Week 8 : 11/3/2020
13/3/2020

2.4 Matrix Transformations

Image from http://uk.ign.com/articles/2009/06/22/transformers-the-many-looks-of-optimus-prime

Idea — Represent the $\boxed{\text{point } (x,y)}$ by the position vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

Select any 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let

matrix multiplication

$$\begin{bmatrix} w \\ z \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \times x + b \times y \\ c \times x + d \times y \end{bmatrix} \to \text{vector}$$

$$\| \quad = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix}$$

■ We have a <u>**transformation**</u> M which maps

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{M} \begin{bmatrix} w \\ z \end{bmatrix}$$

or we can think of this as

$$\boxed{(x, y) \xrightarrow{M} (w, z)}$$

Example 2.24

If $M = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + (-1) \times 3 \\ 0 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

so the point $(2, 3)$ is transformed by M to $(-1, 9)$.

□

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} [a \ b]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [c \ d]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a \times 1 + b \times 0 \\ c \times 1 + d \times 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

Image from

`http://www.cloudworksmg.com/wp-content/uploads/2014/12/whats-big-idea.png`

Consider what happens to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under a matrix transformation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of } M$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of } M$$

← THE key idea →

Result 2.13 A matrix transformation

is completely determined

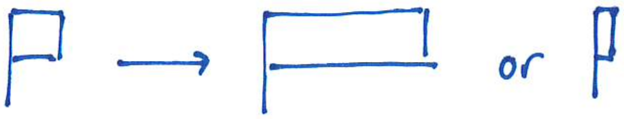
by its action on the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

□

2.4.1 Scaling

stretching (enlarging or shrinking)



Example 2.25 We wish to **scale** (enlarge or shrink) all points by a factor of $a > 0$ in the x-direction only.

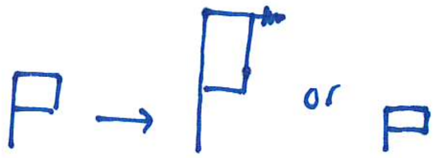
multiplier (scale factor)

So $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $M = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \times x \\ y \end{bmatrix}$

Conclusion: This transformation is a **scaling** in the x-direction. A value of $a > 1$ will enlarge, a value of $0 < a < 1$ will shrink, and $a = 1$ will stay the same.

- Similarly, for $a > 0$ and $b > 0$



  - $M = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ is a **scaling** in the y-direction by a factor of b

    since

$$\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ b \times y \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 0 \\ b \end{bmatrix}$$

- $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a **scaling** *independently* in the x-direction by a factor of a and in the y-direction by a factor of b since

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \times x \\ b \times y \end{bmatrix}$$

- $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the **identity** transformation since

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

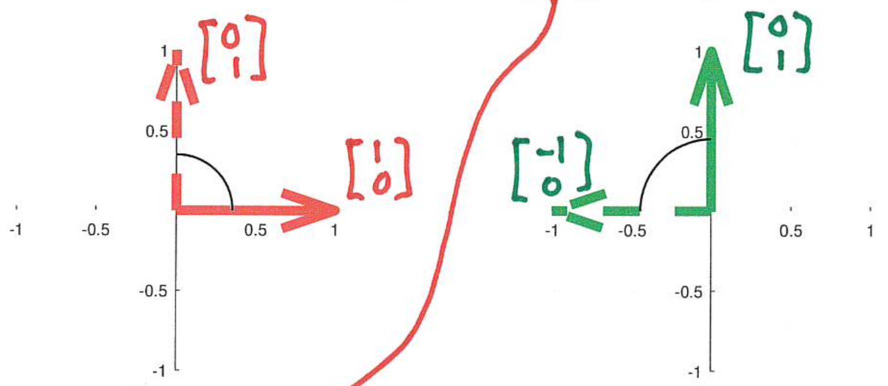
$\square$

▶ A *scaling* transformation can enlarge or shrink. When $a < 0$ or $b < 0$, the enlarging or shrinking is combined with a reflection.

