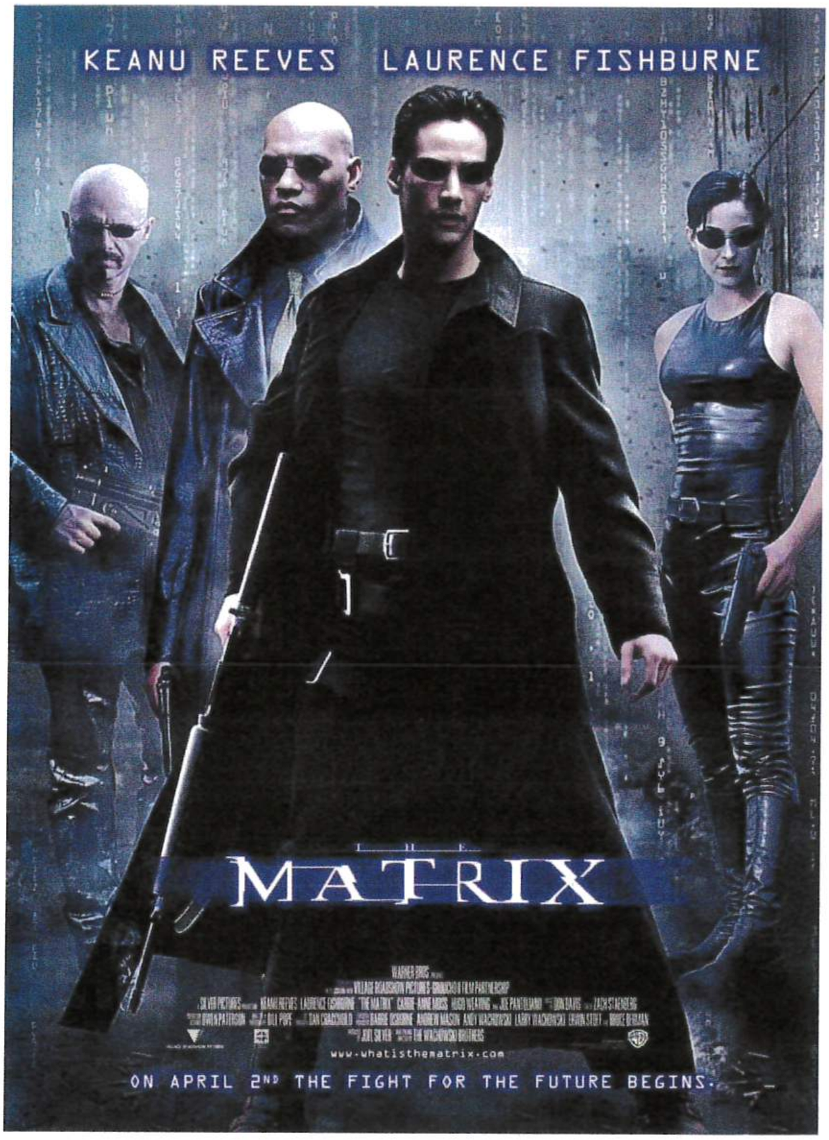
Linear Algebra

Week 7: 4/3/2020
6/3/2020

2.3 What is The Matrix?



The Matrix (1999 film)

Image from http://www.impawards.com/1999/posters/matrix_ver1_xlg.jpg

Definition 2.12 A **matrix** is a rectangular array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

□

2D array

Example 2.13 *Examples of matrices*

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 0 & 3 \\ 0 & -1 & 2 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 8 & 2 \\ -3 & 7 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 8 & 17 & 12 \\ 7 & -8 & 0 \\ 5 & 8 & 1 \end{bmatrix}$$

Conventions —

- We usually use a capital letter to denote a matrix, e.g., A.
- Enclose a matrix in $\left[\quad \right]$ or $\left(\quad \right)$ or $\left(\quad \right)$ but **not** $\left| \quad \right|$.

■ The **size** (or **order**) of a matrix is "$m \times n$" where

- m is the number of **rows** and
- n is the number of **columns**.

If a matrix has the same number of rows as it has columns it is said to be **square**.

Example 2.14

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a 2×1 matrix (2 rows, 1 column)

$\begin{bmatrix} 1 & 5 & 0 & 3 \\ 0 & -1 & 2 & 9 \end{bmatrix}$ is a 2×4 matrix (2 rows, 4 columns)

$\begin{bmatrix} 8 & 2 \\ -3 & 7 \\ 0 & 5 \end{bmatrix}$ is a 3×2 matrix (3 rows, 2 columns)

$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ is a square matrix (2 rows, 2 columns)

▶ We will mostly look at 2×2 matrices (2 rows and 2 columns).

