2.5 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are among the most useful topics in linear algebra. They are used in several areas of mathematics, physics, mechanics, electrical and nuclear engineering, hydrodynamics and aerodynamics, etc. In fact, it is rather odd to find an applied area of science where eigenvalues are never used.

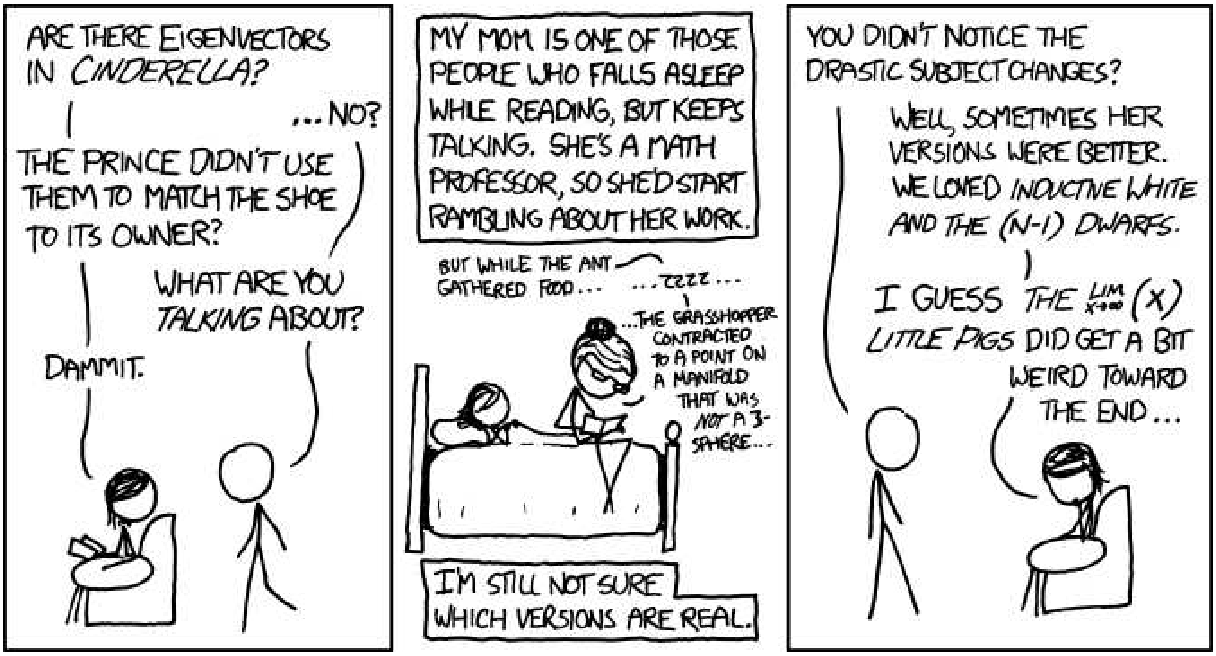

Image from https://xkcd.com/872/

2.5.1 What are Eigenvalues and Eigenvectors?

Suppose A is an $n \times n$ (square) matrix.

- Then $A\boldsymbol{v}$ is usually unrelated to $\boldsymbol{v}$.
- A very interesting case arises when $A\boldsymbol{v}$ happens to be a scalar multiple of $\boldsymbol{v}$, i.e., **parallel** to $\boldsymbol{v}$.
- So, geometrically, $\boldsymbol{v}$ and $A\boldsymbol{v}$ are on the same line through the origin.
- In such a case we will call $\boldsymbol{v}$ an "*eigenvector*" of A and the proportionality constant an "*eigenvalue*" of A.

Definition 2.17 Let A be an $n \times n$ (square) matrix.

A *nonzero* vector $\boldsymbol{v}$ is an **eigenvector** of A if for some scalar λ

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

The scalar λ (which may be zero) is called an **eigenvalue** of A *corresponding to* the eigenvector $\boldsymbol{v}$. □

Example 2.34 Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Show that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors of A.

