Lecture to Summer Schools

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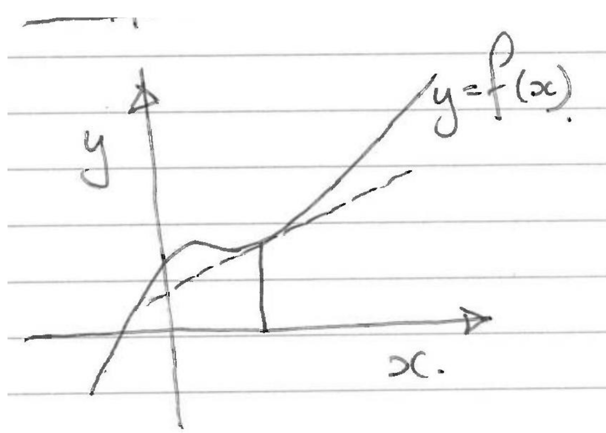
A long time ago

First need to start with some of the basic mathematical tools.

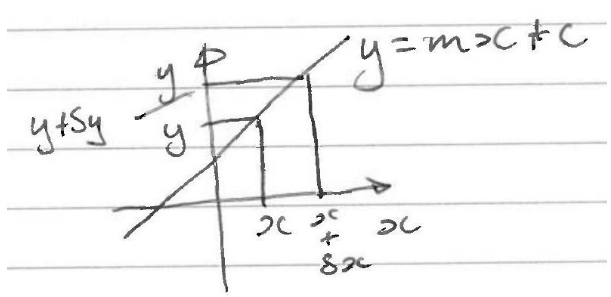
1 Differentiation



Differentiation is just finding the gradient of a line at any point.

1.1 Consider some examples

Let's start with the simple case of a straight line.

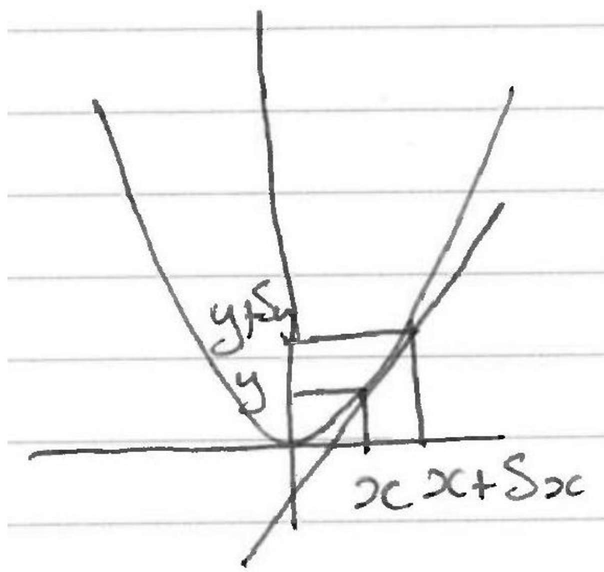


$$\begin{aligned}\frac{\delta y}{\delta x} &= \frac{f(x+\delta x)-f(x)}{\delta x}\\ &= \frac{m(x+\delta x)+c-(mx+c)}{\delta x}\\ &= \frac{mx+m\delta x+c-mx-c}{\delta x}\\ &= m\end{aligned}$$

No Surprise!

Note constant has no effect.

Lets try $y = x^2$.



$$\begin{aligned}\frac{\delta y}{\delta x} &= \frac{(x+\delta x)^2 - x^2}{\delta x}\\ &= \frac{x^2 + 2x\delta x + \delta x^2 - x^2}{\delta x}\\ &= \frac{2x\delta x + \delta x^2.}{\delta x}\\ &= 2x + \delta x.\end{aligned}$$

$$\text{in } \mathcal{L}_{\delta x \to 0} \frac{\delta y}{\delta x} \to \frac{dy}{dx} = 2x(+\cancel{\delta x}^{\,0})$$

$$\therefore \frac{dy}{dx} = 2x \quad \text{for } y = x^2.$$

Can show that for $y = x^n$ (where n is +ve or -ve or fractional) is $\frac{dy}{dx} = nx^{n-1}$. Also differentiation is additive

