ALMOST SURE CONVERGENCE OF COVER TIMES FOR ψ-MIXING SYSTEMS

BOYUAN ZHAO

Abstract. Given a topologically transitive system on the unit interval, one can investigate the *cover time*, *i.e.*, time for an orbit to reach certain level of resolution in the repeller. We introduce a new notion of dimension, namely the *stretched Minkowski dimension*, and show that under mixing conditions, the asymptotics of typical cover times is determined by Minkowski dimensions when they are infinite, or by stretched Minkowski dimensions otherwise. For application, we show that for countably full-branched affine maps, results using the usual Minkowski dimension fail to produce a finite limit whilst our the stretched version can. In addition, cover times of irrational rotations are explicitly calculated as counterexamples due to the absence of mixing.

1. Introduction

Let $([0,1], f, \mu)$ be a probability preserving system where $f : [0,1] \to [0,1]$ is topologically transitive. Then one can investigate the *cover time* for points in the repeller Λ, i.e. given $x \in \Lambda$ let

$$\tau_r(x) := \inf\left\{k : \forall y \in \Lambda, \exists j \le k : y \in d(f^j(x), y) < r\right\}.$$

The first quantitative result of expected cover times was obtained for Brownian motions: $\mathbb{E}[\tau_r]$ was calculated in [M] and later on generalised in recent works [BJK] and [JT] for iterative function systems and one dimensional dynamical systems respectively. In [BJK], an almost sure convergence for $-\log\tau_r/\log r$ was demonstrated for *chaos games* associated to (finite) iterated function systems, assuming the invariant measure μ supported on the attractor satisfies rapid mixing conditions. All results suggests that the asymptotic behaviour of τ_r is crucially linked to the Minkowski dimensions: let $M_\mu(r) := \min_{x\in\mathrm{supp}(\mu)} \mu(B(x,r))$, the *upper and lower Minkowski dimensions* of μ are defined by

$$\overline{\dim}_M(\mu) := \limsup_{r\to 0}\frac{\log M_\mu(r)}{\log r},\quad \underline{\dim}_M(\mu) := \liminf_{r\to 0}\frac{\log M_\mu(r)}{\log r}$$

and simply write $\dim_M(\mu)$ when the two quantities coincide. This dimension-like quantity reflects the exponential decay rate of the minimal measure of a ball at scale r, and is closely related to the hitting time of such balls. The following theorem is our first result giving an almost sure asymptotic growth rate of cover times.

Theorem 1.1. *Let $([0,1], f, \mu)$ be a probability preserving system where f is topologically transitive, Markov and piecewise expanding. Then for μ-a.e. x in the repeller,*

$$\limsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r} \ge \overline{\dim}_M(\mu),\quad \liminf_{r\to 0}\frac{\log\tau_r}{-\log r} \ge \underline{\dim}_M(\mu).$$

If (f,μ) is exponentially ψ-mixing, then for μ-almost every $x \in \Lambda$, the inequalities above are improved to

$$\limsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r} = \overline{\dim}_M(\mu),\quad \liminf_{r\to 0}\frac{\log\tau_r}{-\log r} = \underline{\dim}_M(\mu).$$

2020 *Mathematics Subject Classification.* 37A25, 37D25, 37E05, 37E10.

Key words and phrases. Cover time, exponentially ψ-mixing, irrational rotations, Minkowski dimension.

We remark that systems with finite Minkowski dimensions, or at least $\overline{\dim}_M(\mu) < \infty$, are fairly common. In particular, it is true if the invariant measure in question is *doubling.*

Remark 1.2. *Suppose μ is doubling,* i.e., *there exists constant $D > 0$ such that for all $x \in [0,1]$ and $r > 0$, $D\mu(B(x,r)) \geq \mu(B(x,2r))$, then $\overline{\dim}_M(\mu) < \infty$.*

Proof. For each $n \in \mathbb{N}$ let $x_n \in \operatorname{supp}(\mu)$ be such that $\mu(B(x_n, 2^{-n})) = M_\mu(2^{-n})$, then by the doubling property,

$$M_\mu\left(2^{-n}\right) = \mu\left(B\left(x_n, 2^{-n}\right)\right) \geq D^{-1}\mu\left(B\left(x_n, 2^{-n+1}\right)\right) \geq D^{-1}M_\mu\left(2^{-n+1}\right) = D^{-1}\mu\left(B(x_{n-1}2^{-n+1}\right),$$

and reiterating this one gets $M_\mu(2^{-n}) \geq D^{-n+1}M_\mu(1/2)$, in other words

$$\frac{\log M_\mu(2^{-n})}{-n\log 2} \leq \frac{-(n-1)\log D + \log M_\mu(1/2)}{-n\log 2}.$$

Then as for all $r > 0$, there is unique $n \in \mathbb{N}$ such that $2^{-n} < r \leq 2^{-n+1}$, and $\frac{\log 2^{-n}}{\log 2^{-n+1}} = 1$,

$$\limsup_{r\to 0} \frac{\log M_\mu(r)}{\log r} = \limsup_{n\to\infty} \frac{\log M_\mu\left(2^{-n}\right)}{-n\log 2} \leq \frac{\log D}{\log 2} < \infty. \qquad \square$$

However, the Minkowski dimensions are not always finite due to non-doubling behaviours, or more extreme decay of $M_\mu(r)$ (see Example 3.2). Hence we need a new notion of dimension, invariant under scalar multiplication (*i.e.*, replacing $M_\mu(r)$ by $M_\mu(cr)$ for any $c > 0$ the limit does not change), to capture such decay rate in r.

Definition 1.3. Define the upper and lower *stretched Minkowski dimensions* by

$$\overline{\dim}^s_M(\mu) := \limsup_{r\to 0} \frac{\log|\log M_\mu(r)|}{-\log r}, \quad \underline{\dim}^s_M(\mu) := \liminf_{r\to 0} \frac{\log\log|M_\mu(r)|}{-\log r}.$$

Those quantities should be of independent interests. With this definition, we include a wider range of systems comparing to [BJK, Theorem 2.1], for which $M_\mu(r)$ decays at stretched-exponential rates, so instead we investigate the limits of the form $\frac{\log\log\tau_r(x)}{-\log r}$. Our second theorem below deals with almost sure cover times of these cases.

Theorem 1.4. *Let $([0,1], f, \mu)$ be a probability preserving system where f is topologically transitive, Markov and piecewise expanding. If $\overline{\dim}_M(\mu) = \infty$, but $\overline{\dim}^s_M(\mu), \underline{\dim}^s_M(\mu) < \infty$ then for μ-almost every $x \in \Lambda$,*

$$\liminf_{r\to 0} \frac{\log\log\tau_r(x)}{-\log r} \geq \underline{dim}^s_M(\mu), \quad \limsup_{r\to 0} \frac{\log\log\tau_r(x)}{-\log r} \geq \overline{dim}^s_M(\mu) \tag{1.1}$$

If (f,μ) is exponentially ψ-mixing, then for μ-almost every $x \in \Lambda$,

$$\liminf_{r\to 0} \frac{\log\log\tau_r(x)}{-\log r} = \underline{dim}^s_M(\mu), \quad \limsup_{r\to 0} \frac{\log\log\tau_r(x)}{-\log r} = \overline{dim}^s_M(\mu). \tag{1.2}$$

Layout of the paper. We delay the proofs of the main theorems to Section 4. Several examples that satisfy Theorem 1.1 and Theorem 1.4 will be discussed in Section 3. In Section 5 we will also prove that for irrational rotations, which are known to have no mixing behaviour, Theorem 1.4 fails for almost every point when the rotations are of type η (see Definition 5.1) for some $\eta > 1$. Lastly in Section 6 we show that similar results hold for flows under some natural conditions.

