Week 6 : 26/2/2020
28/2/2020

2.2 How are Vectors Useful?

Recap (Week 5) — Vectors in 2D

- *Scalar* (magnitude only) vs *vector* (magnitude and direction).
- Vector addition (add corresponding components).
- Scalar multiplication (multiply each component).
- *Algebra* view vs *geometry* view (arrows) of vectors in 2D, e.g., parallelogram law of vector addition and stretching of vectors.

Why learn Linear Algebra?

- Read "10 Powerful Applications of Linear Algebra in Data Science" at https://www.analyticsvidhya.com/blog/2019/07/10-applications-linear-algebra-data-science/

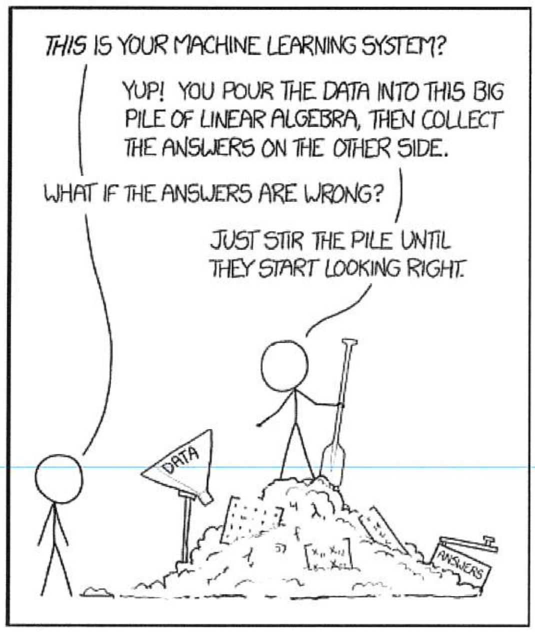


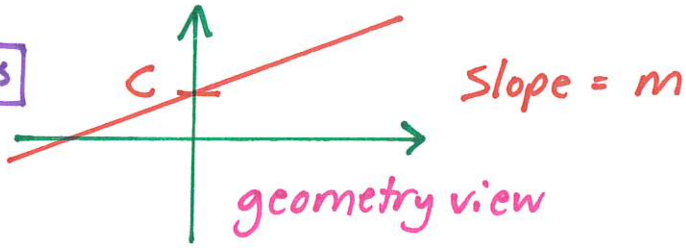
Image from https://xkcd.com/1838/

Speak the language of machine learning.

linear algebra

probability

Line: $y = mx + c$ $\boxed{\text{vs}}$



algebra view

geometry view

2.2.1 Vector Equations for Lines and Curves

instead use **vector** equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + t \begin{bmatrix} c \\ d \end{bmatrix}$ as t goes from $-\infty$ to ∞

Question —

How are lines and curves related to vectors?

↳ difficulty with vertical lines → infinite slope

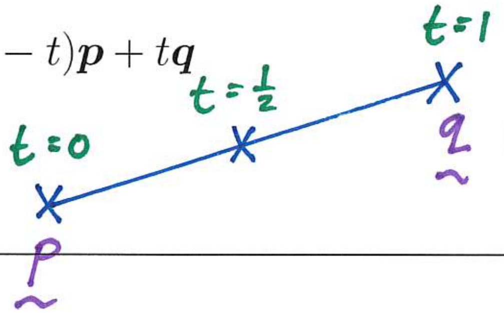
Definition 2.7 A **line segment** is defined by

two endpoints (vectors), $\boldsymbol{p}$ and $\boldsymbol{q}$.

The line segment is the set of all points (vectors)

$$\boldsymbol{a} = (1-t)\boldsymbol{p} + t\boldsymbol{q}$$

as t goes from 0 to 1. □



Notes —

- This is called a **parametric** form for a line segment, as each point (vector) on the line segment corresponds to one value of the *parameter* t.
- A given point (vector) $\boldsymbol{b}$ is $\boxed{\textbf{exactly}}$ on the line if and only if there is a value for t with $0 \leq t \leq 1$ such that

$$\boldsymbol{b} = (1-t)\boldsymbol{p} + t\boldsymbol{q}$$

need to find t

Example 2.9 Consider the particular line segment with endpoints

$$\boldsymbol{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

- When $t = \frac{1}{3}$,

$$\left(1-\tfrac{1}{3}\right)\begin{bmatrix}1\\2\end{bmatrix} + \tfrac{1}{3}\begin{bmatrix}7\\-1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{4}{3}\end{bmatrix} + \begin{bmatrix}\frac{7}{3}\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

so $(3,1)$ is a point on the line segment.

- When $t = \frac{1}{2}$,

$$\left(1-\tfrac{1}{2}\right)\begin{bmatrix}1\\2\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}7\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} + \begin{bmatrix}\frac{7}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}4\\\frac{1}{2}\end{bmatrix}$$

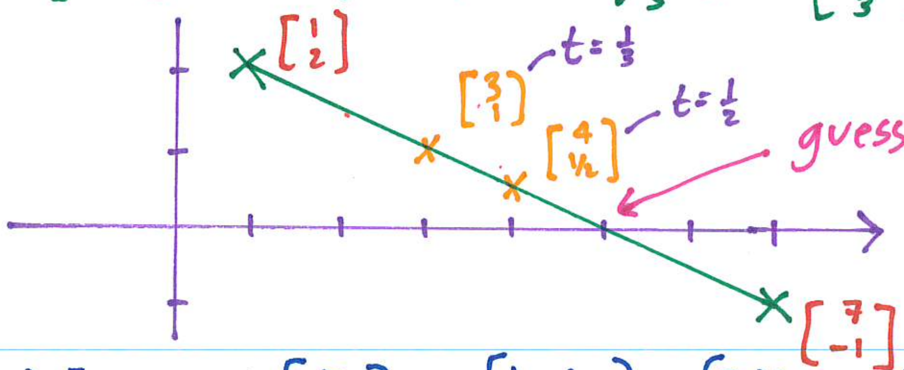
so $(4,\frac{1}{2})$ is a point on the line segment.

$\square$

cannot rely on eyeballing

Practice Problem. Show that the point $(5,0)$ lies **exactly** on the line segment in the previous example.

$$\left(1-\tfrac{2}{3}\right)\begin{bmatrix}1\\2\end{bmatrix} + \tfrac{2}{3}\begin{bmatrix}7\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\end{bmatrix} + \begin{bmatrix}\frac{14}{3}\\-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}$$



$\boxed{\text{or}}$ $$(1-t)\begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}7\\-1\end{bmatrix} = \begin{bmatrix}1-t\\2-2t\end{bmatrix} + \begin{bmatrix}7t\\-t\end{bmatrix} = \begin{bmatrix}1+6t\\2-3t\end{bmatrix}$$

so $1 + 6t = 5 \Rightarrow 6t = 4 \rightarrow t = \frac{2}{3}$

$2 - 3t = 0 \Rightarrow 3t = 2 \rightarrow t = \frac{2}{3}$

agree

Bézier Curves

Bézier curves were invented by Pierre Bézier (1910–1999), an engineer working for the Rénault car company in the 1970s. He originally used the curve in the computer aided design of car bodies. Due to its relative simplicity and flexibility, the Bézier curve has become one of the most popular curves for 2D and 3D graphics ever since.

