Excercise-12

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Exercise

Theory Question Parameter inference with exectation maximization.

Consider a linear Gaussian model with the following structure:

- $x_0 \sim \mathcal{N}(m_0(\theta), P_0(\theta))$
- $x_k \sim \mathcal{N}(A(\theta)x_{k-1}, Q(\theta))$
- $y_k \sim \mathcal{N}(H(\theta)x_k, R(\theta))$

To-do: Estimate θ for some data $y_{1:N}$ an alternate to maximizing the likelihood (discussed in lecture) is the EM-Algorithm. It works as follows:

1. Start from an initial guess $\theta^{(0)}$.
2. For $n = 0, 1, 2, \ldots$ do:
   (a) **E-Step:** Compute:

$$\mathcal{Q}\left(\theta, \theta^{(n)}\right) := \int p(x_{0:N}|y_{0:N}, \theta^{(n)}) \log p(x_{0:N}, y_{0:N}|\theta) dx_{0:N} \tag{1}$$

   (b) **M-Step:** Compute: $\theta^{(n+1)} := \arg\max_\theta \mathcal{Q}(\theta, \theta^{(n)})$

In a considered case of linear Gaussian state space model, $\mathcal{Q}(\theta, \theta^{(n)})$ can be computed analytically. It is of the form:

$$\begin{aligned}
\mathcal{Q}(^{\theta,\theta^{(n)}}) = \\
& -\frac{1}{2}\log|2\pi P_0(\theta)| - \frac{N}{2}\log|2\pi Q(\theta)| - \frac{N}{2}\log|2\pi R(\theta)| \\
& -\frac{1}{2}\operatorname{tr}\left(P_0^{-1}(\theta)\left[P_0^{(s)} + (m_0^{(s)} - m_0(\theta))(m_0^{(s)} - m_0(\theta))^T\right]\right) \\
& -\frac{1}{2}\operatorname{tr}\left(Q^{-1}(\theta)\left[\Sigma - CA(\theta)^T - A(\theta)C^T + A(\theta)\Phi A(\theta)^T\right]\right) \\
& -\frac{1}{2}\operatorname{tr}\left(R^{-1}(\theta)\left[D - BH(\theta)^T - H(\theta)B^T + H(\theta)\Sigma H(\theta)^T\right]\right)
\end{aligned} \tag{2}$$

Where the following quantities are computed form the results of the Rauch-Tung-Striebel smoother run with parameter values $\theta^{(n)}$:

$$\Sigma = \frac{1}{N}\sum_{k=1}^{N}\left(P_k^{(s)} + m_k^s (m_k^s)^T\right)$$
$$\Phi = \frac{1}{N}\sum_{k=1}^{N}\left(P_{k-1}^{(s)} + m_{k-1}^s (m_{k-1}^s)^T\right)$$
$$B = \frac{1}{N}\sum_{k=1}^{N}\left(y_k (m_k^{(s)})^T\right)$$
$$C = \frac{1}{N}\sum_{k=1}^{N}\left(P_k^{(s)} G_{k-1}^T + m_k^{(s)} (m_{k-1}^s)^T\right)$$
$$D = \frac{1}{N}\sum_{k=1}^{N}\left(y_k y_k^T\right)$$

Exercise: Let $\theta = (A, Q)$, that is the model parameters are exactly the full transition model matrices. Derive closed form updates for both A and Q by computing the M-step.

(a) Compute A: We use the fact that trace is a linear operator.

$$\begin{aligned}
&\frac{\partial Q}{\partial A} = 0 \\
&\implies \frac{\partial}{\partial A}\left[-\frac{N}{2}\operatorname{tr}\left(Q^{-1}(\Sigma - CA^T - AC^T + A\Phi A^T)\right)\right] = 0 \\
&\implies \frac{N}{2}\frac{\partial}{\partial A}\operatorname{tr}\left[-Q^{-1}AC^T - Q^{-1}CA^T + Q^{-1}A\Phi A^T\right] = 0 \\
&\implies \frac{\partial}{\partial A}\operatorname{tr}\left[-Q^{-1}AC^T - Q^{-1}CA^T + Q^{-1}A\Phi A^T\right] = 0 \\
&\implies \frac{\partial}{\partial A}\operatorname{tr}\left[Q^{-1}A\Phi A^T\right] = \frac{\partial}{\partial A}\operatorname{tr}\left[Q^{-1}AC^T + Q^{-1}CA^T\right]
\end{aligned} \tag{3}$$

Now consider the LHS:

$$\frac{\partial}{\partial A}\operatorname{tr}\left[Q^{-1}A\Phi A^T\right] \tag{4}$$

This is known matrix differential form of the trace.

$$\frac{\partial}{\partial X_d}\operatorname{tr}\left[A_d X_d B_d X_d^T C_d\right] = A_d^T C_d^T X_d B_d^T + C_d A_d X_d B_d$$

Where $A_d = Q^{-1}$, $B_d = \Phi$, $C_d = I$ and $X_d = A$.

