 from the dealer what was the original price of the car, to the nearest dollar?

8. Calculate the simple interest charged on \$880 borrowed for 9 months, at the rate of $12\frac{1}{2}$ % per annum.

9. A car dealer borrows \$2 500 to be paid back at a 5% simple interest rate. He used this money to buy a car for resale. Find the price this car should be sold at, if the dealer is to make a profit of 20% after 6 months of borrowing.

C. TAXES

Taxation is one way governments raise funds for public services like building roads, medical services and other services. People who are gainfully employed are legally obliged to pay taxes to the state. Usually governments decide what rate to use each year. Generally the Tax headings are as follows:

a) **PAY AS YOU EARN (PAYE)**

This is deducted from salaries or wages on a daily, weekly, fortnightly, monthly or yearly basis. Generally, the taxable income is arrived at as follows

$$\left(\text{Basic Salary} + \text{Allowances}\right) - (\text{Exemptions} + \text{Deductions}) = \text{Taxable Income}$$

Exemptions include tax-free bands, tax-free bonus, and some allowances.

Deductions include pension and medical contributions. Governments usually give a credit for a disability.

b) **VALUE ADDED TAX (VAT)**

This is levied on sales of goods and/or services and it is usually collected from the producer or manufacturer. Some goods, like basic foodstuffs, and some non-profit making services, like certain colleges and non-governmental organisations (NGO) are exempted from tax.

There are many more tax headings but we will restrict our discussion to the two mentioned above.

The table below illustrate PAYE rates on weekly and monthly basis.

Table 27.4 Weekly Table for PAYE

TAX BRACKET			RATES
	to	35,11	0%
35,12	"	117,02	20%
117,03	"	234,04	25%
234,05	"	351,06	30%
351,07	"	702,13	35%
702,14	and	above	37,5%

Table 27.5 Monthly Table for PAYE

TAX BRACKET			RATES
	to	150	0%
151	"	500	20%
501	"	1000	25%
1 001	"	1500	30%
1 501	"	3 000	35%
3 001	and	above	37,5%

Consider the examples below:

Use the given tables 27.4 or 27.5, where applicable.

1. Mr Matuku's monthly salary is $1 250. Calculate his HIV/AIDS levy, given that he has no tax free allowances or any deductions.
(HIV/AIDS Levy = 3% of PAYE)

Solution

$$\text{PAYE} = \frac{30}{100} \times 1250$$

$$\text{HIV/AIDS Levy} = \frac{3\cancel{0}}{10\cancel{0}} \times 125\cancel{0} \times \frac{3}{10\cancel{0}}$$

$$= \frac{1125}{100}$$

$$= \$11,25$$

2. Mr Moyo is paid weekly wages of $572.
He has the following deductions before tax:

Pension	$3,62
Medical Aid	$6,71
Insurance	$2,76

Mr Moyo also enjoys the following tax-free allowances every week. Transport $6
Housing $10

Hint
Aids Levy is a percentage of PAYE.

He is levied a 3% HIV/AIDS Levy. (HIV/AIDS Levy = 3% of PAYE)
Calculate his: a) PAYE.
b) Aids Levy.
c) his take home salary.

Solution

a) Deductions $= \$(3,62 + 6,71 + 2,76)$
$= \$13,09$

Taxable Income $= 572,00$
$- 13,09$
$\$558,91$

Hint
$558,91 falls under 35% Tax in the Weekly Table.

$\therefore$ PAYE $= \dfrac{35}{100} \times 558,91$

$= \$195,62$

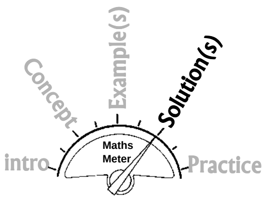
b) $\therefore$ Aids Levy $= \dfrac{3}{100} \times 195,62$

$= \$5,87$

c) Take home $= 572 + 6 + 10 - (3,62 + 6,71 + 2,76 + 195,62 + \$5,87)$
$= \$373,47$

3. Spiwe buys a pair of shoes from Bata at $15, for resale. She puts a 20% mark-up on the shoes. VAT is charged at 15%
a) Calculate the selling price of the shoes, including VAT.
However, when buying the shoes from Bata, Spiwe was charged VAT at 15%. This means she is entitled to a refund from the state.
b) Find how much refund Spiwe is entitled to.

Solution



a) Selling Price before VAT $= 15 + \frac{20}{100} \times \$15$

$= \$15 + \$3$

$= \$18$

VAT $= \frac{15}{100} \times \$18$

$= \$2,70$

$\therefore$ Price VAT inclusive $= \$(18 + 2,70)$

$= \$(20,70)$

Common Errors

Never find percentage of something not basic e.g. Selling Price e.g. 15% of $15 has no meaning

Hint

This includes 15% VAT.

b) Amount paid $= \$15$

Price before VAT $= \frac{100}{115} \times 15$

$= 13,043$

$= \$13,04$

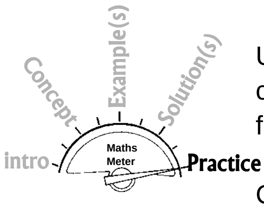
$\therefore$ VAT to Spiwe $= \$(15 - 13,04)$

$= \$1,96$

Refund = VAT after selling – VAT to Bata

$= \$2,70 - \$1,96$

$= \$0,74$

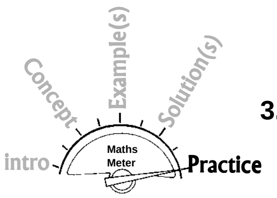


Use the Tax Tables 27.4 and 27.5, on page 218 to make the calculation required. For this exercise all VAT is 15% and all basic foodstuffs i.e. bread, meat, salt, cooking oil and soap are tax-free

Calculations are to the nearest cent.

1. Ms Phiri receives a monthly salary of $165.
The following are her deductions
Pension: $4,40
Insurance: $2,70
She pays a 3% HIV/AIDS levy every month.
Calculate her net salary.

2. Ms Phiri, in question (1) above, worked overtime during a certain month and earned $85 more than her usual basic salary.
Calculate her net salary for the month she worked overtime.



3. Mr Kunda earns a $30.00 weekly wage. One month his overtime totalled $120.00.
However, Mr Kunda is disabled, and he enjoys a $900 per year credit.

He also has the following deductions before tax:
Pension: $5,75
Medical Aid: $11,50
Insurance: $6,25
Calculate the PAYE he was paid for this month.

4. Apart from the 3% HIV/AIDS levy, Mr Kunda, in question 3 above, has a $30 weekly instalment and payment for the lounge suite he is buying through Hire Purchase. Calculate his net salary for the month discussed, in question 3 above.

Tendai was window shopping in the city when he saw the following advertisement splashed over a shop's window:

DISCOUNTS TO CLEAR

10% on all shirts

15% on all jackets

Prices exclude VAT

Fig 27.1

Tendai went into the shop and made some purchases.
Use the above information to answer questions 5 and 6.

5. Tendai bought 2 shirts marked $15 each and a jacket marked $35. Calculate the total amount paid.

6. If Tendai bought 1 shirt marked $15 and a pair of trousers marked $12, how much would he pay altogether?

7. Mai Farai went shopping in town. She put the following items in her trolley:
4 × 2kg salt @ $1,15 per kg
4 bars of soap @ $3,00 each.
2 × 2 litre tins cooking oil @ $3,00 each
4 dish towels @ $3,50 each
1 pair tennis shoes @ $12 a pair and
1 neck tie @ $8 each.
How much change did she get from a $100 note?

D. BILLS

A bill is an amount of money owed for goods supplied or services rendered.
Most people living in urban areas are familiar with bills. They can be electricity, water or phone bills. Many people do not understand the figures on these bills. Below is a typical telephone bill, supposedly from Tel One of Zimbabwe.

TELONE	TAX INVOICE	VAT No. 10001509
Mari Phil	Exchange	Account No.
P. B 701	UMP	160419
UMP		
Date of Account	Last date of payment	
May 2009	23 Jun 2009	

Date	Account Details	Amount
	Balance B/F	156,50
07 May 2009	Receipt 001076	100,00
	Overdue	56,50
	Interest@ 0%	0,00
	Subtotal	56,50
	UMP 3203	
	Rentals June 2009	10,00
27 May 2009	MTRD Units: 01293 01110 (183)	38,43
	Subtotal	48,43
	VAT @ 15%	6,66
	AMOUNT DUE	**111,59**

Fig 27.2

From the Invoice in Fig 27.2.

- The Account has arrears of $156,50
  (B/F means Brought Forward)(B/F also means not paid on a previous bill/s)
- A payment of $100 was made on May 7 reducing the B/F balance to $56,50 due.
- No interest was charged on the arrears but usually interest is charged on all overdue payments.
- Metered (MTRD) units are:
  Current reading (01293) – last reading (01110) = 183
- VAT is charged on the rentals charged + Units charged. (The first subtotal was charged VAT) What is the cost of 1 unit?
  $= \left(\frac{3843}{183} = 21c\right)$

Use the relevant tables to answer the questions which follow.

