Excercise-11

Batuhan, Oezcoemlekci (Matrikelnummer: 6300476)
Aakarsh, Nair (Matrikelnummer: 6546577)

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Exercise

(a) Derive f, that is a formula for the update mean m_t depending on only the previous filtering mean m_{t-1} and not the prediction mean m_t^-.

(b) Now we note that the function for g for the filter covariance does not depend on the data. In fact, the Kalman filter covariance taks on the form of the so-called *discrete-time algabraic Riccati equation(DARE),*. i.e an implicit equation of the form:

$$P_t = C^T P_{t-1} C - (C^T P_{t-1} U)(Z + U^T P_{t-1} U)^{-1}(U^T P_{t-1} C) + N$$

Derive the exact form of the Riccati equation by inserting the predicted covariance matrix into the updated covariance matrix (similar to what you did with the mean). Identify the relationship between (C, U, Z, N) and (A, Q, H, R).

Hints:

- Make sure that there is no m^- or P^- hidden in some auxilliary variables.
- You an write the update step for the covariance matrix $P_t = (I - KH)P_t^-$ where K is the Kalman gain and I is the idntity matrix. This should simplify things.

Answer:

We start with the equations we know that is:

$$P_t^- = AP_{t-1}A^T + Q \tag{1}$$

as well as :

$$P_t = (I - KH)P_t^- \tag{2}$$

We know the kalman gain is given by:

$$K = P_t^- H^t (HP_t^- H^t + R)^{-1} \tag{3}$$

We simplify the equation 2 by first muliplying through then using the Woodburry identity which is given as:

$$(A_w + U_w C_w V_w)^{-1} = A_w^{-1} - A_w^{-1} U_w (C_w^{-1} + V_w A_w^{-1} U_w)^{-1} V_w A_w^{-1}$$

Thus we get:

$$P_t = P_t^- - P_t^- H^T (H P_t^- H^T + R)^{-1} H P_t^- \tag{4}$$

We note that the left hand side follows the form of the woodburry identity where $A_w = (P_t^-)^{-1}$, $U_w = H^T$, $V_w = H$, $C = R^{-1}$.

Thus we can simplify the LHS of 4 to

$$P_t = ((P_t^-)^{-1} + H^T R^{-1} H)^{-1} \tag{5}$$

Now substituting 1 into 5 we get:

$$P_t = ((Q + A P_{t-1} A^T)^{-1} + H^T R^{-1} H)^{-1} \tag{6}$$

Using woodburry identity with $A_w = Q$ $U_w = A$, $C_w = P_{t-1}$, $V = A^T$ simplifying the inner inverse we get:

$$P_t = \left((Q^{-1} - Q^{-1} A (P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1}) + H^T R^{-1} H\right)^{-1} \tag{7}$$

We can re arrange the terms collecting terms without P_{t-1} to get:

$$P_t = \left((Q^{-1} + H^T R^{-1} H) - Q^{-1} A (P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1}\right)^{-1} \tag{8}$$

Let $K_1 = (Q^{-1} + H^T R^{-1} H)$, then we note that we have, $A_w = K_1$, $U_w = Q^{-1} A$, $C_w = -(P_{t-1}^{-1} + A^T Q^{-1} A)^{-1}$ and $V_w = A^T Q^{-1}$, Thus we can open the inverse to get:

$$P_t = K_1^{-1} - K_1^{-1} (Q^{-1} A) \left(-(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A\right)^{-1} (A^T Q^{-1}) K_1^{-1} \tag{9}$$

Consider the inner inverse term:

$$K_2 := (-(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A)^{-1} \tag{10}$$

Simplifying it we get:

$$K_2 = -\left(P_{t-1}^{-1} + A^T (Q^{-1} - Q^{-1} K_1^{-1} Q^{-1}) A\right)^{-1} \tag{11}$$

Where $A_w = P_{t-1}^{-1}$, $U_w = A^T$, $C_w = (Q^{-1} - Q^{-1} K_1^{-1} Q^{-1})$ and $V_w = A$, using woodburry identity we get:

$$K_2 = -\left(P_{t-1} - P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1}\right) \tag{12}$$

$$K_2 = -P_{t-1} + P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1} \tag{13}$$

Substituting 13 into 9 we get:

$$P_t = K_1^{-1} - K_1^{-1}(Q^{-1}A)\left(-P_{t-1} + P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1}\right)(A^TQ^{-1})K_1^{-1} \tag{14}$$

Thus P_t becomes:

$$\begin{aligned} P_t = \\ & K_1^{-1}(Q^{-1}A)P_{t-1}(A^TQ^{-1})K_1^{-1} \\ -(K_1^{-1}(Q^{-1}A)P_{t-1}A^T)((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}(AP_{t-1}(A^TQ^{-1})K_1^{-1}) \\ & + K_1^{-1} \end{aligned} \tag{15}$$

Comparing with the equation

$$P_t = C^TP_{t-1}C - (C^TP_{t-1}U)(Z + U^TP_{t-1}U)^{-1}(U^TP_{t-1}C) + N$$

We see that $C = A^TQ^{-1}K_1^{-1}$ and $C^T = K_1^{-1}Q^{-1}A$, $N = K_1^{-1}$, $Z = (Q^{-1} - Q^{-1}K_1Q^{-1})^{-1}$, $U^T = A$ and $U = A^T$

Substituting the value of $K_1 = (Q^{-1} + H^TR^{-1}H)$

We get:

- $C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1}$
- $C^T = (Q^{-1} + H^TR^{-1}H)^{-1}Q^{-1}A$
- $Z = (Q^{-1} - Q^{-1}(Q^{-1} + H^TR^{-1}H)Q^{-1})^{-1}$
- $U^T = A, U = A^T$

We still need to show that the terms C and C^T are transposes of each other Consider, $C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1}$

Let $A_w = Q^{-1}$, $U_w = H^T$, $C_w = R^{-1}$, $V_w = H$ then we get

$$\begin{aligned} C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1} &= A^TQ^{-1}(Q - QH^T(R + HQH^T)^{-1}HQ) \\ &= A^T - A^TH^T(R + HQH^T)^{-1}HQ \end{aligned}$$

Consider $C^T = (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A$, where

$$\begin{aligned} C^T = (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A &= (Q - QH^T (R + HQH^T)^{-1} HQ) Q^{-1} A \\ &= A - QH^T (R + HQH^T)^{-1} HA \end{aligned} \quad (16)$$

We note that R, Q is are PSD as they are covariance matrices of normal distributions, with $R = R^T$ and $Q = Q^T$ respectively. Moreover the matrix HQH^T is also symmetric as because assuming an eigendecompistion $H^T(L\Lambda L^T)H$ of Q then

$$\begin{aligned} HQH^T = (H^T(L\Lambda L^T)H)^T &= (H^T(L\Lambda L^T)^T(H^T)^T \\ &= H(L\Lambda L^T)H^T) = HQH^T \end{aligned} \quad (17)$$

Thus we see that $(R + HQH^T)^{-1}$ is symmetric as :

$$((R + HQH^T)^{-1})^T \;=\; (R^T + HQH^T)^{-1} \;=\; (R + HQH^T)^{-1} \quad (18)$$

Thus taking the transpose of C^T in equation 16 yields

$$\begin{aligned} (C^T)^T &= (A - QH^T(R + HQH^T)^{-1} HA)^T \\ &= A^T - A^T H^T (R + HQH^T)^{-1} HQ^T \\ &= A^T - A^T H^T (R + HQH^T)^{-1} HQ = C \end{aligned} \quad (19)$$