Week 5 : 19/2/2020
21/2/2020

Key background for AI, machine learning and data science.

Topic 2

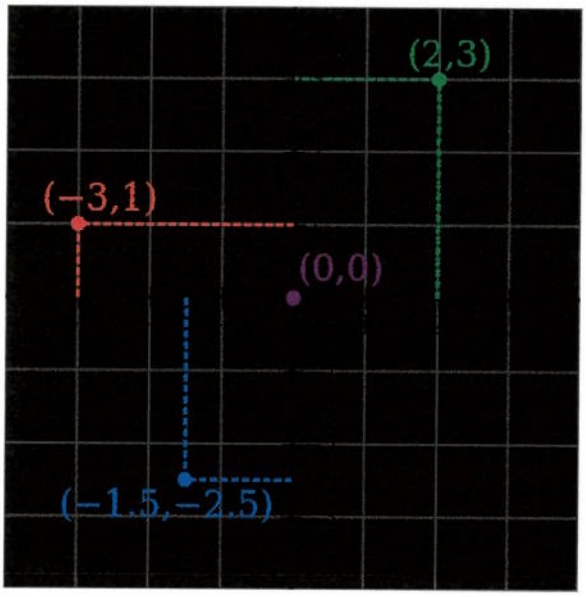
$\boxed{\text{Linear Algebra}}$ for Data Scientists

2.1 Vectors in 2D

Image from http://fluffware.tumblr.com/post/4580822773/axes

2.1.1 Cartesian Coordinates

The **Cartesian coordinate system** (named after the famous mathematician René Descartes) is an assignment of *ordered pairs* (a, b) to points in a plane (called the *coordinate plane* or the xy-plane).



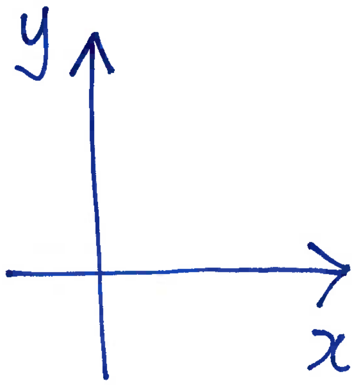


Image from `https://en.wikipedia.org/wiki/Cartesian_coordinate_system`

(x, y) world

Note —

- The coordinates (x, y) are called the **Cartesian coordinates** of the point.
- The Cartesian coordinate system is simply

  two real number lines

  drawn at a $90°$ angle.
- The **origin** is the point $(0, 0)$.

History.

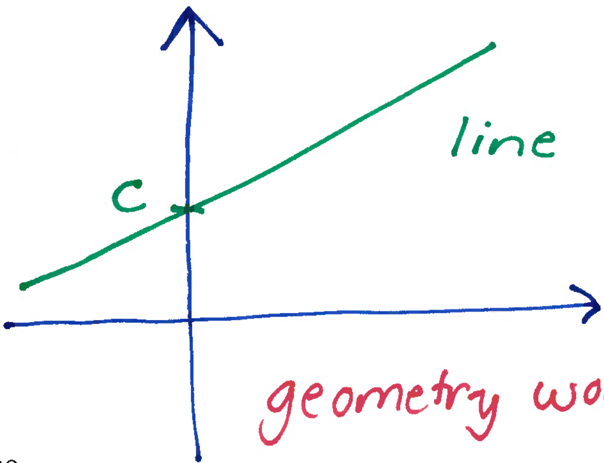
René Descartes (1596–1650). French philosopher and mathematician.

"I think, therefore I am."
– Descartes

Image from http://en.wikipedia.org/wiki/Descartes

The invention of Cartesian coordinates in the 17th century by Descartes revolutionised mathematics by providing the first systematic link between Euclidean geometry and algebra. Using the Cartesian coordinate system, geometric shapes (such as curves) can be described by algebraic equations involving the coordinates of the points lying on the shape.

$y = mx + c \quad \longleftrightarrow$



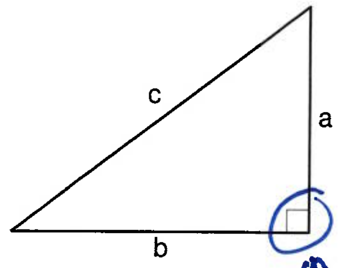
algebra world

geometry world

Result 2.1 *Theorem of Pythagoras*

For a **right-angled triangle** (where one of the corners makes an angle of $\frac{\pi}{2}$ radians or 90 degrees)

$$a^2 + b^2 = c^2$$



□

Proof.