Definition 2.8 A **quadratic Bézier curve** is defined by

- two endpoints ($\boldsymbol{p}$ and $\boldsymbol{q}$) ■
- one control point ($\boldsymbol{c}$) □

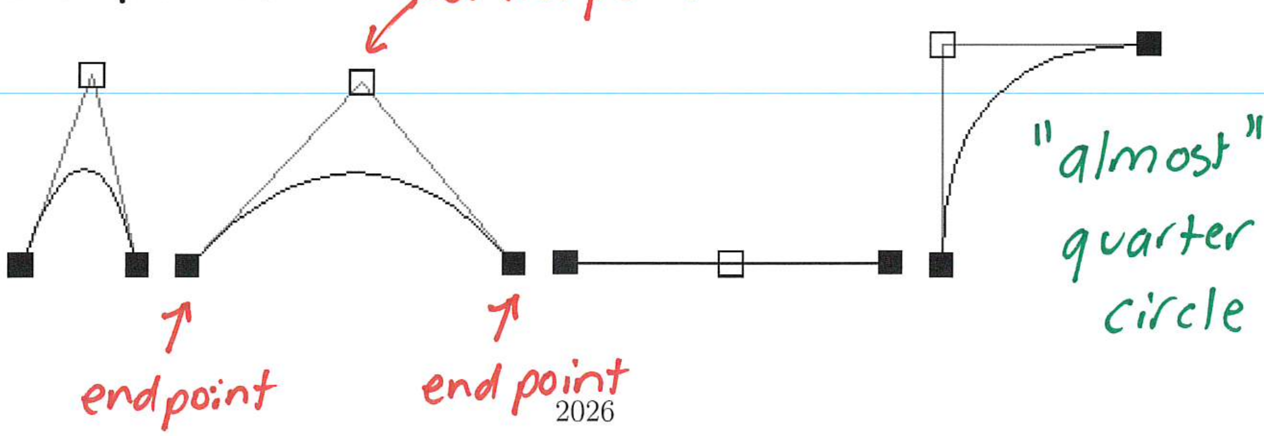
The control point defines the **slope** of the curve at the endpoints.

The curve is the set of all points (vectors)

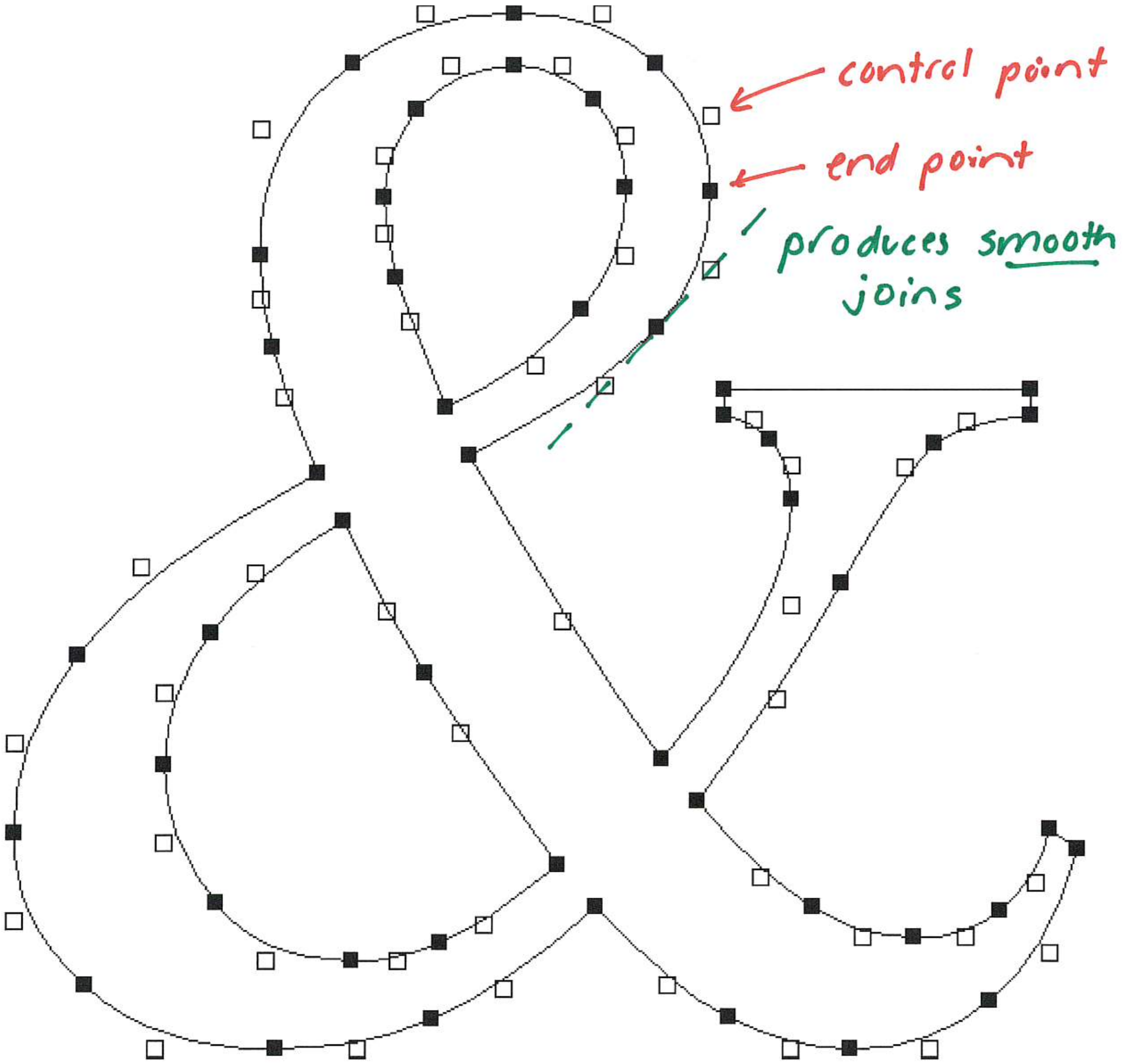
$$\boldsymbol{a} \;=\; (1-t)^2\boldsymbol{p} + (2 \times (1-t) \times t)\boldsymbol{c} + t^2\boldsymbol{q}$$