So $\frac{d}{dx}(x^3 + 2x^2 + 15x + 6) = 3x^2 + 4x + 15$

Other examples:

$$\frac{d\sqrt{x}}{dx} = \frac{dx^{1/2}}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d(1/x)}{dx} = \frac{dx^{-1}}{dx} = -1x^{-2} = -\frac{1}{x^2}$$

Other functions:

$$\frac{d\sin x}{dx} = \cos x.$$

$$\frac{d\cos x}{dx} = -\sin x.$$

Also $\frac{d\sin f(x)}{dx} = f'(x)\cos f(x)$ where $f'(x) = \frac{df(x)}{dx}$

$$\text{so } \frac{d\sin \omega x}{dx} = \omega\cos\omega x.$$

Special example.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\frac{de^x}{dx} = e^x$$
$$\frac{de^{f(x)}}{dx} = f'(x)e^{f(x)}.$$

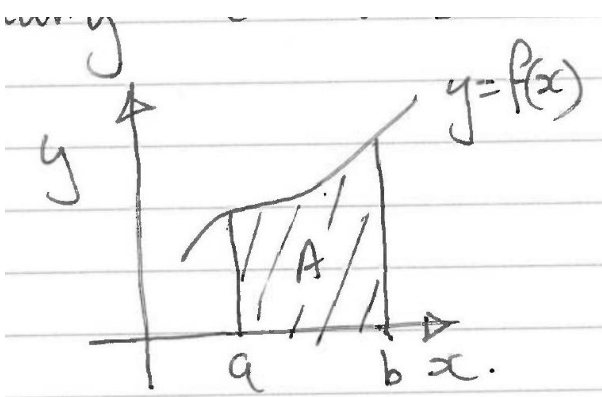
Note $\ln(e^x) = x$ also $\frac{d\ln(x)}{dx} = \frac{1}{x}$

2 Integration

Is the opposite to differentiation.

$$\frac{dy}{dx} = f(x)$$
$$\text{Old fashioned "S"} \rightarrow \int dy = \int f(x)dx.$$
$$y = \int f(x)dx \leftarrow \text{ indefinite integral}$$

Actually a measure of the area.



$$A = \int_a^b f(x)dx.$$

Definite integral.

3 Solving (Ordinary) Differential Equations (O.D.E.)

Note indefinite integral only defined up to a constant so if $\frac{dy}{dx} = x^2$.

$$\int dy = \int x^2 dx$$
$$= \frac{x^3}{3} + \underline{\underline{c}}$$

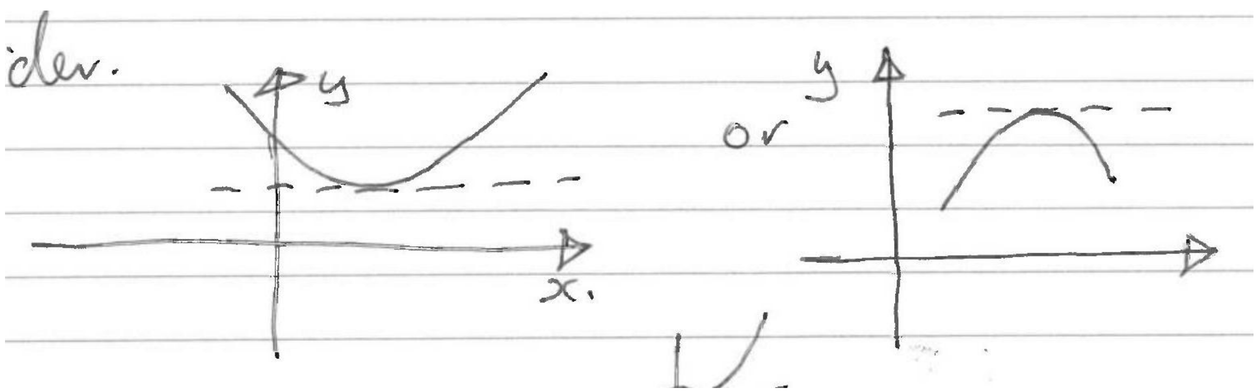
4 Second Order O.D.E.s

Differential of a differential.

$$\frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2 f(x)}{dx^2} = f''(x)$$

