A Brief Introduction to Tensor-Train Interpolation

The tensor-train cross interpolation [1] of tensor $A(u_0, ..., u_{n-1})$ is of the form

$$\sum_{\mathcal{I},\mathcal{J}} \boldsymbol{A}(u_0, \mathcal{J}_1)[\boldsymbol{p}_0(\mathcal{I}_0, \mathcal{J}_1)]^{-1} \left(\prod_{\alpha=1}^{n-2} \boldsymbol{A}(\mathcal{I}_{\alpha-1}, u_\alpha, \mathcal{J}_{\alpha+1})[\boldsymbol{p}_\alpha(\mathcal{I}_\alpha, \mathcal{J}_{\alpha+1})]^{-1} \right) A(\mathcal{I}_{n-2}, u_{n-1}), \tag{1}$$

where $\mathcal{I} = \{\mathcal{I}_0, \ldots, \mathcal{I}_{n-2}\}$ and $\mathcal{J} = \{\mathcal{J}_1, \ldots, \mathcal{J}_{n-1}\}$. For each α, $\mathcal{I}_\alpha$ is a set of grouped indices, $\mathcal{I}_\alpha = \{(u_0^0, \ldots, u_\alpha^0), \ldots\}$ and $\mathcal{J}_\alpha = \{(u_\alpha^0, ..., u_{n-1}^0), \ldots\}$. $\mathcal{I}_\alpha$ and $\mathcal{J}_{\alpha+1}$ have the same length so that $\boldsymbol{p}_\alpha$ is square across each node. We require the nestness condition that,

$$\forall \alpha, \quad u_{\le\alpha} \in \mathcal{I}_\alpha \implies u_{\le\alpha-1} \in \mathcal{I}_{\alpha-1}, \quad u_{\ge\alpha-1} \in \mathcal{J}_{\alpha-1} \implies u_{\ge\alpha} \in \mathcal{J}_\alpha. \tag{2}$$

Eq. (1) can be graphically represented as matrix-product states (MPS) Algorithm for constructing the interpolation is consists of two parts. The first part is adding new pivots with the maximum-volume principle for each node. We require f to evaluate the tensor component and the position of the node, α. The next step is to connect all the nodes while preserving the nestness condition. We require initial rank-1 I and J, which may be generated from random numbers. Also, tolerance and number of sweeps.

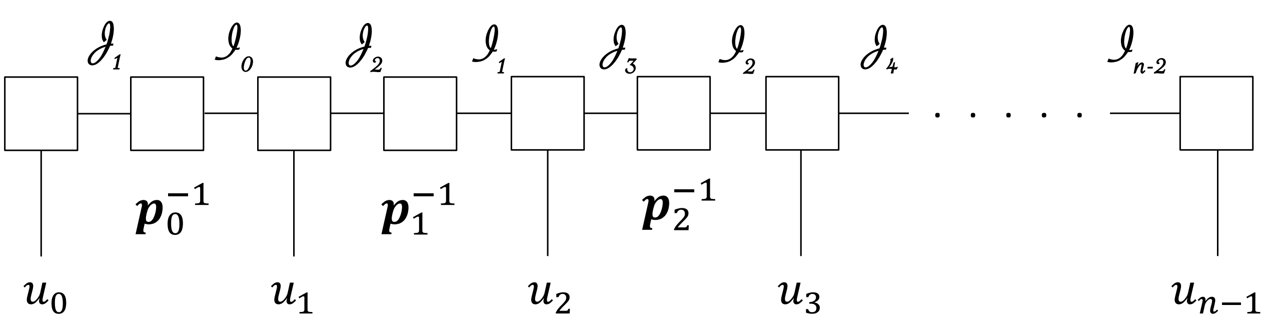

FIG. 1. Graphic Representation of tensor-train interpolation in Eq. (1). Three-leg blocks are $\boldsymbol{A}(\mathcal{I}_{\alpha-1}, u_\alpha, \mathcal{J}_{\alpha+1})$. Two-leg blocks are inverse pivots matrices We chose the convention as $\mathcal{I}_{-1} = \varnothing$, $\mathcal{J}_n = \varnothing$.

Algorithm 1: The maximum-volume principle on a node

Data: f, α, $\mathcal{I}_\alpha$, $\mathcal{J}_{\alpha+1}$, $\mathcal{I}_{\alpha-1}$, $\mathcal{J}_{\alpha+2}$
Result: new pivot, error estimate
$\boldsymbol{p} \leftarrow f(\mathcal{I}_\alpha, \mathcal{J}_{\alpha+1})$;
$\epsilon \leftarrow$ empty list;
pivots $\leftarrow$ empty list;
for $x_b \in \mathcal{I}_{\alpha-1},\ x_t \in \{0,1\},\ y_b \in \mathcal{J}_{\alpha+2},\ y_h \in \{0,1\}$ **do**
    $x \leftarrow x_b \oplus x_t$
    $y \leftarrow y_h \oplus y_b$
    $F \leftarrow f(x, y)$
    $\boldsymbol{v}_1 \leftarrow f(x, \mathcal{J}_{\alpha+1})$
    $\boldsymbol{v}_2 \leftarrow f(\mathcal{I}_\alpha, y)$
    Add $|F - \boldsymbol{v}_1 \cdot (\boldsymbol{p}^{-1} \cdot \boldsymbol{v}_2)|$ to ϵ
    $\boldsymbol{p}' \leftarrow \begin{pmatrix} \boldsymbol{p} & \boldsymbol{v}_2^{\mathrm{T}} \\ \boldsymbol{v}_1 & F \end{pmatrix}$
    if $\boldsymbol{p}'$ *is fully-ranked* **then**
        Add (x, y) to pivots
    end
end
new pivot $\leftarrow$ argmax $[\epsilon(x, y)]$ for $(x, y) \in$ pivots;
error estimate $\leftarrow \max(\epsilon)$

Algorithm 2: The Cross Interpolation

Data: f, $\mathcal{I}$, $\mathcal{J}$, tol, nswp
Result: Expanded I and J
$\epsilon \leftarrow$ list of zeros of length $n-1$;
while $i < nswp$ **do**
    for $\alpha \in \{0, \ldots, n-2\}$ **do**
        Apply `maxvol` to the current node
        $\epsilon[\alpha] \leftarrow$ error estimate
        if error estimate > tol and new pivot is not empty **then**
            add new pivot to I_α, $J_{\alpha+1}$
        end
    end
    for $\alpha \in \{n-2, \ldots, 0\}$ **do**
        Repeat the above
    end
    if $\max[\epsilon] < tol$ **then**
        exit
    end
end

* boyuanshi0607@gmail.com

[1] D. V. Savostyanov, Quasioptimality of maximum-volume cross interpolation of tensors, Linear Algebra and its Applications **458**, 217 (2014).