1(a) Show that

$$N(x; a, A)\, N(x; b, A) = N(a; b, A+B)\, N(x; (A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b), (A^{-1}+B^{-1})^{-1})$$

We start by showing that their multiplicative factors are equivlent

LHS

$$(2\pi)^{-D/2} \det(A)^{-1/2} * (2\pi)^{-D/2} \det(B)^{-1/2} = (2\pi)^{-D} \det(A)^{-1/2} \det(B)^{-1/2}$$

$$= (2\pi)^{-D} \det(AB)^{-1/2}$$

Using $\det(A)\det(B) = \det(AB)$

RHS :

$$(2\pi)^{-D/2} \det(A+B)^{-1/2} (2\pi)^{-D/2} \det((A^{-1}+B^{-1})^{-1})^{-1/2}$$

$$= (2\pi)^{-D} \det(A+B)^{-1/2} \det((A^{-1}+B^{-1})^{-1})^{-1/2}$$

$$= (2\pi)^{-D} \det((A+B)(A^{-1}+B^{-1})^{-1})^{-1/2}$$

using the Wood berry identity $(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A = A(I-(A+B)^{-1}A)$

$$= (2\pi)^{-D} \left[\det\left((A+B)\, A\, (I-(A+B)^{-1}A)\right)\right]^{-1/2}$$

$$= (2\pi)^{-D} \left[\det(A+B) \det(A) \det(I-(A+B)^{-1}A)\right]^{-1/2}$$

$$= (2\pi)^{-D} \det(A)^{-1/2} \left[\det((A+B) - (A+B)(A+B)^{-1}A)\right]^{-1/2}$$

$$= (2\pi)^{-D} \det(A)^{-1/2} \left[\det(A+B-A)\right]^{-1/2}$$

$$= (2\pi)^{-D} \det(A)^{-1/2} \det(B)^{-1/2}$$

$$= (2\pi)^{-D} \det(AB)^{-1/2}$$

Thus the multiplying exponent in both RHS and LHS are equal.

We now compare the exponent term of LHS & RHS ignoring the $-1/2$ factor for simplicity

LHS : $(x-a)^T A^{-1} (x-a) + (x-b)^T B^{-1} (x-b)$

opening the symetric multiplicative terms we have.

$$(x-a)^T (A^{-1}x - A^{-1}a) + (x-b)^T (B^{-1}x - B^{-1}b)$$

$$\Rightarrow x^T A^{-1} x - x^T A^{-1} a - a^T A^{-1} x + a^T A^{-1} a + x^T B^{-1} x - x^T B^{-1} b - b^T B^{-1} x + b^T B^{-1} b$$

$\Rightarrow$ combining $x^T \underset{\smile}{.*} x$ terms using multiplicative & distributive laws.

$$\Rightarrow x^T (A^{-1} + B^{-1}) x - x^T (A^{-1}a + B^{-1}b) - (a^T A^{-1} + b^T B^{-1}) x + a^T A^{-1} a + b^T B^{-1} b$$

we know that A, B are positive semi-definite & symetric as they are covariance matrices. $\Rightarrow (A^{-1})^T = A^{-1}$

$$\Rightarrow \underbrace{x^T (A^{-1} + B^{-1}) x}_{\text{LHS \#1}} - \underbrace{x^T (A^{-1}a + B^{-1}b)}_{\text{LHS \#2}} - \underbrace{(A^{-1}a + B^{-1}b)^T x}_{\text{LHS \#3}} + \underbrace{a^T A^{-1} a}_{\text{LHS \#4}} + \underbrace{b^T B^{-1} b}_{\text{LHS \#5}}$$

We label the terms and argue that they arise equivalently on RHS as well.

RHS : Consider the exponent term ignoring the common $-1/2$ factor of RHS we see. we have.

we consider the exponet arising from $N(a; b, A+B)$, with label RHS#1

RHS#1

$$(a-b)^T (A+B)^{-1} (a-b)$$

$$\Rightarrow (a-b)^T \left[ (A+B)^{-1} a - (A+B)^{-1} b \right]$$

$$\Rightarrow \underbrace{a^T (A+B)^{-1} a}_{RHS\#1.1} - \underbrace{a^T (A+B)^{-1} b}_{RHS\#1.2} - \underbrace{b^T (A+B)^{-1} a}_{RHS\#1.3} + \underbrace{b^T (A+B)^{-1} b}_{RHS\#1.4}$$

Next we consider the terms arising in the exponent for

$$N\left(x;\ (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b),\ (A^{-1}+B^{-1})^{-1}\right)$$