2. Setup

Say $f : [0,1] \to [0,1]$ is a *piecewise expanding Markov map* if there is a finite or countable index set $\mathcal{A}$ and a collection of subintervals $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ with disjoint interiors covering the domain of f, such that:
(1) for any $a \in \mathcal{A}$, $f_a := f|_{P_a}$ is injective and $f(P_a)$ a union of elements in $\mathcal{P}$;
(2) there is a uniform constant $\gamma > 1$ such that for all $a \in \mathcal{A}$, $|Df_a| \geq \gamma$.

There is a shift system associated to f: let M be an $\mathcal{A}\times\mathcal{A}$ matrix such that $M_{ab} = 1$ if $f(P_a)\cap P_b \neq \emptyset$ and 0 otherwise. The map f is *topologically transitive* if for all $a, b \in \mathcal{A}$, there exists k such that $M^k_{ab} > 0$. Let Σ denote the space of all *infinite admissible words*, i.e.

$$\Sigma := \left\{x = (x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{N}_0} : M_{x_k,x_{k+1}} = 1,\ \forall\, k \geq 0\right\}.$$

A natural choice of metric on Σ is $d_s(x, y) := 2^{-\inf\{j\geq 0:\, x_j\neq y_j\}}$ and we define the projection map $\pi : \Sigma \to [0,1]$ by

$$x = \pi\,(x_0, x_1, \dots) \text{ if and only if } x \in \bigcap_{i=0}^{\infty} f^{-i}P_{x_i}.$$

The dynamics on Σ is the left shift $\sigma : \Sigma \to \Sigma$ given by $\sigma(x_0, x_1, \dots,) = (x_1, x_2, \dots)$, then π defines a semi-conjugacy $f\circ\pi = \pi\circ\sigma$, and the corresponding symbolic measure $\tilde{\mu}$ of μ is given by $\mu = \pi_*\tilde{\mu}$, i.e. for all Borel-measurable set $B \in \mathcal{B}([0,1])$, $\mu(B) = \tilde{\mu}\left(\pi^{-1}B\right)$.

Denote $\mathcal{P}^n := \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$, each $P \in \mathcal{P}^n$ corresponds to an $n-$cylinder in Σ: let $\Sigma_n \subseteq \mathcal{A}^n$ denote all finite words of length n and for any $\mathbf{i} \in \Sigma$, the n-cylinder defined by $\mathbf{i}$ is

$$[\mathbf{i}] = [i_0, \dots, i_{n-1}] := \{y \in \Sigma : y_j = i_j,\ j = 0, \dots, n-1\},$$

then $\pi[i_0, i_1, \dots, i_{n-1}] = \bigcap_{i=0}^{n-1} f^{-i}P_{i_i} =: P_{\mathbf{i}}$.

The *repeller* of f is the collection of points with all their forward iterates contained in $\mathcal{P}$, namely

$$\Lambda := \left\{x \in [0,1] : f^k(x) \in \bigcup_{a\in\mathcal{A}} P_a \text{ for all } k \geq 0.\right\}$$

Furthermore, (f, μ) is required to have the following mixing property.

Definition 2.1. Say μ is *exponentially ψ-mixing* if there are $C_1, \rho > 0$ and a monotone decreasing function $\psi(k) \leq C_1 e^{-\rho k}$ for all $k \in \mathbb{N}$, such that the corresponding symbolic measure $\tilde{\mu}$ satisfies: for all $n, k \in \mathbb{N}$, $\mathbf{i} \in \Sigma_n$ and $\mathbf{j} \in \Sigma^* = \bigcup_{l\geq 1}\Sigma_l$,

$$\left|\frac{\tilde{\mu}([\mathbf{i}] \cap \sigma^{-(n+k)}[\mathbf{j}])}{\tilde{\mu}[\mathbf{i}]\tilde{\mu}[\mathbf{j}]} - 1\right| \leq \psi(k).$$

3. Examples

Theorem 1.4 is applicable to the following systems.

Example 3.1. Finitely branched Gibbs-Markov maps: let f be a topologically transitive piecewise expanding Markov map with $\mathcal{A}$ finite. f is said to be *Gibbs-Markov* if for some *potential* $\phi : \Sigma \to \mathbb{R}$

which is *locally Hölder* with respect to the symbolic metric d_s, there exists $G > 0$ and $P \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, all $x = (x_0, x_1, \dots) \in \Sigma$,

$$\frac{1}{G} \le \frac{\tilde{\mu}([x_0, \dots, x_{n-1}])}{\exp\left(\sum_{j=0}^{n-1} \phi(\sigma^j x) - nP\right)} \le G.$$

For maps of this kind, $|Df|$ is uniformly bounded so for each ball at scale r, it is possible to approximate any ball with finitely many cylinders of the same depth (see for example the proof of [JT, Lemma 3.2]), and by the Gibbs property of $\tilde{\mu}$, the asymptotic decay rate converges so $\dim_M(\mu)$ exists and is finite. Since Gibbs measures are exponentially ψ-mixing (see [Bow, Proposition 1.14]), by Theorem 1.1, we have

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu)$$

for μ-almost every x in the repeller.

In the example below, $\overline{\dim}_M(\mu) = \infty$ therefore the generalised Minkowski dimension is needed.

Example 3.2. Similar to [JT, Example 7.4], consider the following class of infinitely full-branched maps: pick $\kappa > 1$ and set $c = \zeta(\kappa) = \sum_{n\in\mathbb{N}} \frac{1}{n^\kappa}$. Let $a_0 = 0$, $a_j = \sum_{j=1}^{n} \frac{1}{cj^\kappa}$ and define

$$f : \forall\, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ f(x) = cn^\kappa (x - a_{n-1}) \text{ for } x \in [a_{n-1}, a_n) =: P_n.$$

Then f is an infinitely full-branched affine map, and we can associate this map with a full-shift system on $\mathbb{N}$: $x = \pi(i_0, i_1, \dots)$ if for all $j \ge 1$, $f^j(x) \in P_{i_j}$.
Let $\delta > 1$ and define $\tilde{\mu}$ the finite Bernoulli measure such that

$$\tilde{\mu}([i_0, \dots, i_{n-1}]) = \prod_{j=0}^{n-1} \delta^{-i_j},$$

then the push-forward measure $\mu = \pi_* \tilde{\mu}$ has $\mu(P_n) = \delta^{-n}$.

Proposition 3.1. $\overline{\dim}_M(\mu) = \infty$, *but* $\dim^s_M(\mu) = \frac{1}{\kappa-1}$.