Visual "proof by rearrangement"

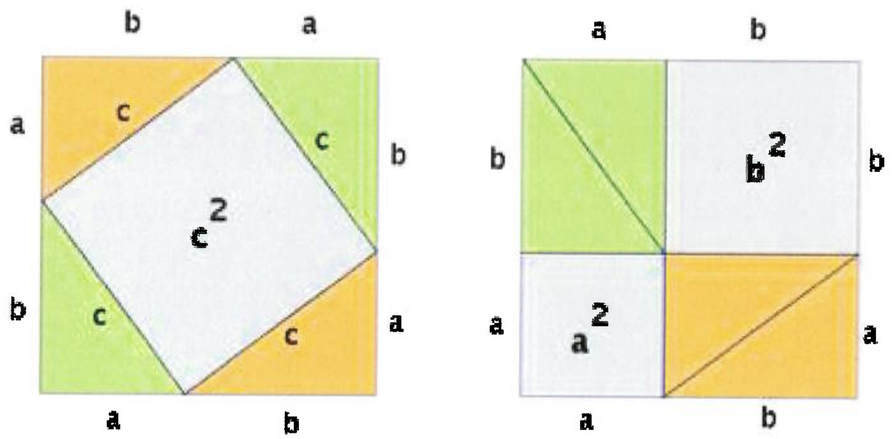
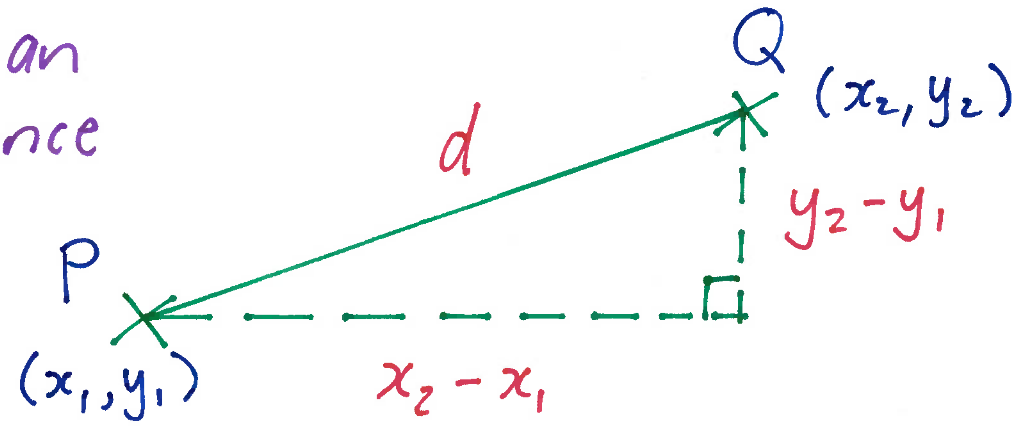


Image from `http://math.stackexchange.com/questions/675522/whats-the-intuition-behind-pythagoras-theorem` ■

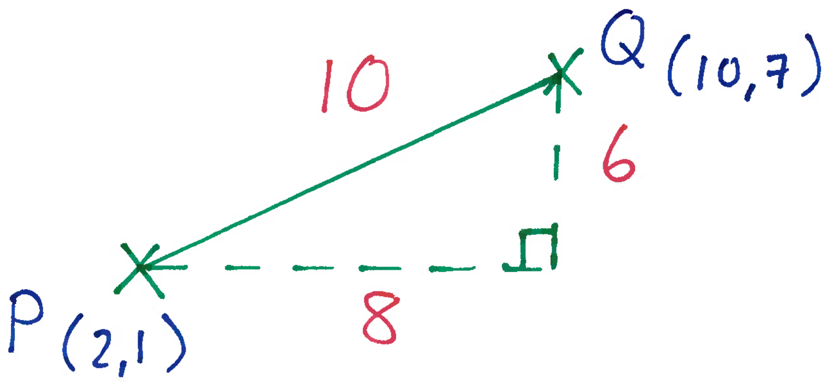
Suppose we have two points (x_1, y_1) and (x_2, y_2). The $\boxed{\text{distance}}$ between these points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Euclidean distance



Example 2.1 Find the distance between points $P(2, 1)$ and $Q(10, 7)$.