as t goes from 0 to 1. □

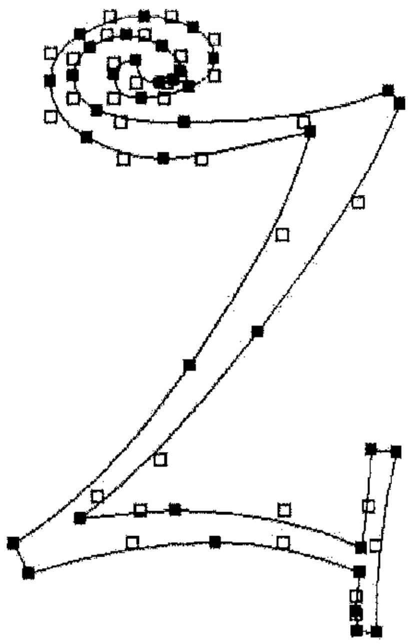
Example 2.10



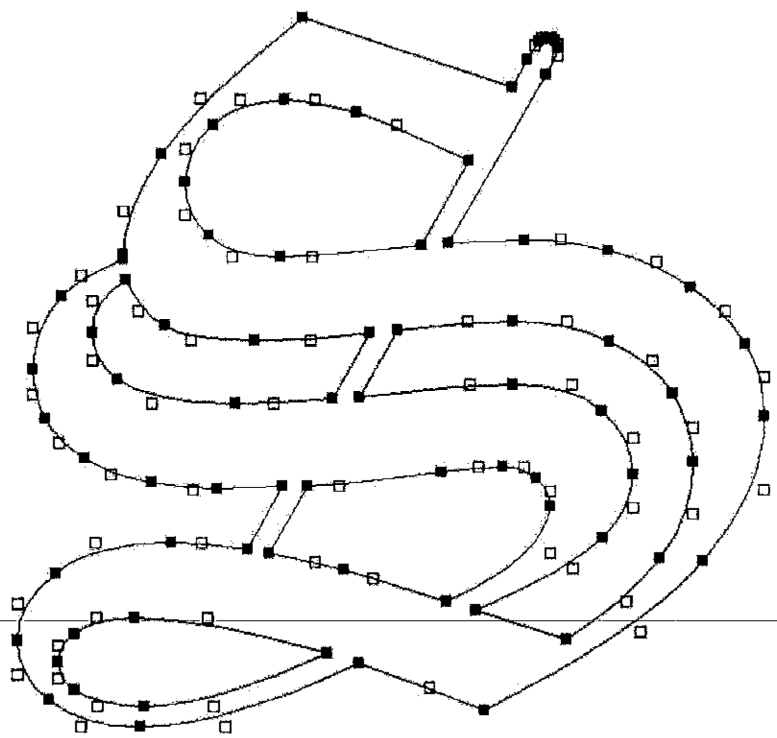
For example, the individual characters (or **glyphs**) in a particular font are often designed and defined using line segments and quadratic Bézier curves.



Times New Roman '&'



Curlz MT ‘z’



Old English Text MT ‘S’

2.2.2 Review of Directions and Angles

So far, we have studied points and vectors in Cartesian coordinates. It would be useful to be able to translate easily from Cartesian coordinates to magnitude and direction, and vice versa.

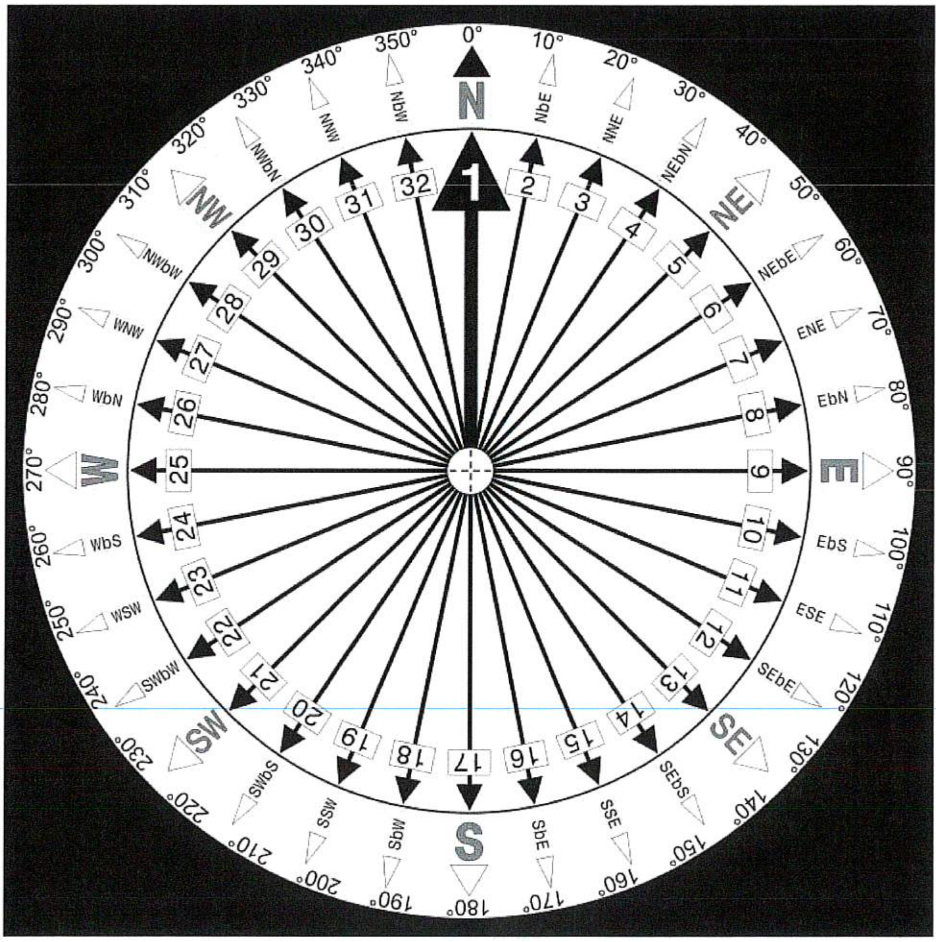
Compass Directions

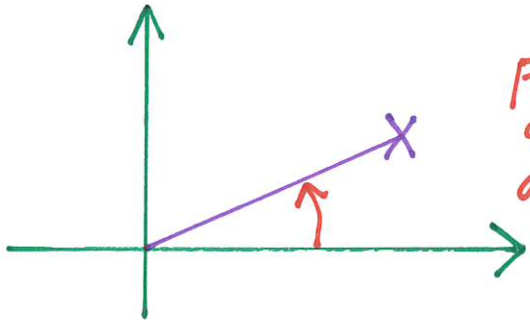
In navigation and astronomy, directions are often given by

↳ sailing

compass bearings

which measure the angle **clockwise** from North in units of **degrees**.





positive angles are anticlockwise

Angles

We interpret an angle $\angle AOB$ as a **rotation** about O of OA to a position specified by OB.

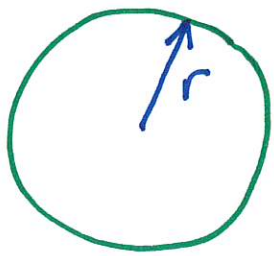
- A counterclockwise rotation produces a **positive angle**, whereas as clockwise rotation give a **negative angle**.
- Lower-case Greek letters such as α ("alpha"), β ("beta") and θ ("theta") are often used to denote angles.
- In navigation and astronomy, angles are measured in degrees, but in mathematics and computer science it is best to use units called **radians** because of the way they simplify later calculations.
- An angle of **degree** measure 1° corresponds to

$$\frac{1}{360} \text{ of a complete counterclockwise revolution.}$$

- An angle of **radian** measure 1 corresponds to

$$\frac{1}{2\pi} \text{ of a complete counterclockwise revolution.}$$

circle



$$\frac{\text{angle in radians}}{2\pi} = \frac{\text{angle in degrees}}{360}$$

$$2\pi \text{ radians} = 360^\circ$$

circumference $= 2\pi r$

- The radian measure of an angle is the **length of the arc** on the **unit circle** (centre at $(0,0)$ and radius 1).