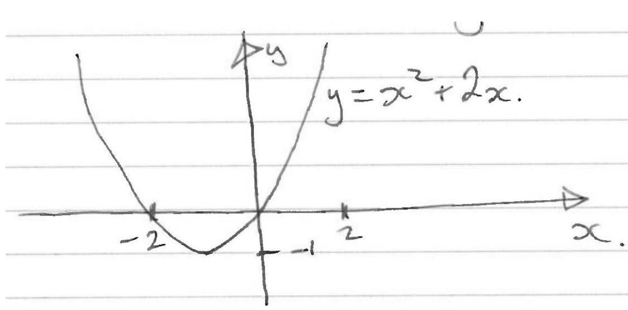
4.1 Maxima and minima

Consider $\frac{dy}{dx}$ for the following cases:



In all case $\frac{dy}{dx} = 0$
if $\frac{d^2y}{dc^2}$ is +ve it is a min
is -ve it is a max
is O can be max or min, or point of inflection.

For example $y = x^2 + 2x$



$$\frac{dy}{dx} = 2x + 2 = 0 \text{ a min.}$$
$$\therefore \quad x = -1$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2$$

i.e. +ve so a minimum

5 Now Consider a physical situation

5.1 Newton's Second Law

$$\mathbf{F} = m\mathbf{a}$$

F Depends on the physics, m is just a constant, so let's consider **a** (will also ignore that it is a vector so dropthe bold font).

$$a = \text{rate of change of velocity} = \frac{dv}{dt}$$

but v is rate of change of position.

$$v = dx/dt$$
$$\therefore \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$
$$\therefore \quad F = m\frac{d^2x}{dt^2} = m\ddot{x}$$

Aside

Can always write a second order O.D.E. and two first order O.D.E.s

$$\frac{d^2x}{dt^2} = F(x)$$

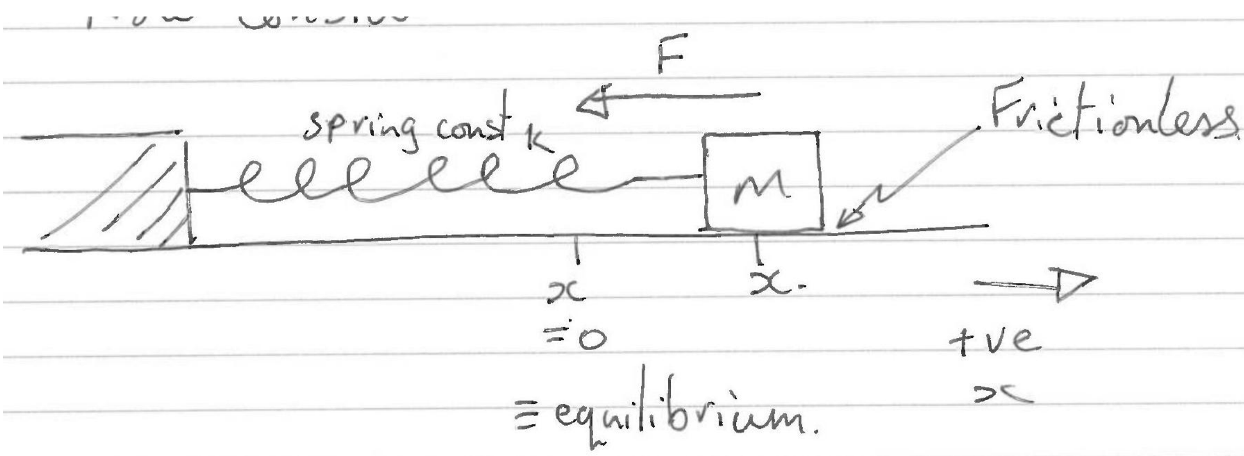
is the same as

$$\left.\begin{aligned} \tfrac{dv(x)}{dt} &= F(x) \\ \tfrac{dx}{dt} &= v(x) \end{aligned}\right\}$$

Something that will be very useful when you are simulating physics.