Proof. For each $r > 0$, $M_\mu(r)$ is found near 1, then along the sequence $r_n = \frac{1}{2c} \sum_{j\ge n} j^{-\kappa} \approx \frac{1}{2c(\kappa-1)n^{\kappa-1}}$, the ball that realises $M_\mu(r_n)$ is contained in $\bigcup_{j=n}^{\infty} P_j$, hence

$$\delta^{-n} \le M_\mu(r_n) \le \frac{\delta^{-n}}{1 - \delta^{-1}},$$

therefore

$$\overline{\dim}_M(\mu) \ge \limsup_{n\to\infty} \frac{n \log \delta}{(\kappa - 1)\log n} = \infty,$$

whereas for all n,

$$\frac{\log n}{(\kappa-1)\log n} \le \frac{\log|\log M_\mu(r_n)|}{-\log r_n} \le \frac{\log n + \log\log \delta}{(\kappa-1)\log n + \log(2c(\kappa-1))}.$$

As for all $r > 0$, there is unique $n \in \mathbb{N}$ such that $r_{n+1} \le r < r_n$ while $\lim_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} = 1$, one concludes with $\dim^s_M(\mu) = \frac{1}{\kappa-1}$. $\square$

As in [JT, Example 7.4] it is very difficult for the system to cover small neighbourhoods of 1 so [BJK, Theorem 2.1] fails to produce a finite limit for $\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r}$, but since $\tilde{\mu}$ is Bernoulli hence ψ-mixing, our main theorem asserts that

$$\lim_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} = \frac{1}{\kappa - 1} \quad \mu\text{-a.e.}$$

4. Proof of Theorem 1.4

The proofs in this section are adapted from those of [BJK, Proposition 3.1, 3.2]. We will only demonstrate the proofs for Theorem 1.4, *i.e.*, the asymptotics are determined by stretched Minkowski dimensions; the proofs for Theorem 1.1 are obtained by replacing all stretched exponential sequences in the proofs below by some exponential sequence, *e.g.* for a given constant $s \in \mathbb{R}$, $e^{\pm n^s}$ will be replaced by $2^{\pm ns}$.

Assuming the inequalities in (1.1), we first prove the (1.2) which requires the exponentially ψ-mixing condition.

Remark 4.1. *Assuming the conditions of Theorem 1.4, we will prove that the statements hold along the subsequence $r_n = n^{-1}$ such that for each $r > 0$ there is a unique $n \in \mathbb{N}$ with $r_{n+1} < r \leq r_n$ while $\lim_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} = 1$ (if $\overline{\dim}_M(\mu)$ or $\underline{\dim}_M(\mu)$ are finite we choose $r_n = 2^{-n}$ instead), and since $\log \tau_r(x)$ is increasing as $r \to 0$,*

$$\limsup_{n\to\infty} \frac{\log\log \tau_{r_n}(x)}{-\log r_n} = \limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r},$$

and similarly for liminf's.

4.1. Proof of (1.2).

Proposition 4.2. *Suppose (f, μ) is exponentially ψ-mixing, and the upper stretched Minkowski dimension of μ, $\overline{\dim}^s_M(\mu)$, is finite, then for μ-almost every $x \in \Lambda$,*

$$\limsup_{n\to\infty} \frac{\log\log \tau_r(x)}{-\log r} \leq \overline{\dim}^s_M(\mu).$$

Proof. Let $\varepsilon > 0$, and for simplicity denote $\overline{\alpha} := \overline{\dim}^s_M(\mu)$.

For any finite k-word $\mathbf{i} = x_0, \dots, x_{k-1} \in \Sigma_k$, let $\mathbf{i}^- = x_0, \dots, x_{k-2}$, i.e. $\mathbf{i}$ dropping the last digit. Recall that for each $\mathbf{i} \in \Sigma^*$, $P_\mathbf{i} = \pi[\mathbf{i}]$, and we define

$$\mathcal{W}_r := \{\mathbf{i} \in \Sigma^* : \operatorname{diam}(P_\mathbf{i}) \leq r < \operatorname{diam}(P_{\mathbf{i}^-})\}.$$

By expansion, for each $n \in \mathbb{N}$, the lengths of the words in $\mathcal{W}_{n^{-1}}$ are bounded from above, hence we can define

$$L(n) := \frac{\log n}{\log \gamma} + 1 \geq \max\{|\mathbf{i}| : \mathbf{i} \in \mathcal{W}_{n^{-1}}\}.$$

Given $y \in [0, 1]$ and $r > 0$ such that $B(y, r) \subset \operatorname{supp}(\mu)$, define the corresponding symbolic balls by

$$\tilde{B}(y, r) := \{[\mathbf{i}] : \mathbf{i} \in \mathcal{W}_r, P_\mathbf{i} \cap B(y, r) \neq \emptyset\}.$$

Note that if for some $x \in P_\mathbf{i}$ and $[\mathbf{i}] \in \tilde{B}(y, r)$, $d(x, y) \leq r + diam(P_\mathbf{i}) \leq 2r$ therefore

$$B(y, r) \subset \pi\tilde{B}(y, r) \subset B(y, 2r).$$

Let $\mathcal{Q}_n$ be a cover of Λ with balls of radius $r_n = 1/2n$, denote the collection of their centres by $\mathcal{Y}_n$, and $\#\mathcal{Q}_n = \#\mathcal{Y}_n \leq n$. Let $\tau(\mathcal{Q}_n, x)$ be the minimum time for the orbit of x to have visited each element of $\mathcal{Q}_n$ at least once,

$$\tau(\mathcal{Q}_n, x) := \min\left\{k \in \mathbb{N} : \text{ for all } Q \in \mathcal{Q}_n, \text{ exists } 0 \leq j \leq k : f^j(x) \in Q\right\}.$$

Then $\tau_{1/n}(x) \leq \tau(\mathcal{Q}_n, x)$ for all n and all x since for all $y \in \Lambda$, there is $Q \in \mathcal{Q}_n$ and $j \leq \tau(\mathcal{Q}_n, x)$ such that $f^j(x), y \in Q$ hence $d(f^j(x), y) \leq 1/n$. Let $\varepsilon > 0$ be an arbitrary number and for each $k \in \mathbb{N}$, set $L'(k) = L(k) + \frac{1}{\rho}\left(k^{\overline{\alpha}+\varepsilon} - \log C_1\right)$, then

$$\begin{aligned}
&\mu\left(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right) \leq \mu\left(x : \tau(\mathcal{Q}_n, x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right)\\
&= \mu\left(x : \exists y \in \mathcal{Y}_n : f^j(x) \notin B(y, 1/2n),\, \forall j \leq e^{n^{\overline{\alpha}+\varepsilon}} L'(2n)\right)\\
&\leq \mu\left(x : \exists y \in \mathcal{Y}_n : f^{jL'(4n)}(x) \notin B(y, 1/2n),\, \forall j \leq e^{n^{\overline{\alpha}+\varepsilon}}\right)\\
&= \mu\left(\bigcup_{y\in\mathcal{Y}_n} \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(4n)}(x) \notin B(y, 1/2n)\right) \leq \sum_{y\in\mathcal{Y}_n} \mu\left(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(2n)}(x) \notin B(y, 1/2n)\right)
\end{aligned} \tag{4.1}$$

