Due: April 9

To be completed in groups of no more than four students

Q1. Consider a three-state security market and the following return matrix:

$$R = \begin{pmatrix} 1.005 & 1.005 & 1.005 \\ 0.97 & 1.01 & 1.05 \end{pmatrix}.$$

Furthermore, suppose that the probabilities of the three states are given by the vector $p = (0.25, 0.35, 0.4)$.

a) What are two (positive) stochastic discount factors consistent with the returns?

b) If a new security with payoff $X_3 = (100, 100, 110)$ is introduced, what two prices for this security exclude arbitrage?

c) This part (only) is <u>optional</u>. Can you determine all prices P_3 that exclude arbitrage?

d) Suppose that $P_3 = 100$. Find an SDF.

e) Can you find two risky portfolios that have the same expected return, but different return variance?

f) Similarly, find a risky portfolio that has the same expected return as the risk-free bond. Compute the covariance of this return with the SDF you computed in part d).

g) For each of the three returns, compute (i) the covariance of the return with the SDF and (ii) the excess expected return (i.e., in excess of the risk-free rate). Perform the same exercise for a portfolio with weights $(0.2, 0.3, 0.5)$ in the three assets.

h) Plot the four pairs of quantities calculated above on a two-dimensional graph. What shape does the plot appear to have?

Q2. Consider a one-period environment in which a risk-free asset with return $R_0 = 1.025$ is available, as well as three risky returns with the following first two moments:

$$\mathrm{E}[\vec{R}] = \begin{pmatrix} 1.10 \\ 1.12 \\ 1.07 \end{pmatrix}, \qquad \mathrm{Var}(\vec{R}) = \begin{pmatrix} 0.0600 & 0.0377 & 0.0259 \\ 0.0377 & 0.0950 & 0.0285 \\ 0.0259 & 0.0285 & 0.0700 \end{pmatrix}.$$

Consider now investing a $W = 2.5$M amount by taking $W_1 = 1$M, $W_2 = 2$M, and $W_3 = -1.5$M, with the remainder invested at the risk-free rate.

a) What are the portfolio weights in the four assets?

b) What are the expected return and expected excess return of the portfolio?

c) What are the variance and standard deviation of the return?

d) Consider now a portfolio given by weights w' with the risky weights being twice the weights w: $(w'_1, w'_2, w'_3) = 2 \times (w_1, w_2, w_3)$. What are the expected excess return and standard deviation for this portfolio?

e) For a given return R, the quantity $(E[R] - R_0)/\sigma(R)$ is known as the *Sharpe ratio* of the return. What are the Sharpe ratios of portfolios w and w'?

f) Consider now the return associated with a risky-investment weight $\hat{w} = (0.3, 0.3, 0.3)$. What are the covariance and correlation of the returns R and $\hat{R}$?

Q3. This question is optional. Consider an investor who consumes at dates 0 and 1 and perceives utility

$$U(C_0, C_1) = \log(C_0) + \delta \mathrm{E}[\log(C_1)],$$

with $\delta = 0.95$, from a consumption plan (C_0, C_1) (remember, at time 0, when decisions are made, C_1 is not known). Let us assume that the investor is currently planning to consume $C_0 = 100$, respectively $C_1 = (100, 110, 110)$. There are three states of the world, to which the investor assigns probabilities $p = (0.4, 0.2, 0.4)$. We also consider below two securities with returns $R_1 = (1.1, 1.1, 1.1)$ and $R_2 = (1, 1.15, 1.25)$ and a third security with payoff $X_3 = (100, 100, 110)$.

a) Taking the consumption plan as given, what is the agent's marginal utility for consumption in all states? Consequently, what is the agent's stochastic discount factor? Equivalently, what are her risk-neutral probabilities?

b) Suppose that the only asset available is the third one, paying X_3. What should its price P_3 be for the agent to choose to take a zero position in this asset?

c) Consider now only the returns R_1 and R_2 (security 3 no longer exists). For each of the two assets, would the agent be better off taking a (small) long or short position in the asset?

   [Hint: Consider a small trade, e.g., consume ε less at 0, invest this difference in asset i, and therefore consume εR_i more at 1. Would such a deviation increase the agent's utility for $\varepsilon > 0$ or $\varepsilon < 0$? Remember that, for small $\varepsilon > 0$, and a function f that reasonably behaved, $f(x + \varepsilon) > f(x)$ if $f'(x) > 0$. Alternatively: How does the market price compare to the value the agent places on the asset's payoff, give her consumption plan?]

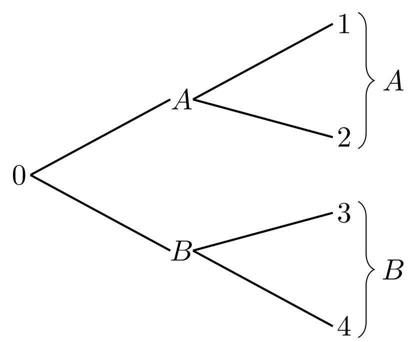
d) Here the point is to illustrate how agents trade to adjust consumption until their marginal utilities become valid SDFs.
   Suppose now that the agent trades freely in the two assets to maximize her utility. That is, she buys and sells optimally chosen amounts of the two securities. What are these amounts, and the resulting consumption plan?

[Hint: Let θ_i be the amount invested in each of the two securities. The objective of the investor is to solve

$$\max_{\theta_1,\theta_2} \log(C_0 - \theta_1 - \theta_2) + \delta \mathrm{E}[\log(C_1 + \theta_1 R_1 + \theta_2 R_2)].$$

Because the objective is concave in θ_1 and θ_2, the necessary first-order conditions are also sufficient.]