$90^\circ = \frac{\pi}{2}$ radians , $180^\circ = \pi$ radians

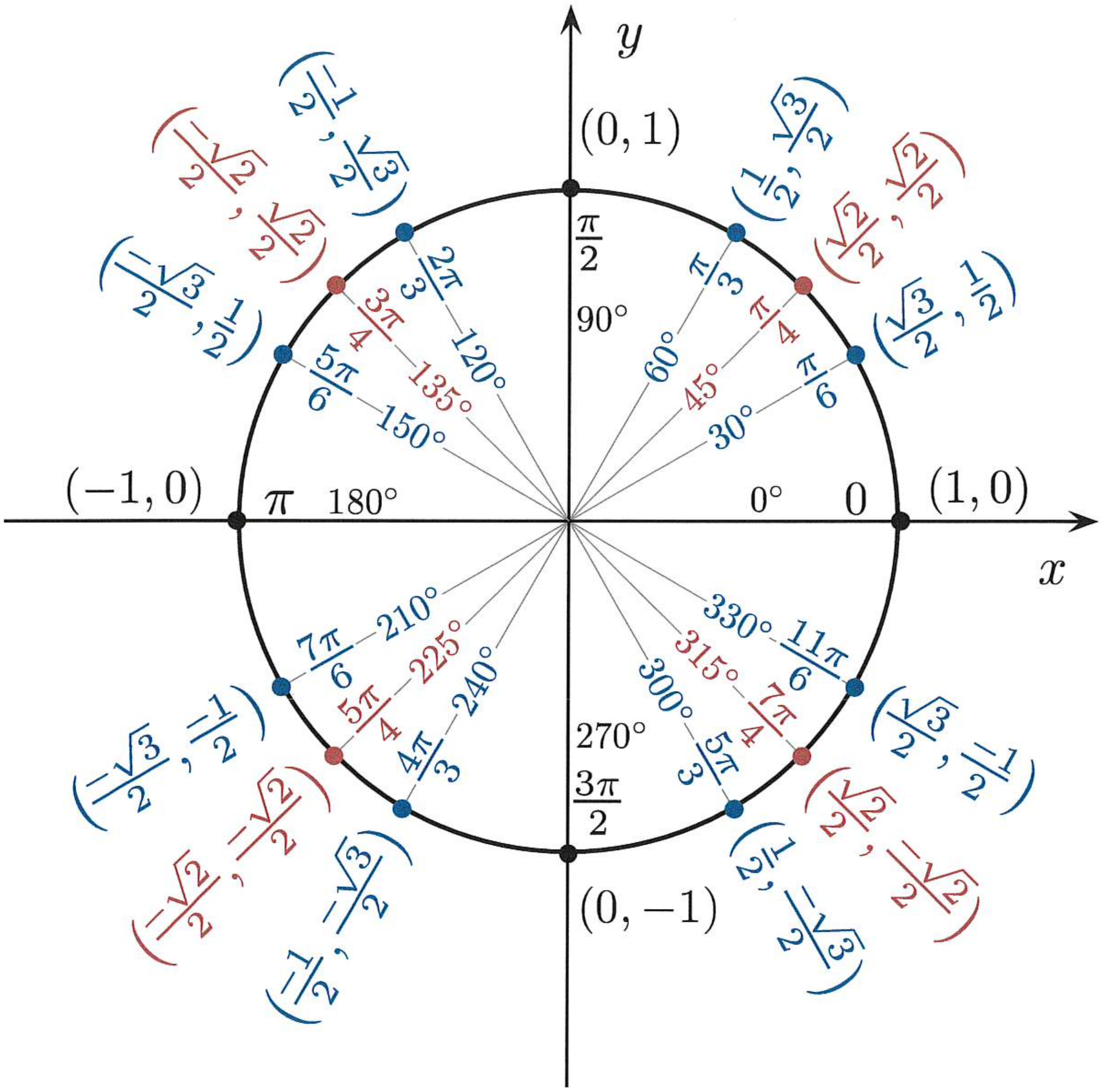


Image from `https://upload.wikimedia.org/wikipedia/commons/4/4c/Unit_circle_angles_color.svg`

Trigonometric Functions

Trigonometry helps us understand angles, triangles, and circles through the use of special **trigonometric functions**.

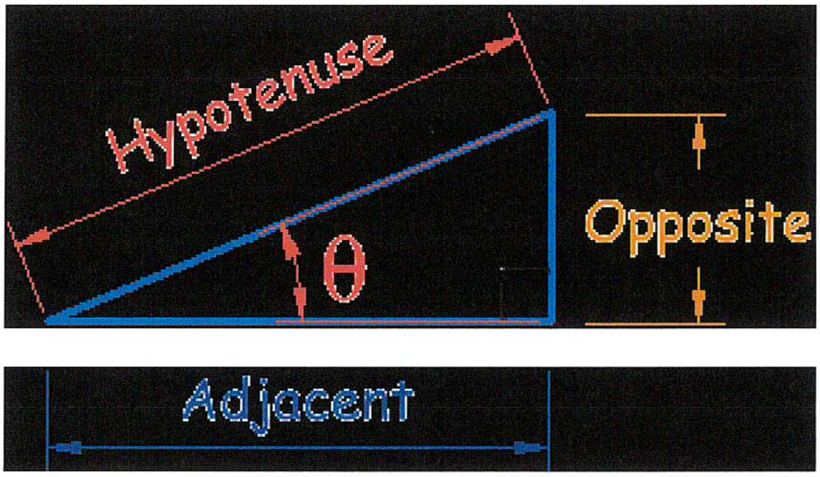


Image from https://upload.wikimedia.org/wikipedia/commons/3/39/Trigonometry_triangle.png

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

Remember: SOH — CAH — TOA

Definition 2.9 *The three (main) trigonometric functions*

Consider the circle with radius r centred at $(0,0)$.

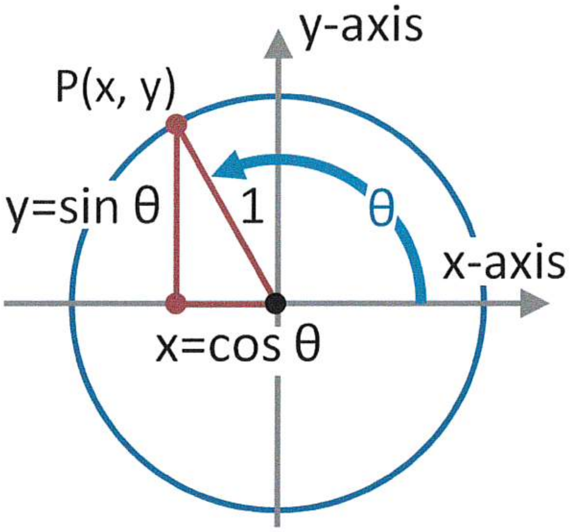


Image from Wikipedia

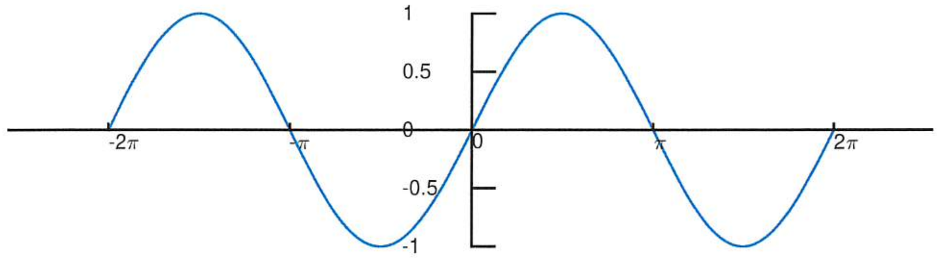
* $\boxed{\text{sine:} \quad \sin\theta = \frac{y}{r} \qquad \text{cosine:} \quad \cos\theta = \frac{x}{r}}$ *