As $\pi\left(\tilde{B}(z,r)\right) \subseteq B(z, 2r)$ for all z and all $r > 0$, using the exponentially ψ-mixing property of $\tilde{\mu}$, *i.e.*, $\psi(k) \leq C_1 e^{-\rho k}$ for all $k \in \mathbb{N}$, by our choice of $L'(4n)$,

$$\begin{aligned}
&\sum_{y\in\mathcal{Y}_n} \mu\left(x : \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(4n)}(x) \notin B(y, 1/2n)\right) \leq \sum_{y\in\mathcal{Y}_{k+1}} \tilde{\mu}\left(x : \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} \sigma^{-jL'(4n)}(\pi^{-1}x) \notin \tilde{B}\left(y, 1/4n\right)\right)\\
&\leq \left(1 + \psi\left(\frac{1}{\rho}\left((4n)^{\overline{\alpha}+\varepsilon} - \log C_1\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y\in\mathcal{Y}_{k+1}} \left(1 - \tilde{\mu}\left(\tilde{B}\left(y, \frac{1}{2n}\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}}\\
&\leq \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y\in\mathcal{Y}_{k+1}} \left(1 - \mu\left(B\left(y, \frac{1}{4n}\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}}
\end{aligned} \tag{4.2}$$

By definition of $\overline{\alpha}$, for all n large such that $\frac{\varepsilon}{4}\log n \geq \left(\overline{\alpha} + \frac{\varepsilon}{4}\right)\log 4$, there is

$$\log\left(-\log M_\mu\left(\frac{1}{4n}\right)\right) \leq \overline{\alpha} + \varepsilon/4(\log 4n) \leq (\overline{\alpha} + \varepsilon/2)\log n$$

so for all $y \in \operatorname{supp}(\mu)$ and all n large,

$$\mu\left(B\left(y, \frac{1}{4n}\right)\right) \geq e^{-n^{\overline{\alpha}+\varepsilon/2}} \geq \frac{e^{-n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}.$$

As for all $u \in \mathbb{R}$ and large k, $(1 + \frac{u}{k})^k \approx e^u$, combining (4.1) and (4.2), for some uniform constant $C_2 > 0$,

$$\begin{aligned}
\mu\left(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right) &\leq \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y\in\mathcal{Y}_{k+1}} \left(1 - e^{-n^{\overline{\alpha}+\varepsilon/2}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}}\\
&\leq \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} n\left(1 - \frac{e^{n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \leq C_2 \exp\left(\log n - e^{n^{\varepsilon/2}}\right)
\end{aligned}$$

which is clearly summable over n. Then by Borel Cantelli, for all n large enough $\tau_{1/n}(x) \leq e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)$. Since $\log L'(4n) \approx (\overline{\alpha} + \varepsilon)\log n \ll n^{\overline{\alpha}+\varepsilon}$, we have for $\mu-$a.e. $x \in \Lambda$,

$$\limsup_{n\to\infty} \frac{\log\log \tau_{1/n}(x)}{\log n} \leq \limsup_{n\to\infty} \frac{\log\log\left(e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right)}{\log n} \leq \overline{\alpha} + \varepsilon.$$

By Remark 4.1 this upper bound for lim sup holds for all sequences decreasing to 0 and as $\varepsilon > 0$ was arbitrary, sending it to 0 one obtains that for $\mu-$a.e. $x \in \Lambda$,

$$\limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} = \limsup_{n\to\infty} \frac{\log\log \tau_{1/n}}{\log n} \le \overline{\alpha}.$$ □

Proposition 4.3. *Suppose (f,μ) is exponentially ψ-mixing and the lower stretched Minkowski dimension of μ, $\underline{\dim}^s_M(\mu)$, is finite, then for μ-a.e. $x \in \Lambda$,*

$$\liminf_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \le \underline{\dim}^s_M(\mu).$$

Proof. Again for simplicity, denote $\underline{\alpha} := \underline{dim}^s_M(\mu)$. Let $\varepsilon > 0$ and by definition of liminf there is a subsequence $\{n_k\}_k \to \infty$ such that for all k,

$$\frac{\log(-\log M_\mu(1/n_k))}{\log n_k} \le \underline{\alpha} + \varepsilon,$$

then repeating the proof of Proposition 4.2 by replacing n by n_k everywhere, one gets that for $\mu-$almost every x,

$$\liminf_{k\to\infty} \frac{\log\log \tau_{1/n_k}(x)}{\log n_k} \le \underline{\alpha} + \varepsilon.$$

Again send $\varepsilon \to 0$, and use the fact that liminf over the entire sequence is no greater than the liminf along any subsequence, the proposition is proved. □

4.2. **Proof of the inequalities** (1.1)**.**

Proposition 4.4. *For μ-almost every $x \in \Lambda$,*

$$\liminf_{n\to\infty} \frac{\log\log \tau_r(x)}{-\log r} \ge \underline{\dim}^s_M(\mu).$$

Proof. We continue to use the notation $\underline{\alpha} = \underline{\dim}^s_M(\mu)$. Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha}$ for all large n there exists $y_n \in \text{supp}(\mu)$ such that $\mu(B(y_n, 1/n)) \le e^{-n^{\overline{\alpha}-\varepsilon}}$. Let

$$T(x,y,r) := \inf\left\{j \ge 0 : f^j(x) \in B(y,r)\right\},$$

so for all $n \in \mathbb{N}$ and all x, $\tau_{1/n}(x) \ge T(x, y_n, 1/n)$. Then by invariance,

$$\begin{aligned}
&\mu\left(x : \tau_{1/n}(x) < e^{n^{\overline{\alpha}-\varepsilon}}/n^2\right) \le \mu\left(x : T(x,y_n,1/n) < e^{n^{\overline{\alpha}-\varepsilon}}/n^2\right) \\
&= \mu\left(x : \exists\, 0 \le j < e^{n^{\overline{\alpha}-\varepsilon}}/n^2 :\ f^j(x) \in B(y_n, 1/n)\right) \le \bigcup_{j=0}^{e^{n^{\overline{\alpha}-\varepsilon}}/n^2} \mu\left(x : f^j(x) \in B(y_n,1/n)\right) \\
&\le \sum_{j=0}^{e^{n^{\overline{\alpha}-\varepsilon}}/n^2} \mu\left(f^{-j}B\left(y_n, \frac{1}{n}\right)\right) \le \frac{e^{n^{\overline{\alpha}-\varepsilon}}}{n^2} e^{-n^{\overline{\alpha}-\varepsilon}} = \frac{1}{n^2},
\end{aligned}$$

which is summable. By Borel-Cantelli, since $2\log n \ll n^{\underline{\alpha}-\varepsilon}$, for μ-almost every x

$$\liminf_{n\to\infty} \frac{\log\log \tau_{1/n}(x)}{\log n} \ge \underline{\alpha} - \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary one can send it to 0. □