Notes —

- A **vector** is any $m \times 1$ matrix, i.e., a matrix with only one column, e.g.,

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- The square 2×2 matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the **identity matrix**.

3x3 matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Two matrices are **equal** if
  - they are of the same size and
  - their corresponding elements are equal.

similar to vectors

- The **negative** of a matrix A is found by multiplying all the elements by -1.

- In **scalar multiplication**, each element of the matrix is multiplied by a scalar (number), i.e.,

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} k \times a & k \times b \\ k \times c & k \times d \end{bmatrix}$$

- In **matrix addition**, elements in corresponding positions are added, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Warning! Add matrices of the **same size only**. The two matrices must have the *same number of rows* and the *same number of columns*.

Properties of matrix addition —

- Commutative, i.e., $A + B \;=\; B + A$
- Associative, i.e., $A + (B + C) \;=\; (A + B) + C$

Example 2.15 Let $A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 3 \\ 0 & -1 & 9 \end{bmatrix}$

Does $A = B$? Yes

Example 2.16 Let $A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 3 \\ 0 & -2 & 9 \end{bmatrix}$

Does $A = B$? No

different

Example 2.17 Let $A = \underset{}{\overset{2\times 3}{\begin{bmatrix} 1 & 5 & 3 \\ 0 & -1 & 9 \end{bmatrix}}}$ and $B = \overset{2\times 2}{\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}}$

Does $A = B$? No since different sizes.

Example 2.18 If

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

then

$$3A \;=\; \begin{bmatrix} 3 \times 3 & 3 \times (-1) \\ 3 \times 2 & 3 \times 0 \end{bmatrix} \;=\; \begin{bmatrix} 9 & -3 \\ 6 & 0 \end{bmatrix}$$

Example 2.19

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ -1 & 5 \end{bmatrix}$$

Example 2.20

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$$

is **not allowed** because they do not have the same size.

Google : "symmetric" matrix.

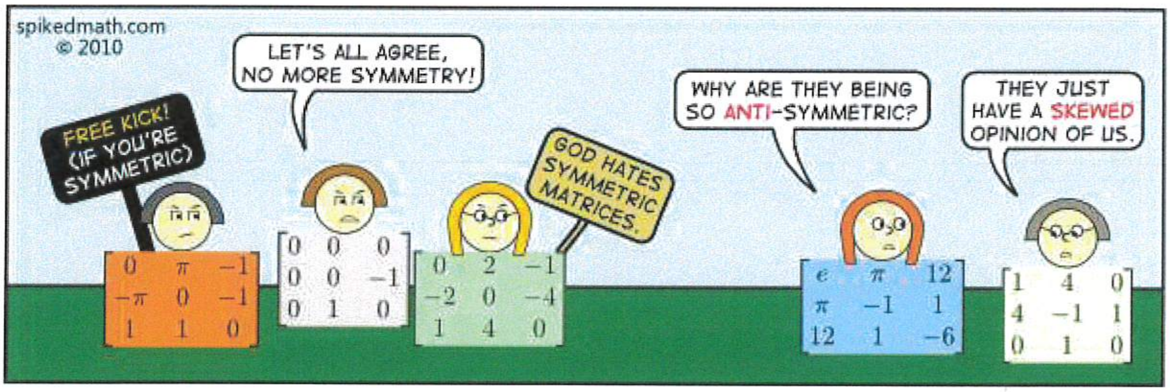


Image from http://spikedmath.com/323.html

Image from

http://www.cloudworksmg.com/wp-content/uploads/2014/12/whats-big-idea.png

* Key skill *

Question —

How do we **multiply** matrices?

1×2 matrix   2×1 matrix

$\boxed{\text{Case I}}$ Suppose A is a single **row** and B is a single **column**.

If $A = \begin{bmatrix} a & b \end{bmatrix}$ and $B = \begin{bmatrix} x \\ y \end{bmatrix}$ then define

$$C = AB = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} a \times x + b \times y \end{bmatrix}$$

1×1 matrix (a scalar)

which is a 1×1 matrix (1 row, 2 column), i.e., it is a scalar (number).

▶ This is similar to a *vector dot product*.

Example 2.21 Let $A = \begin{bmatrix} 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$$\begin{aligned} C = AB &= \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ &= 4 \times 3 + 2 \times 7 \qquad = 12 + 14 \\ &= 26 \end{aligned}$$

- We have multiplied elements in the **row of A** with corresponding elements in the **column of B** and **added** the results together.

$$\begin{bmatrix} \longrightarrow \end{bmatrix} \begin{bmatrix} \Big\downarrow \end{bmatrix}$$

left finger moves across

right finger moves down

muscle memory

$\boxed{\text{Case II}}$ Suppose A is a 2×2 matrix and B is a single column.