Thus we have we use the symmetry of the postive semi-definte matrix inverse $Q^{-1} = (Q^{-1})^T$ to simplify the LHS:

$$\begin{aligned}
\frac{\partial}{\partial A}\operatorname{tr}\left[Q^{-1}A\Phi A^T\right] &= Q^{-T} I^T A\Phi^T + IQ^{-1}A\Phi \\
&= Q^{-T}A\Phi^T + Q^{-1}A\Phi \\
&= Q^{-T}A\Phi^T + Q^{-1}A\Phi \\
&= 2Q^{-1}A\Phi
\end{aligned} \tag{5}$$

For the RHS we can we can simplify the first term using:

$$\frac{\partial}{\partial X_d} \operatorname{tr}\left[A_d X_d B_d\right] = A_d^T B_d^T \qquad (6)$$

With $A_d = (Q^{-1})$ and $X_d = A$ and $B_d = C^T$ for the first term we get:

$$\frac{\partial}{\partial A} \operatorname{tr}\left[Q^{-1} A C^T\right] = Q^{-T} C \quad = Q^{-1} C \quad (7)$$

And for the second term we can use the following matrix identity we get:

$$\frac{\partial}{\partial X_d} \operatorname{tr}\left[A_d X_d^T\right] = A_d \qquad (8)$$

With $A_d = (Q^{-1})C$ and $X_d = A$ the second term simplifies to:

$$\frac{\partial}{\partial A} \operatorname{tr}\left[Q^{-1} C A^T\right] = Q^{-1} C \quad (9)$$

Thus taken together , using the linearity of the trace operator we can separate the terms we get:

$$\begin{aligned} \frac{\partial}{\partial A} \operatorname{tr}\left[Q^{-1} A C^T + Q^{-1} C A^T\right] &= Q^{-1} C + Q^{-1} C \\ &= 2 Q^{-1} C \end{aligned} \qquad (10)$$

Thus LHS and RHS together becomes:

$$\begin{aligned} 2Q^{-1} A \Phi &= 2 Q^{-1} C \\ Q^{-1} A \Phi &= Q^{-1} C \\ A \Phi &= C \\ A &= C \Phi^{-1} \end{aligned} \qquad (11)$$

(b) Compute Q:

We again compute the critical point by computing the matrix derivative with respect to Q and setting it to zero. Canceling the the constant from the linear operators for trace and derivative.

Let

$$Z = \left[\Sigma - C A^T - A C^T + A \Phi A^T\right]$$

$$
\begin{aligned}
\frac{\partial}{\partial Q}\left[-\frac{N}{2}\log|2\pi Q| - \frac{N}{2}\operatorname{tr}\left(Q^{-1}Z\right)\right] &= 0 \\
\frac{\partial}{\partial Q}\left[-\log|2\pi Q| - \frac{\partial}{\partial Q}\operatorname{tr}\left(Q^{-1}Z\right)\right] &= 0 \\
\frac{\partial}{\partial Q}\left[-\log|2\pi Q|\right] - \left[\frac{\partial}{\partial Q}\operatorname{tr}\left(Q^{-1}Z\right)\right] &= 0 \quad (12)\\
\frac{\partial}{\partial Q}\left[-\log|Q|\right] - \left[\frac{\partial}{\partial Q}\operatorname{tr}\left(Q^{-1}Z\right)\right] &= 0 \\
-\left[\frac{\partial}{\partial Q}\operatorname{tr}\left(Q^{-1}Z\right)\right] &= \frac{\partial}{\partial Q}\left[\log|Q|\right]
\end{aligned}
$$

Consider the LHS, we use the following matrix differential form of the trace.

$$
\frac{\partial}{\partial X_d}\operatorname{tr}(A_d X_d^{-1} B_d) = -X_d^{-T} A_d^T B_d^T X_d^{-T} \qquad (13)
$$

With $A_d = I$, $B_d = Z$ and $X_d = Q$, and the fact that $Q^{-T} = Q^{-1}$ being inverse of PSD metrix we get:

$$
\frac{\partial}{\partial Q}(Q^{-1}Z) = -Q^{-T}Z^T Q^{-T} = -Q^{-1}Z^T Q^{-1} \qquad (14)
$$

We also note that $Z = Z^T$ as we have that Σ and Φ are symmetric matrices.

$$
\begin{aligned}
Z^T &= (\Sigma - CA^T - AC^T + A\Phi A^T)^T & &= (\Sigma^T - AC^T - CA^T + A\Phi^T A^T) \\
&= (\Sigma - CA^T - AC^T + A\Phi A^T) & &= Z
\end{aligned}
\qquad (15)
$$

Thus we get the LHS to be :

$$
\frac{\partial}{\partial X_d}\operatorname{tr}(Q^{-1}Z) = -Q^{-1}ZQ^{-1} \qquad (16)
$$

For the RHS we use the matrix identity:

$$
\frac{\partial}{\partial X_d}\log|X_d| = X_d^{-T} \qquad (17)
$$

With $X_d = Q$ we get the RHS to be:

$$
\frac{\partial}{\partial Q}\log|Q| = Q^{-T} = Q^{-1} \qquad (18)
$$

Taken together we get

$$\begin{aligned} -Q^{-1}ZQ^{-1} &= -Q^{-1} \\ \implies Q^{-1}ZQ^{-1} &= Q^{-1} \\ \implies ZQ^{-1} &= I \end{aligned} \tag{19}$$

Thus from the uniqueness of the inverse we get

$$Q = Z = (\Sigma - CA^T - AC^T + A\Phi A^T) \tag{20}$$

References