Similar to Proposition 4.2 and Proposition 4.3,

Proposition 4.5. *For μ-almost every x,*

$$\limsup_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}\geq \overline{\dim}_M^s(\mu)$$

Proof. Let $\varepsilon>0$, then by definition of limsup there exists a subsequence $\{n_k\}_k\to\infty$ such that for all k,

$$\frac{\log\log\left(-M_\mu(1/n_k)\right)}{\log n_k}\geq\overline{\alpha}-\varepsilon.$$

Then repeating the proof of Proposition 4.4 along $\{n_k\}_k$, one gets that for μ-almost every x:

$$\limsup_{k\to\infty}\frac{\log\log\tau_{1/n_k}(x)}{\log n_k}\geq\overline{\alpha}-\varepsilon,$$

then sending $\varepsilon\to 0$,

$$\limsup_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}\geq\limsup_{k\to\infty}\frac{\log\log\tau_{1/n_k}(x)}{\log n_k}\geq\overline{\alpha}. \qquad \square$$

5. Irrational rotations

The proof of (1.2) requires an exponentially ψ-mixing rate which is a strong mixing condition, and it is natural to ask if the same asymptotic growth in Theorem 1.4 remains the same under different mixing conditions, e.g. exponentially ϕ-mixing and α-mixing, or even polynomial ψ-mixing. Although these questions are unresolved, in this section we will show that the limsup and liminf of the asymptotic growth rate can differ if the system is not mixing at all.

Let $\theta\in(0,1)$ be an irrational number and $T(x)=x+\theta\ (\mathrm{mod}\ 1)$, and μ the one-dimensional Lebesgue measure on $[0,1)$. Then (T,μ) is an ergodic probability preserving system with $\dim_M(\mu)=1$.

Definition 5.1. For a given irrational number θ, the *type* of T_θ is given by the following number

$$\eta=\eta(\theta):=\sup\left\{\beta:\liminf_{n\to\infty}n^\beta\|n\theta\|=0\right\},$$

where for every $r\in\mathbb{R}$, $\|r\|=\min_{n\in\mathbb{Z}}|r-n|$.

Remark 5.2. *(See* [K]*) For every $\theta\in(0,1)$ irrational, $\eta(\theta)\geq 1$ and $\eta(\theta)=1$ almost everywhere, but there exists irrational number with $\eta(\theta)\in(1,\infty]$, e.g. the Liouville numbers.*

For any irrational number $\theta\in(0,1)$ there is a unique continued fraction expansion

$$\theta=[a_1,a_2,\dots]:=\cfrac{1}{a_1+\frac{1}{a_2+\dots}}$$

where $a_i\geq 1$ for all $i\geq 1$. Set $p_0=0$ and $q_0=1$, and for $i\geq 1$ choose $p_i,q_i\in\mathbb{N}$ coprime such that

$$\frac{p_i}{q_i}=[a_1,\dots,a_i]=\cfrac{1}{a_1+\frac{1}{\dots\frac{1}{a_i}}}.$$

Definition 5.3. The a_i terms are called the *i-th partial quotient* and p_i/q_i the i-th *convergent*. In particular, (see [K])

$$\eta(\theta)=\limsup_{n\to\infty}\frac{\log q_{n+1}}{\log q_n}.$$

Theorem 5.4. *For any irrational rotation T_θ with $\eta(\theta) > 1$,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu) = 1 < \eta = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ \mu\text{-a.e.}$$

The proof of this theorem relies on algebraic properties of the number $\eta(\theta)$, and for simplicity write η from now on.

Lemma 5.5. [KS, Fact 1, Lemma 7]

(a) $q_{i+2} = a_{i+2}q_{i+1} + q_i$ *and* $p_{i+2} = a_{i+2}p_{i+1} + p_i$*,*
(b) $1/(2q_{i+1}) \leq 1/(q_{i+1} + q_i) < \|q_i\theta\| < 1/q_{i+1}$ *for* $i \geq 1$*,*
(c) If $0 < j < q_{i+1}$*, then* $\|j\theta\| \geq \|q_i\theta\|$*.*
(d) for $\varepsilon > 0$*, there exists uniform* $C_\varepsilon > 0$ *such that for all* $j \in \mathbb{N}$*,* $j^{\eta+\varepsilon}\|j\theta\| > C_\varepsilon$

The following propositions are largely based on [KS, Proposition 6, Proposition 10].

Proposition 5.6. *For μ-a.e. x,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \eta. \tag{5.1}$$

Proof. First we prove the following simple claim.

Claim. *The function $\varphi(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}$ is constant μ a.e.*

Proof of Claim. Suppose $\tau_r(x) = k$, and for any $y \neq x$, if there exists z such that for all $0 \leq j \leq k$, $\left|T^j y - z\right| \geq r$, then for all $0 \leq j \leq k$

$$\left|T^j y - x + x - z\right| = \left|T^j x - (x + z - y)\right| \geq r$$

contradicting $\tau_r(x) = k$, then by symmetry $\tau_r(x) = \tau_r(y)$, in particular $\tau_r(x) = \tau_r(Tx)$ so $\varphi \circ T = \varphi$, and μ is (uniquely) ergodic implies φ is constant almost everywhere. □

By [KS, Proposition 10], for almost every x, y

$$\limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \geq \eta,$$

where $W_E(x) := \inf\{n \geq 1 : T^n x \in E\}$ denotes the waiting time of x before visiting E. Hence there exists a set of strictly positive measure consisting of points that satisfy

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \geq \eta$$

since for all $y \in [0, 1)$, $\tau_r(x) \geq W_{B(y,r)}(x)$, then by the claim above this holds for almost every x. □

Proposition 5.7. *For μ-a.e. x,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \eta.$$

Proof. Let $\mathcal{Q}_n := \{[2^{-n}j, 2^{-n}(j+1)) : j = 0, \ldots, 2^n - 1\}$ and $\tau(\mathcal{Q}_n, x)$ the minimum time for x to have visited each element of $\mathcal{Q}_n$. By Lemma 5.5 (a) and (c), $\{\|q_i\theta\|\}_i$ is a decreasing sequence, and there for each $n \in \mathbb{N}$ exists a minimal j such that $\|q_j\theta\| < 2^{-n} \leq \|q_{j-1}\theta\|$, write $j = j_n$.

By [KS, Proposition 6] for all n, there is $\mu\left(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}\right) = 0$. Notice that for all $a, b \in [0,1)$,

$$\mu\{W_{[a,a+b)}(x) = k\} = \mu\left\{\{x : W_{[0,b)}(x) = k\} + a\right\} = \mu\left\{W_{[0,b)}(x) = k\right\} \tag{5.2}$$

since $\mu = Leb$ is translation invariant. Then by (5.2)

$$\begin{aligned}
&\mu\left\{\tau\left(\mathcal{Q}_n, x\right) > q_{j_n} + q_{j_n-1}\right\} = \mu\left\{x : \forall Q \in \mathcal{Q}_n : W_Q(x) > q_{j_n} + q_{j_n-1}\right\} \\
&= \mu\left(x : \bigcup_{Q \in \mathcal{Q}_n} \{W_Q(x) > q_{j_n-1} + q_{j_n}\}\right) \leq \sum_{Q \in \mathcal{Q}_n} \mu\left(W_Q > q_{j_n-1} + q_{j_n}\right) \\
&= \sum_{j=0}^{2^n-1} \mu\left(W_{[2^{-n}j, 2^{-n}(j+1))} > q_{j_n} + q_{j_n-1}\right) = \sum_{j=0}^{2^n-1} \mu\left(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}\right) = 0
\end{aligned}$$

hence by Borel-Cantelli, for all n large enough, $\tau_{2^{-n+1}}(x) \leq (q_{j_n} + q_{j_n-1})$ for μ-a.e $x \in [0,1)$.