If $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\quad B = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$C = AB = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \\ \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a \times x + b \times y \\ c \times x + d \times y \end{bmatrix}$$

which is a 2×1 matrix (2 rows, 1 column).

- We have multiplied elements in the **first row of A**
  with corresponding elements in the **column of B**
  and **added** the results together
  to give the **first element of C**.
- We have multiplied elements in the **second row of A**
  with corresponding elements in the **column of B**
  and **added** the results together
  to give the **second element of C**.

$$\begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix} \begin{bmatrix} \big\downarrow \end{bmatrix}$$

2 × 2 matrix times vector

Example 2.22 Let $A = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ then

$$C = AB = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \\ \begin{bmatrix} -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 3 + 5 \times 1 \\ (-1) \times 3 + 4 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

This is what we will do next week when we look at matrix transformations.

point (3, 1) ⟶ point (11, 1)

$\boxed{\text{Case III}}$ Consider where both A and B are 2×2 matrices.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & w \\ y & z \end{bmatrix}$ then

$$C = AB = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} ax + by & aw + bz \\ cx + dy & cw + dz \end{bmatrix}$$

which is a 2×2 matrix.

- We have multiplied elements in **row i** of A
  with corresponding elements in **column j** of B
  and **added** the results together
  to give the **element in row i and column j** of C.

$$\begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix} \begin{bmatrix} \downarrow & \downarrow \end{bmatrix}$$

matrix times matrix

Example 2.23 Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix}$ then

$$C = AB = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \end{bmatrix} & \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \\ \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \end{bmatrix} & \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 5 + (-2) \times (-7) & 1 \times 6 + (-2) \times 8 \\ 3 \times 5 + 4 \times (-7) & 3 \times 6 + 4 \times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 14 & 6 - 16 \\ 15 - 28 & 18 + 32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -10 \\ -13 & 50 \end{bmatrix}$$

Practice Problem.

$$D = \begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{array}{cc} & \begin{array}{cc} \text{column 1} & \text{column 2} \end{array} \\ \begin{array}{c} \text{row 1} \\ \\ \\ \text{row 2} \end{array} & \begin{bmatrix} \boxed{\overset{\text{row 1}}{\begin{bmatrix} 5 & 6 \end{bmatrix}} \overset{\text{col 1}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}} & \boxed{\overset{\text{row 1}}{\begin{bmatrix} 5 & 6 \end{bmatrix}} \overset{\text{col 2}}{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}} \\ \\ \boxed{\overset{\text{row 2}}{\begin{bmatrix} -7 & 8 \end{bmatrix}} \overset{\text{col 1}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}} & \boxed{\overset{\text{row 2}}{\begin{bmatrix} -7 & 8 \end{bmatrix}} \overset{\text{col 2}}{\begin{bmatrix} -2 \\ 4 \end{bmatrix}}} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \boxed{5 \times 1 + 6 \times 3} & \boxed{5 \times (-2) + 6 \times 4} \\ \\ \boxed{-7 \times 1 + 8 \times 3} & \boxed{-7 \times (-2) + 8 \times 4} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{23} & \boxed{14} \\ \\ \boxed{17} & \boxed{46} \end{bmatrix} \neq \begin{bmatrix} 19 & -10 \\ -13 & 50 \end{bmatrix}$$

Question —

What is special about $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & w \\ y & z \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} x & w \\ y & z \end{bmatrix}$$

Similar to number 1 in usual multiplication.

► So $MI = M$ and $IM = M$ which is why I is called the **identity** matrix.

Properties of Matrix Multiplication —

- **Not** commutative (in general), i.e.,

$$\text{in general} \qquad \boxed{AB \neq BA}$$

Warning!

e.g., we have shown that

$$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & -10 \\ -13 & 50 \end{bmatrix}$$

but

$$\begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 14 \\ 17 & 46 \end{bmatrix}$$

- Associative, i.e.,

$$(AB)C = A(BC)$$

This will be important later when we look at composite transformations.

Summary

- The *size* (or *order*) of a matrix with m rows and n columns is "$m \times n$".
- A *vector* is any $m \times 1$ matrix (has only one column).
- A *square* matrix has the same number of rows as columns.

linear algebra

- *Scalar multiplication* — each element of the matrix is multiplied by a scalar (number).
- *Matrix addition* — elements in corresponding positions are added.

$$A + B = B + A$$

- *Matrix multiplication* —

$$\begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix}\begin{bmatrix} \downarrow \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \longrightarrow \\ \longrightarrow \end{bmatrix}\begin{bmatrix} \downarrow & \downarrow \end{bmatrix}$$

$$AB \neq BA \qquad (AB)C = A(BC) \qquad A(B+C) = AB+AC$$

- For any 2×2 matrix M and the *identity* matrix I, we have $MI = M$ and $IM = M$.