5.2 Oscillators (Simple Harmonic Motion)

Now consider



$$F = -kx$$

- because it is always in a direction to return to the equilibrium position

$$2^{\text{nd}} \text{ law} \qquad m\tfrac{d^2x}{dt^2} = F = -kx.$$

So lets try
$x = \cos\omega t$

$$\frac{dx}{dt} = -\omega \sin\omega t$$
$$\frac{d^2x}{dt^2} = -\omega^2 \cos\omega t$$

So

$$-m\omega^2 \cos\omega t = -kx \cos\omega t$$
$$\Rightarrow \omega^2 = k/m$$
$$\text{or } \omega = \sqrt{\mathrm{k}/m}.$$

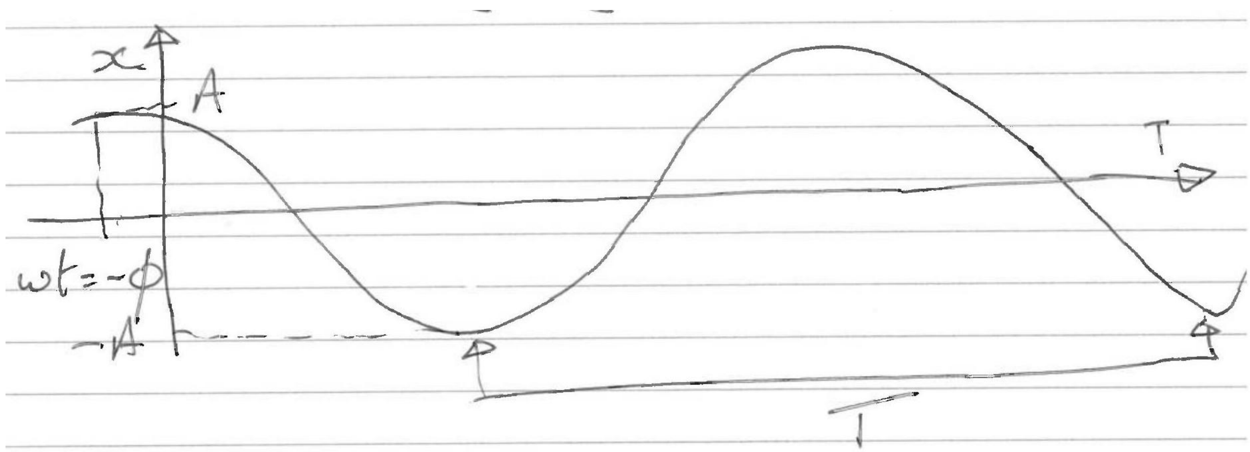
So what sort of motion is this? Well, in fact could have been. more general and had.

$$x = A\cos(\omega t + \phi)$$

where A is the amplitude, and ϕ is a phase.
Now

$$\frac{dx}{dt} = -A\omega\cos(\omega t + \phi)$$

so everything still works!



$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= 2\pi\sqrt{m/k}.
\end{aligned}
$$

So frequency ν

$$\nu = \omega/2\pi$$

Note: Doesn't depend on amplitude.

5.2.1 Superposition

Could add solutions together.

e.g. $x = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2)$

A_1, A_2, ϕ_1 and ϕ_2 can have any value but ω is fixed by the physics.

5.2.2 Now lets consider energy

$$
\begin{gathered}
\text{if } x = A\cos(\text{cot} + \overset{0}{\cancel{\phi}}) = A\cos(\omega t) \text{ (setting } \phi \text{ to 0 with no loss of generality)} \\
\frac{dx}{dt} = -A\omega\sin\omega t \\
K.E = \frac{1}{2}mv^2 = \frac{A^2\omega^2 m\sin^2\omega t}{2}
\end{gathered}
$$

So K.E. Change throughout motion.
Now lets consider P.E. ($U(x)$)
When is $U = 0$?
Don't forget always offset so really ask when is U minimum?
Clearly when a equilibrium.
So calculate U

$F = -dU/dx$ (discuss this in terms of a valley).

$$U = -\int_0^x F dx = +\int_0^x kx dx.$$

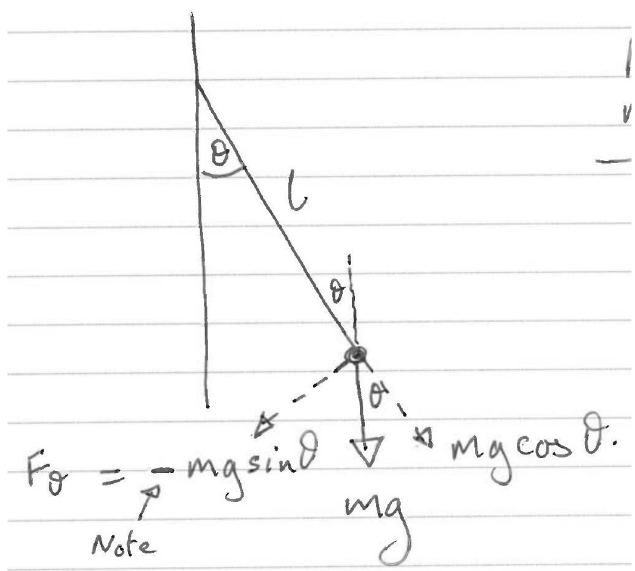
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$$

Note that S.H.M. parabolic potential $\mathbf{U = \frac{1}{2}kx^2}$.