What are the corresponding eigenvalues?

Solution.

- Because

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  we can wee that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A

  with corresponding eigenvalue $\lambda = 3$.

- Also

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

  so we can wee that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of A

  with corresponding eigenvalue $\lambda = -2$.

$\square$

Notes —

- Note that any nonzero scalar multiple of an eigenvector $\boldsymbol{v}$ is also an eigenvector because if $\boldsymbol{w} = r\boldsymbol{v}$ then

$$A\boldsymbol{w} = A(r\boldsymbol{v}) = r(A\boldsymbol{v}) = r(\lambda\boldsymbol{v}) = \lambda(r\boldsymbol{v}) = \lambda\boldsymbol{w}$$

  Furthermore, $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same eigenvalue.

- In Example 2.34 we see that the nonzero vectors on the lines ℓ_1 and ℓ_2 determined by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors of A.

- The linear transformation $A\boldsymbol{x}$ **stretches** the vectors along ℓ_1 by a factor of $\lambda = 3$.

- The vectors along ℓ_2 are first **reflected** about the origin and then **stretched** by a factor of 2.

Example 2.35 Find the eigenvalues and eigenvectors of A geometrically.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that $A\boldsymbol{x}$ is the <u>**reflection**</u> of $\boldsymbol{x}$ about the line $y = x$.

Solution.

The only vectors that remain on the same line are all vectors along the lines $y = x$ and $y = -x$.

These without the origin are the only eigenvectors.

For $\boldsymbol{v}$ along $y = x$ we have $A\boldsymbol{v} = 1\boldsymbol{v}$, so $\boldsymbol{v}$ is an eigenvector with corresponding eigenvalue 1.

For $\boldsymbol{v}$ along $y = -x$ we have $A\boldsymbol{v} = -1\boldsymbol{v}$, so $\boldsymbol{v}$ is an eigenvector with corresponding eigenvalue -1.

$\square$

2.5.2 Diagonalisation

Let A be a square matrix.

- The **main diagonal** of A is the entries on the diagonal from top-left to bottom-right.
- A is **upper triangular** if all entries below the main diagonal are zero.
- A is **lower triangular** if all entries above the main diagonal are zero.
- A is **diagonal** if all entries above and below the main diagonal are zero.

Result 2.18 The eigenvalues of a triangular (or diagonal) matrix are the entries on its main diagonal. $\square$

Definition 2.19 Let A and B be two $n \times n$ (square) matrices. We say that

$$B \;\underline{\textbf{is similar to}}\; A$$

if there exists an invertible matrix P such that

$$B = P^{-1}AP \qquad \square$$

Terminology —

- If an $n \times n$ matrix A is similar to a *diagonal* matrix D, it is

  <u>diagonalisable</u>.

- This means that there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is a diagonal matrix D, i.e.,

$$P^{-1}AP = D$$

- The process of finding matrices P and D as described is called *diagonalisation*.
- We say that P and D *diagonalise* A.

Example 2.36 Show that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

is diagonalised by

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$P^{-1} = \frac{1}{1-0}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

□

▶ Note that

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

also diagonalise A. *Check this for yourself.*

Result 2.20 Let A be an $n \times n$ (square) matrix.

If A is diagonalisable with $P^{-1}AP = D$, then

- the columns of P are eigenvectors of A and
- the diagonal entries of D are the corresponding eigenvalues.

$\square$

Result 2.21 Any $n \times n$ matrix A with n distinct eigenvalues is diagonalisable. $\square$

Summary

- Let A be an $n \times n$ (square) matrix.
  - A *nonzero* vector $\boldsymbol{v}$ is an **eigenvector** of A if for some scalar λ

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

  - The scalar λ (which may be zero) is called an **eigenvalue** of A *corresponding to* the eigenvector $\boldsymbol{v}$.
- Diagonalisation of a square matrix A means

$$P^{-1}AP = D$$

where D is a diagonal matrix and

  - eigenvectors as columns of P
  - corresponding eigenvalues on the diagonal of D