tangent: $\tan\theta = \dfrac{y}{x}$

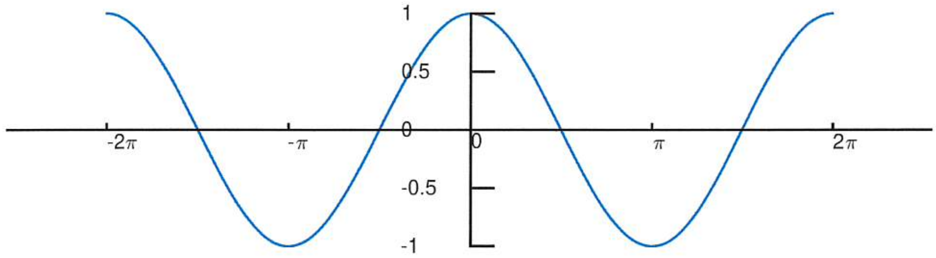
□

Graphs of Trigonometric Functions

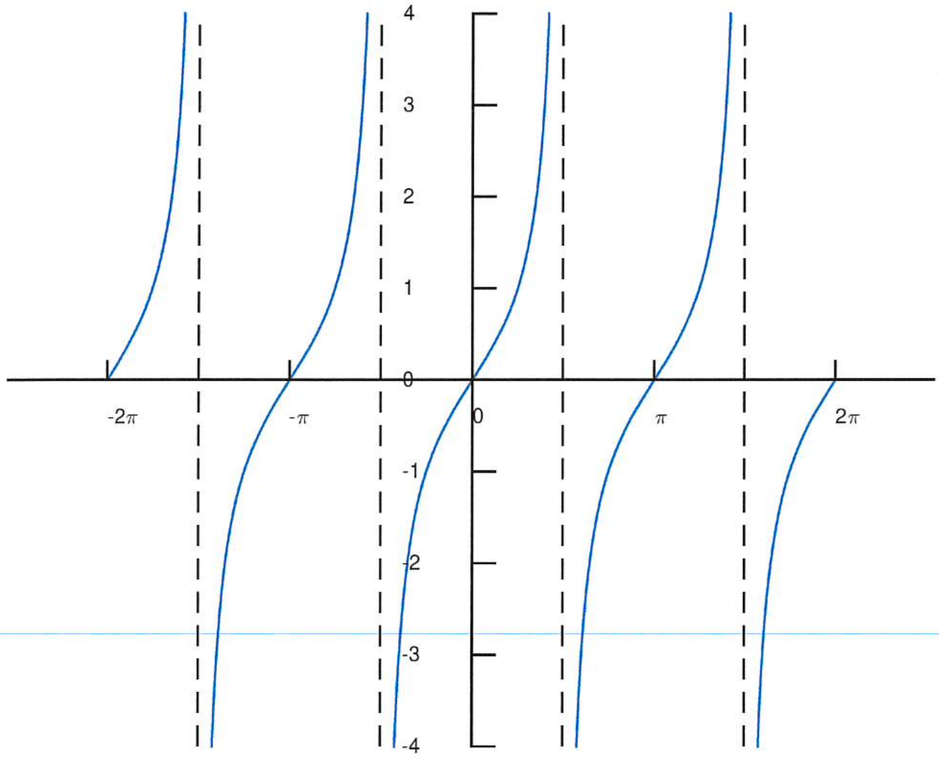
$\sin\theta$



$\cos\theta$



$\tan\theta$



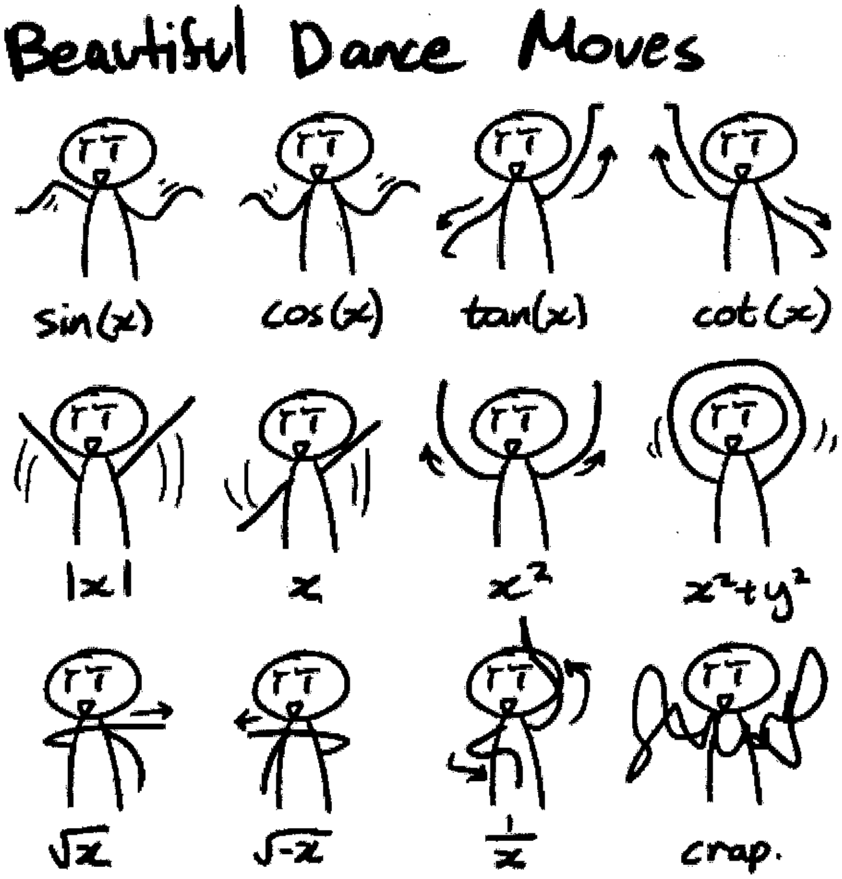


Image from

http://www.calculushumor.com/uploads/1/2/0/2/12023481/1701393_orig.jpg

Vector has magnitude and direction
$\downarrow$ r $\downarrow$ θ

Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O along the positive x-axis.

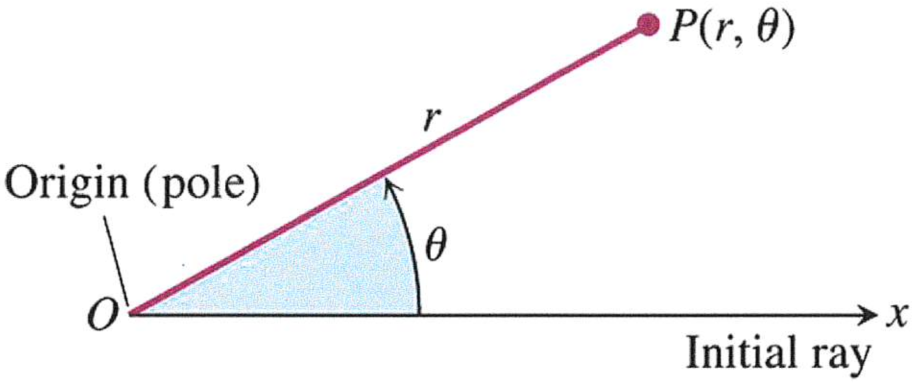


Image from Hass 1/e p588

Notes —

- Then each point P can be located by assigning to it a

polar coordinate pair (r, θ)

- r gives the directed distance from O to P.
- θ gives the directed angle from the positive x-axis to OP.
- As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise.