Table 27.6 **RATES**

Item	Rate	Amount
Land	1,166 cents in the $	
Improvements	0,0458 cents in the $	
Refuse Removal	Fixed at	$26,75
Sewage Charge	Fixed at	$33,40

Table 27.7 **Electricity**

Customer Type	Code	Monthly Fixed Charge	Energy Charged cents/kwh
Domestic	E1	\$10,00	7,63 cents for first 300kwh 10,05 cents on balance
Small Scale Industrial Area	E2	\$20,50	11,55c
Commercial Industrial Farming Area	E3	\$35,70	9,45c

N.B. Surcharge of 10% is charged on each Bill

Table 27.8 **Water Authority**

Customer Type	Code	Rate (per month per cubic metre)
Residential	W1	First 15m^3 at 33,6c/m^3 Next 30m^3 at 45,3c/m^3 The rest at 69,9c/m^3
Commercial and Industrial	W2	43,7C/m^3

Telephone

Table 27.9 G. CELLULAR PHONES (G.C.P)

Customer	FROM	TO	Charge per minute
Peak Hours	G.C.P G.C.P G.C.P	G.C.P Other Landlines	22c 23c 21c
After Hours	G.C.P G.C.P G.C.P	G.C.P Other Landlines	21c 22c 20c

Use either table 27.7 or 27.8 and 27.9 to answer the following questions.

1. Check if the following bill is correct.
Energy used 350kwh under E1
Total Bill \$41,71

2. Ms Ono used 70m^3 of water, in a certain month. Find how much she would have to pay:
a) under W1. b) under W2.

3. Mr Brian uses a G. Cellular Phone (GCP) line. During a certain month, he made the following calls:
170 minutes to G.C.P during peak hours.
70 minutes to landlines after hours.
166 minutes to other cellular services during peak hours.
Calculate his total bill for that month.

Solutions

1. Energy charge:

$300 \times 7{,}63$	=	2 289,00
$50 \times 10{,}05$	=	502,50
Subtotal	=	2 791,50
Fixed charge	=	1 000
Subtotal	=	3 791,5
10% Surcharge	=	379,2
	=	4170,7
Thus Total Bill	=	\$41,71

The Bill is correct.

2. a) Billing under W1

$15m^3 \times 33{,}6c$	=	504,0
$30m^3 \times 45{,}3c$	=	1 359,0
$25m^3 \times 69{,}9c$	=	1 747,5
	=	3 610,5
Total	=	\$36,11

b) Under W2

$70m^3 \times 43{,}7c$	=	3059,0
Total	=	\$30,59

3. Mr. Brian's telephone bill

$170min \times 22c$	=	3 740
$70min \times 20c$	=	1 400
$166min \times 23c$	=	3 818
Total Bill	=	\$89,58

1. Check if the following bills are correct.

a) Energy used under E1 = 405 kwh
Total Bill = \$47,79

b) Water Bill
$35m^3$ under W1
Total Bill = \$17,80

c) Energy used under E3 = 556 kwh
Total Bill = \$96,10

2. Ms Kennedy lives in a suburb where her total land is worth \$2 650 and has improvements worth \$14 591.
Calculate Ms Kennedy's rates bill.

3. The following is a Telephone bill for the month of May for Jack who uses a GCP contract line.

Table 27.10

Description	Peak	Amount (cents)
To G.C.P	87 min Peak Hours	1 914
To G.C.P	a min After Hours	819
To other lines	43 min Peak Hours	b
To other lines	27 min After Hours	594
To Landlines	c min Peak Hours	609
To Landlines	36 min After Hours	720
	Total	d

a) Find (i) a (ii) b (iii) c (iv) d

b) How much change does Jack receive from \$60, after paying the bill.

4. GCP charges 15% interest on all overdue amounts. In the month of May mentioned in (3), Jack's account was \$14,60 overdue.
Calculate the grand total Jack was expected to pay in the month of May from question 3.

Hint

Rates + Electricity + Water, Bills

5. Mr Nyathi has a commercial stand worth \$3 058 and the improvements he has made, so far, are worth \$12 676. He received a rates bill, which included electricity and water charges for that month. The statement showed that Mr Nyathi had used 379 kwh energy and 55m^3 of water, that month.
Calculate the total bill he received.

6. Below is an incomplete water bill for a household from a certain town.

Table 27.11

Description	**Previous Reading**	**Present Reading**	**Consumption (Units)**	**Rate cents/Unit**	**Cost \$ c**
Water	7 974,7	8 168,2	x	12,4	y
Fixed Charge					11 80
Amount due					z

Find the value of a) x
b) y
c) z

7. A certain private electricity authority charges for its energy supply as follows:
(All calculations being to the nearest cent)

Fixed monthly charge $16,50
Energy charges *per unit*
first 200 units 17,3cents
the rest 36,5cents

Fig 27.3 is an incomplete electricity bill, under this authority for the month of October.

Description	**Previous Reading**	**Present Reading**	**Consumption (Units)**	**Cost $ c**
Energy	6 7869	6 8148		
Fixed Charge				16 50
Sub-total				
Development Levy (5% of sub-total)				
Total				

Fig 27.3

Calculate for the month of October:
a) the number of units used.
b) the cost of the electricity used.
c) the Development Levy that was paid.
d) the total amount due for this month.

The tables below gives the rates of postage, for Zimbabwe and to send letters and parcels, by airmail, to countries in Africa, Europe and the rest of the world.

Table 27.12 **Letters**

Destination	**Weight of Letter (wg)**	**Cost $ c**
Zimbabwe	$0 < w \leqslant 20$	25c
Zone A	$20 < w \leqslant 100$	50c
	$100 < w \leqslant 250$	75c
	$250 < w \leqslant 500$	1 00
	$500 < w \leqslant 1000$	1 50
	$1000 < w \leqslant 2000$	2 00

Table 27.13

	COST		
WEIGHT OF LETTER (wg)	**AFRICA (Zone B)**	**EUROPE Zone (C)**	**REST OF WORLD (Zone D)**
$0 < w \leqslant 10$	50c	75c	$1,00
$10 < w \leqslant 50$	75c	$1,00	$1,25
$50 < w \leqslant 100$	$1,00	$1,25	$1,50
$100 < w \leqslant 250$	$1,25	$1,50	$2,00
$250 < w \leqslant 500$	$2,00	$2,00	$4,00
$500 < w \leqslant 1000$	$2,50	$3,00	$4,00
$1000 < w \leqslant 2000$	$3,10	$4,00	$5,00

Table 27.14 **Parcels**

WEIGHT	ZIMBABWE	AFRICA	EUROPE	REST OF WORLD
First 1kg	$3,00	$10,00	$15,00	$20,00
Each additional kg	$1,00	$5,00	$5,00	$5,00

Use tables 27.12, 27.13 or 27.14 to answer questions 8 and 9.

8. Zandile wants to post the following:
2 letters within Zimbabwe, one weighing 15g and the other 148g.
1 letter to Europe, weighing 39g and
1 parcel to Australia, weighing 4kg.
How much is she to pay for postage?

9. One day Sipho posted the following items:
2 parcels to Malawi, each weighing less than 1kg, and 5 letters (2 to Europe, weighing between 250g and 500g each, and 3 to Egypt, weighing between 10g and 50g each.)
On the same day, Ann posted 2 parcels, 1 to Nigeria weighing $3\frac{1}{2}$ kg, and the other one to Jamaica, weighing $2\frac{1}{2}$ kg.
Calculate who paid the most and by how much.

Hint

Any fraction of a kg is rounded up

10. At the end of a certain working day, Lundi Post Office in Zimbabwe, processed the following for postage items.

Table 27.15

ITEM	WEIGHT	DESTINATION
16 letters	less than 20g	Zimbabwe
7 letters	between 100 and 250g	Zimbabwe
5 letters	between 50g and 100g	Europe
1 Parcel	3kg	South Africa
1 Parcel	$2\frac{1}{2}$ kg	Zimbabwe
1 Parcel	3kg	Japan
2 Parcels	4kg each	France
1 Parcel	6kg	Britain
1 Parcel	$5\frac{1}{2}$ kg	New Zealand

Calculate the amount of money the Post Office made on postage on this day.

E. INSURANCE AND MORTGAGES

People and their properties are exposed to various risks e.g. car accidents – the car needs to be insured against damage and the driver against death or injury. Insurance is a financial precaution against possible loss or damage. A periodic payment for insurance is called a **premium**. This is quoted in various ways although, it is usually payable annually or monthly.

Car insurance companies usually offer discounts called **No claims bonuses**, for no claims made over an agreed period of time.