2.4.2 Rotations and Reflections

first column of M   second column of M

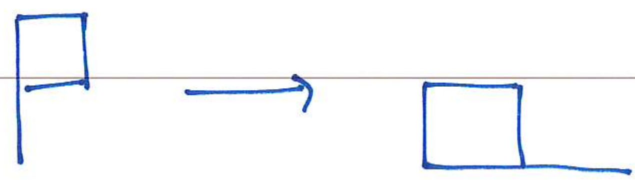
Example 2.26 Suppose $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



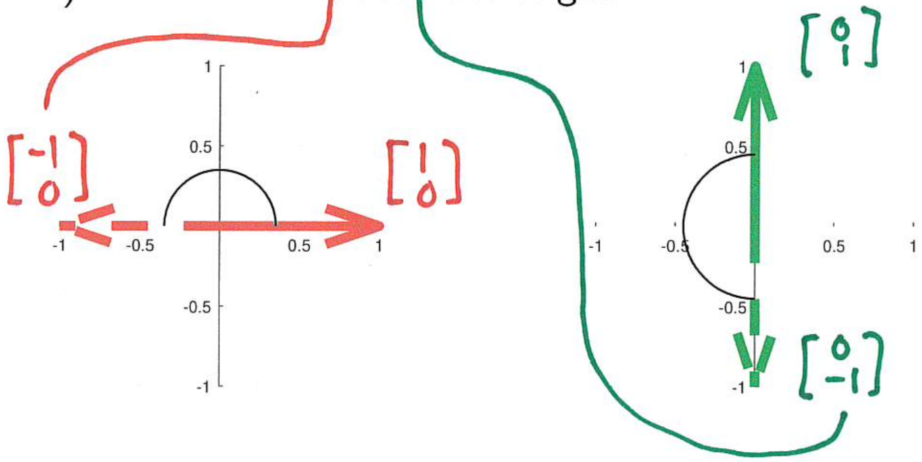
then $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

Conclusion: This transformation is a **rotation** of 90 degrees (or $\frac{\pi}{2}$ radians) anticlockwise about the origin, i.e., $(0,0)$ is the fixed point of the rotation.

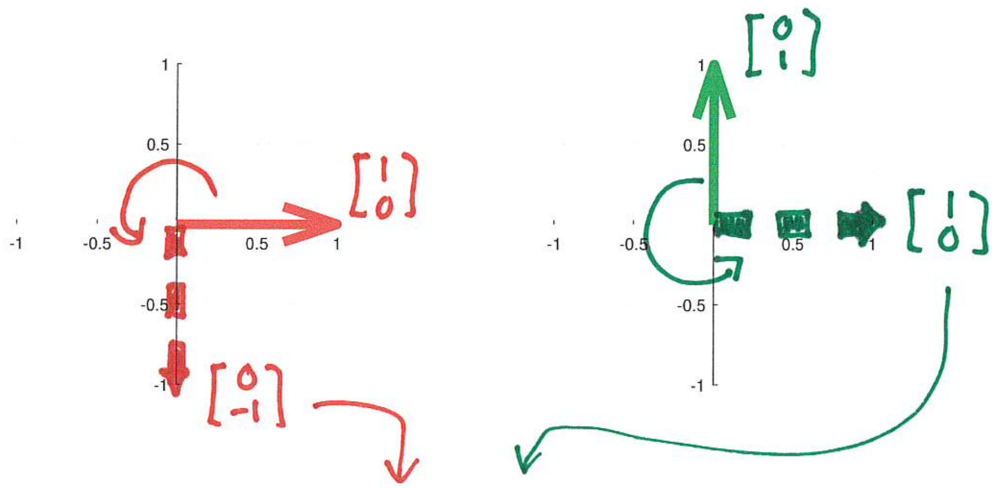
- Remember that **positive** angles are **anticlockwise**.



► Similarly, $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a **rotation** of 180 degrees (or π radians) anticlockwise about the origin.