Q4. This question is optional. The goal to illustrate the details of the multi-period model in the simplest non-trivial setting possible. Thus, consider a 4-state world, $s \in \{1, 2, 3, 4\}$, where the state is known at date 2. The nature of the question is even more pedagogical than that of other questions. At date 1, everyone learns whether the true state of the world is in set $A = \{1, 2\}$ or in the complement, $B = \{3, 4\}$. The following diagram is meant to capture this set-up intuitively.



Consider two securities traded in this environment. Security 1 pays a dividend of one at both times 1 and time 2, for sure. Its (ex-dividend) prices are: $S_{1,1}(A) = 0.98$, $S_{1,1}(B) = 0.96$, and $S_{1,0} = 1.88$. Security 2 pays dividends $D_{2,2}(1) = 2$, $D_{2,2}(2) = 2.5$, $D_{2,2}(3) = 2$, $D_{2,2}(4) = 1.5$, $D_{2,1}(A) = 2$, and $D_{2,1}(B) = 2$, and has prices $S_{2,1}(A) = 2.1$, $S_{2,1}(B) = 1.6$, and $S_{2,0} = 3.6$. (To be clear about the notation, the first subscript indicates the security, the second subscript indicates the time, and the quantity in the parenthesis the event in which the value obtains at the given time.)

Finally, the probabilities of the four states are $p = (0.2, 0.3, 0.25, 0.25)$.

a) What is the probability that at time 1 event A is realized, i.e., $p(A)$? $p(B)$? Conditional on being in event A at time 1, what is the conditional probability of the true state being 2, $p(2|A)$? $p(3|A)$?

b) Let us now describe pricing in this economy in terms of risk-neutral probabilities and stochastic discount factors. First, let us note that, at every node of the tree, by trading in the two securities any payoff profile can be obtained at the two possible subsequent nodes. This means that the market is (dynamically) complete.

By definition, a risk-neutral probability q satisfies appropriate pricing equations at all times in all events. In this case, we have three such restrictions (one at time time 0 and two at time 1). In particular, at time 1 in each event E we must have

$$\begin{aligned} S_{i,1}(E) &= \mathrm{E}^Q\left[(D_{i,2}(s)+S_{i,2}(s))/R^0_{1,2}(E)|E\right] = \mathrm{E}^Q\left[D_{i,2}(s)|E\right]/R^0_{1,2}(E) \\ &= \sum\nolimits_s q(s|E)D_{i,2}(s)/R^0_{1,2}(E), \end{aligned}$$

where I used that $S_{i,2}=0$ — otherwise there's an arbitrage! — and denoted the risk-free return between 1 and 2 by $R^0_{1,2}(E)$. Note that the latter quantity may depend on the event E. A risk-free security may or not be traded in event E, but market completeness ensures that at least such a portfolio can be traded.

With the given dividend and price data, the conditional probabilities $q(s|E)$ can all be computed.

At time 0, the restriction on q can be written as a one-period restriction,

$$\begin{aligned} S_{i,0} &= \mathrm{E}^Q\left[(D_{i,1}(E)+S_{i,1}(E))/R^0_{0,1}\right] \\ &= \sum\nolimits_E q(E)(D_{i,1}(E)+S_{i,1}(E))/R^0_{0,1}, \end{aligned}$$

or as a full two-period restriction:

$$S_{i,0} = \mathrm{E}^Q\left[D_{i,1}(E)/R^0_{0,1} + (D_{i,2}(E)+S_{i,2}(E))/R^0_{0,2}(E)\right],$$

where $R^0_{0,2}(E) = R^0_{0,1}R^0_{1,2}(E)$ is the return to investing one unit at the risk-free at time 0 and rolling over the proceeds from 1 to 2. Note that this depends on the event realized at 1.

Task: Compute the risk-neutral probability q.

c) Perform the same exercise for the stochastic discount factor. That is, compute the process m_t such that

$$m_t S_{i,t} = \mathrm{E}\left[m_{t+1}(D_{i,t+1}+S_{i,t+1})|\mathcal{F}_t\right]$$

for all the possible information realizations by time t. Note that multiplying the process m by some constant will maintain the validity of the equations, which means that we can normalize the process if we so wish, e.g., by setting $m_0=1$. Just like the prices and dividends given above, the process must be measurable with respect to $\mathcal{F}_t$.

Q5. This question is optional. It is also quite mechanical. In class, we defined the risk-neutral probability measure, or equivalent martingale measure, by requiring that

$$G_t = \sum\nolimits_{s=0}^t \left(R^0_{0,s}\right)^{-1} D_s + \left(R^0_{0,t}\right)^{-1} S_t$$

be a martingale under Q:

$$\mathrm{E}_s^Q[G_t] = G_s$$

for all $0 \le s \le t \le T$. Intuitively, this is saying that all the value obtained from one share of the asset, from time 0 to time t, when "discounted" to time 0, is not predictable: its average change, conditional on information to date (time s), is zero. (That is, $\mathrm{E}_s^Q[G_t - G_s] = 0$.)

Show that this definition is equivalent to the following relation holding for all $t \ge 0$ and $t < \tau < T$:

$$S_t = \mathrm{E}_t^Q\left[\sum\nolimits_{s=t+1}^{\tau} \left(R_{t,s}^0\right)^{-1} D_s + (R_{t,\tau}^0)^{-1} S_\tau\right].$$

Note that, for $\tau = t+1$, we obtain the one-period relation

$$S_t = \mathrm{E}_t^Q\left[\left(R_{t,t+1}^0\right)^{-1}(D_t + S_{t+1})\right].$$