Let $\varepsilon > 0$, and by Lemma 5.5 there exists C_ε such that

$$\log\left(q_{j_n} + q_{j_n-1}\right) \leq \log\left(2q_{j_n}\right) \leq \log\frac{2}{\|q_{j_n}\theta\|} \leq (\eta + \varepsilon)\log q_{j_n} + \log 2 - \log C_\varepsilon$$

again by Lemma 5.5 and our choice of j_n, for all n large enough, up to a uniform constant

$$\log\tau_{2^{-n+1}}(x) \leq \log(q_{j_n} + q_{j_n-1}) < (\eta + \varepsilon)\log q_{j_n} \leq -(\eta+\varepsilon)\log\|q_{j_n-1}\theta\| \leq (\eta+\varepsilon)n\log 2$$

hence $\limsup_{n\to\infty}\frac{\log\tau_{2^{-n}}(x)}{n\log 2} \leq \eta + \varepsilon$ for μ-almost every x, and send ε to 0 the proposition is proved since for each $r < 0$ there is a unique $n \in \mathbb{N}$ for which $2^{-n} < r \leq 2^{-n+1}$. □

Proposition 5.8. *For μ-almost every $x \in [0,1)$,*

$$\liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r} = 1.$$

Proof. Let $\varepsilon > 0$, and using the same arguments in the last proof, i.e. cover time is greater than the hitting time of the ball of smallest measure at scale r, then along the sequence $r_n = 2^{-(n+1)}$, one gets for all $[a - r_n, a + r_n) \subset [0,1)$, there is

$$\begin{aligned}
&\sum_{n\geq 1}\mu\left(\tau_{r_n}(x) < 2^{n(1-\varepsilon)}\right) \leq \sum_{n\geq 1}\mu\left(W_{[a-2^{-n-1}, a+2^{-n-1})}(x) < 2^{n(1-\varepsilon)}\right) \\
&\leq \sum_{n\geq 1}\sum_{k=0}^{2^{n(1-\varepsilon)}}\mu\left(T^{-k}[a - 2^{-n-1}, a + 2^{-n-1})\right) = \sum_{n\geq 1}2^{n(1-\varepsilon)}2^{-n} = \sum_{n\geq 1}2^{-\varepsilon n} < \infty.
\end{aligned}$$

Since for each r there is a unique n such that $r_n < r \leq r_{n-1}$ while $\lim_n\frac{\log r_n}{\log r_{n-1}} = 1$ so by Borel Cantelli,

$$\liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r} = \liminf_{n\to\infty}\frac{\log\tau_{2^{-n}}(x)}{n\log 2} \geq 1 - \varepsilon,$$

and sending ε to 0 the proposition is proved.

For the upper bound of liminf, recall that $\tau\left(\mathcal{Q}_n, x\right) \geq \tau_{2^{-n}}(x)$, we can repeat the proof of Proposition 5.7, apart from that this time we choose $\{2^{-n_i}\}_i$ according to $\{q_i\}_{i\in\mathbb{N}}$: for each i, choose $n_i \in \mathbb{N}$ to be the smallest number such that

$$\|q_{i+1}\theta\| < 2^{-n_i} \leq \|q_i\theta\|,$$

hence as in Proposition 5.7,

$$\mu\left(\tau\left(\mathcal{Q}_{n_i},x\right)>q_{i+1}+q_i\right)\leq\sum_{Q\in\mathcal{Q}_{n_i}}\mu\left(W_Q>q_{i+1}+q_i\right)=0.$$

Again by Lemma 5.5 (b), $q_{i+1}+q_i\leq 2q_{i+1}\leq\frac{2}{\|q_i\theta\|}<2^{n_i+1}$ by our choice of n_i, so $\lim_{i\to\infty}\frac{\log(q_i+q_{i+1})}{n_i\log 2}\leq 1$, therefore for μ-a.e. x,

$$\liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\leq\liminf_{i\to\infty}\frac{\log\tau_{2^{-n_i}}(x)}{n_i\log 2}\leq\liminf_{i\to\infty}\frac{\log\tau\left(\mathcal{Q}_{n_i},x\right)}{n_i\log 2}\leq 1.\qquad\square$$

6. Cover time for flows

In this section we prove results analogous to Theorem 1.1 and [BJK, Theorem 2.1] regarding cover times for the same class of flows discussed in [RT, §4].

Let $\{f_t\}_t$ be a flow on a metric space $(\mathcal{X},d_{\mathcal{X}})$ preserving an ergodic probability measure ν, i.e. $\nu\left(f_t^{-1}A\right)=\nu(A)$ for every $t\geq 0$ and A measurable. Let Ω denote the non-wandering set and define the cover time of x at scale r by

$$\tau_r(x):=\inf\left\{T>0:\,\forall y\in\Omega,\,\exists t\leq T:\,d(f_t(x),y)<r\right\}.$$

We will assume the existence of a Poincaré section $Y\subset\mathcal{X}$, and let $R_Y(x)$ denote the first hitting time to Y, i.e. $R_Y(x):=\inf\{t\geq 0:f_t(x)\in Y\}$ so we can define the Poincaré map $F=f_{R_Y}$ and let μ be the induced measure on Y, $\mu=\frac{1}{\overline{R}}\nu|_Y$, where $\overline{R}=\int R(x)d\nu(x)\in(0,\infty)$. Additionally, assume the following conditions are satisfied:

(H1) $\dim_M(\mu)$ exists and is finite for (F,μ),
(H2) (Y,F,μ) is Gibbs-Markov so Theorem 1.1 is applicable for μ-almost every $y\in Y$.
(H3) $\{f_t\}_t$ has bounded speed: there exists $K>0$ such that for all $t>0$, $d(f_s(x),f_{s+t}(x))<Kt$.
(H4) $\{f_t\}_t$ is topologically mixing and there exists $T_1>0$ such that

$$\bigcup_{0<t\leq T_1}f_t(Y)=\mathcal{X}.\tag{6.1}$$

(H5) There exists

$$C_f:=\sup\left\{\operatorname{diam}(f_t(I))/\operatorname{diam}(I):I\text{ an interval contained in }Y,0<t\leq T_1\right\}\in(0,\infty)$$

Remark 6.1. *The last condition is satisfied when (H3) holds and the flow is, for example, Lipschitz, i.e. there exists $L>0$ such that for all $x,y\in\mathcal{X}$,*

$$d_{\mathcal{X}}(f_t(x),f_t(y))\leq L^t d_{\mathcal{X}}(x,y).$$

Theorem 6.2. *Let (f_t,ν) be a probability preserving flow satisfying conditions (H1)-(H5), then for ν-almost every $x\in\Omega$,*

$$\liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\geq\underline{dim}_M(\nu)-1.\tag{6.2}$$

Furthermore, if $\overline{\dim}_M(\nu)=\dim_M(\mu)+1$,

$$\lim_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=\overline{\dim}_M(\mu)\quad\nu\text{-almost everywhere}\tag{6.3}$$

Proof of (6.2). This part is again analogous to that of Proposition 4.3 and [RT, Theorem 4.1].