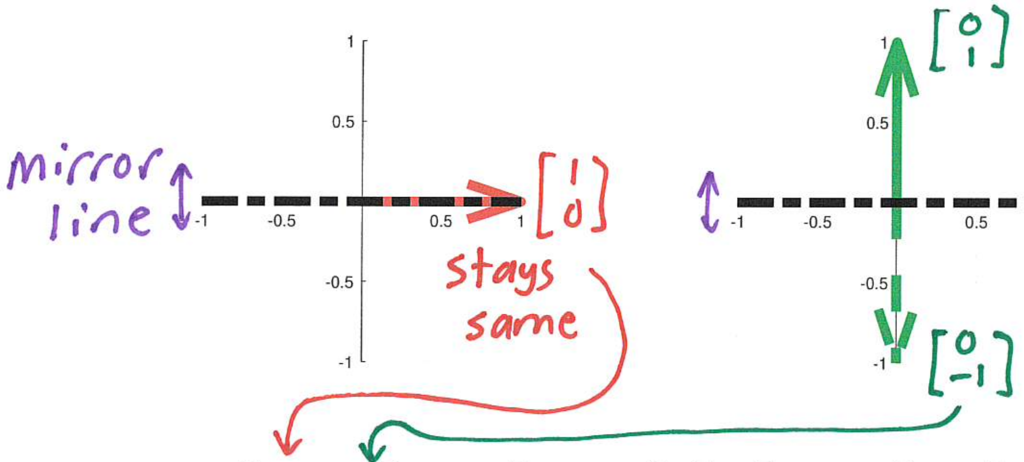


Practice Problem. Write down a single matrix that represents a **rotation** of 270 degrees (or $\frac{3\pi}{2}$ radians) anticlockwise about the origin.



$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

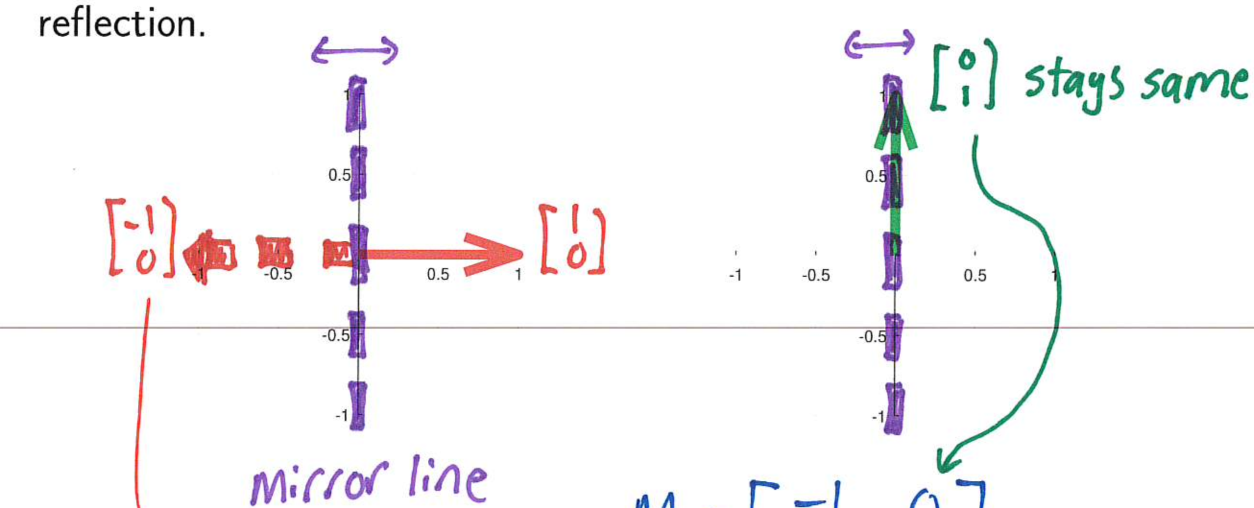
Example 2.27 Suppose $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



then $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$

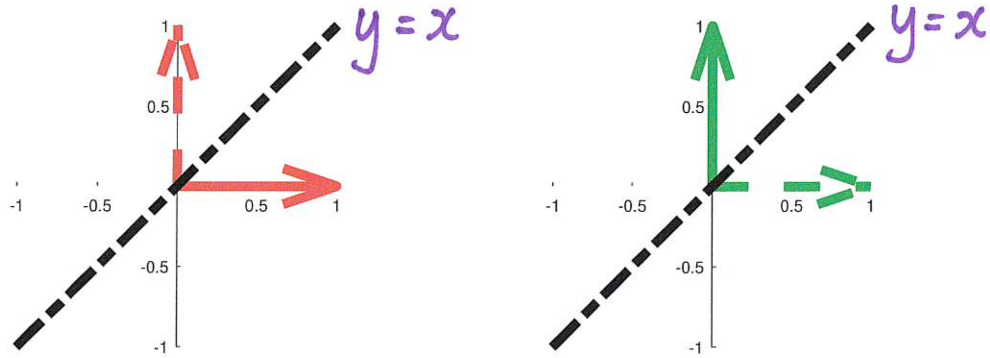
Conclusion: This transformation is a **reflection** in the x-axis, i.e., the x-axis is the *mirror line* of the reflection.

