Week 3: 5/2/2020
7/2/2020

1.3 Discrete Random Variables

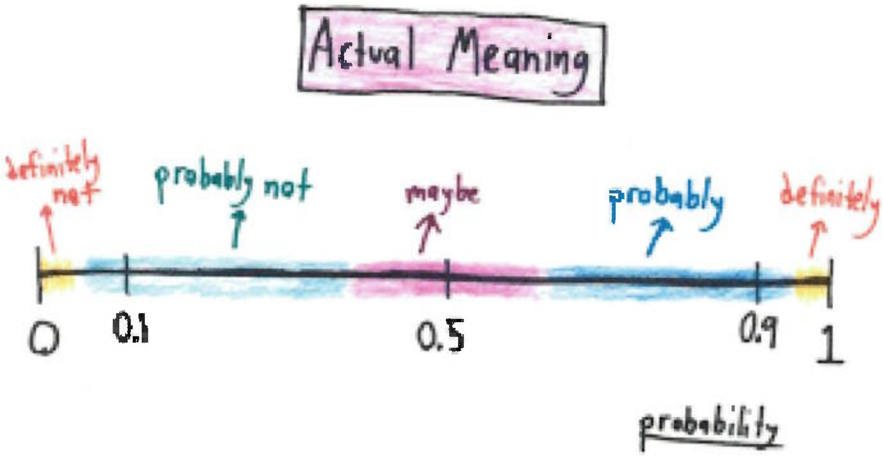
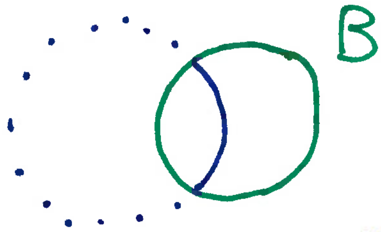


Image from `statpics.blogspot.com/2015/10/professional-probability.html`

Recap (Week 1) —

- *Experiment* (or trial) is any operation or procedure whose outcome cannot be predicted with certainty.
- *Sample space* (S) consists of all possible outcomes associated with the experiment.
- *Event* (A) is some subset of the sample space, i.e., $A \subseteq S$.
- $0 \leq P(A) \leq 1$ for each event $A \subseteq S$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A, B
- If A and B are *mutually exclusive* then $A \cap B = \emptyset$
- $P(\overline{A}) = 1 - P(A)$ for any event A



Recap (Week 2) —

event

- *Conditional probability* $P(A\,|\,B) \;=\; \dfrac{P(A\cap B)}{P(B)}$

  sample space

  so $\quad P(A\cap B) \;=\; P(A\,|\,B)\times P(B)$

- If $P(A\,|\,B) = P(A)$ then events A and B are *independent*.

- *Chain rule* — multiply probability along the branches in a probability tree diagram

$$P(A\cap B) \;=\; P(A)\times P(B\,|\,A)$$
$$P(A\cap B\cap C) \;=\; P(A)\times P(B\,|\,A)\times P(C\,|\,A\cap B)$$

- *Bayes' Theorem* (flip the probability tree)

$$P(B\,|\,A) \;=\; \frac{P(A\,|\,B)\times P(B)}{P(A)}$$

use a <u>probability tree</u> to solve problems

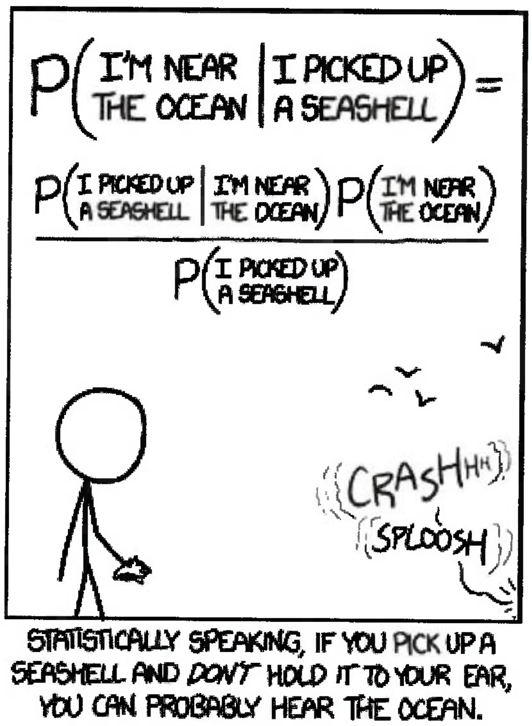


Image from https://xkcd.com/1236/

1.3.1 What is a Probability Distribution?

- The **sample space** (S) consists of *all possible outcomes* associated with the experiment.
- A **random variable** (X) assigns a