A **mortgage** is a loan given specifically for the purchase of land and/or any buildings which are on the land. This business is usually done by building societies and these own the property until the loan on it is fully paid.
Monthly repayments are usually made which include the interest charged plus repayment, of the capital borrowed.

Consider the following examples:

1. The premium for insuring a house worth \$23 000 is \$5,30 per \$1 000 value. Find the premium to be paid.

Solution

1. Premium $= 23 \times 5{,}30$
$= \$121{,}90$

2. A building society charges a monthly repayment of $9,65 for each $1 000 borrowed.

a) Calculate the monthly repayment on a mortgage of $11 500, giving the answer to the nearest cent.

b) Calculate the total repayment over 25 years.

c) If this monthly repayment includes 13,25% interest, calculate the capital repayment on each $1000.

Solution

a) Monthly repayment $= 9{,}65 \times 11{,}5$
$= 110{,}975$
$= \$110{,}98$

b) Total over 25 years $= 110{,}975 \times 12 \times 25$
$= \$33\,292{,}50$

c) $\therefore$ Capital repayment $= \dfrac{100}{113{,}25} \times 9{,}65$

$= \$8{,}52$

Hint

Use unapproximated monthly repayment.

*****

Capital (100%) + Interest (13,25%) = $9,65 (113,25%)

3. A car worth $2 500 is insured at $15 for every $100, annually. A no claims bonus is given, at 20% of premiums paid if no claims are made over a period of 3 years. Calculate how much the owner got from the insurance company if no claims were made in the first 3 years.

Solution

Premiums in 3 years $= \dfrac{15}{1\cancel{00}} \times \$25\cancel{00} \times 3$

$= \$1\,125$

$\therefore$ No claims bonus $= \dfrac{2\cancel{0}}{10\cancel{0}} \times \$2500$

$= \$500$

The table 27.16 overleaf shows the premiums for insuring building and contents in the different surburbs of a certain town.

Table 27.16

AREA	Buildings Per $1000	Contents Per $1000
High Density	$1,30	$5,70
Medium Density	$2,65	$8,85
Low Density	$3,10	$12,05

Use the Table 27.16 above, to answer questions 1 – 7.

1. Find the premium for insuring a house (building) worth $10 600 in the medium density area.

2. Find the premium for insuring a house worth $6 860 and contents worth $2 400, in the high density area.

3. Mr and Mrs Penny live in the low density area. Their valuable contents include a lounge suite worth $3 550, a home theatre worth $765, 3 bedroom suites worth $1 800 altogether, 2 rings worth $700 each, a carpet worth $260 and a watch worth $130. Find the premium they would pay if they insure the contents listed above.

4. Mr and Mrs Penny's house/(building), in question 3, is worth $34 000. They want to insure the house and some contents. One insurance company says items, with a value below 15% of the total value of the contents intended to be insured, cannot be listed and insured.

a) State, the items which qualify, to be listed and insured, by this company.

b) Find the premium for the house and the listed contents.

5. Ms Nkomo lives in the medium density area and her house is worth $13 300 with contents worth $5 760. A discount of 2% is given on the contents, if the windows are burglar-barred, and 5% discount is allowed on the building, if it has a ceiling. Find the premium for insuring the house and contents if

a) no discount is allowed.

b) both discounts are allowed.

c) the contents discount only, is allowed.

6. A building society sets a monthly repayment of $12,54 for each $1000 borrowed. Calculate the monthly repayment on a mortgage of $11 660, giving the answer to the nearest cent.

7. The building society in question 6, charges an interest of $12\frac{1}{2}$% per annum. How much interest is paid on each $1 000 borrowed:

a) per year? b) per month?

8. Murewa Rural Council sold a commercial stand to As-One-Greens (Pvt) Ltd, for $6 780. The company obtained a mortgage of 60%, of the price of the stand.

a) Find the amount of the mortgage loan.
This loan is to be paid over 10 years at an interest rate of $7\frac{1}{2}$ % per annum.

b) Calculate the total monthly repayment.

c) What is the monthly repayment for each $1 000?

9. A man buys a luxury car for $8 500 and insures it at $4,70 on each $1 000, annually. He realises that depreciation is at 20% annually, and the insurance rate increases by 70c every year.

a) How much insurance is the man expected to pay over 3 years?

b) The Insurance Company decides to give the man a 'no claims bonus' of 30% in the third year. Calculate the actual amount he pays in the third year.

10. Ms Anne wants to take out a loan to buy herself a second hand car. She estimates what it will cost her to run the car for one year. She lists the expenses as follows:
Loan repayment at $65 per month.
Premium for insurance at $120.
Road tax at $90 annually.
A service charge of $200 and $14 for petrol weekly.

a) How much does her estimates for one year of motoring amount to?

b) How much is this monthly?

F. INTEREST

When you borrow money, you are likely to pay it back with **interest**. When you bank money in a savings bank account, it accrues interest. Two types of interest will be discussed here.

a) Simple Interest (I)

Components of this type are

Principal (P) – the money invested or borrowed.

Time (T) – the period one is to invest or borrow the principal before repaying.

Rate (R) – the percentage of interest usually given per annum (year).

$$\text{Simple Interest} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100} \quad \left(\frac{PRT}{100}\right)$$

This formula can be manipulated to produce other formulae like:

$$\text{Principal (P)} = \frac{100I}{RT}$$

$$\text{Time (T)} = \frac{100I}{PR}$$

$$\text{Rate (R)} = \frac{100I}{PT}$$

Consider the following examples:

1. Find the simple interest earned on \$350, at 9% per annum, for 3 years.

2. Find the time, in years, that \$250 will earn \$30 simple interest, at 6% per annum.

Solution

1. Given:

Principal	–	\$350
Rate	–	9%
Time	–	3 years

$$\therefore \text{ Simple Interest} = \frac{35\cancel{0} \times 9 \times 3}{10\cancel{0}}$$

$$= \$94{,}50$$

2. Given:

Principal	–	\$250
Interest	–	\$30
Rate	–	6%

$$\therefore \text{ Time} = \frac{100 \times 30}{250 \times 6} \text{ using } \frac{100I}{PR}$$

$$= 2 \text{ years}$$

There are two very important features of simple interest.

(i) The principal is fixed over a period of time, so interest earned is the same annually. In example (1) the interest was \$31,50 annually, for 3 years.

(ii) The denominator 100, in the formula, comes from the rate, as in example (1) 9% = $\frac{9}{100}$

If the rate was $16\frac{2}{3}$%, then Interest would be

$$\frac{\text{Principal} \times 50 \times \text{Time}}{300} \quad \text{since} \left(16\frac{2}{3}\% = \frac{50}{300}\right)$$

b) **Compound Interest**

This type of interest is not fixed over time. The principal for the first year changes in the second year and the principal in the second year changes in the third year and so forth. What creates the change is that the interest earned in one year, is added to the principal before calculating the interest for the next year, hence compound interest. The illustration below may clarify the issue.

Year 1 Interest = Principal $\times$ Rate

Year 2 (Principal = Principal for year 1 + Interest earned)
$\therefore$ Interest = Year 2 Principal $\times$ Rate

Year 3 (Principal = Principal for year 2 + Interest earned)
$\therefore$ Interest = Year 3 Principal $\times$ Rate and so forth.

3. Let us use the same figures as in example 1, for simple interest to calculate the compound interest.

Solution

3. Year 1: Principal – \$350

$$\therefore \text{Interest} = \$350 \times \frac{9}{100} \times 1 = \$31{,}50$$

Year 2: Principal = \$350 + \$31,50 = \$381,50

$$\therefore \text{Interest} = 381{,}50 \times \frac{9}{100} = \$34{,}34$$

Year 3: Principal = \$381,50 + \$34,34 = \$415,84

$$\therefore \text{Interest} = \$415{,}84 \times \frac{9}{100} = \$37{,}43$$

$\therefore$ Interest in 3 years = \$31,50
\$34,34
+ \$37,43
\$103,27

or
Total Amount for final year – Initial Principal
i.e. (415,84 + 37,43) – 350 = \$103,27
Notice this interest is calculated yearly (thus time is 1 year) so the formula is simply, Principal $\times$ Rate.

4. Calculate the compound interest earned on \$560, at $12\frac{1}{2}\%$, for 4 years.

Solution

4. Year 1: Principal = \$560

$\therefore$ Interest $= \$560 \times \dfrac{25}{200} = \$70$

Year 2: Principal = \$560 + 70 = \$630

$\therefore$ Interest $= \$630 \times \dfrac{25}{200} = \$78{,}75$

Year 3: Principal = \$630 + \$78,75 = \$708,75

$\therefore$ Interest $= \$708{,}75 \times \dfrac{25}{200} = \$88{,}59$

Year 4: Principal \$708,75 + \$88,59 = \$797,34

$\therefore$ Interest $= \$797{,}34 \times \dfrac{25}{200} = \$99{,}67$

$\therefore$ Total Interest earned = \$70,00
\$78,75
\$88,59
+\$99,67
\$337,01

Calculate the missing items in the table below.