Practice Problem. Write down a single matrix that represents a **reflection** in the y-axis, i.e., the y-axis is the *mirror line* of the reflection.



$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

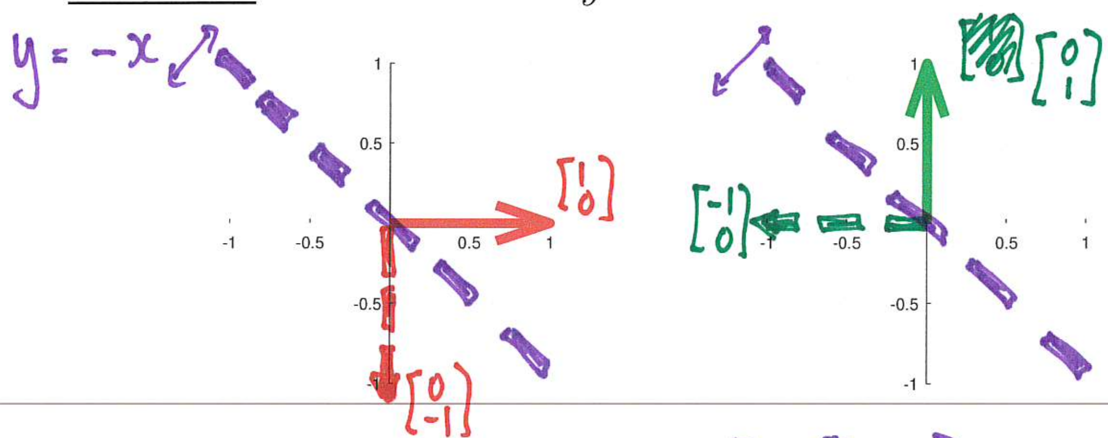
Example 2.28 Consider a **reflection** in the *mirror* line $y = x$.



Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

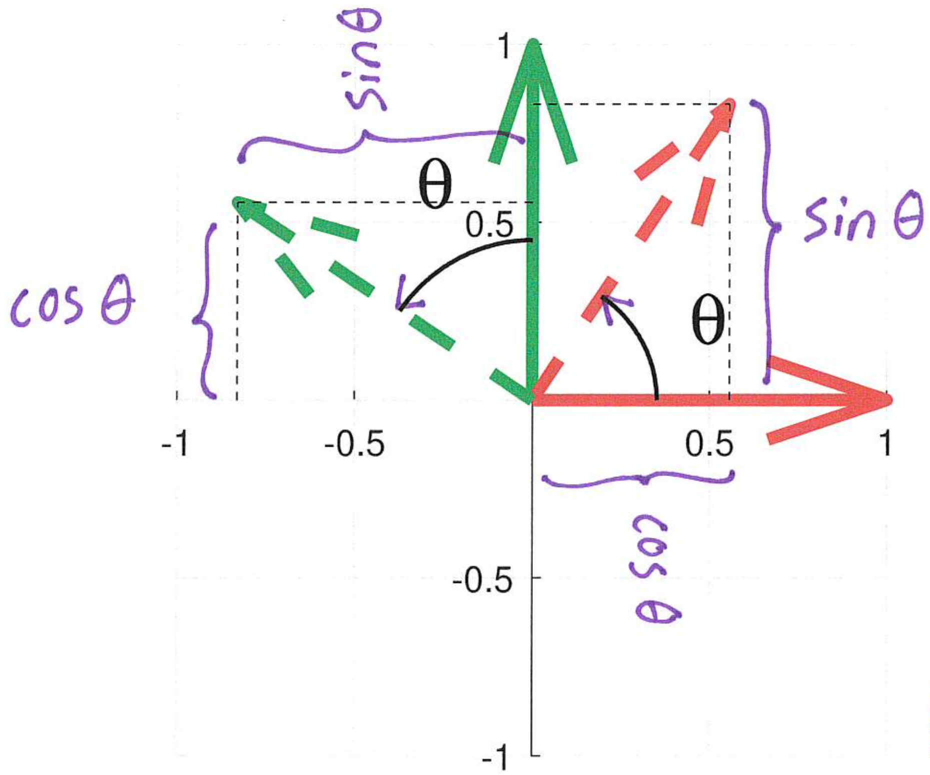
Practice Problem. Write down a single matrix that represents a **reflection** in the *mirror* line $y = -x$.