Solution.

$$\begin{aligned}
\text{distance} &= \sqrt{(10-2)^2 + (7-1)^2} \\
&= \sqrt{8^2 + 6^2} \\
&= \sqrt{64 + 36} \\
&= \sqrt{100} \\
&= 10
\end{aligned}$$

□

2.1.2 What is a Vector?

single number

A **scalar** is a quantity that is characterised

solely by **magnitude**

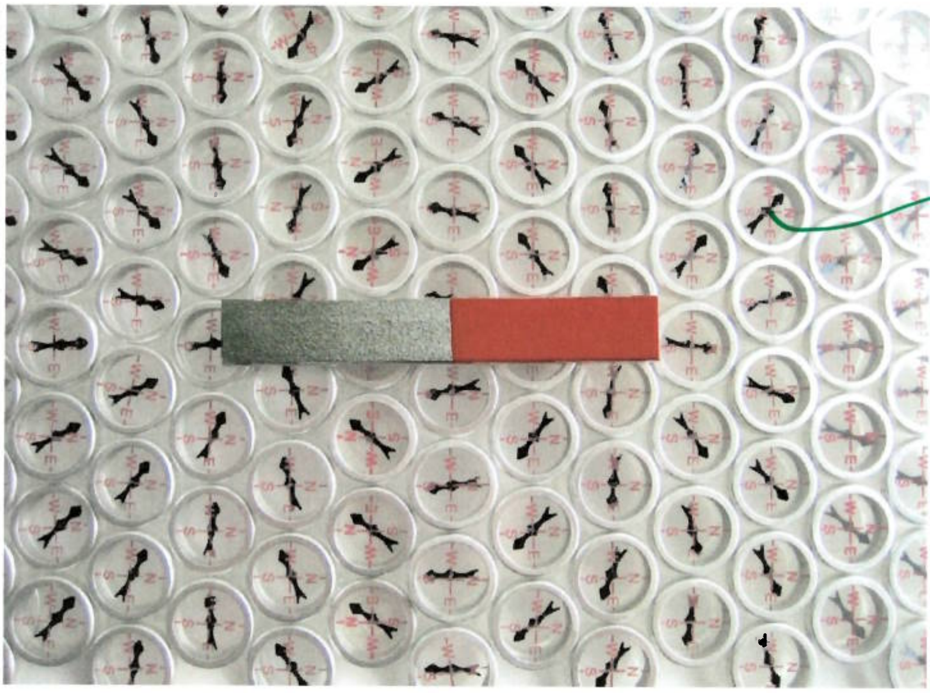
such as mass, time, temperature, distance, speed, pressure, work, energy, or voltage. The value of a scalar is an ordinary number.

A **vector** is a quantity that is characterised

by both **magnitude** and **direction**

such as displacement, force, momentum, velocity, acceleration or magnetic field. A vector is represented by boldface type, such as $\boldsymbol{a}$, or as a pair of points with an arrow above, such as $\overrightarrow{OA}$.

Example: wind has strength and direction.



small compasses give direction of the magnetic field at each location

Image from `http://www.webnode.me/bar-magnet-compass.html`

Point: (x, y)

two dimensions

Definition 2.2 A **vector** (in 2D) is an ordered pair of numbers which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

The numbers x and y are called **components** of the vector.

↳ elements □

Example 2.2 Some examples of vectors. (in 2D)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ \pi \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

□

zero vector

Vectors in 3D:

$$\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.1.3 Geometric Representation of Vectors

algebra

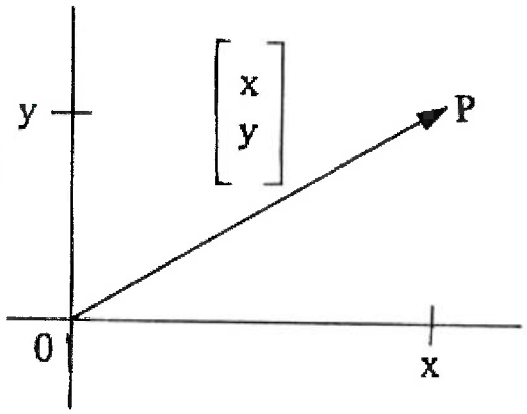
The *vector* $\begin{bmatrix} x \\ y \end{bmatrix}$ may be represented geometrically as the

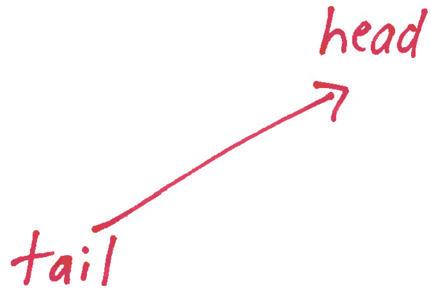
directed line segment (arrow) geometry

from the *point* $O(0,0)$ to the *point* $P(x,y)$.

This directed line segment is called the **position vector** of P and is denoted by $\overrightarrow{OP}$. *The direction of* $\overrightarrow{OP}$ *points towards* P.

