1. Theory Question: In the lecture we encountered the plot reproduced here, showing a structured classification problem. It was pointed out that the most salient feature of this problem is that each class seems to have its own generative distribution.

i.e the blue points have a different distribution than the green ones, (although both distributions happen to overlap).

Such situations indicate an anti-causal relationship.

("the label causes the inputs")

and it might seem that discriminative modeling paradigm adopted in the lecture is not appropriate.

The goal of this week's theory is to realise that the situation is a bit more subtle than that.

For the purpose of this exercise, we will assume that there are two classes C_1 and C_2, defining probability distributions

$p(x|C_1)$, $p(x|C_2)$ over the inputs, and

classes are drawn with the probability

$$p(C) = [p(C_1), p(C_2) = 1 - p(C_1)]$$

(a) Given a new input x, how would you compute the posterior $p(C_1|x)$? Show that it can be written as a logistic function.

Ans: $p(C_1|x) = \dfrac{1}{1+e^{-a(x)}}$ where $a(x) = \ln \dfrac{p(C_1|x)}{p(C_2|x)}$

Consider the RHS

$$\Rightarrow \frac{1}{1+e^{-a(x)}} = \frac{1}{1+e^{-\ln\left(\frac{p(C_1|x)}{p(C_2|x)}\right)}} = \frac{1}{1+\frac{p(C_2|x)}{p(C_1|x)}} = \frac{p(C_1|x)}{p(C_1|x)+p(C_2|x)}$$

using Bayes rule

$$\Rightarrow \frac{p(C_1|x)}{\frac{p(x|C_1)\,p(C_1)}{p(x)} + \frac{p(x|C_2)\,p(C_2)}{p(x)}}$$

$$\Rightarrow p(C_1|x)\left[\frac{p(x)}{p(x|C_1)p(C_1) + p(x|C_2)p(C_2)}\right]$$

From law of total probability $p(x) = p(x|C_1)p(C_1) + p(x|C_2)p(C_2)$

$$\Rightarrow p(C_1|x)\left[\frac{p(x)}{p(x)}\right] = p(C_1|x)$$