$$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

2.4.3 General Rotations and Reflections

Idea — We wish to construct a transformation matrix M which represents a general **rotation** of θ (radians) anticlockwise about the origin.



$\cos\theta = \dfrac{x}{r}$

$\sin\theta = \dfrac{y}{r}$

but $r = 1$

Suppose $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

then (x, y) in Cartesian coordinates

must correspond to

(r, θ) in polar coordinates

so $x = r \times \cos\theta$ and $y = r \times \sin\theta$

but we also know that $r = 1$.

Therefore $x = \cos\theta$ and $y = \sin\theta$ so

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

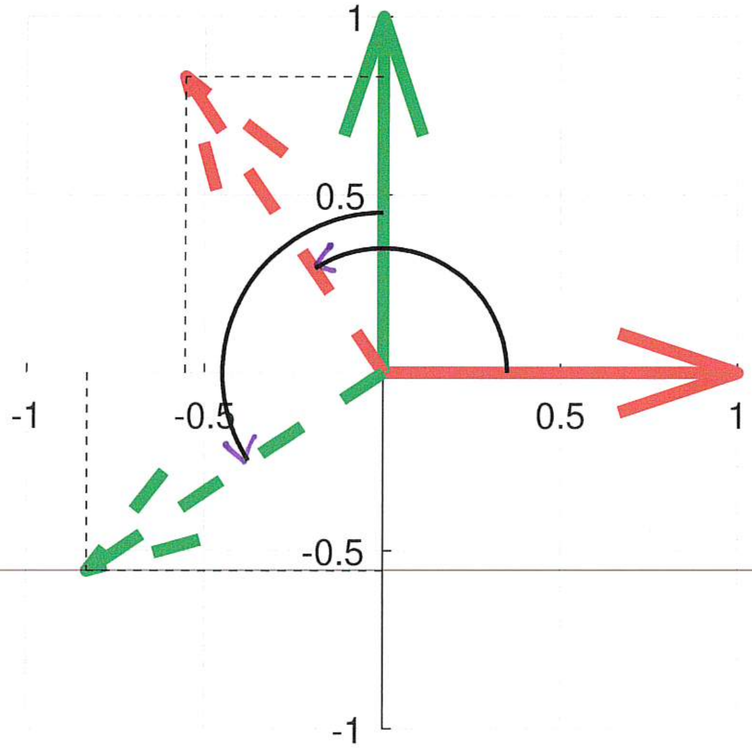
Similarly

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

giving

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 2.29 Consider a rotation of $\frac{11\pi}{16}$ radians anticlockwise about the origin.