So

$$\begin{aligned}
E_{\text{Tot}} &= K.E. + P.E. \\
&= \frac{1}{2}A^2\omega^2 m\sin^2(\omega t) + \frac{1}{2}A^2 k\cos^2\omega t \\
&= \frac{1}{2}A^2\omega^2 m\big(\cancelto{1}{\sin^2(\omega t) + \cos^2(\omega t)}\big) \\
&= \frac{1}{2}A^2\omega^2 m.
\end{aligned}$$

5.3 Now lets consider a pendulum



Ask about resolving forces.

Rotational movement equivalent of N.2.

So let's consider the rotational displacement $x_\theta = l\theta$ so then:

$$v_\theta = \dot{x_\theta} = l\dot{\theta}$$
$$a_\theta = \ddot{x_\theta} = l\ddot{\theta}$$

So now consider rotational Newton 2

$$F_\theta = ma_\theta = m\ddot{x_\theta} = ml\ddot{\theta}$$

Rotation equivalent of Force is torque τ

$$\tau = lF_\theta$$

Also, the rotational equivalent of mass is moment of inertia $I = ml^2$ so this leads to the rotational equivalent to Newton's 2^{nd} law:

$$\tau = I\ddot{\theta}$$

where I is the moment of inertia, τ is the torque and $\ddot{\theta}$ is the angular acceleration

so now lets look at the forces on a pendulum

$$\begin{aligned} I =& ml^2 \\ \tau =& lF_\theta \end{aligned}$$

$$lF_\theta = ml^2\ddot{\theta}$$

When $\theta << 1\text{rad} \sin\theta \approx \theta$

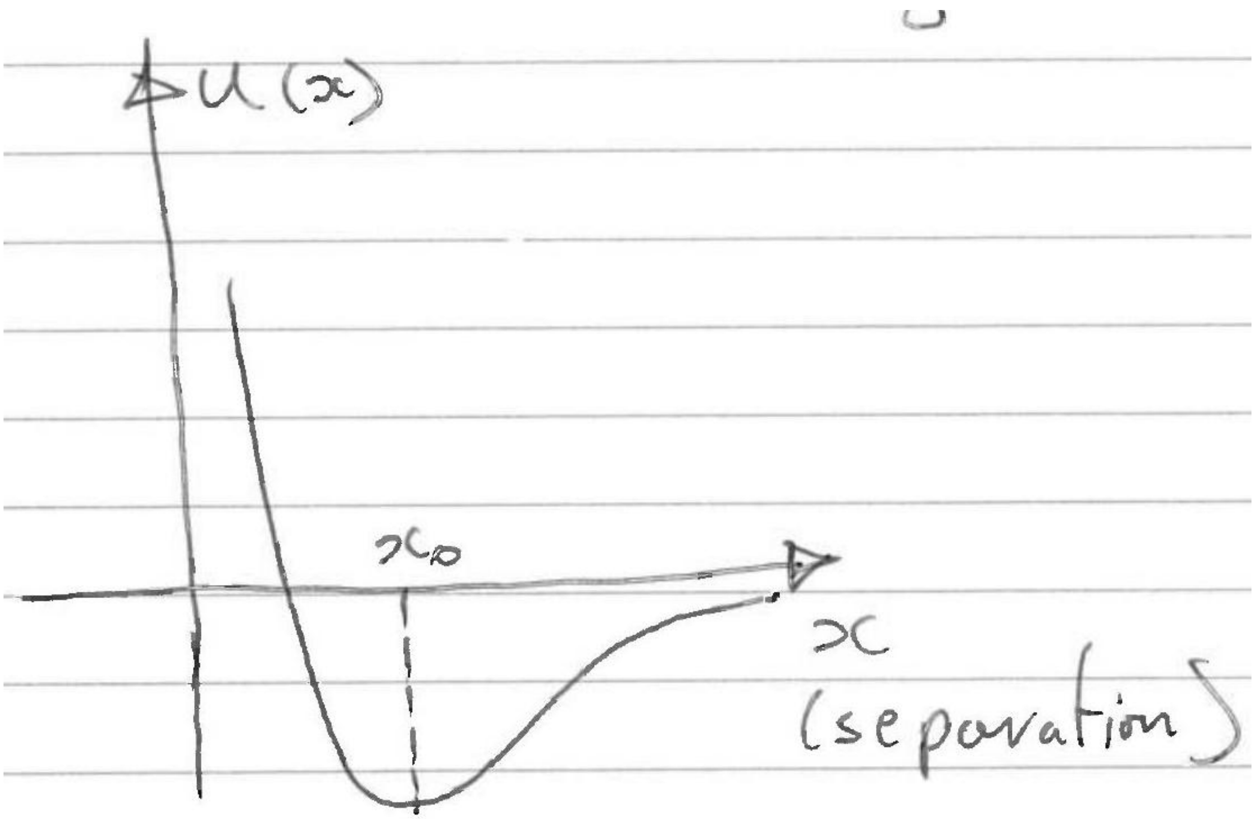
$$\begin{aligned} -gm\sin\theta &= ml\ddot{\theta} \\ -gm\theta &= ml\ddot{\theta} \\ \ddot{\theta} &= -\frac{g}{l}\theta \end{aligned}$$