	Principal	Rate	Time	Simple Interest	Compound Interest
1.	\$200	12%	2 years		
2.	\$340	$6\frac{1}{2}\%$	4 years		
3.		10%	3 years	\$25,50	
4.	\$525		4 years	\$105	
5.	\$1 500	$5\frac{1}{3}\%$	10 months		
6.	\$3 200	$12\frac{1}{2}\%$	9 months		

E. MONEY

The world today is awash with money of different currencies and trade accomodates the different currencies in circulation. This is possible because money is treated like any other commodity and is being sold and bought. Thus foreign currencies (money which does not belong to a particular country) can be bought and sold (just like shoes can be bought and sold) between countries or people.
To facilitate this trade of money, exchange rates need to be established. The following are exchange rates used on a certain day.

Table 27.17 Rate of Exchange

Currencies	Rate of Exchange
R / $	7,46
R / P	0,88
R / £	11,95
R / €	0,89
$ / €	1,46
£ / €	1,59

KEY
R (South African Rand)
$ (US Dollar)
P (Botswana Pula)
€ (European Euro)
£ (British Pound)
Also note that
R/$ 7,46 means
$1 = R7,46

Fig 27.4

Consider the following conversions using the above rates of exchange:

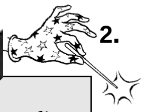
1. P176 to Rands
2. £5 to a) Rands (R) b) Pulas (P)
3. R120 to a) Pounds (£) b) Dollar ($)

Solutions

1. R1 = P0,88
   ? more = P176

$$\frac{176}{0{,}88} = \frac{\cancel{17600}^{200}}{\cancel{88}}$$

$$= R200$$



Hint

Since there is no direct comparison between £ and (P), go through the Rand.

2. a) £1 = R11,95 from the table above

$\therefore$ £5 = 5 × 11,95

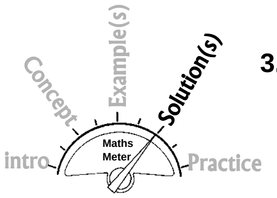
= R59,75

b) £5 = 59,75

P1 = R0,88

$\therefore$ £5 $= \dfrac{59,75}{0,88}$

= P67,90



3. a) £1 = R11,95

$\therefore$ R120 $= \dfrac{120}{11,95}$

= £10,04

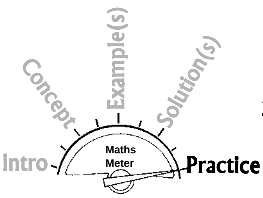
b) \$1 = R7,46

$\therefore$ R120 $= \dfrac{120}{7,46}$

= £16,09

Notice that calculations, in this case, are to 2 decimal places, in line with what is in conversion table 27.17.

Use the exchange rate table 27.17 on page 235 to answer the following:



1. Convert R50 to: a) Pulas b) Euros c) Dollars d) Pounds

2. Convert €150 to: a) Rand b) Pulas c) Dollars d) Pounds

3. Convert \$2 to: a) Pounds b) Euros c) Pulas d) Rands

4. Convert £135 to: a) Dollars b) Rands c) Euros d) Pulas

5. Convert P1000 to: a) Rands b) Euros c) Pounds d) Dollars

6.

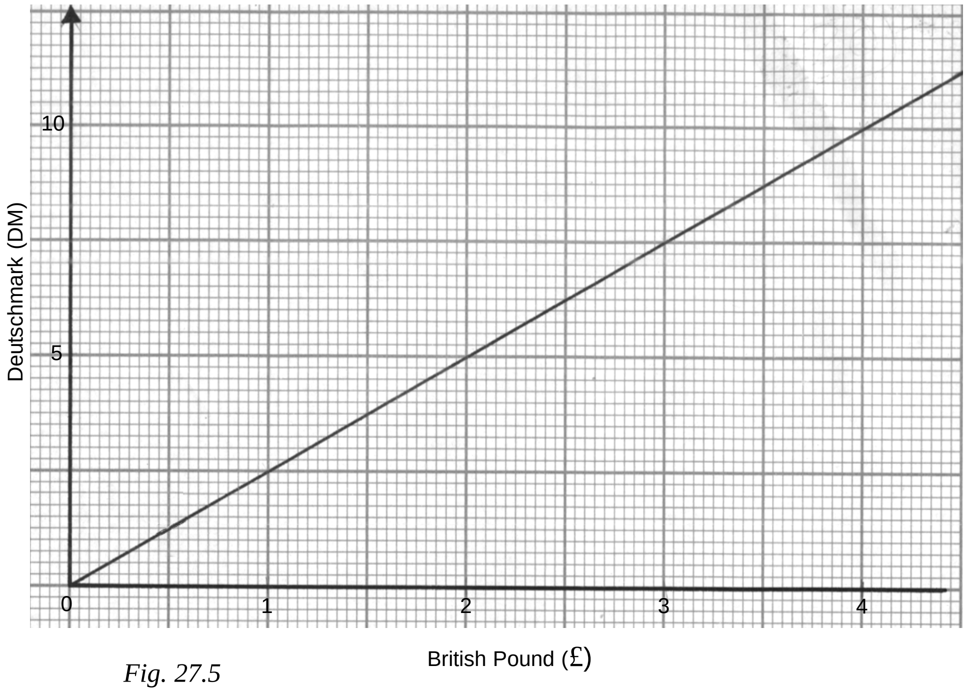


Fig. 27.5

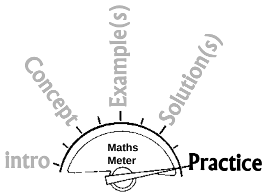


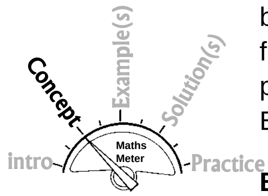
Fig 27.5 is a conversion graph between German Deutschmark (DM) and British Pound Sterling (£).

Use the graph to convert:

a) £1 to DM b) 7,5DM to £

c) £2,50 to DM d) £3,10 to DM

e) 1,25DM to £ f) 11DM to £



The previous section assumed that there is just one exchange rate between two currencies. In reality, dealers use two rates, one for buying and the other one for selling. In any Bank and there will be a billboard displaying the two rates. Notice that these change from day to day and vary slightly from Bank to Bank, just like the price of a packet of rice varies from dealer to dealer.

Below is a typical billboard with conversion display.

Bank Exchange Rates (To the US Dollar $)

CURRENCY	BUYING	SELLING
Australia (AUD)	0,8806	0,8894
Botswana (BWP)	0,1509	0,1524
Canada (CAD)	1,0757	1,0865
European Union (EURO)	1,4551	1,4697
Japan (JPY)	89,3261	90,2239
Kenya (KES)	74,4260	75,1740
South Africa (ZAR)	7,3439	7,4177
Sweden (SEK)	6,9494	7,0192
Switzerland (CHF)	1,0234	1,0388
United Kingdom (GBP)	1,5967	1,6128

Fig 27.6

This means

(i) If you go to the bank with Rands to exchange for US Dollars the bank will buy your Rands at R7,3439 for \$1.

(ii) If you go to the bank with US Dollars to exchange for Rands the bank will sell you its Rands at R7,4177 for \$1.

For their service (i.e. facilitating your buying and/or selling of forex) banks normally charge 1% commission, unless otherwise stated.

Consider the examples below:

1. How much will the bank pay in US\$ for my 1 000 Yen?

Solution

1. 1000 Yen $= \dfrac{1000}{89{,}3261}$

$= \$11{,}19$

Hint

The Bank is buying Mr Sadza's Shillings at KES74,4260 to \$1

2. Mr Sadza exported maize worth 67780 Kenyan Shillings. How much, in US Dollars, does he get from his bank if the bank charges 1% commission?

Solution

2. KES67780 $= \dfrac{67780}{74{,}4260} = \$910{,}70$

less 1% Commission $= -\$9{,}11$

$\therefore$ Mr Sadza received $= \$901{,}59$

Hint

The Bank is selling its Pulas to Ms Ncube at P0,1524 to \$1.

3. Ms Ncube has US\$217 and wants to change this to Pulas. Find how much she gets, if the bank charges $1\frac{1}{2}$% commission.

Solution

3. \$217 to Pulas $= 217 \times 0{,}1524 = \text{P}33{,}07$

less $1\frac{1}{2}$ % Commission $= \frac{3}{200} \times 33{,}07 = -\text{P } 0{,}50$

Amount received $= \text{P}32{,}57$

Use the Bank Exchange Rates in Fig 27.6 on page 237.

No commission is charged for transactions in numbers 3 – 6.

1. Convert \$890 to: a) AUD b) GBP

2. Convert CAD776 to: a) JPY b) CHF

3. How much will the bank pay in Euros, for \$70,55?

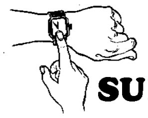
4. David's brother sent his mother \$500, through the bank, from abroad. How much is this in:
a) SEK? b) BWP? c) CHF?