$\underline{p}$ means "p bold"

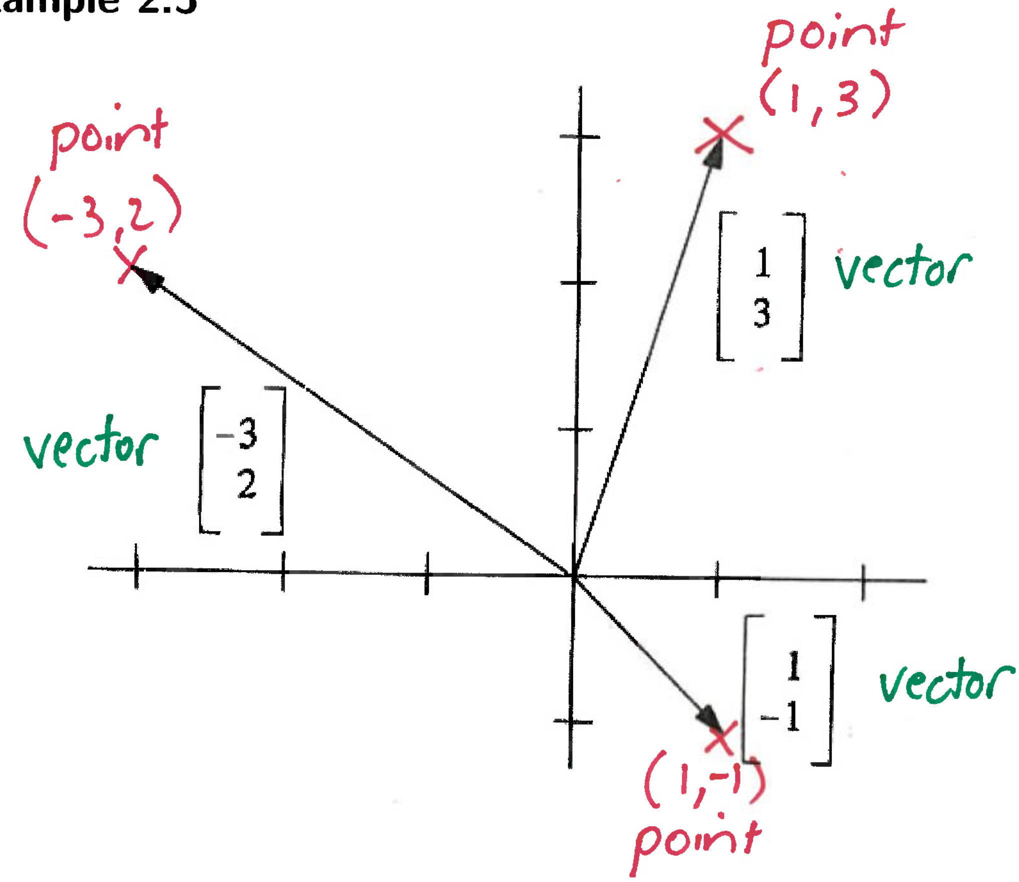




Conventions —

- Upper case letters P, Q, etc, denote **points**. The special point $O(0,0)$ is the **origin** of the coordinate system.
- Lower case bold letters $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{u}$, $\boldsymbol{v}$, etc, denote **vectors**. The special vector $\mathbf{0}$ is the **zero vector** with all components zero (it has magnitude of 0 and arbitrary direction).
- Lower case non-bold letters k, t, etc, denote **scalars**, i.e., ordinary numbers.

Example 2.3



► Position vectors give a one-to-one correspondence between points and position vectors. But position vectors are tied to the origin.

Free Vectors

In geometry and physics it is convenient to be able to interpret **any** directed line segment as a vector.

- The vector $\begin{bmatrix} x \\ y \end{bmatrix}$ may be represented geometrically by the

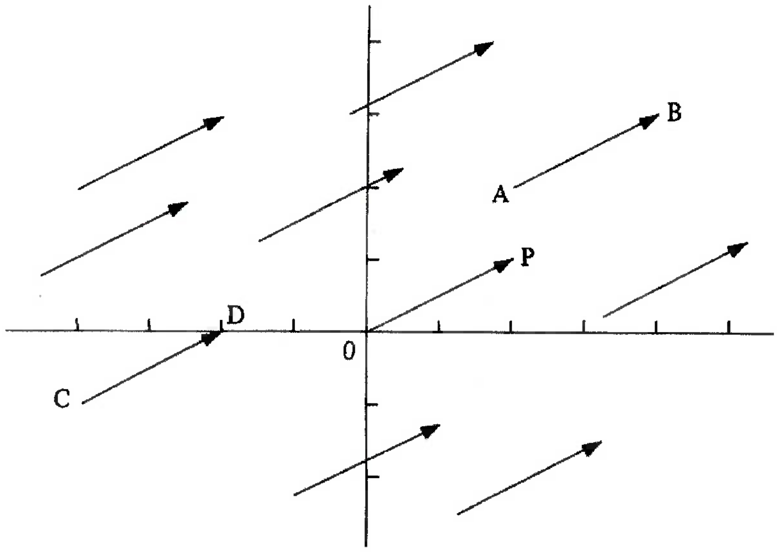
  directed line segment from (a, b) to $(a + x, b + y)$

  for **any** initial point (a, b).