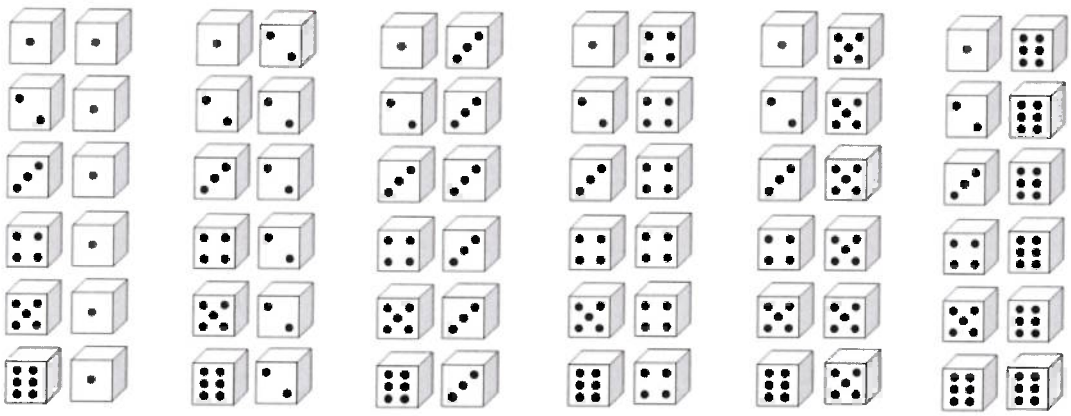
  value to each *outcome* in the sample space.
- A random variable is best described by its

  probability distribution

  which gives the values that X may assume and the probability of observing each value.

dice 1 dice 2

Example 1.13 Roll a pair of dice. The sample space is



36 outcomes

Image from

`http://www.tc3.edu/instruct/sbrown/swt/pic/dice-combinations3d-bw.png`

- Let the random variable X be

"the number of times the number 1 faces up".

- *What values can X take?*

  Since the number 1 can be face up on **none** or **one** or **both** of the dice, the random variable X can take on the values 0 or 1 or 2 only.

- *What is the probability of X taking these values?*

← dice 2 →

dice 1 ↕	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	1	0	0	0	0	0
5	1	0	0	0	0	0
6	1	0	0	0	0	0

value of X for each of the 36 possible outcomes

see zero  25 times

see one  ten times

see two  once

We would write

$$P(X=0) = \frac{25}{36} \qquad P(X=1) = \frac{10}{36} \qquad P(X=2) = \frac{1}{36}$$

or summarise the probability distribution in a table

x_i	0	1	2
$P(X = x_i)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

□

Think of a discrete set as having "gaps" between values (numbers or categories).

Notes —

- A **discrete random variable** assigns values to a discrete set, usually a subset of the integers.
- Random variables — use capital letters, e.g., X, Y, Z
- Observed values — use lower case letters, e.g., x, y, z.
- We write statements like "$P(X = x) = \frac{1}{4}$".
  Here the A is the event "$X = x$" and $P(A) = \frac{1}{4}$.
- The **probability distribution** of a discrete random variable, X, gives the probabilities of all the possible values of X.
- For many situations it will not be necessary to make a list of all probabilities in order to specify a probability distribution, because some simple formula (called the **probability function**) can be found. ↳ see Geometric and Binomial (later)
- When all the possible values of a random variable have been considered, the sum of the probabilities must be **one**.
- A probability distribution must assign probabilities to outcomes so that

  (1) $0 \leq P(X = x_i) \leq 1$ for all outcomes x_i in sample space

  (2) $\sum_i P(X = x_i) = 1$

- A probability distribution may be represented graphically in the form of a **bar chart**.

outcomes are
equally likely

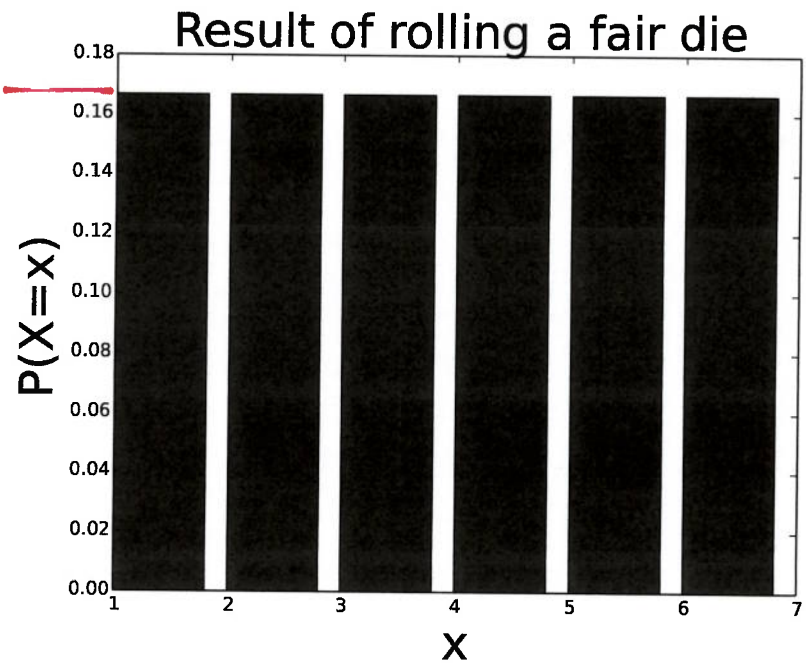
Example 1.14 Let X be the random variable defined as