5. Calculate how much David's mother (in question 4) will receive in each case, if the bank charges 1% commission.

6. A leather bag in a Zimbabwean Courier shop, is priced at \$23,50. A tourist from Sweden has SEK5420 and wants to buy this bag. How much, in SEK does she have to part with, if the bank charges $1\frac{1}{2}$ % commission for exchanging services?

7. Mr Smith had \$350 in his wallet. He decides to change it into Euros and spends half of it on luxuries. He then changed the remainder into GBP.
If the bank charged him 1% commission on each exchange, how much did he lose?

8. Ms Mangana bought KES1000 thinking that the exchange rate was \$1 to KES74,43. The exchange rate was actually \$1 to KES74,33. If the bank commission was 1%, did Ms Mangana pay more or less than she expected?



SUMMARY

1. Gross income is one's earnings before deductions.

2. Net income or take-home pay is one's income, after deductions.

3. Selling Price = Cost Price + Profit
or Cost Price – loss
* *Do not find percentage of Selling Price.*

4. Discount is an amount allowed to a customer for buying cash.

5. Hire Purchase uses deposit and/or installments. It is more expensive than a cash purchase.

6. Depreciation is the loss of value over a period of time. It happens usually to vehicles buildings and furniture (capital goods).

7. Taxes: PAYE – deducted from salaries,
VAT – levied on sales of goods and services.

8. Insurance is a financial precaution against possible loss/damage.

9. Premium – a periodic payment for insurance.

10. Mortgage – a loan given specifically for the purchase of land and/or buildings.

11. Money (in different currencies) is bought and sold using exchange rates.

EXAM PRACTICE 27

Hints

- Read and understand the key words in the problem. Take note of when to add, subtract etc.

- Usually the problem gives the stages to be followed in answering the question. Don't jump to conclusions. Go through the expected stages.

- Percentage is always of the basic figure
  e.g. Cost price not Selling Price.
  It is the original figure not the figure after some increase or decrease etc.
  Do not jump to finding the percentage of whatever figure is given in the problem, first check if this figure is given in the problem. Check if this figure is the basic or original one first.

- Bills give basic charges usually in *cents* e.g. 16,30 is often taken carelessly as \$16,30. **Watch out**.

- Note the use of 'each' in charges. This means *multiplying* the units by the charge given.

- Be on the alert when dealing with forex i.e. when is the bank buying from you or selling to you? The examples below will help clarity points to note on types of questions which come from this chapter.

Consider the following examples:

1. Fig 27.7 below shows an incomplete telephone bill.

	$	c
Fixed monthly Rental	10	50
Cost of metered calls from 021627 to 021877 at 21c per unit		
Subtotal		
VAT at 15%		
Total Amount Due		

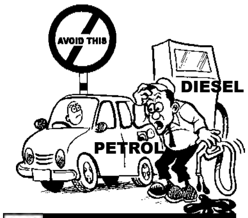
Fig 27.7

Calculate a) the number of units used.
b) the subtotal.
c) the total amount due.

Solution

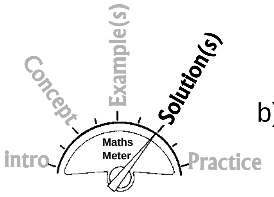
a) Number of units = Difference in the meter readings

i.e.
$$\begin{array}{r} 021877 \\ -\,021627 \\ \hline 250 \\ \hline \end{array}$$



Common Errors

Answer given as 000250. This is more of a reading than a number. The zeros before the 2 are not needed.



b) Subtotal = Fixed Rental + Cost of Metered Calls

Cost of metered calls $= 250 \times 21c$

$= \$52,50$

$\therefore$ Subtotal $= \$10,50$

$$\begin{array}{r} \$10,50 \\ 52,50 \\ \hline \$63,00 \\ \hline \end{array}$$

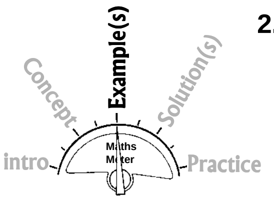
c) Amount due = Subtotal + 15% VAT

Thus VAT $= \frac{15}{100} \times 63$

$= \$9,45$

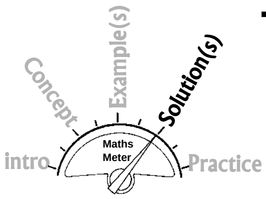
$\therefore$ Total Amount due $= \$63,00$

$$\begin{array}{r} \$63,00 \\ 9,45 \\ \hline \$72,45 \\ \hline \end{array}$$



2. a) Mr Mlambo borrowed \$7 000 from a bank to be repaid in 10 months. If he paid back a total amount of \$8 050, calculate the bank's annual simple interest rate.

b) A car depreciated at 20% per annum. In 2002 a car was valued at \$3 248. What was its value in 2001?



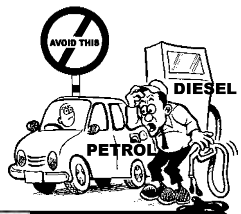
Solution

a) Simple Interest $= 8\,050 - 7\,000$

$= \$1\,050$

Rate $= \dfrac{10\cancel{0} \times 1050 \times 12}{700\cancel{0} \times 1\cancel{0}}$

$= 18\%$



Common Error

$\frac{100 \times 1050}{7000 \times 10}$

Remember 10 is in months not years.

Common Error

$\frac{20 \times 3248}{100} + 3248$

This has no meaning at all since the 20% is already in the 3 248!

Hint

In 2001 the value is bigger than 2002 value. This is because of the depreciation effect

***Tip**: Use simple proportion*

b) In Percentages 2001 – 100%

2002 – 80%

$\therefore$ 3 248	80
more	100

2001 value $= \frac{100}{80} \times 3248$

$= \$4\,060$

Now do the following:

1. The following is an extract from a mortgage loan account.

Table 27.18

DATE	TRANSACTION	DEBIT	CREDIT	BALANCE
31/01/09	Balance brought forward			3 000
02/02/09	Interest charged on loan for the month of February	40		3 040
28/02/09	Loan repayment		340	x
01/03/09	Interest charged on loan for the month of March	y		

Calculate a) the percentage interest rate charged *per annum.*

b) the value of x

c) the value of y

2. a) Convert 14m/s to km/h

b) In February, Zodwa received a basic wage of \$80 per month. She usually receives a commission of $2\frac{1}{2}$% on all sales over \$500.

(i) Find Zodwa's total income for the month of February, if she sold goods worth \$1 700.

(ii) Zodwa's February basic wage is a result of a 20% increase from the January wage.
What was Zodwa's basic wage in January?

3. In a certain town, customers can pay for their gas bills using one of two tariffs given below.

a) a credit Tariff which charges 1,57c per kwh plus a standing charge of 10,3c per day or

b) the Domestic Prepayment Tariff, which charges 2,02c per kwh for the first 1 500kwh plus 1,57 per kwh for additional kwhs plus a standing charge of \$6.

The Shades family used 2 910kwh in one month (30 days).
Which method is cheaper for them to use, and by how much?

4. The table below is an extract of the rates of postage in Zimbabwe and the given regions of the world.

Table 27.19

COUNTRY OF DESTINATION	COST OF LETTERS	COST OF PARCELS
Zimbabwe (Zone A)	Up to 20g . . . . .25c Over 20g up to 100g . . 50c	First 1kg . . . .\$3.00 Each additional kg . . \$1.00
Any other Country in Africa (Zone B)	Up to 10g . . . . 50c Over 10g up to 50g . . 75c	First 1kg . . . .\$10.00 Each additional kg . . \$5.00
Europe (Zone C)	Up to 10g . . . . 75c Over 10g up to 50g . . \$1.00	First 1kg . . . .\$15.00 Each additional kg . . \$5.00

Find the cost of sending:

a) a letter, weighing 85g, to Lupane (Zone A)

b) a parcel, weighing 5kg, to Lagos (Zone B)

c) 2 letters, one weighing 8g and the other one 40g, and a parcel, weighing 6kg, to Italy (Zone C).

5. A hawker borrows \$750 which is to be repaid over 3 months, at a simple interest rate of 20% per annum. The hawker used the money to buy items for resale. At the end of 3 months, the hawker had made a profit of 10%. Find the value of the hawker's sales during the 3 month period.

6. On a certain day, the following were the exchange rates of a certain Bank to the US Dollar.

Table 27.20

Currency	Buying	Selling
Pula (Bwp)	0,1509	0,1524
Euro (E)	1,4551	1,4697
Rand (ZAR)	7,3439	7,4177
Pound (GBP)	1,5967	1,6125

a) On this day, state, with reasons the currency which was:
   (i) the weakest of the four.
   (ii) stronger than the US dollar.