$(r, \theta) \longleftrightarrow (x, y)$

world world

- *Equations relating Polar and Cartesian coordinates*

$$x = r \times \cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r \times \sin\theta \qquad \tan\theta = \frac{y}{x}$$

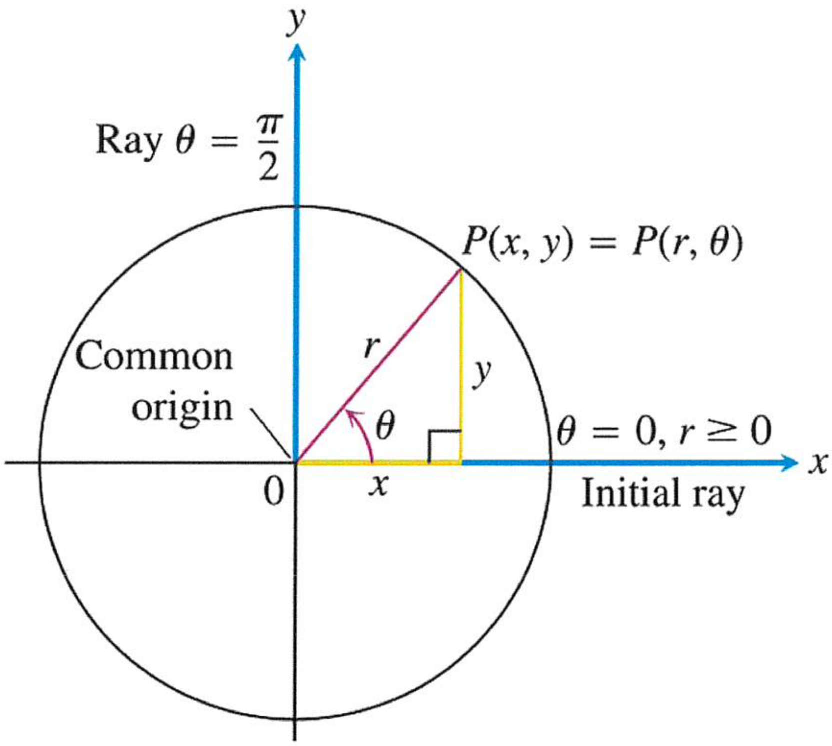


Image from Hass 1/e p589

- Instead of using $\tan\theta = \frac{y}{x}$
  in a programming language (Python, R, etc)
  we use $\theta = \text{atan2}(y, x)$

2.2.3 Vector “Dot” Product

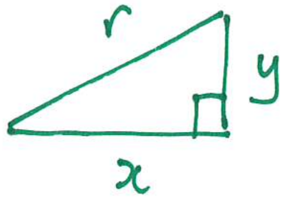
sometimes called “scalar” product

Question —

Can we meaningfully **multiply** vectors?

- If P is the point (x, y) in 2D, then the line segment OP has length $\sqrt{x^2 + y^2}$.

  Pythagoras

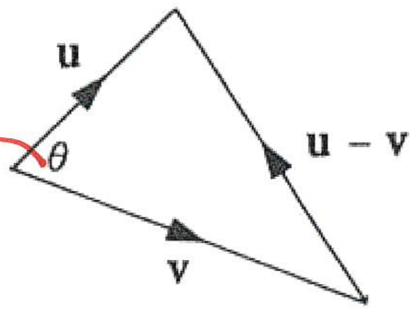
  

  We say that the vector $\boldsymbol{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ has **magnitude**

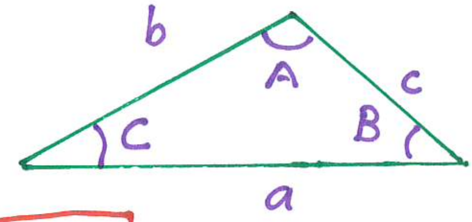
  $$|\boldsymbol{p}| = \sqrt{x^2 + y^2}$$

- The $\boxed{\text{angle}}$ between two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be found using the **cosine rule**.

want to find θ



- Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.



$c^2 = a^2 + b^2 - 2ab\cos C$

Cosine Law

- The $\boxed{\text{cosine rule}}$ (from trigonometry) gives

$$\cos\theta = \frac{|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - |\boldsymbol{u}-\boldsymbol{v}|^2}{2|\boldsymbol{u}||\boldsymbol{v}|}$$

- But

$|\underset{\sim}{u}|$ = magnitude of $\underset{\sim}{u}$

$$\begin{aligned} &|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - |\boldsymbol{u}-\boldsymbol{v}|^2 \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 \\ &= 2(u_1 v_1 + u_2 v_2) \end{aligned}$$

$$\boxed{\cos\theta = \frac{u_1 v_1 + u_2 v_2}{|\boldsymbol{u}||\boldsymbol{v}|}}$$

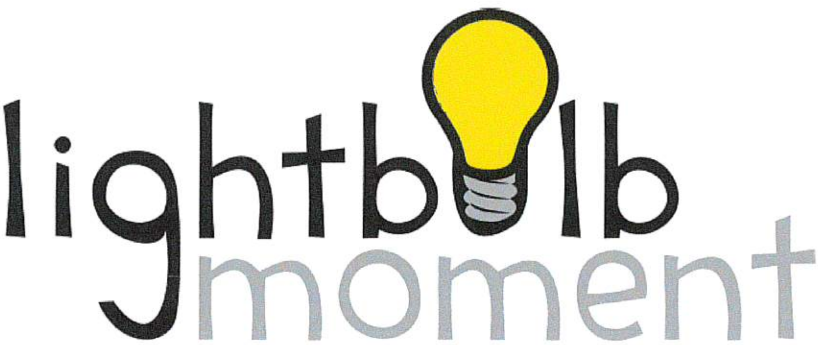


Image from http://www.coroflot.com/

This quantity

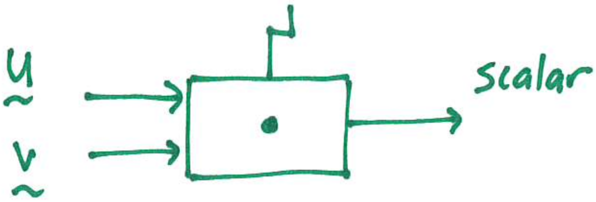
$$u_1 v_1 + u_2 v_2$$

seems to be very useful.

It looks like

multiplying the corresponding components
(so some kind of vector multiplication)

<u>but</u> then summing over all the components.