$\therefore$ S. H.M. with $w = \sqrt{g/l}$.
or $T = 2\pi\sqrt{l/g}$.

- Note independent of mass (and amplitude as long as $\sin\theta \approx \theta$)

5.4 So why is S. H.M Everywhere?

Consider any potential that has an equilibrium.
E.g. Attraction between neutral molecules in gas



Consider particle held by potential around x_0

Taylor expansion

$$
\begin{aligned}
u\left(x_0+\delta x\right)=u\left(x_0\right)+\frac{1}{1!}\frac{du(x_0)}{dx}\delta x &+ \frac{1}{2!}\frac{d^2u(x_0)}{dx^2}\delta x^2 \\
&+ \frac{1}{3!}\frac{d^3u\left(x_0\right)}{dx\delta^3}\delta x^3 \\
&+ \cdots
\end{aligned}
$$

But $\frac{du(x_0)}{dx}=0$

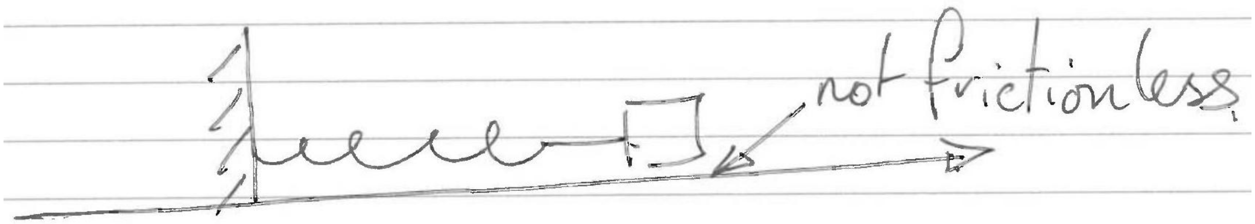
$$
\text{So } u\left(x_0+\delta x\right)\approx U\left(x_0\right)+\frac{1}{2}\frac{d^2u\left(x_0\right)}{dx^2}\delta x^2
$$

$U(x_0)$ is just a const offset and can be set to 0 so we have a parabolic. Note that $\frac{1}{2}\frac{d^2u(x_0)}{dx^2}$ is also a constant that depends on the physics.

$\therefore$ S.H.M. everywhere.

6 Just consider 2 more cases

6.1 Damped S.H.M.



$$
\begin{aligned}
&\text{Typically friction is } -bv \\
&\text{So} \\
&F_{\text{Tot}} = -kx - bv = -kx - b\dot{x}
\end{aligned}
$$

From N2.