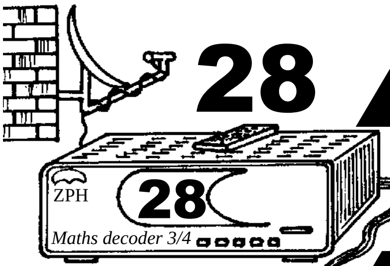
b) State which exchange rate the bank would use (Buying or Selling), if you took:
   (i) Pulas to the bank for US Dollars.
   (ii) US Dollars to the bank for Pounds.
   (iii) Euros to the bank for US Dollars.

7. a) In 2009, the value of a farmhouse was \$7 500. In 2010 the value decreased by 8,5%
Calculate the value of the farmhouse in 2010.

b) If the 2009 value of \$7 500 was a 10% increase on the 2008 value, find the value of the farmhouse in 2008, to the nearest cent.

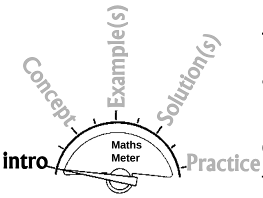
8. Calculate: a) the simple interest.
and b) the compound interest, on \$900, at 11% per annum, for 3 years.

9. A businessman borrowed \$5 500, at 8% per annum, for 2 years. Calculate the amount he has to pay back, if the interest gained was: a) simple interest.
b) compound interest.



28 Use of calculators

An introduction to Scientific Calculators



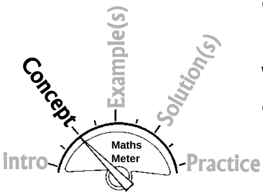
The past four decades have witnessed a revelation in the use and dependence on computers in general and the calculator in particular. Prior to the 70's the computations in mathematics were enhanced by the use of tables of logarithms and slide rules. With the widespread availability of calculators, the use of logarithm tables has become less and less popular. Slide rules are no longer used. Calculators come in a large array of different types, sizes and prices.

Table 1

Calculators

Non-scientific Calculators | Scientific calculators | Graphing calculators

In the 'O' Level course for which this book is intended, the most appropriate type is the **scientific calculator**. Most of these scientific calculators use **algebraic logic** (AL) and not **Reverse Polish Notation** (RPN). In this chapter we explain the use of calculators with algebraic logic which are most popular and required by the course. Although calculators vary among manufacturers and models we recommend the following for this course: 'Sharp, Casio, Radio Shack and Texas instruments" Fig 28.1.



Sharp Dal *Texas instruments* *Casio*

Fig 28.1

A typical scientific calculator is illustrated in Fig 28.2. Also note that this introduction is a guide and not intended to take the place of **your own user's manual**.

TYPICAL SCIENTIFIC CALCULATOR OUTLOOK

Keys and their uses. In the diagram below, only the necessary 2nd Function notations for this level have been shown above the main keys.

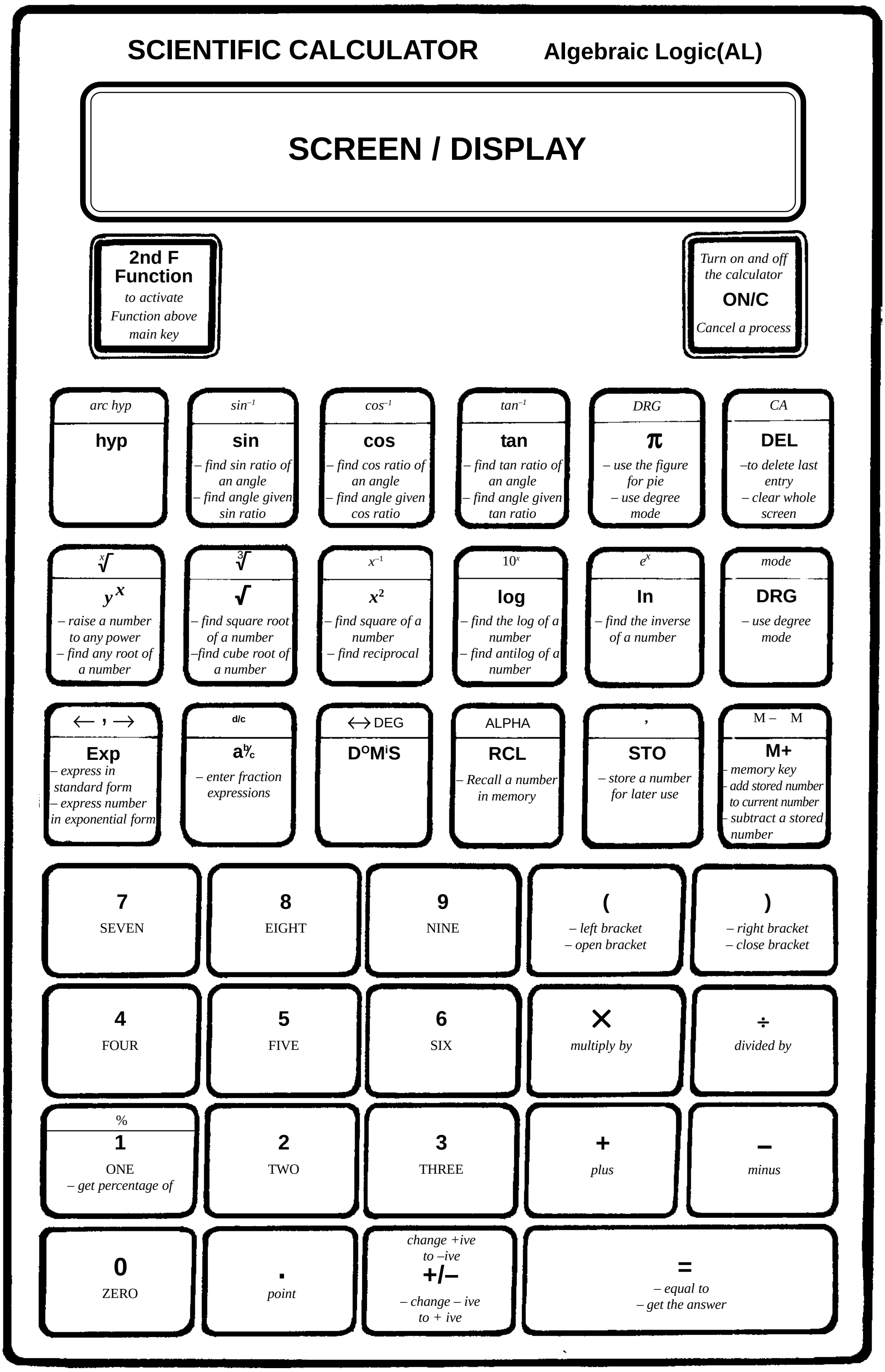
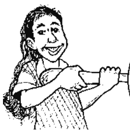


Fig 28.2



Syllabus Expectations

By the end o f this chapter the student should be able to use a scientific calculator to:

1. Perform arithmetic operations (+, -, x, ÷).
2. Find the square root or cube root of any number.
3. Find the square or cube of any number.
4. Raise a number to any power.
5. Find any root of a number.
6. Find the natural logarithm and the logarithm of a number.
7. Find the reciprocal of a number.
8. Store and recall a number using the memory key.
9. Enter a negative number on the display.
10. Use the 2^{nd} function key in general.
11. Write a number in exponential form using the exponential key.
12. Find the Tan, Cos and Sin of an angle.
13. Find arc Tan, arc Cos and arc Sin.

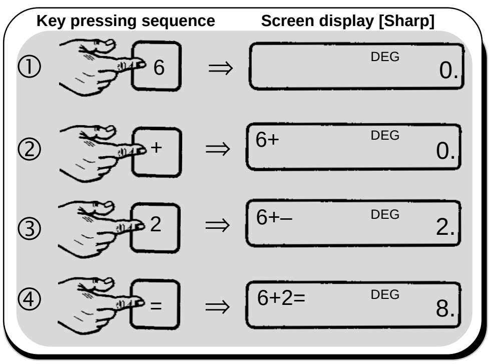
ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that pupils are able to:

- Perform arithmetic operations of simple numbers manually.
- Understand the following definitions square, cube, power, square root, cube root, exponential, reciprocal, pie, tan, Cosine, Sine, inverse and 2nd function.

CONCEPT 1: To perform an arithmetic operation [6 + 2 =]

The important operation keys to note are the +, -, x, ÷ and =. You first enter the numbers or signs the way you write or read the arithmetic operation. Proceed as shown in the box:



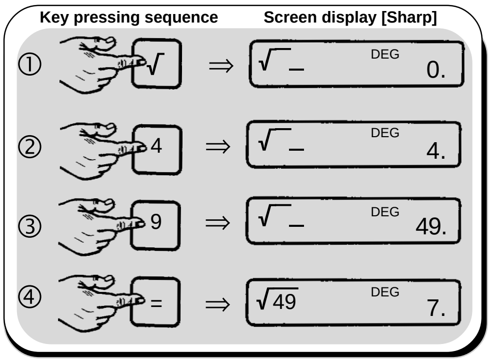
Tip

The key marked +/– allows you to change the sign of a number on display. Hence to enter –2 use the key strokes 2+/–2.