Definition 2.10 *Dot product of vectors*

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then

$$\boldsymbol{u} \bullet \boldsymbol{v} \;=\; u_1 v_1 + u_2 v_2$$

Warning! $\boldsymbol{u} \bullet \boldsymbol{v}$ is an ordinary number (**not** a vector). □

- Remember that if $\boldsymbol{a} = \begin{bmatrix} x \\ y \end{bmatrix}$ then $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$.

  Then the **magnitude** of $\boldsymbol{a}$, written $|\boldsymbol{a}|$, is simply

$$|\boldsymbol{a}| \;=\; \sqrt{\boldsymbol{a} \bullet \boldsymbol{a}}$$

- Vector $\boldsymbol{u}$ is called a **unit vector** if $|\boldsymbol{u}| = 1$, i.e., it has magnitude of 1.

- The angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boxed{\cos\theta \;=\; \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}}$$

  and therefore

$$\boldsymbol{u} \bullet \boldsymbol{v} \;=\; |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$$

  which is the usual form we see in physics and engineering applications.

Idea —

- Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-zero ($|\boldsymbol{u}| \neq 0$ and $|\boldsymbol{v}| \neq 0$) and

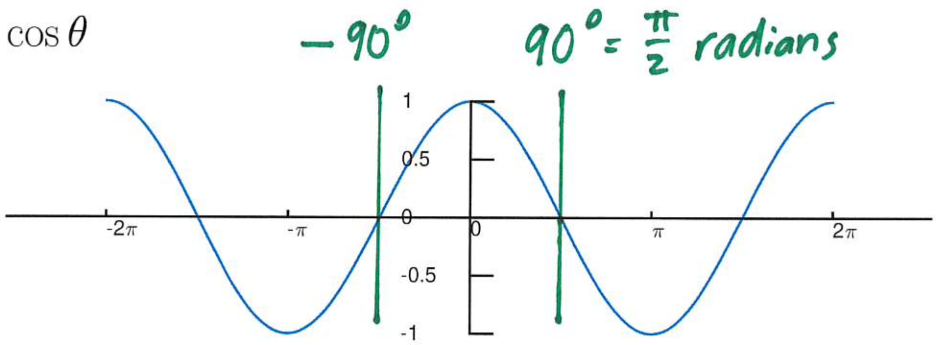
$$\boldsymbol{u} \bullet \boldsymbol{v} = 0$$

- Then

$$|\boldsymbol{u}||\boldsymbol{v}| \cos\theta = 0$$

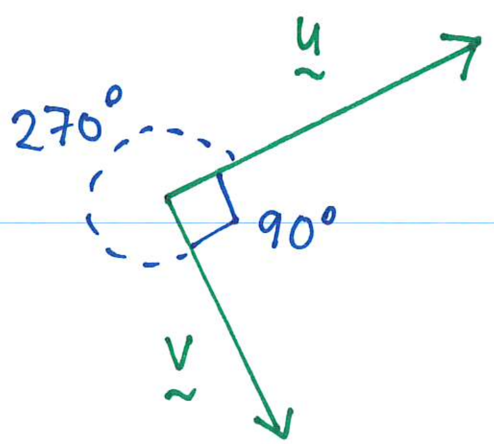
- Since $|\boldsymbol{u}| \neq 0$ and $|\boldsymbol{v}| \neq 0$, we must have

$$\cos\theta = 0$$



- Therefore $\theta = \frac{\pi}{2}$ or $90°$, i.e., $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** (perpendicular).

→ $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$



Definition 2.11 Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** if

$$\boldsymbol{u} \bullet \boldsymbol{v} = 0$$

$\square$

Example 2.11 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Then

$$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 4 \end{bmatrix} = 2 \times (-2) + 1 \times 4 = -4 + 4 = 0$$

scalar

Therefore the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *orthogonal.*

$\square$

One more useful idea —

- Let $k = |\boldsymbol{a}|$.
- Then the scalar product $(\frac{1}{k})\,\boldsymbol{a}$ is a unit vector with the same direction as $\boldsymbol{a}$.

2.2.4 Vectors in 3D

A **vector** (in 3D) is an ordered triple of numbers written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- The numbers x, y and z are called **components** of the vector.
- We have a similar geometric representation in 3D (position vectors and free vectors).
- The definitions of equality, vector addition, and scalar multiplication, etc, are all similar to the 2D cases.
- The vector equation for a line is the same.

$$\boldsymbol{v}(t) = \boldsymbol{p} + t\boldsymbol{u}$$

- If Q is the point (x, y, z) in $\mathbb{R}^3$, then the line segment OQ has length $\sqrt{x^2 + y^2 + z^2}$.
- We say that the vector $\boldsymbol{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has **magnitude**

$$|\boldsymbol{q}| = \sqrt{x^2 + y^2 + z^2}$$

- If $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

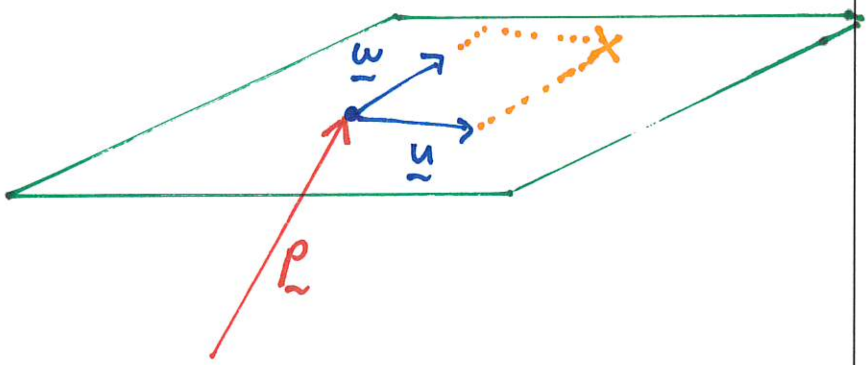
$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and all the properties for dot products in 2D follow.

■ *Vector equation for a plane* (in 3D)

Suppose non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ are not parallel, i.e., $\boldsymbol{u} \neq k\boldsymbol{w}$. A vector equation for the plane through P containing $\boldsymbol{u}$ and $\boldsymbol{w}$ is

$$\boldsymbol{v}(s,t) = \boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{w}$$



Notes —

- For any point P and any non-zero non-parallel vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ there is a **unique** plane through P containing $\boldsymbol{u}$ and $\boldsymbol{w}$.
- We call $\boldsymbol{u}$ and $\boldsymbol{w}$ the **direction vectors** for the plane.
- The vector $\boldsymbol{p}$ is the position vector of some point in the plane.
- The vector $\boldsymbol{v}$ is the position vector of a point in the plane. The notation $\boldsymbol{v}(s,t)$ emphasises that s and t are variables or parameters.
- As s and t range through the set of real numbers, the end-point of the vector $\boldsymbol{p} + s\boldsymbol{u} + t\boldsymbol{w}$ traces out the whole plane.

Example 2.12 Find a vector equation for the plane containing the points

$$P(-2, 4, 1) \qquad Q(1, 0, 2) \qquad R(3, -1, 1)$$

Solution.

Let $\quad \boldsymbol{u} = \overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$

Let $\quad \boldsymbol{w} = \overrightarrow{PR} = \boldsymbol{r} - \boldsymbol{p} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$

$$\boldsymbol{v}(s, t) = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}$$

$\square$

A **normal vector** for a plane is a vector that is orthogonal (perpendicular) to every vector in that plane.