- When (a, b) is not the origin, the directed line segment is called a **free vector**. This gives many possible geometric representations of the same vector.

Example 2.4 The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ may be represented:

- by its position vector $\overrightarrow{OP}$
- or by the directed line segment $\overrightarrow{AB}$ from $A(2, 2)$ to $B(4, 3)$
- or by the directed line segment $\overrightarrow{CD}$ from $C(-4, -1)$ to $D(-2, 0)$
- or by any one of many more possibilities.



"free" vectors

Vectors can be free or position vectors $\square$ geometrically $\longrightarrow$ no worry as to which.

2.1.4 Equality of Vectors

■ Two vectors are **equal** if and only if they have the same components.

- If $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$ then $\boxed{a = c}$ and $\boxed{b = d.}$
- If $a = c$ and $b = d$ then $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$.

Example 2.5 Does $\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$?

$\times$

$\checkmark$

$\boxed{\text{No}}$ since $2 \neq -2$.

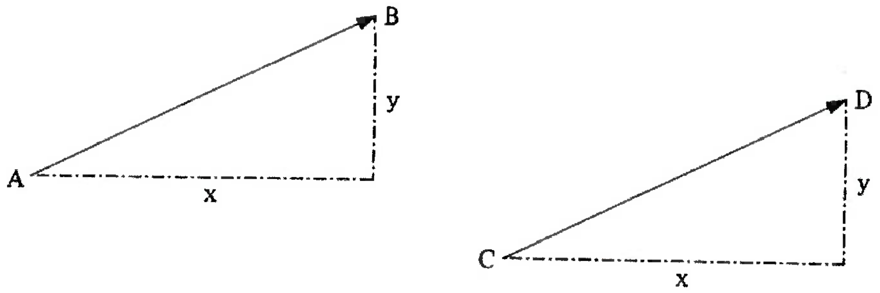
□

Question —

When do two directed line segments represent the same vector?

We know that two vectors are **equal**, that is $\boldsymbol{a} = \boldsymbol{b}$ when the corresponding components are the same.

Consider the points A, B, C, and D.



The directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector $\begin{bmatrix} x \\ y \end{bmatrix}$ if and only if

(1) slope of $\overrightarrow{AB}$ = slope of $\overrightarrow{CD} = \dfrac{y}{x}$

(2) length of $\overrightarrow{AB}$ = length of $\overrightarrow{CD} = \sqrt{x^2 + y^2}$

(3) the arrows point in the same direction.

Same slope
Same length
$\nearrow \neq \swarrow$

> *Conclusion:* Two vectors are **equal** if they have the same magnitude (length) and the same direction (slope and arrows point same way).

Two fundamental ideas in Linear Algebra:
(1) vector addition , (2) scalar multiplication

2.1.5 Vector Addition

$$\begin{matrix} \text{vector} \to \\ \text{vector} \to \end{matrix} \boxed{+} \to \text{vector}$$

Definition 2.3 *Vector Addition*

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

defining what this symbol means

□

Example 2.6

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+4 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Practice Problem. Suppose

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

(a) $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

(b) $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\boldsymbol{w} + \boldsymbol{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