Fix some $y \in \mathcal{X}$ and consider the random variable

$$S_{T,r}(x) := \int_0^T \mathbf{1}_{B(y,r)}(f_t(x))dt,$$

and observe that by bounded speed property

$$\{x : \exists 0 \le t \le T \text{ s.t. } d(f_t(x), y)) < r\} \subset \{S_{T,2r}(x) > r/K\}$$

since if $d(f_s(x), y) < r$ for some s, then for all $t < r/K$, $d(f_{t+s}(x), y) < 2r$.

Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha} = \underline{dim}_M(\nu)$ for all large n there exists $y_n \in \Omega$ such that $\nu(B(y_n, 2^{-n+1})) \le 2^{n(\alpha-\varepsilon)}$. Recall that

$$T(x, y, r) := \inf\{t \ge 0 : f_t(x) \in B(y, r)\},$$

and similarly for all $n \in \mathbb{N}$ and all x, $\tau_n(x) \ge T(x, y_n, 2^{-n})$. Leting $r_n = 2^{-n}$, by Markov's inequality, for any $\mathcal{T}_n > r_n$,

$$\begin{aligned}
&\nu\left(x : \tau_n(x) < T_n\right) \le \nu\left(x : T(x, y_n, r_n) < \mathcal{T}_n\right) \\
&= \nu\left(x : \exists\, 0 \le t < \mathcal{T}_n : \ f_t(x) \in B(y_n, r_n)\right) \\
&\le \nu\left(x : S_{\mathcal{T}_n,2r_n}(x) > r_n/K\right) \le K{r_n}^{-1} \int_0^{\mathcal{T}_n} \int \mathbf{1}_{B(y_n,2r_n)}(f_t(x))d\nu(x)dt \\
&\le K{r_n}^{-1}\mathcal{T}_n\nu(B(y_n, 2r_n)) \le K\mathcal{T}_n r_n^{\underline{\alpha}-\varepsilon-1} = K\mathcal{T}_n 2^{-n(\underline{\alpha}-\varepsilon-1)},
\end{aligned}$$

hence choosing $\mathcal{T}_n = 2^{n(\underline{\alpha}-\varepsilon-1)}/n^2$ the estimate above is summable. By Borel-Cantelli, for ν-almost every x

$$\liminf_{n\to\infty} \frac{\log \tau_n(x)}{n \log 2} \ge \underline{\alpha} - 1 - \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary one can send it to 0, and by Remark 4.1 the lower bound extends to. ☐

Note that the proof of lower bound is independent of the mixing properties of (F, μ). For upper bound, first notice that the cover time of the Poincaré map is comparable to the cover time of the flow.

Lemma 6.3. *Let*

$$\tau_r^F(x) := \min\{n \in \mathbb{N}_0 : \forall y \in Y, \exists 0 \le j \le n : d(y, F^j x) < r\} \in \mathbb{N}.$$

There exists $\lambda = \frac{1}{C_f}$ *for* C_f *defined in (H5) such that* $\tau_r(x) \le T_1 + \sum_{j=0}^{\tau^F_{\lambda r}(x)} R_Y(F^j x)$.

Proof. This is adapted from the proof of [JT, Lemma 6.4] and [RT, Theorem 2.1]. F is by assumption Gibbs-Markov so one can find $\mathcal{P}$, a natural partition of Y using cylinder sets with respect to F such that for each $P \in \mathcal{P}$: (a) $\operatorname{diam}(P) \le r/C_f$, and (b) for all $0 < t \le T_1$, $f_t(P)$ is connected. Suppose $\tau^Y_{r/C_f}(x) = k$, then the orbit $\{x, F(x), \dots, F^k(x)\}$ must have visited every element of $\mathcal{P}$, and by (6.1) for each $y \in \Omega$ there is $P \in \mathcal{P}$ and $0 < s \le T_1$ such that $y \in f_s(P)$ and hence there exists $j \le k$ such that $d(f^s(F^j(x)), y) \le C_f|P| < r$. Then set $\lambda = 1/C_f$ the lemma is proved. ☐

Proof of (6.3). Now assume $\overline{\dim}_M(\nu) = \dim_M(\mu)+1$ and (6.2), we will show that $\limsup_{r\to 0} \log \tau_r(x)/-\log r \le \overline{\dim}_M(\nu) - 1$ for ν-a.e. x. Let $\xi > 0$ be arbitrary and set

$$U_{\xi,N} := \left\{x \in Y : |R_n(x) - n\overline{R}| \le \xi n, \forall n \ge N\right\},.$$

where $R_n(x) = \sum_{j=0}^{n-1} R_Y(F^j(x))$. By ergodicity $\lim_N \mu(U_{\xi,N}) = 1$ so for N large enough $\nu(U_{\xi,N}) > 0$ hence by invariance,

$$\lim_{N\to\infty} \nu\left(\bigcup_{t=0}^{\xi N} f_{-t}(U_{\xi,N})\right) = 1. \tag{6.4}$$

Let $\varepsilon > 0$ be arbitrary, for each ν typical $x \in \mathcal{X}$ on can pick t^* such that $f_{t^*} \in Y$ and for all sufficiently small $r > 0$ by Theorem 1.4 applied to the Poincaré map and Lemma 6.3 we have the following two inequalities,

$$\frac{\log(\tau_{\lambda r}(x) - T_1)}{-\log r} \leq \frac{\log\left((\overline{R}+\xi)\tau_r^F(f_{t^*}x)\right)}{-\log r}, \quad \frac{\log \tau_r^F(f_{t^*}x)}{-\log r} \leq \dim_M(\mu) + \varepsilon.$$

Then as $\lambda, \overline{R}$ are constants and ε is arbitrary, for ν-almost every x,

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \dim_M(\mu) = \overline{\dim}_M(\nu) - 1. \qquad \square$$

6.1. **Example: suspension semi-flows over topological Markov shifts.** $\mathcal{A}$ be a finite alphabet and M an $\mathcal{A}\times\mathcal{A}$ matrix with $\{0,1\}$ entries, we will consider two-sided topological Markov shift systems $(\Sigma, \sigma, \phi, \mu)$, where

$$\Sigma := \left\{x = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}} \mid \text{for all } j\text{: } x_j \in \mathcal{A} \text{ and } M_{x_j,x_{j+1}} = 1\right\},$$

σ denotes the usual left shift, ϕ is a locally-Hölder potential and μ is the unique Gibbs measure with respect to ϕ, and assume that $\dim_M(\mu)$ is well defined. Define the symbolic metric on Σ by $d(x,y) = 2^{-x\wedge y}$, where

$$x \wedge y = \sup\{k \geq 0 : x_j = y_j,\ \forall |j| < k\}.$$

An n-cylinder in this setting is given by $[x_{-(n-1)}, \dots, x_0, \dots, x_{n-1}] := \{y \in \Sigma,\ y_j = x_j, \forall |j| < n\}$, and it is a well-known fact that balls in Σ are precisely the cylinder sets. The left-shift map σ is bi-Lipschitz with Lipschitz constant $L = 2$.