Consider a plane containing the point $P = (x_0, y_0, z_0)$ and let

$$\boldsymbol{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be a normal vector to the plane. All other vectors perpendicular to the plane are scalar multiples of $\boldsymbol{n}$ (since they are parallel to $\boldsymbol{n}$).

A point $Q(x, y, z)$ lies on this plane if the vector $\overrightarrow{PQ}$ is perpendicular to $\boldsymbol{n}$, i.e.,

$$\overrightarrow{PQ} \bullet \boldsymbol{n} = 0$$

But

$$\overrightarrow{PQ} \bullet \boldsymbol{n} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a(x - x_0) + b(y - y_0) + c(z - z_0)$$

So $Q(x, y, z)$ lies in the plane if

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This equation is called the **normal equation of the plane**.

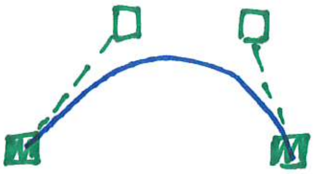
Note that this can be rearranged as

useful equation

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$\boxed{ax + by + cz = d} \qquad \text{looks more familiar}$$

► Two planes are **parallel** if their normal vectors are parallel vectors.



cubic Bézier curve

2 end points

2 control points

We can also model 3D surfaces using Bézier patches, e.g., the Utah teapot (see `https://en.wikipedia.org/wiki/Utah_teapot`) is a famous example from Computer Graphics.

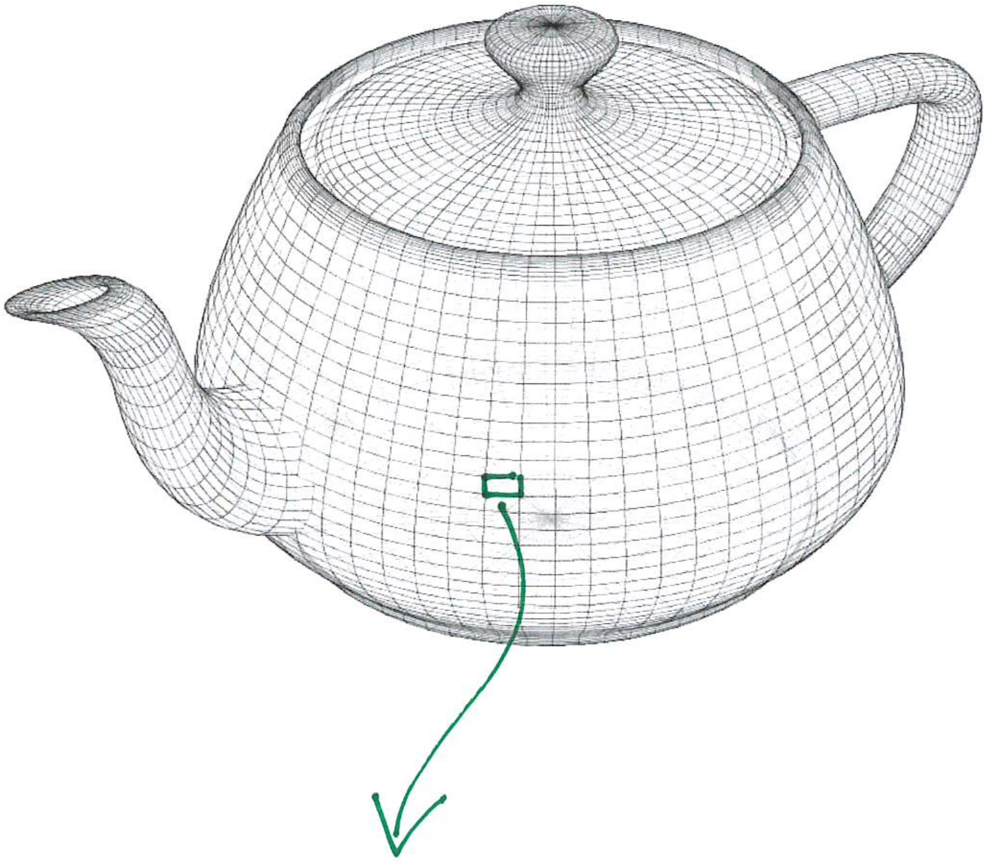
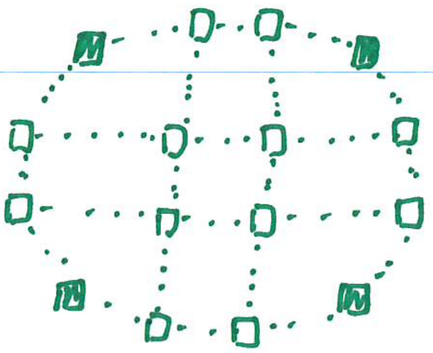


Image from `https://github.com/rm-hull/wireframes`



cubic Bézier patch

4 corner points

12 control points

Summary

- A *line segment* is defined by two endpoints (vectors), $\boldsymbol{p}$ and $\boldsymbol{q}$.

  The line segment is the set of all points (vectors)

  $$\boldsymbol{a} = (1-t)\boldsymbol{p} + t\boldsymbol{q}$$

  as t goes from 0 to 1.

- A *quadratic Bézier curve* is defined by two endpoints (vectors), $\boldsymbol{p}$ and $\boldsymbol{q}$, and one control point (vector), $\boldsymbol{c}$.

  The curve is the set of all points (vectors)

  $$\boldsymbol{a} = (1-t)^2\boldsymbol{p} + (2 \times (1-t) \times t)\boldsymbol{c} + t^2\boldsymbol{q}$$

  as t goes from 0 to 1.

  The control point defines the **slope** of the curve at the endpoints.

- Compass directions
  - North $\uparrow$, East $\rightarrow$, South $\downarrow$, West $\leftarrow$
  - bearing in degrees clockwise from North

- Angles (degrees and radians)
  - positive is anticlockwise, negative is clockwise
  - 2π radians is 360 degrees

$$\frac{\text{angle in radians}}{2\pi} = \frac{\text{angle in degrees}}{360}$$

- Trigonometric functions

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

- Polar coordinates (r, θ)

$$\begin{aligned} x &= r \times \cos\theta & \qquad r &= \sqrt{x^2 + y^2} \\ y &= r \times \sin\theta & \qquad \tan\theta &= \frac{y}{x} \end{aligned}$$

- If $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then the

vector **dot** product

of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\begin{aligned} \boldsymbol{u} \bullet \boldsymbol{v} &= u_1 v_1 + u_2 v_2 \\ \boldsymbol{u} \bullet \boldsymbol{v} &= |\boldsymbol{u}||\boldsymbol{v}| \cos\theta \\ \cos\theta &= \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} \end{aligned}$$

- Magnitude of a vector is $|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \bullet \boldsymbol{u}}$
- Orthogonal (perpendicular) vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have $\boldsymbol{u} \bullet \boldsymbol{v} = 0$
- Unit vector $\boldsymbol{u}$ has $|\boldsymbol{u}| = 1$
- Vector dot product in 3D

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- Planes in 3D

$$\boldsymbol{v}(s, t) = \boldsymbol{p} + s\,\boldsymbol{u} + t\,\boldsymbol{w}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$