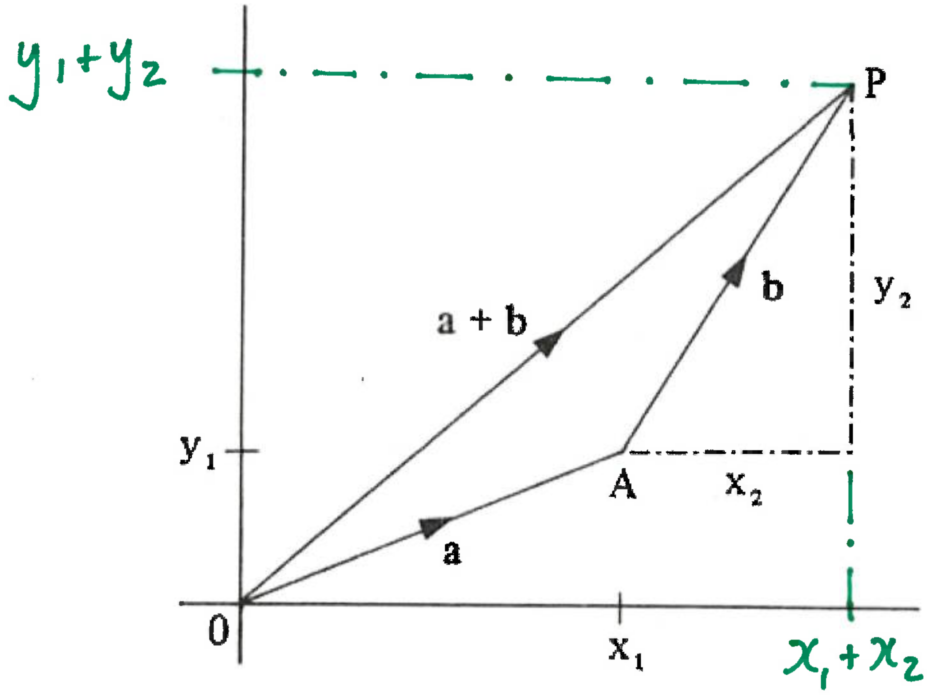
□

Question —

What does vector addition look like?

- Let $\boldsymbol{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.
- Then $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$.
- Represent $\boldsymbol{a}$ by its position vector $\overrightarrow{OA}$ so $A = (x_1, y_1)$.
- Let $P = (x_1 + x_2, y_1 + y_2)$ so that $\overrightarrow{OP}$ represents $\boldsymbol{a} + \boldsymbol{b}$.
- Then $\overrightarrow{AP}$ represents the vector $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ which is $\boldsymbol{b}$.

$$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$$



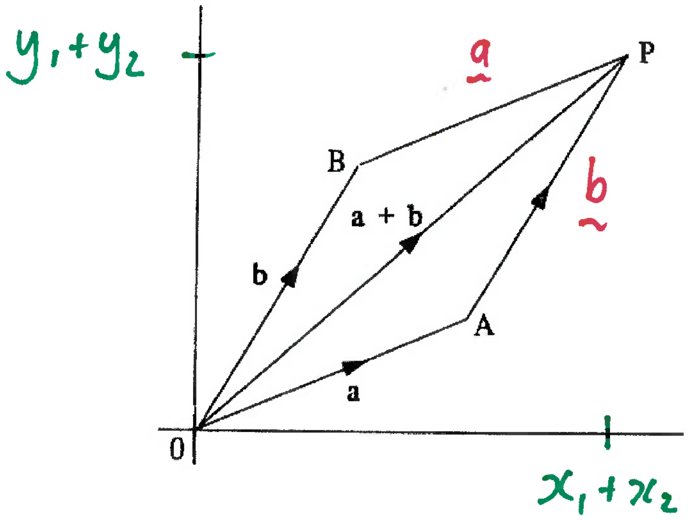
Question —

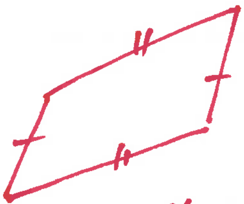
Does $\quad \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$?

Answer — Geometrically

Parallelogram Law of Vector Addition

$$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OB} + \overrightarrow{BP}$$





parallelogram

Answer — Algebraically

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ y_2 + y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Conclusion: Vector addition obeys the **commutative law**:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

"Algebra" is the study of structure.

2.1.6 Scalar Multiplication

Definition 2.4 *Scalar Multiplication*

$$k\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \times x \\ k \times y \end{bmatrix}$$

scalar vector vector

□

Example 2.7

$$3\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \times (-1) \\ 3 \times 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

□

Practice Problem. Suppose

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

(a) $3\boldsymbol{v} = 3\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 12 \end{bmatrix}$

(b) $2\boldsymbol{u} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(c) $-2\boldsymbol{w} = -2\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$

(d) $3\boldsymbol{v} + 2\boldsymbol{u} = \begin{bmatrix} -9 \\ 12 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 16 \end{bmatrix}$