CONCEPT 2: To find the square root of any number. [$\sqrt{49}$ =]

To find the square root of any number:

Identify the square root key marked $\sqrt{\ }$ or $\sqrt{x}$. Proceed as shown in the box:

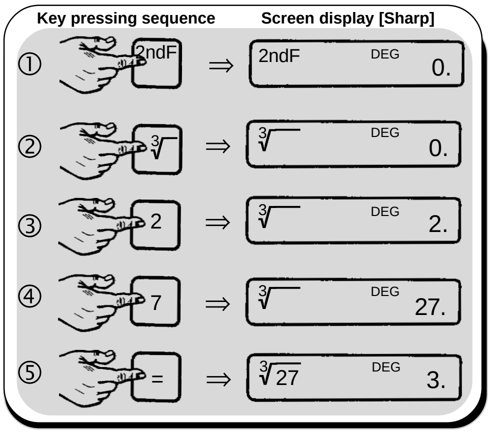


Tip

Experiment with your calculator to find out which method it uses.

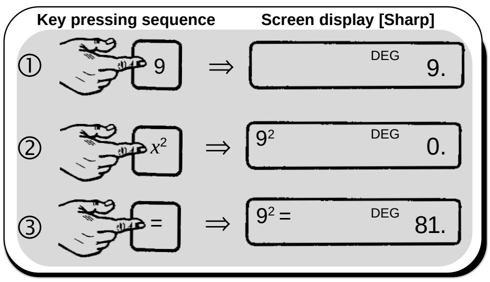
Concept 3: To find cube root of any number. [$\sqrt[3]{27}$ =]

Identify the cube root key marked $\sqrt[3]{\ }$. Proceed as shown in the box:



Concept 4: To find the square of any number. [9^2 =]

Identify the squaring key marked (x^2) and proceed as shown in the box:

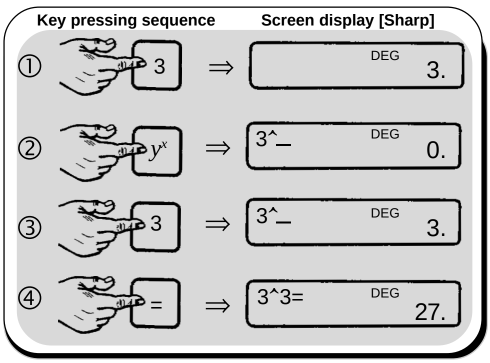


Hint

Note that the square root key and the squaring key maybe found on the same key with one of them being a second function.

Concept 5: To find the cube of any number. [3^3 =]

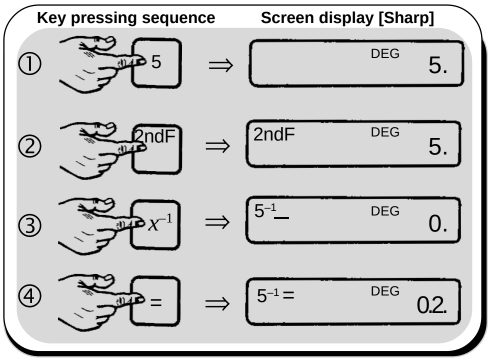
Identify the cubing key marked. Proceed as shown in the box:



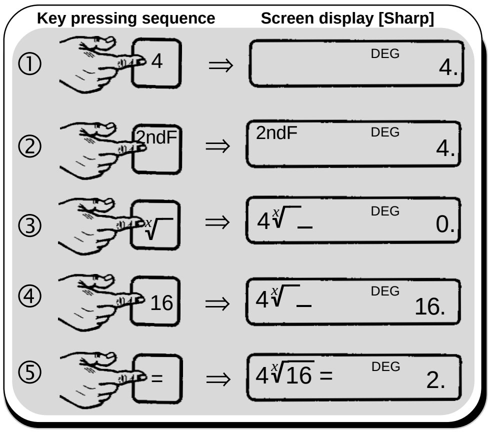
Tip

Again experiment with your calculator to find if you need to use the 2^{nd} function key.

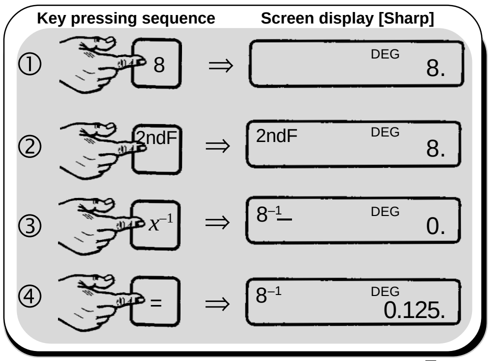
Concept 6: Raising a number to any power. $[2^4 =]$
Identify the exponential key marked x^y or y^x. This keystroke allows you to raise a number to any power. Proceed as shown in the box:



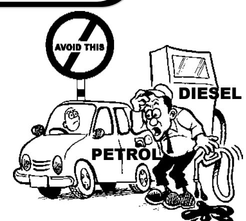
Concept 7: Finding any root of a number. $[\sqrt[4]{16} =]$
Identify the root key marked or $\sqrt[y]{x}$ or $\sqrt[x]{y}$ and experiment with it. Some calculators do not have this sign. In this case you may use the inverse key in conjuction with the exponential key e.g. to find the fifth root of 32 using the following keystrokes. 32 inv x^y 5 = 2
Proceed as shown in the box:



Concept 8: To find the reciprocal of a number. $[8^{-1} =]$
To find the reciprocal use the key marked $\frac{1}{x}$ or x^{-1}.Proceed as shown in the box:



NOTE that the when two number have a product = 1, they are called reciprocals, in this case, 0,125 is a reciprocal of 8 and vice-versa.
Inverse operations "undo" each other e.g. squaring and taking the square root are inverse operations. Experiment this with your calculator.

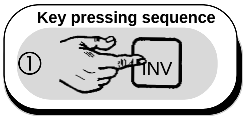


Common Error

Student confuse the reciprocal key $\frac{1}{x}$ and the inverse key inv.

Concept 9: Use of the inverse key. Inv.

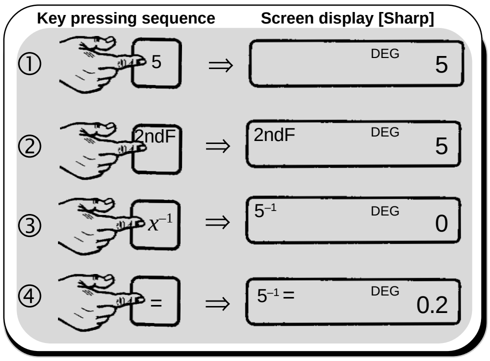
The use varies among different calculators. Please refer to your Manual. Proceed as shown in the box:



Concept 10: Use of the second function key 2ndF [5^{-1} =]

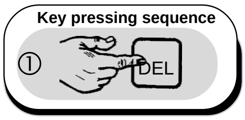
This key is usually marked as 2^{nd}F or Shift and is always used in conjuction with another key. The 2^{nd}F key is used to activate a function which is printed above an operation key and not the key itself. Most calculators use different colours for the 2^{nd}F key and corresponding functions above the keys.

For example if you want to find the reciprocal function, ($\frac{1}{x}$) is printed above another key. Proceed as shown in the box:



Concept 11: Use of the clearing or delete key

This key is usually marked CE or C or DEL. This key allows you to delete the last entry entered into the display. Continuous pressing the delete key will clear the entire operation on display. Experiment this with your calculator.



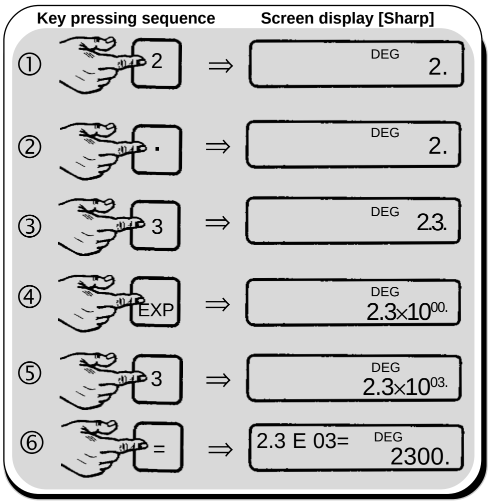
Tip

Note that when very large or small numbers are obtained as answers most scientific calculators express such answers in scientific notation (standard form)

Concept 12: Entering numbers in standard form into the calculator.

Any number can be expressed in standard form i.e. in the form a x 10^b e.g. 2300 = 2,3 x10^3 proceed as follows. 2.3 exp 3

Again this varies among different calculators models so experiment with your calculator.