"the result of rolling a $\boxed{\text{fair}}$ six-sided die".

then

$$P(X = x) = \tfrac{1}{6} \qquad \text{for } x = 1, 2, 3, 4, 5, 6$$

$\frac{1}{6}$



Sample space :

Values assigned by random variable X	1	2	3	4	5	6
Probabilities	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example 1.15 Roll a pair of dice and let the random variable Y be the sum of the numbers that face up.

dice 2

dice 1

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

y_i	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

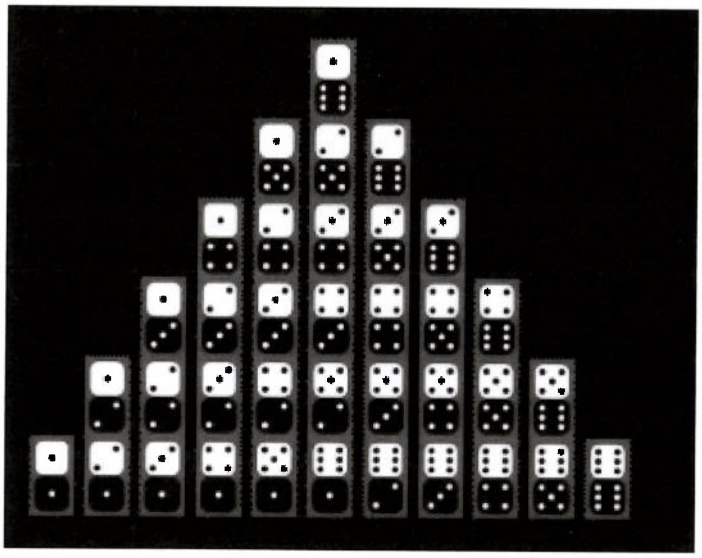
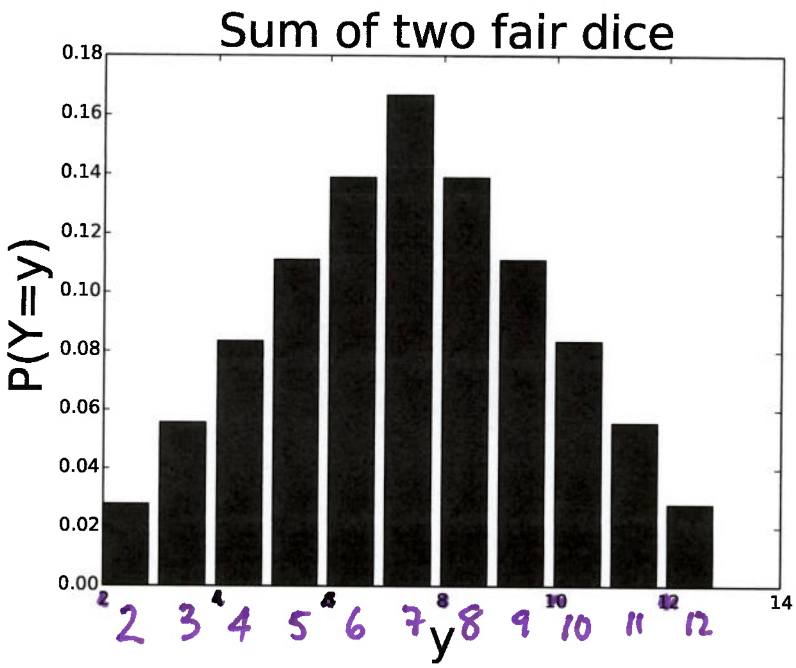


Image from https://study.com/academy/lesson/geometric-distribution-definition-equations-examples.html



Sample space : ..., ..., ..., ..., ..., ...
(page 1037)

Values assigned by random variable X: $1+1=2$, $1+2=3$, $1+3=4$, $1+4=5$, $1+5=6$, ...

Probabilities: $\frac{1}{36}$, $\frac{1}{36}$, $\frac{1}{36}$, $\frac{1}{36}$, $\frac{1}{36}$, ...

Then group all ways to make $2, 3, 4, \ldots, 12$

1.3.2 Mean and Variance

The **mean** of a random variable is its "expected value" (or average value).

