

Excercise-11

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Exercise

- (a) Derive f , that is a formula for the update mean m_t depending on only the previous filtering mean m_{t-1} and not the prediction mean m_t^- .
- (b) Now we note that the function for g for the filter covariance does not depend on the data. In fact, the Kalman filter covariance takes on the form of the so-called *discrete-time algebraic Riccati equation* (DARE), i.e an implicit equation of the form:

$$P_t = C^T P_{t-1} C - (C^T P_{t-1} U)(Z + U^T P_{t-1} U)^{-1} (U^T P_{t-1} C) + N$$

Derive the exact form of the Riccati equation by inserting the predicted covariance matrix into the updated covariance matrix (similar to what you did with the mean). Identify the relationship between (C, U, Z, N) and (A, Q, H, R) .

Hints:

- Make sure that there is no m^- or P^- hidden in some auxilliary variables.
- You can write the update step for the covariance matrix $P_t = (I - KH)P_t^-$ where K is the Kalman gain and I is the identity matrix. This should simplify things.

Answer:

We start with the equations we know that is:

$$P_t^- = AP_{t-1}A^T + Q \quad (1)$$

as well as :

$$P_t = (I - KH)P_t^- \quad (2)$$

We know the kalman gain is given by:

$$K = P_t^- H^t (HP_t^- H^t + R)^{-1} \quad (3)$$

We simplify the equation 2 by first multiplying through then using the Woodbury identity ¹ which is given as:

¹ Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831. [Online; accessed 16-July-2023]

$$(A_w + U_w C_w V_w)^{-1} = A_w^{-1} - A_w^{-1} U_w (C_w^{-1} + V_w A_w^{-1} U_w)^{-1} V_w A_w^{-1}$$

Thus we get:

$$P_t = P_t^- - P_t^- H^T (H P_t^- H^T + R)^{-1} H P_t^- \quad (4)$$

We note that the left hand side follows the form of the Woodbury Identity² where $A_w = (P_t^-)^{-1}$, $U_w = H^T$, $V_w = H$, $C = R^{-1}$.

Thus we can simplify the LHS of 4 to

$$P_t = ((P_t^-)^{-1} + H^T R^{-1} H)^{-1} \quad (5)$$

Now substituting 1 into 5 we get:

$$P_t = ((Q + A P_{t-1} A^T)^{-1} + H^T R^{-1} H)^{-1} \quad (6)$$

Using woodbury identity with $A_w = Q$, $U_w = A$, $C_w = P_{t-1}$, $V = A^T$ simplifying the inner inverse we get:

$$P_t = \left((Q^{-1} - Q^{-1} A (P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1}) + H^T R^{-1} H \right)^{-1} \quad (7)$$

We can re arrange the terms collecting terms without P_{t-1} to get:

$$P_t = \left((Q^{-1} + H^T R^{-1} H) - Q^{-1} A (P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1} \right)^{-1} \quad (8)$$

Let $K_1 = (Q^{-1} + H^T R^{-1} H)$, then we note that we have, $A_w = K_1$, $U_w = Q^{-1} A$, $C_w = -(P_{t-1}^{-1} + A^T Q^{-1} A)^{-1}$ and $V_w = A^T Q^{-1}$, Thus we can open the inverse to get:

$$P_t = K_1^{-1} - K_1^{-1} (Q^{-1} A) \left(-(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A \right)^{-1} (A^T Q^{-1}) K_1^{-1} \quad (9)$$

Consider the inner inverse term:

$$K_2 := -(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A \quad (10)$$

Simplifying it we get:

$$K_2 = - \left(P_{t-1}^{-1} + A^T (Q^{-1} - Q^{-1} K_1^{-1} Q^{-1}) A \right)^{-1} \quad (11)$$

Where $A_w = P_{t-1}^{-1}$, $U_w = A^T$, $C_w = (Q^{-1} - Q^{-1} K_1^{-1} Q^{-1})$ and $V_w = A$, using woodbury identity we get:

² Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831. [Online; accessed 16-July-2023]

$$K_2 = - \left(P_{t-1} - P_{t-1} A^T ((Q^{-1} - Q^{-1} K_1 Q^{-1})^{-1} + A P_{t-1} A^T)^{-1} A P_{t-1} \right) \quad (12)$$

$$K_2 = -P_{t-1} + P_{t-1} A^T ((Q^{-1} - Q^{-1} K_1 Q^{-1})^{-1} + A P_{t-1} A^T)^{-1} A P_{t-1} \quad (13)$$