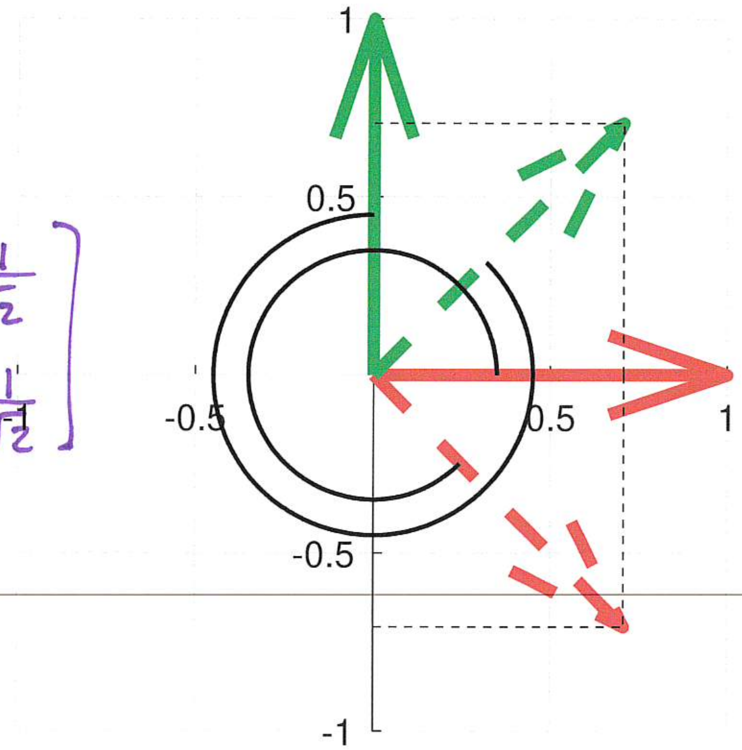
R can help us here. Note that the R function `cbind` glues together columns to make a matrix.

```
> theta = 11*pi/16
> M = cbind(c(cos(theta),sin(theta)),
            c(-sin(theta),cos(theta)))
> M
           [,1]       [,2]
[1,] -0.5555702 -0.8314696
[2,]  0.8314696 -0.5555702
```

□

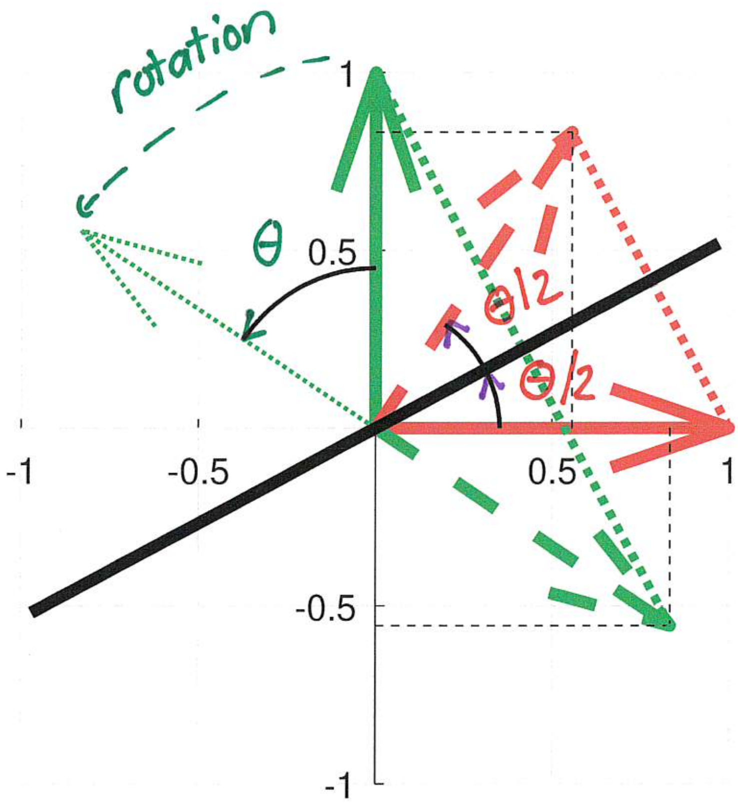
Practice Problem. Give that $\cos(\frac{7\pi}{4}) = \frac{1}{\sqrt{2}}$ and $\sin(\frac{7\pi}{4}) = -\frac{1}{\sqrt{2}}$, write down a transformation matrix M that represents a rotation of $\frac{7\pi}{4}$ radians anticlockwise about the origin.

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



Idea — Suppose we wish to construct a transformation matrix M which represents a **reflection** in *any* given **mirror line**.

The mirror line is specified by rotating the x-axis by $\frac{\theta}{2}$ (radians) clockwise about the origin.



