Exercise Sheet #7

due on Monday, 19 June 2023, 10am sharp

1. **Theory Question:** This theory exercise is about properties of block matrices, i.e. matrices of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A and D are square blocks of arbitrary size. You can assume that all inverses below exist.

   (a) **Inverse of a block matrix:** In the lecture, the following statement about the inverse of M was provided without proof:

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BQ^{-1}CA^{-1} & -A^{-1}BQ^{-1} \\ -Q^{-1}CA^{-1} & Q^{-1} \end{bmatrix},$$

   where $Q \coloneqq D - CA^{-1}B$ denotes the so-called *Schur complement*. Prove this statement.

   (b) **Block LU decomposition:** The LU decomposition of M is a decomposition into a lower triangular matrix L and an upper triangular matrix U such that $M = LU$. Show that the LU decomposition of M is given by

$$L = \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ C(A^{-\frac{1}{2}})^\top & Q^{\frac{1}{2}} \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} (A^{\frac{1}{2}})^\top & A^{-\frac{1}{2}}B \\ 0 & (Q^{\frac{1}{2}})^\top \end{bmatrix},$$

   where $A^{\frac{1}{2}}$ and $Q^{\frac{1}{2}}$ are the Cholesky factors of A and Q (assuming A and Q are s.p.d.), i.e. both are lower triangular matrices such that $A^{\frac{1}{2}}(A^{\frac{1}{2}})^\top = A$ and $Q^{\frac{1}{2}}(Q^{\frac{1}{2}})^T = Q$. $A^{-\frac{1}{2}}$ is the inverse of $A^{\frac{1}{2}}$.

   Note: If M is symmetric (i.e. $C = B^\top$), $U = L^\top$. Thus, $LL^\top$ is the Cholesky decomposition of M in this case.

   (c) **Determinant of a block matrix:** Show that

$$\det(M) = \det(A)\det(Q).$$

   Hint: You can use that

$$\det(\begin{bmatrix} P & Q \\ 0 & S \end{bmatrix}) = \det(\begin{bmatrix} P & 0 \\ R & S \end{bmatrix}) = \det(P)\det(S)$$

   and the multiplicativity of the determinant, i.e. $\det(P \cdot Q) = \det(P)\det(Q)$.

2. **Practical Question:** can be found in `Ex07.ipynb`