Concept 13: To find the natural logarithm of a number i.e. the power to which 10 must be raise to get the number.[log 100 =]
Proceed as shown in the box:

	Key pressing sequence		Screen display [Sharp]
①	log	⇒	log_ DEG 0.
②	1	⇒	log_ DEG 1.
③	0	⇒	log_ DEG 10.
④	0	⇒	log_ DEG 100.
⑤	=	⇒	log100 = DEG 2.

Concept 14: To find the anti-logarithm of a number.
[Antilog of 2]
Identify the 10^x key the use the 2nd function to manipulate the problem.Proceed as shown in the box:

	Key pressing sequence		Screen display [Sharp]
①	2ndF	⇒	2ndF DEG 0.
②	10^x	⇒	10^_ DEG 0.
③	2	⇒	10^_ DEG 2.
④	=	⇒	10^2 = DEG 100.

Concepts 15: The pie key
This is an irrational number and is required mostly in the mathematics of a circle. Its value on most calculators = 3,1415927. It is found by pressing the π key. Experiment with your calculator to see if you can get its value
.

	Key pressing sequence		Screen display [Sharp]
①	π	⇒	π – DEG 0.
②	=	⇒	π= 3,141592654 DEG 4.

Concept 16: The Trigonometric keys [sin30⁰ =]

Sin^{-1}	Cos^{-1}	Tan^{-1}
$\boxed{Sin}$	$\boxed{Cos}$	$\boxed{Tan}$

These are used to find ratios of angles and vice versa. Above each of the three trigonometric keys there is also the sin^{-1}, cos^{-1} and tan^{-1} function with the 2nd function key. Two concepts must be understood:

	Key pressing sequence		Screen display [Sharp]
①	sin	⇒	sin_ DEG 0.
②	3	⇒	sin_ DEG 3.
③	0	⇒	sin_ DEG 30.
④	=	⇒	sin30 = DEG 0.5.

a) To find the ratio given an angle e.g. find Sin 30^0, Cos30^0, Tan 30^0 .
b) To find the angle give the ratio

c) E.g. Sinx^0 = 0.5, Cos x^0 = 0.5, Tan x^0 = 0.5
Use your calculators and experiment to see if you get the following.
Sin 30^0 = 0.5
Cos 30^0 = 0.8660
Tan 30^0 = 0.5774

Hint

The chapters on solving the right-angled triangle will help you understand more about trigonometric ratios.

Sin x = 0.5 $\Rightarrow$ $x = 30^0$
Cos x = 0.5 $\Rightarrow$ $x = 60^0$
Tan x = 0.5 $\Rightarrow$ $x = 26.57^0$

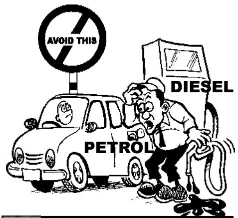
Concept 17: Use of the brackets

Opening and closing brackets may also be used with the scientific calculators. The round bracket are often on the relevant keys i.e. (and) Consider the following 2x(2+3 – 1) -4 you proceed as shown in the box:

	Key pressing sequence		Screen display [Sharp]
①	2	⇒	DEG 2.
②	×	⇒	2×_ DEG 0.
③	(	⇒	2×(_ DEG 0.
④	2	⇒	2×(_ DEG 2.
⑤	+	⇒	2×(2+_ DEG 0.
⑥	3	⇒	2×(2+_ DEG 3.
⑦	–	⇒	2×(2+3–_ DEG 0.
⑧	1	⇒	2×(2+3–_ DEG 1.
⑨	)	⇒	2×(2+3–1)_ DEG 0.
⑩	–	⇒	2×(2+3–1)–_ DEG 0.
⑪	4	⇒	2×(2+3–1)–_ DEG 4.
⑫	=	⇒	2×(2+3–1)–4= DEG 4.

Concept 18: Use of the $\boxed{\text{DRG}}$ key

D on the key stands for degree mode.
R on the key stands for radian mode.
G on the key stands for gradient mode
For almost all our computations in this course ensure that the screen is in degree (D) mode. Rarely do we need to operate the calculator when its in radian (R) or gradient G form.



Common Error

Always ensure the screen is in degree mode. If its another mode you don't get the correct answer (Experiment this with your calculator).

Concept 19: Memory key, $\boxed{\text{M}}$

This key allows you to store a number for later use. Three aspects are associated with memory.

a) Storing a number in the memory by using the key labelled M or Sto.
b) To add to or subtract from the value currently in memory by using the key labelled M+ and M- keys respectively,
c) Recalling or retrieve the value in memory by using the key labelled MR RM or RCL.
   Supposing you want to store the number 8 in memory
   Step 1 enter the number 8.
   Step 2 press the memory key.

You may then perform other calculators and when you need to use 8 you may retrieve it by pressing the memory recall key.
You may experiment with your calculator on how your can add to or subtract from the value currently in memory.

Hint

The word exponential is used to describe various situations in Maths e.g., exponential number, Exponential form , exponential graph. Discuss this with your teacher.

Concept 20: The $\boxed{e^x}$ key

e is a special number in maths = 2.7183. Although its use is beyond the scope of this book, this key should not be confused with the EXP key.

1. Using your scientific calculator, write down the relevant keys for the following operations:
   a) To switch on your calculator.
   b) To switch off your calculator.
   c) To delete the last digit appearing on your screen.
   d) To change your calculator from degree mode to radian mode and vice-versa.
   e) To find the square of a number.

2. Evaluate the following using the calculator
   a) $\left(\frac{7}{3}\right)^2 + 1.92 \times 10^2$

b) $\dfrac{4\pi \times 10^3}{\text{Log } 20 + 36} + \dfrac{\sqrt{52}}{\sqrt{36}}$

c) $(8400020 \times 4^{-3}) + (9\pi)^2 - 6.002$

d) $\dfrac{48 \times 102 + 3^5}{5^3 + 3 \times 10^4}$ e) $\dfrac{0.00047 \times 270\ 000}{420 \times 10–3 \times \sqrt[5]{50}}$

3. Find the following using the calculator:

a) Sin 46^0	b) Sin 120^0	c) Sin 36^0
d) Cos 56^0	e) Cos 150^0	f) Cos 0^0
g) Tan 67^0	h) Tan 162^0	i) Tan 45^0

4. Evaluate the following using the calculator:

(i) Find the sine of the following

a) 0.6679	b) 0.1112	c) 1.000
d) 0.8660	e) 0.4679	

(ii) Find the cosine of the following:

a) 0.8946	b) 0.7121	c) 0.4136
d) 1.000	e) 0.0000	

(iii) Find the tangent of the following:

a) 0.3789	b) 2.7113	c) 1.4978
d) 3.369	e) 6.117	

5. Manipulate the following using the scientific calculator:

a) $\dfrac{[(-3) - 2\pi \times 4^5]^2 - 436.00 \times 1.022 \times 10^2}{[4\pi + (-3)(5)4] + (3.6)^2}$

b) $\dfrac{6213678 - 47778}{(4 \times 10^2)^2 - 3.6 - 4 \times (-6)3}$

c) $\dfrac{420 - [6.2 + 1.6^2]^3}{\text{Sin } 140^0 - \text{Cos}^2 30^0}$

d) $\dfrac{\text{Tan}^2\ 40^2 + \text{Sin}^2\ 100}{\text{Cos } 160^0 \times \sqrt[4]{42}}$

e) $\dfrac{(4 \times 10^{-3})^2 + (-3)^5 + (3.3)^2}{4\pi \times 10^{-6} - 4921 + 3 - \frac{2}{5}}$

f) $\dfrac{4\frac{1}{2} + 8 - \frac{2}{3} - (0.1115)^3}{\sqrt{(25.6 + 37) - (0.22)5}}$

SUMMARY

1. This chapter is only a guide and is not intended to take the place of your owner's manual. The best summary for this chapter is to read your manual thoroughly to understand it.

2. Give special attention to the following operation keys as you read your user's manual or experiment with your calculator.

(i) Arithmetic operation keys **+ – x ÷ =**
(ii) Clearing key **C** or **Ce** .
(iii) 2nd function key.
(iv) Memory key and its associates **M** or **Sto**.
(v) Square root key $\sqrt{\ }$ or $\sqrt{x}$
(vi) Squaring key x^2
(vii) Reciprocal key $\frac{1}{x}$
(viii) Exponential key x^y or y^x
(ix) Root key $\sqrt[y]{x}$ or $\sqrt[x]{y}$
(x) Change sign key **+/–**
(xi) Log key **Log**.
(xii) Anti-log key **10^x**.
(xiii) The trigonometric ratio keys **Sin Cos Tan** as well as **Sin^{-1} Cos^{-1} Tan^{-1}**.
(xiv) The pie key π or **pie**.
(xv) The **DRG** keystroke.
(xvi) The **e^x** key.