

Excercise-12

Batuhan, Oezcoemlekci (Matrikelnummer: 6300476)

Aakarsh, Nair (Matrikelnummer: 6546577)

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Exercise

Theory Question Parameter inference with expectation maximization.

Consider a linear Gaussian model with the following structure:

- $x_0 \sim \mathcal{N}(m_0(\theta), P_0(\theta))$
- $x_k \sim \mathcal{N}(A(\theta)x_{k-1}, Q(\theta))$
- $y_k \sim \mathcal{N}(H(\theta)x_k, R(\theta))$

To-do: Estimate θ for some data $y_{1:N}$ an alternate to maximizing the likelihood (discussed in lecture) is the EM-Algorithm. It works as follows:

1. Start from an initial guess $\theta^{(0)}$.
2. For $n = 0, 1, 2, \dots$ do:
 - (a) **E-Step:** Compute:

$$\mathcal{Q}(\theta, \theta^{(n)}) := \int p(x_{0:N} | y_{0:N}, \theta^{(n)}) \log p(x_{0:N}, y_{0:N} | \theta) dx_{0:N} \quad (1)$$

- (b) **M-Step:** Compute: $\theta^{(n+1)} := \arg \max_{\theta} \mathcal{Q}(\theta, \theta^{(n)})$

In a considered case of linear Gaussian state space model, $\mathcal{Q}(\theta, \theta^{(n)})$ can be computed analytically. It is of the form:

$$\begin{aligned} \mathcal{Q}(\theta, \theta^{(n)}) = & -\frac{1}{2} \log |2\pi P_0(\theta)| - \frac{N}{2} \log |2\pi Q(\theta)| - \frac{N}{2} \log |2\pi R(\theta)| \\ & - \frac{1}{2} \text{tr} \left(P_0^{-1}(\theta) \left[P_0^{(s)} + (m_0^{(s)} - m_0(\theta))(m_0^{(s)} - m_0(\theta))^T \right] \right) \\ & - \frac{1}{2} \text{tr} \left(Q^{-1}(\theta) \left[\Sigma - CA(\theta)^T - A(\theta)C^T + A(\theta)\Phi A(\theta)^T \right] \right) \\ & - \frac{1}{2} \text{tr} \left(R^{-1}(\theta) \left[D - BH(\theta)^T - H(\theta)B^T + H(\theta)\Sigma H(\theta)^T \right] \right) \end{aligned} \quad (2)$$

Where the following quantities are computed from the results of the Rauch-Tung-Striebel smoother run with parameter values $\theta^{(n)}$:

$$\Sigma = \frac{1}{N} \sum_{k=1}^N \left(P_k^{(s)} + m_k^s (m_k^s)^T \right)$$

$$\Phi = \frac{1}{N} \sum_{k=1}^N \left(P_{k-1}^{(s)} + m_{k-1}^s (m_{k-1}^s)^T \right)$$

$$B = \frac{1}{N} \sum_{k=1}^N \left(y_k (m_k^{(s)})^T \right)$$

$$C = \frac{1}{N} \sum_{k=1}^N \left(P_k^{(s)} G_{k-1}^T + m_k^{(s)} (m_{k-1}^s)^T \right)$$

$$D = \frac{1}{N} \sum_{k=1}^N \left(y_k y_k^T \right)$$

Exercise: Let $\theta = (A, Q)$, that is the model parameters are exactly the full transition model matrices. Derive closed form updates for both A and Q by computing the M-step.

(a) Compute A : We use the fact that trace is a linear operator.

$$\begin{aligned} \frac{\partial Q}{\partial A} &= 0 \\ \Rightarrow \frac{\partial}{\partial A} \left[-\frac{N}{2} \text{tr} \left(Q^{-1} (\Sigma - CA^T - AC^T + A\Phi A^T) \right) \right] &= 0 \\ \Rightarrow \frac{N}{2} \frac{\partial}{\partial A} \text{tr} \left[-Q^{-1} AC^T - Q^{-1} CA^T + Q^{-1} A\Phi A^T \right] &= 0 \\ \Rightarrow \frac{\partial}{\partial A} \text{tr} \left[-Q^{-1} AC^T - Q^{-1} CA^T + Q^{-1} A\Phi A^T \right] &= 0 \\ \Rightarrow \frac{\partial}{\partial A} \text{tr} \left[Q^{-1} A\Phi A^T \right] &= \frac{\partial}{\partial A} \text{tr} \left[Q^{-1} AC^T + Q^{-1} CA^T \right] \end{aligned} \quad (3)$$

Now consider the LHS:

$$\frac{\partial}{\partial A} \text{tr} \left[Q^{-1} A\Phi A^T \right] \quad (4)$$

This is known matrix differential form of the trace.

$$\frac{\partial}{\partial X_d} \text{tr} \left[A_d X_d B_d X_d^T C_d \right] = A_d^T C_d^T X_d B_d^T + C_d A_d X_d B_d$$

Where $A_d = Q^{-1}$, $B_d = \Phi$, $C_d = I$ and $X_d = A$.

Thus we have we use the symmetry of the postive semi-definite matrix inverse $Q^{-1} = (Q^{-1})^T$ to simplify the LHS:

$$\begin{aligned} \frac{\partial}{\partial A} \text{tr} \left[Q^{-1} A\Phi A^T \right] &= Q^{-T} I^T A\Phi^T + I Q^{-1} A\Phi \\ &= Q^{-T} A\Phi^T + Q^{-1} A\Phi \\ &= Q^{-T} A\Phi^T + Q^{-1} A\Phi \\ &= 2Q^{-1} A\Phi \end{aligned} \quad (5)$$

For the RHS we can we can simplify the first term using:

$$\frac{\partial}{\partial X_d} \text{tr} [A_d X_d B_d] = A_d^T B_d^T \quad (6)$$

With $A_d = (Q^{-1})$ and $X_d = A$ and $B_d = C^T$ for the first term we get:

$$\frac{\partial}{\partial A} \text{tr} [Q^{-1} A C^T] = Q^{-T} C = Q^{-1} C \quad (7)$$

And for the second term we can use the following matrix identity we get:

$$\frac{\partial}{\partial X_d} \text{tr} [A_d X_d^T] = A_d \quad (8)$$

With $A_d = (Q^{-1})C$ and $X_d = A$ the second term simplifies to:

$$\frac{\partial}{\partial A} \text{tr} [Q^{-1} C A^T] = Q^{-1} C \quad (9)$$

Thus taken together , using the linearity of the trace operator we can separate the terms we get:

$$\begin{aligned} \frac{\partial}{\partial A} \text{tr} [Q^{-1} A C^T + Q^{-1} C A^T] &= Q^{-1} C + Q^{-1} C \\ &= 2Q^{-1} C \end{aligned} \quad (10)$$

Thus LHS and RHS together becomes:

$$\begin{aligned} 2Q^{-1} A \Phi &= 2Q^{-1} C \\ Q^{-1} A \Phi &= Q^{-1} C \\ A \Phi &= C \\ A &= C \Phi^{-1} \end{aligned} \quad (11)$$

(b) Compute Q :

References