(c) Finally use the above results to show the following two statements. which help interpretting the covariance matrix and its inverse (the precision matrix.

(a) A zero entry in the i,j^{th} position of the covariance matrix Σ implies that x_i and x_j are marigimally independent

(b) A zero-entry in the i,j^{th} position of the precision matrix $\Lambda = \Sigma^{-1}$ implies that x_i and x_j are conditionally independent given all other variables x_k for $k \neq i,j$.

(a) Without loss of generality, we can assume that $(i,j) = (1,2)$, as we can permute the variables without changing the distribution.

we have been given that $\Sigma_{12} = 0$ and from the symmetry of covariance matrix we know $\Sigma_{21} = 0$

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left\{ \frac{-1}{2} (x-\mu)^T \Sigma (x-\mu) \right\} \quad ; x \in \mathbb{R}^D$$

consider the matrix $A = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_{D-2} \end{bmatrix}$; $A = A^T$

As we know the affine projection $p(z := Ax) = \mathcal{N}(x; A\mu, A\Sigma A^T)$

thus we have $z = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_D \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

The new mean $\mu' = A\mu = \begin{bmatrix} \mu_x \\ \mu_y \\ 0 \\ 0 \\ \vdots \end{bmatrix}_{D\times 1}$

The new covariance $\Sigma' = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_{D\times 2} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1D} \\ \vdots & & \ddots & \\ \Sigma_{D1} & & \cdots & \Sigma_{DD} \end{bmatrix} \begin{bmatrix} I_2 & O_2 \\ O_2 & O_{D\times 2} \end{bmatrix}$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 & 0 \\ \Sigma_{21} & \Sigma_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \end{bmatrix}$$

Thus the joint distribution of x_1 x_2 is given by

$$p\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

as we have $\Sigma_{21} = \Sigma_{12} = 0$

$$p(x_1 | x_2) = \mathcal{N}(x_1 ; \mu_{x_1} + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$
$$= \mathcal{N}(x_1 ; \mu_{x_1} ; \Sigma_{11}) = p(x_1)$$

and $p(x_2|x_1) = p(x_2)$ by symmetry.

$$p(x_1, x_2) = p(x_1|x_2)\, p(x_2)$$
$$= p(x_1)\, p(x_2)$$

thus x_1 and x_2 are marginally independent. ∎

(b) A zero-entry in the i, j^{th} position of the precision matrix $\Lambda = \Sigma^{-1}$ implies that x_i and x_j are conditionally independent given all other variables x_k for $k \neq i, j$.

Answer: Without loss of generality we have been given