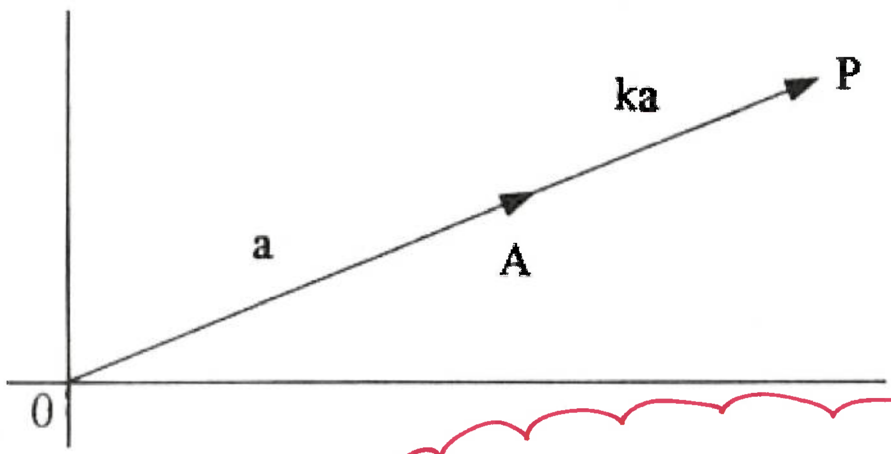
(e) $(3\boldsymbol{v} + 2\boldsymbol{u}) + \boldsymbol{w} = \begin{bmatrix} -7 \\ 16 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix}$

$\square$

Question —

What does scalar multiplication look like?

- Let $\overrightarrow{OA} = \boldsymbol{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\overrightarrow{OP} = k\boldsymbol{a} = \begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix}$ with $k \neq 0$.
- If $x_1 \neq 0$, then the slope of the line segment OP is $\dfrac{ky_1}{kx_1} = \dfrac{y_1}{x_1}$ which is the slope of OA, so $\boxed{\boldsymbol{a} \text{ and } k\boldsymbol{a} \text{ lie along the same line.}}$ same slope
- If $x_1 = 0$, $\begin{bmatrix} 0 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ ky_1 \end{bmatrix}$ also lie along the same line.



Idea: vector is stretched or shrunk.

Notes —

- If k is a *positive scalar* and $\boldsymbol{a}$ is a vector then $k\boldsymbol{a}$ is a vector in the same direction as $\boldsymbol{a}$ but k times as long.
- If k is a *negative scalar* and $\boldsymbol{a}$ is a vector then $k\boldsymbol{a}$ is a vector in the opposite direction as $\boldsymbol{a}$ but $|k|$ times as long.
- Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are **parallel** if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \neq 0$.

Idea of a "negative": 5 + what = 0 ?

$\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ + what = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

2.1.7 Negative of a Vector

Definition 2.5 *Negative of a vector.*

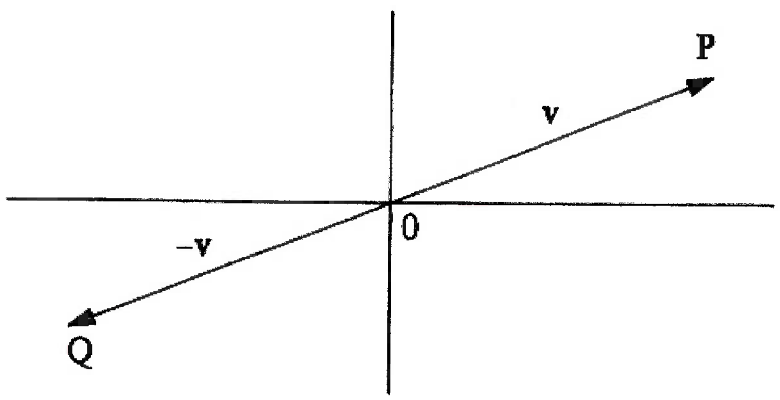
Let "$-\boldsymbol{v}$" mean "$(-1)\boldsymbol{v}$"

If $\boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $-\boldsymbol{v} = \begin{bmatrix} -a \\ -b \end{bmatrix}$ □

Suppose $\overrightarrow{OP} = \boldsymbol{v}$ and $\overrightarrow{OQ} = -\boldsymbol{v}$.

Then $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ have the same magnitude but point in opposite directions.