Let $\varphi : \Sigma \to \mathbb{R}$ be a Lipschitz roof function, define the space

$$Y_\varphi := \{(x,s) \in \Sigma \times \mathbb{R}_{\geq 0} : 0 \leq s \leq \varphi(x)\} / \sim$$

where $(x, \varphi(x)) \sim (\sigma(x), 0)$ for all $x \in I$. The suspension flow Ψ over σ is the function acts on Y_φ by

$$\Psi_t(x,s) = (\sigma^k(x), v),$$

where $k, v \geq 0$ are determined by $s + t = v + \sum_{j=0}^{k-1} \varphi(\sigma^j(x))$. The invariant measure ν for the flow Ψ on Y_φ satisfies the following: for every $g : Y_\varphi \to \mathbb{R}$ continuous,

$$\int g d\nu = \frac{1}{\int_\Sigma \varphi d\mu} \int_\Sigma \int_0^{\varphi(x)} g(x,s) ds d\mu(x) \tag{6.5}$$

The metric on Y_φ is the Bowen-Walters distance d_Y (see for example [BW]). Define another metric d_π on Y_φ: for all $(x_i, t_i)_{i=1,2} \in Y_\varphi$,

$$d_\pi((x_1,t_1),(x_2,t_2)) := \min\left\{\begin{matrix} d(x,y) + |s-t|, \\ d(\sigma x, y) + \varphi(x) - s + t, \\ d(x, \sigma y) + \varphi(y) - t + s \end{matrix}\right\},$$

then the following proposition says this metric is comparable to the Bowen-Walters distance.

Proposition 6.4. [BS, Proposition 17] *there exists $c = c_\pi$ such that*

$$c^{-1} d_\pi((x_1,t_1),(x_2,t_2)) \leq d_Y((x_1,t_1),(x_2,t_2)) \leq c\, d_\pi((x_1,t_1),(x_2,t_2)).$$

Then the Minkowski dimension of the flow-invariant measure ν is given by

Proposition 6.5. $\dim_M(\nu) = \dim_M(\mu) + 1$.

Proof. The proof is based on the proof of [RT, Theorem 4.3] for correlation dimensions.

By Proposition 6.4 for all $r > 0$,

$$(B(x, r/2c) \times (s - r/2c, s + r/2c)) \cap Y \subset B_Y((x, s), r)$$

where B_Y denotes the ball with respect to the metric d_Y, then for all $(x, s) \in Y_\varphi$, put $\overline{\varphi} = \int_\Sigma \varphi d\mu$, then

$$\nu(B_Y((x, s), r)) \geq \nu(B(x, r/2c) \times \left(s - \frac{r}{2c}, s + \frac{r}{2c}\right)$$

$$\frac{\log \nu(B_Y((x, s), r))}{\log r} \leq \frac{\log\left(\frac{r}{c\overline{\varphi}} \mu\left(B(x, \frac{r}{2c})\right)\right)}{\log r}$$

hence $\overline{\dim}_M(\nu) = \limsup_{r\to 0} \frac{\log \min_{(x,s)\in supp(\nu)} \nu(B_Y((x,s),r)}{\log r} \leq \dim_M(\mu) + 1$.

For lower bound, define

$$B_1 := B(x, cr) \times (s - cr, s + cr), \;\; B_2 := B(\sigma x, cr) \times [0, cr)$$
$$B_3 := \left\{(y, t) : y \in B(\sigma^{-1}x, 2cr), \text{ and } \varphi(y) - cr \leq t \leq \varphi(y)\right\}$$

Then as in the proof of [RT, Theorem 4.3], $B_Y((x, s), r) \subset (B_1 \cup B_2 \cup B_3) \cap Y_\varphi$.

For all $r > 0$ and $(x, s) \in Y_\varphi$ by (6.5), and as μ is σ, σ^{-1} invariant,

$$\nu(B_1 \cap Y_\varphi) = 2cr\mu(B(x, cr))/\overline{\varphi}, \;\; \nu(B_2, Y_\varphi) \leq cr\mu(B(x, cr))/\overline{\varphi}$$
$$\nu(B_3 \cap Y_\varphi) \leq cr\mu(\sigma^{-1}B(x, 2cr))/\overline{\varphi} = cr\mu(B(x, 2cr))/\overline{\varphi}.$$

Therefore

$$\nu(B_Y((x, s), r) \leq \frac{1}{\overline{\varphi}}\left(3r\mu(B(x, cr)) + cr\mu(B(x, 2cr))\right),$$

which is enough to conclude that $\underline{dim}_M(\nu) \geq \dim_M(\mu) + 1$. □

Acknowledgement

I am completing my PhD with the support from *Chinese Scholarship Council.* I am also thankful for various comments and help from my supervisor M. Todd, as well as other comments on Section 6 from J. Rousseau.

References

[BJK] B. Bárány, N. Jurga, I. Kolossváry. *On the Convergence Rate of the Chaos Game*, International Mathematics Research Notices, **2023** 5, (2023) 4456–4500.
[BS] L. Barreira, B. Saussol. *Multifractal analysis of hyperbolic flows*, Comm. Math. Phys **214** (2000), 339-371.
[BW] R.Bowen, P. Walters. *Expansive one-parameter flows*, J. Differential Equations, **12** (1972), 180-193.
[Bow] R. Bowen, *Equilibrium States and The Ergodic Theory of Anosov Diffeomorphisms*, Lect. Notes in Math. **470**, Springer, 1975.
[GRS] S. Gouëzel, J. Rousseau and M. Stadlbauer. *Minimal distance between random orbits*, Probab. Theory Relat. Fields (2024).
[JT] N. Jurga, M. Todd. *Cover times in dynamical systems*, In Press: Isarel Journal of Mathematics, (2024).
[K] A. Khintchine. *Continued Fractions*, Univ. Chicago Press, Chicago, (1964).
[KS] D. Kim and B. Seo. *The waiting time for irrational rotations*, Nonlinearity, **16** 1861 (2003).
[M] P. Matthews. *Covering problems for Brownian motion on spheres*, Ann. Probab. **(1)** 16 (1988) 189–199.

[RT] J. Rousseau, M. Todd. *Orbits closeness for slowly mixing dynamical systems*. Ergodic Theory and Dynam. Systems, **44** 4 (2024), 1192–1208.

Mathematical Institute, University of St Andrews, North Haugh, St Andrews, KY16 9SS, Scotland

Email address: bz29@st-andrews.ac.uk