A similar analysis to the case of a general rotation gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

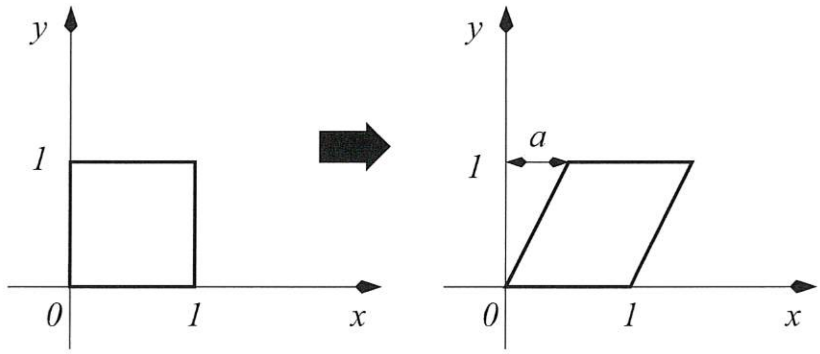
giving

$$M = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

2.4.4 Shear

Idea — For some number a, suppose

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} a \\ 0 \end{bmatrix}$$

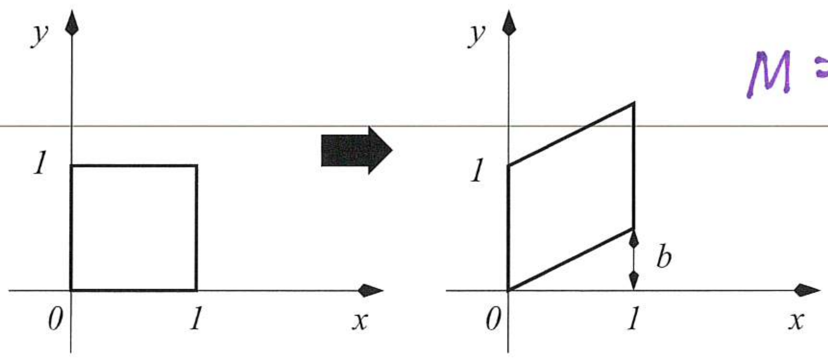


giving

$$M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

This is called a **shear** in the x-direction. If $a > 0$ then we have a shear to the right, and if $a < 0$ then we have a shear to the left.

Practice Problem. What transformation matrix M represents a **shear** in the y-direction as illustrated below?



$M = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$

2.4.5 Composite Transformations (the magic)

Idea — Suppose we wish to perform

- a **rotation** by 90 degrees anticlockwise about the origin $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

followed by

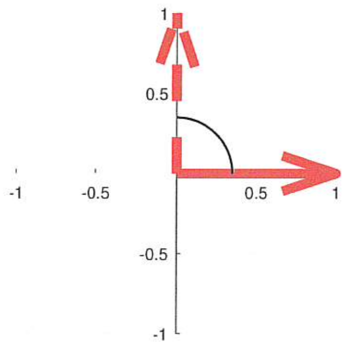
- a **reflection** in the x-axis. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

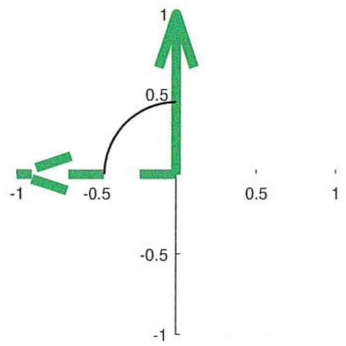
We can find a single matrix for this transformation as follows.

- rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so $(x, y) \xrightarrow{\text{rotation}} (x', y')$

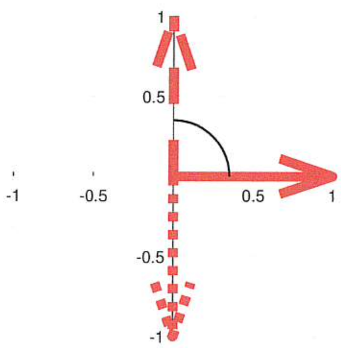


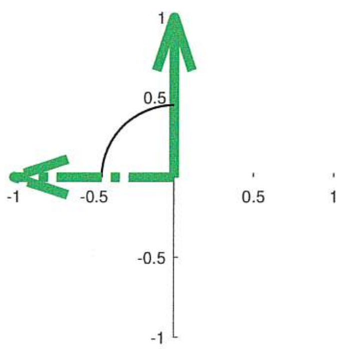


- followed by reflection

$$
\begin{aligned}
\begin{bmatrix} x'' \\ y'' \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\
&= \overset{\text{ref}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \left(\overset{\text{rot}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}
$$