■ A discrete random variable X takes on the values

$$x_1, x_2, x_3, \ldots$$

with probabilities

$$P(X = x_1), P(X = x_2), P(X = x_3), \ldots$$

value probability of seeing that value

The **mean** is defined to be expected value

$$E(X) = \sum_i x_i \times P(X = x_i)$$

= value × probability
+ value × probability
+ ...

This sum might contain an infinite number of terms.

Notes —

- The expected value of a function f of a random variable X is defined to be

$$E(f(X)) = \sum_i f(x_i) \times P(X = x_i)$$

- For example, using $f(x) = x^2$

$$E(X^2) = \sum_i (x_i)^2 \times P(X = x_i)$$

■ The **variance** of a discrete random variable X is defined to be

$$\begin{aligned} \text{var}(X) &= E((X-\mu)^2) \\ &= \sum_i (x_i-\mu)^2 \times P(X=x_i) \end{aligned}$$

where $\mu = E(X)$.

Notes —

- The variance is an

  "average of squared distances (deviations) from the mean"

- An equivalent formula for the variance is

$$\text{var}(X) = E(X^2) - (E(X))^2$$

This is the formula we usually use to solve problems.

Useful

- The **standard deviation** is the square root of the variance.

$$\text{standard deviation}(X) = \sqrt{\text{var}(X)}$$

$E(X)$ = value × probability + value × probability + ...

$E(X^2)$ = value^2 × probability + value^2 × probability + ...

Example 1.16 Let X be the number of times the number 1 comes up when rolling two dice (see page 1038).

value	x_i	0	1	2
probability	$P(X = x_i)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$$
\begin{aligned}
E(X) &= 0 \times \tfrac{25}{36} + 1 \times \tfrac{10}{36} + 2 \times \tfrac{1}{36} = \frac{0+10+2}{36} \\
&= \tfrac{12}{36} = \boxed{\tfrac{1}{3}}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= 0^2 \times \tfrac{25}{36} + 1^2 \times \tfrac{10}{36} + 2^2 \times \tfrac{1}{36} = \frac{0+10+4}{36} \\
&= \tfrac{14}{36} = \boxed{\tfrac{7}{18}}
\end{aligned}
$$

$$
\begin{aligned}
\text{var}(X) &= E(X^2) - (E(X))^2 \\
&= \tfrac{7}{18} - \left(\tfrac{1}{3}\right)^2 = \boxed{\tfrac{5}{18}}
\end{aligned}
$$

□

Example 1.17 Let the random variable Y be the sum of the numbers that face up when rolling a pair of dice (see page 1041).

y_i	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned}
E(Y) &= 2(\tfrac{1}{36}) + 3(\tfrac{2}{36}) + 4(\tfrac{3}{36}) + 5(\tfrac{4}{36}) + 6(\tfrac{5}{36}) + 7(\tfrac{6}{36}) \\
&\quad + 8(\tfrac{5}{36}) + 9(\tfrac{4}{36}) + 10(\tfrac{3}{36}) + 11(\tfrac{2}{36}) + 12(\tfrac{1}{36}) \\
&= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} \\
&= \frac{252}{36} = \boxed{7}
\end{aligned}$$

as we would expect! (symmetry)

$$\begin{aligned}
E(Y^2) &= 2^2(\tfrac{1}{36}) + 3^2(\tfrac{2}{36}) + 4^2(\tfrac{3}{36}) + 5^2(\tfrac{4}{36}) + 6^2(\tfrac{5}{36}) + 7^2(\tfrac{6}{36}) \\
&\quad + 8^2(\tfrac{5}{36}) + 9^2(\tfrac{4}{36}) + 10^2(\tfrac{3}{36}) + 11^2(\tfrac{2}{36}) + 12^2(\tfrac{1}{36}) \\
&= \frac{4+18+48+100+180+294+320+324+300+242+144}{36} \\
&= \frac{1974}{36} = \boxed{\frac{329}{6}}
\end{aligned}$$

$$\mathrm{var}(Y) = E(Y^2) - (E(Y))^2 = \frac{329}{6} - (7)^2 = \boxed{\frac{35}{6}}$$

□

Practice Problem. Consider the random variable X with probability distribution given below.

x_i	2	5
$P(X = x_i)$	0.4	0.6

Find $E(X)$ and $var(X)$.

$$E(X) = 2 \times 0.4 + 5 \times 0.6$$
$$= 0.8 + 3.0$$
$$= \boxed{3.8}$$

$$E(X^2) = 2^2 \times 0.4 + 5^2 \times 0.6$$
$$= 4 \times 0.4 + 25 \times 0.6$$
$$= 1.6 + 15$$
$$= \boxed{16.6}$$

$$var(X) = E(X^2) - (E(X))^2$$
$$= 16.6 - (3.8)^2$$
$$= \boxed{2.16}$$