1(c). Assume that you have found the mode $\hat{f}_x$ of this posterior (i.e the MAP-estimate) and used it to construct an appropriate Gaussian posterior $q(f_x) = \mathcal{N}(f_x ; \hat{f}_x, \Sigma)$ on $f_x$, wher $\Sigma$ is some covariance matrix which does not feature in the following. Show that the MAP estimate for the latent function $f$ which can be written compactly as

$$\mathbb{E}_q [f(\circ)] = m_\circ + k_{\circ x} \nabla \log p(y \mid \hat{f}_x)$$

that is using the gradient of the log-likelihood.

Answer:

① $\hat{f}_x$ is the mode of posterior.

② $q(f_x) = \mathcal{N}(f_x ; \hat{f}_x , \Sigma)$ is the approximate Gaussian posterior

③ Map Estimate of latent function f can be written as

$$\mathbb{E}_q(f(\cdot)) = m(x) + K_{\cdot x} \underline{\underline{\nabla \log}} \; \underline{\underline{p(y|\hat{f}_x)}}$$

④ We know that at $f_x = \hat{f}_x$ the mode.

$$\nabla_{f_x} \log p(f_x | y, X) = 0$$

$$\nabla_{f_x} \log p(y|f_x) + \nabla_{f_x} \log p(f_x) = 0$$

$$\nabla_{f_x} \log p(y|f_x) + \nabla_{f_x} \left[ \frac{1}{2} (f_x - m_x)^T K_{XX}^{-1} (f_x - m_x) \right] = 0$$

$$\nabla_{f_x} \log p(y|f_x) - K_{XX}^{-1} (f_x - m_x) = 0$$

$$\nabla_{f_x} \log (p(y|f_x)) = K_{XX}^{-1} (f_x - m_x)$$

thus $f_x = \hat{f}_x$ we have:

$$\nabla_{f_x} \log (p(y|\hat{f})) = K_{XX}^{-1} (\hat{f} - m_x)$$

⑤ As we know $\hat{f}$ the approximate Gaussian posterior at training points.

$$q(f_X) = \mathcal{N}(f_X; \hat{f}, \Sigma)$$

⑥ The posterior predictions at $f_x$ (Lecture-14, Slide-17)

$$q(f_x | y) = \int p(f_x | \vec{f}_X)\, q(\vec{f}_X)\, d\vec{f}_X$$

$$= \mathcal{N}(f_x; m_x + K_{xX} K_{XX}^{-1}(\hat{f} - m_X), \Sigma_{posterior})$$

Thus $E_q(f(\cdot)) = m(\cdot) + K_{\cdot X} K_{XX}^{-1}(\hat{f} - m_X)$

as expected value is precisely mean of gaussian.

From ④ we have at $f_X = \hat{f}$

$$K_{XX}^{-1}(\hat{f} - m_X) = \nabla_{f_X} \log(p(y | \hat{f}))$$

Thus $E_q(f(\cdot)) = m(\cdot) + K_{\cdot X} \nabla_{f_X} \log(p(y | \hat{f}))$