Substituting 13 into 9 we get:

$$P_t = K_1^{-1} - K_1^{-1} (Q^{-1} A) \left(-P_{t-1} + P_{t-1} A^T ((Q^{-1} - Q^{-1} K_1 Q^{-1})^{-1} + A P_{t-1} A^T)^{-1} A P_{t-1} \right) (A^T Q^{-1}) K_1^{-1} \quad (14)$$

Thus P_t becomes:

$$\begin{aligned} P_t = & K_1^{-1} (Q^{-1} A) P_{t-1} (A^T Q^{-1}) K_1^{-1} \\ & - (K_1^{-1} (Q^{-1} A) P_{t-1} A^T) ((Q^{-1} - Q^{-1} K_1 Q^{-1})^{-1} + A P_{t-1} A^T)^{-1} (A P_{t-1} (A^T Q^{-1}) K_1^{-1}) \\ & + K_1^{-1} \quad (15) \end{aligned}$$

Comparing with the equation

$$P_t = C^T P_{t-1} C - (C^T P_{t-1} U)(Z + U^T P_{t-1} U)^{-1} (U^T P_{t-1} C) + N$$

We see that $C = A^T Q^{-1} K_1^{-1}$ and $C^T = K_1^{-1} Q^{-1} A$, $N = K_1^{-1}$,
 $Z = (Q^{-1} - Q^{-1} K_1 Q^{-1})^{-1}$, $U^T = A$ and $U = A^T$

Substituting the value of $K_1 = (Q^{-1} + H^T R^{-1} H)$

We get:

- $C = A^T Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1}$
- $C^T = (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A$
- $Z = (Q^{-1} - Q^{-1} (Q^{-1} + H^T R^{-1} H) Q^{-1})^{-1}$
- $U^T = A, U = A^T$

We still need to show that the terms C and C^T are transposes of each other Consider, $C = A^T Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1}$

Let $A_w = Q^{-1}$, $U_w = H^T$, $C_w = R^{-1}$, $V_w = H$ then we get

$$\begin{aligned} C &= A^T Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1} = A^T Q^{-1} (Q - Q H^T (R + H Q H^T)^{-1} H Q) \\ &= A^T - A^T H^T (R + H Q H^T)^{-1} H Q \end{aligned}$$

Consider $C^T = (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A$, where

$$\begin{aligned} C^T &= (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A = (Q - Q H^T (R + H Q H^T)^{-1} H Q) Q^{-1} A \\ &= A - Q H^T (R + H Q H^T)^{-1} H A \quad (16) \end{aligned}$$

We note that R, Q are PSD as they are covariance matrices of normal distributions, with $R = R^T$ and $Q = Q^T$ respectively. Moreover the matrix HQH^T is also symmetric as because assuming an eigendecomposition³ $H^T(L\Lambda L^T)H$ of Q then

$$\begin{aligned} HQH^T &= (H^T(L\Lambda L^T)H)^T = (H^T(L\Lambda L^T)^T(H^T)^T) \\ &= H(L\Lambda L^T)H^T = HQH^T \end{aligned} \quad (17)$$

Thus we see that $(R + HQH^T)^{-1}$ is symmetric as :

$$((R + HQH^T)^{-1})^T = (R^T + HQH^T)^{-1} = (R + HQH^T)^{-1} \quad (18)$$

Thus taking the transpose of C^T in equation 16 yields

$$\begin{aligned} (C^T)^T &= (A - QH^T(R + HQH^T)^{-1}HA)^T \\ &= A^T - A^T H^T (R + HQH^T)^{-1} H Q^T \\ &= A^T - A^T H^T (R + HQH^T)^{-1} H Q = C \end{aligned} \quad (19)$$

Thus the final mapping we get is

1. $C = A^T Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1} = A^T - A^T H^T (R + HQH^T)^{-1} H Q$
2. $C^T = (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} A = A - QH^T (R + HQH^T)^{-1} H A$
3. $Z = (Q^{-1} - Q^{-1} (Q^{-1} + H^T R^{-1} H) Q^{-1})^{-1}$
4. $U^T = A, U = A^T$

References

Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831. [Online; accessed 16-July-2023].

Wikipedia contributors. Definite matrix — Wikipedia, the free encyclopedia, 2023b. URL https://en.wikipedia.org/w/index.php?title=Definite_matrix&oldid=1158373968. [Online; accessed 16-July-2023].

³ Wikipedia contributors. Definite matrix — Wikipedia, the free encyclopedia, 2023b. URL https://en.wikipedia.org/w/index.php?title=Definite_matrix&oldid=1158373968. [Online; accessed 16-July-2023]