so $(x, y) \xrightarrow{\text{rotation}} (x', y') \xrightarrow{\text{reflection}} (x'', y'')$





$$
\overset{\text{reflection second}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \overset{\text{rotation first}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} [1 \;\; 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} & [1 \;\; 0]\begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ [0 \;\; -1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} & [0 \;\; -1]\begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{bmatrix}
$$

apply right-to-left

$$
= \begin{bmatrix} 1\times 0 + 0\times 1 & 1\times(-1) + 0\times 0 \\ 0\times 0 + (-1)\times 1 & 0\times(-1) + (-1)\times 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
$$

reflection in $y = -x$

- The <u>**composite transformation matrix**</u> is

$$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

so $(x, y) \xrightarrow{M} (x'', y'')$

Note — the composite transformation is a <u>**product**</u> of the individual matrices.

Warning! —

- We must perform the matrix multiplication in the <u>**correct order**</u>.
- The first transformation matrix is on the <u>**right**</u>.
- Matrix multiplication is <u>**not**</u> commutative.

Example 2.30 Determine a single matrix M that represents the composite transformation composed of applying

- a **reflection** in the x-axis

followed by

- a **rotation** by 90 degrees anticlockwise about the origin.

Solution.

$$M = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation second}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{reflection first}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

□

- Matrix multiplication is <u>not</u> commutative so order does matter.

2.4.6 Inverse of a Transformation

Question —

How can we **undo** a matrix transformation?

Idea —

- We know how to perform a transformation using a 2×2 matrix M, such as a rotation or a reflection.
- We simply need to find another matrix, say A, so that $AM = I$ where I is the **identity** matrix, i.e.,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 2.14 A square matrix M is said to have an **inverse**

written M^{-1}

if we can find a square matrix of the same size such that

$$M^{-1}M = I$$

$\square$

Example 2.31 The inverse of $\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix}$

Check —

$$\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 2 & 1 \times 0 + 0 \times \frac{1}{2} \\ -4 \times 1 + 2 \times 2 & -4 \times 0 + 2 \times \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Definition 2.15

The **determinant** of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a \times d - b \times c$$

□

Example 2.32 If $A = \begin{bmatrix} 3 & 6 \\ -1 & 1 \end{bmatrix}$

then $\det(A) = 3 \times 1 - 6 \times (-1) = 3 + 6 = 9$ □

Result 2.16 Consider any 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- If $\det(M) = ad - bc \neq 0$ then

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $\det(M) = ad - bc = 0$ then A has **<u>no inverse</u>**.

and transformation is a <u>projection</u>. $\square$
(2D → 1D)

Example 2.33 Consider $M = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$

$$\det(M) = 3 \times 6 - 4 \times 2 = 10$$

$$M^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.4 \\ -0.2 & 0.3 \end{bmatrix}$$

$\square$

▶ The standard matrix transformations we have been looking at all have "natural" inverses.

- Scaling

$$M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$$

- Rotation

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

- Reflection

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For **every** reflection matrix M, we have

$$M^{-1} = M$$

reflection applied twice is the identity

- Shear

$$M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$

Summary

A matrix transformation is **completely determined** by its action on

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} b \\ d \end{bmatrix}$ then $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Typical transformations (in 2D) can each be represented by a 2×2 matrix M with $\det(M) \neq 0$.

Identity	$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Scale by a in the x-direction and b in the y-direction	$M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
Reflection in the x-axis	$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection in the line $y = x$	$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Rotate anticlockwise about the origin by an angle θ	$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Rotate anticlockwise about the origin by $\frac{\pi}{2}$	$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Rotate anticlockwise about the origin by π	$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Rotate anticlockwise about the origin by $\frac{3\pi}{2}$	$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Shear in the x-direction	$M = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction	$M = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$

- Composite transformations

  A followed by B

  gives $M = BA$ (applied right-to-left)

  A followed by B followed by C

  gives $M = CBA$ (applied right-to-left)

- For a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $\det(M) = ad - bc$

  If $\det(M) = 0$ then A has no inverse.

  If $\det(M) \neq 0$ then A has exactly one inverse.

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$