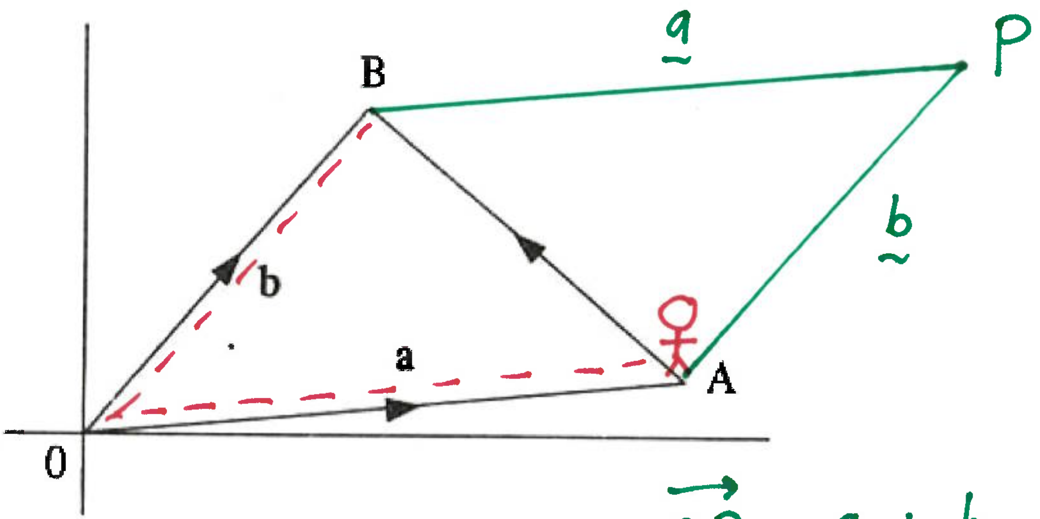
Therefore, $-\boldsymbol{v}$ can also be represented by $\overrightarrow{PO}$.



Conclusion: The negative of a vector points in the opposite direction but has the same magnitude.

2.1.8 Vector Subtraction

In the figure below, we can get from A to B by travelling from A to O and then from O to B.



$\overrightarrow{OP} = \underset{\sim}{a} + \underset{\sim}{b}$

vector from point O to point P

$$\begin{aligned}
\boxed{\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}} \\
&= (-\overrightarrow{OA}) + \overrightarrow{OB} \\
&= -\boldsymbol{a} + \boldsymbol{b} \\
&= \boldsymbol{b} + (-\boldsymbol{a}) \\
&= \boldsymbol{b} - \boldsymbol{a}
\end{aligned}$$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

vector from point A to point B

Definition 2.6 *Vector subtraction*

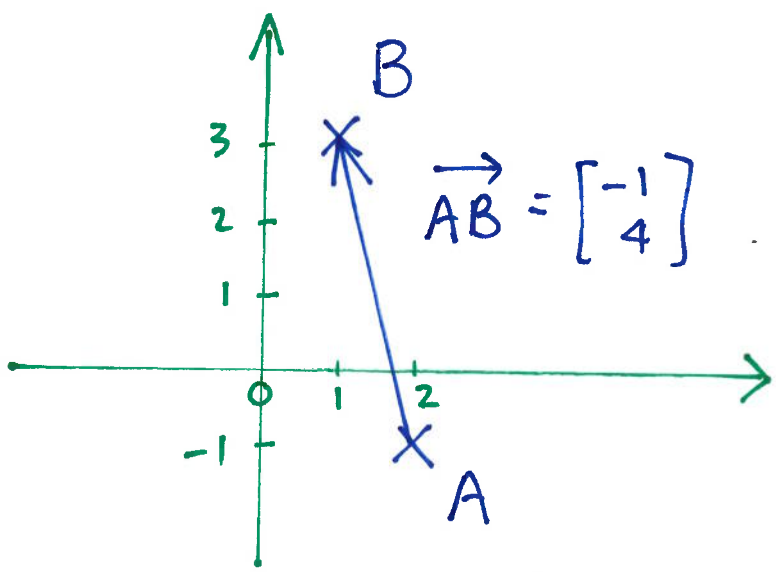
Let "$\boldsymbol{u} - \boldsymbol{v}$" mean "$\boldsymbol{u} + (-\boldsymbol{v})$" $\square$

Example 2.8 For points $A(2, -1)$ and $B(1, 3)$, find $\overrightarrow{AB}$.

Solution.

$$\begin{aligned}
\overrightarrow{AB} &= \boldsymbol{b} - \boldsymbol{a} \\
&= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 4 \end{bmatrix}
\end{aligned}$$

$\square$



Summary

- Cartesian coordinates (x, y)
- Distance between points (x_1, y_1) and (x_2, y_2) is

number

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Pythagoras

- Scalar – magnitude only, e.g., mass
- Vector — magnitude and direction, e.g., force, wind
  - position vector vs free vector (both just called vectors)
- Equality of vectors (corresponding components are equal, same magnitude and direction)
- Vector addition (add corresponding components)
  - parallelogram law of vector addition

$$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OB} + \overrightarrow{BP}$$

- Scalar multiplication (multiply each component)
  - parallel vectors (one is a scalar multiple of the other)
- Negative of a vector: $-\boldsymbol{a} = (-1)\boldsymbol{a}$
  - same magnitude, opposite direction
- Vector subtraction: $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$
- Zero vector denoted $\mathbf{0}$ (all components zero, magnitude zero and arbitrary direction)

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has magnitude of 0
what direction?