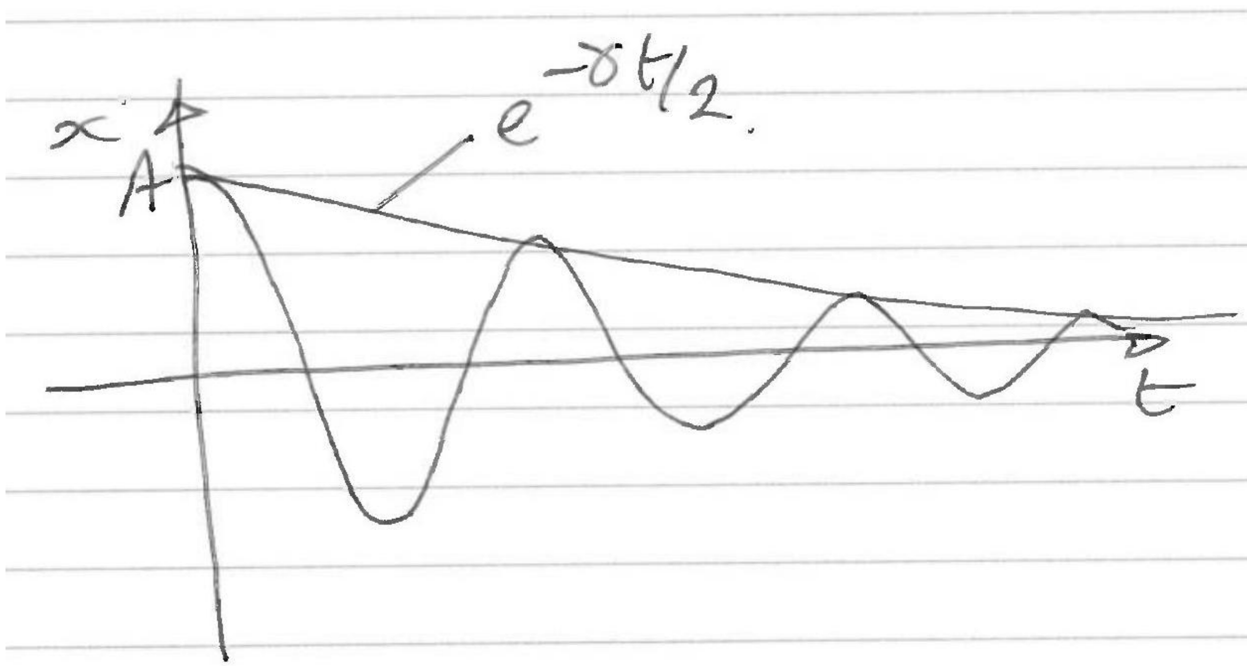
$$
\begin{aligned}
&m\ddot{x} + b\dot{x} + kx = 0 \\
&\text{or } \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \\
&\text{where} \\
&\gamma = b/m \\
&\omega_0 = k/m \text{ "natuval" S. H.M } \omega
\end{aligned}
$$

Now actual behaviour depends on level of damping.
but in light damping case.

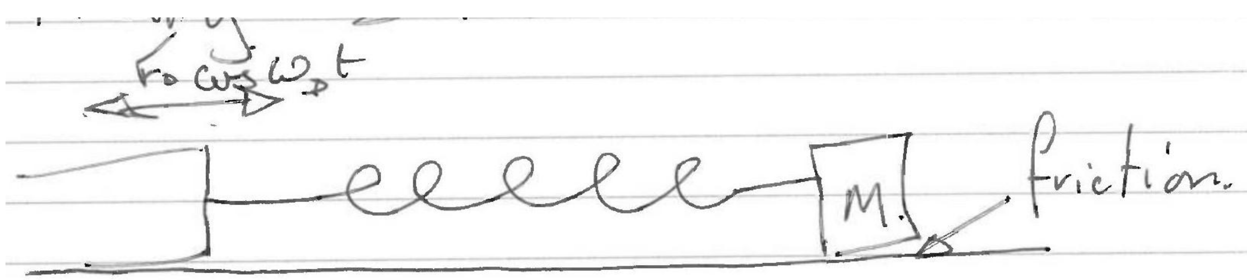
$$x = Ae^{-\gamma t/2}\cos(\omega t + \phi)$$

where

$$
\begin{aligned}
\omega &= \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \\
&= \sqrt{k/m - \frac{b^2}{4m^2}}
\end{aligned}
$$



6.2 Finally Damped and Driven S.H.M.



$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F_D \cos(\omega_D t) \, .$$