RHS#2

$$\left(x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b)\right)^T (A^{-1}+B^{-1}) \left(x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b)\right)$$

$$\Rightarrow \left(x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b)\right)^T \left( (A^{-1}+B^{-1}) x - (A^{-1}a + B^{-1}b) \right)$$

$$\Rightarrow \overbrace{x^T (A^{-1}+B^{-1}) x}^{RHS\#2.1} - \overbrace{x^T (A^{-1}a + B^{-1}b)}^{RHS\#2.2} - \overbrace{(A^{-1}a + B^{-1}b)^T x}^{RHS\#2.3} +$$

$$+ \underbrace{(A^{-1}a + B^{-1}b)^T \left((A^{-1}+B^{-1})^{-1}\right)^T (A^{-1}a + B^{-1}b)}_{RHS\#2.4}$$

we note that

RHS#2.1 = LHS#1

RHS#2.2 = LHS#2

RHS#2.3 = LHS#3

We first simplify RHS#2.4

$$(A^{-1}a + B^{-1}b)^T \left((A^{-1}+B^{-1})^{-1}\right)^T (A^{-1}a + B^{-1}b)$$

① $A^{-1}$ is symmetric positive definite

② $B^{-1}$ is symmetric positive definite.

③ $(A^{-1}+B^{-1})$ is symmetric positive definite.

④ $\left((A^{-1}+B^{-1})^{-1}\right)^T = (A^{-1}+B^{-1})^{-1}$

$$\Rightarrow (A^{-1}a + B^{-1}b)^T (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b)$$

$$\Rightarrow (a^T A^{-1} + b^T B^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b)$$

$$\Rightarrow \left[a^T A^{-1}(A^{-1}+B^{-1})^{-1} + b^T B^{-1}(A^{-1}+B^{-1})^{-1}\right](A^{-1}a + B^{-1}b)$$

Simplifying using the Woodbury identities.

$$(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A = B - B(A+B)^{-1}B$$

$$\Rightarrow \left[a^T A^{-1}(A - A(A+B)^{-1}A) + b^T B^{-1}(B - B(A+B)^{-1}B)\right](A^{-1}a + B^{-1}b)$$

$$= \left[(a^T - a^T(A+B)^{-1}A) + (b^T - b^T(A+B)^{-1}B)\right](A^{-1}a + B^{-1}b)$$

$$= \left[(a^T(A^{-1}a + B^{-1}b) - a^T(A+B)^{-1}A(A^{-1}a + B^{-1}b) - b^T(A+B)^{-1}B(A^{-1}a + B^{-1}b) \right.$$
$$\left. + b^T(A^{-1}a + B^{-1}b)\right]$$

Simplifying.

$$= \left[\overbrace{a^T A^{-1} a}^{\text{RHS 2.4.1}} + \overbrace{a^T B^{-1} b}^{\text{RHS 2.4.3}} - \overbrace{a^T (A+B)^{-1} a}^{\text{RHS 2.4.5}} - \overbrace{a^T (A+B)^{-1} A B^{-1} b}^{\text{RHS 2.4.7}}\right.$$
$$\left.\underbrace{b^T B^{-1} B}_{\text{RHS 2.4.2}} + \underbrace{b^T A^{-1} a}_{\text{RHS 2.4.4}} - \underbrace{b^T (A+B)^{-1} b}_{\text{RHS 2.4.6}} - \underbrace{b^T (A+B)^{-1} B A^{-1} a}_{\text{RHS 2.4.8}}\right]$$

We note that RHS 2.4.1 = LHS 4

RHS 2.4.2 = LHS 5

We next note that : RHS 1.1 + RHS 2.4.5 = 0 $\{a^T(A+B)^{-1}a\}$

RHS 1.4 + RHS 2.4.6 = 0 $\{b^T(A+B)^{-1}b\}$

We collect the remaining RHS terms and show they sum to zero :

$$\Rightarrow \Big[ -\overbrace{a^T(A+B)^{-1}b}^{\text{RHS 1.2}} - \overbrace{b^T(A+B)^{-1}a}^{\text{RHS 1.3}}$$

$$+ \overbrace{a^T B^{-1} b}^{\text{RHS 2.4.3}} - \overbrace{a^T(A+B)^{-1}AB^{-1}b}^{\text{RHS 2.4.7}}$$

$$+ \underbrace{b^T A^{-1} a}_{\text{RHS 2.4.4}} - \underbrace{b^T(A+B)^{-1}BA^{-1}a}_{\text{RHS 2.4.8}} \Big]$$

collecting terms $a^T \cdot * b$ and $b^T \cdot * a$ matching.

$$\Rightarrow a^T\left[B^{-1} - (A+B)^{-1} - (A+B)^{-1}AB^{-1}\right]b$$

$$+ b^T\left[A^{-1} - (A+B)^{-1} - (A+B)^{-1}BA^{-1}\right]a$$

$$\Rightarrow a^T\left[B^{-1} - (A+B)^{-1}\left[I + AB^{-1}\right]\right]b + b^T\left[A^{-1} - (A+B)^{-1}(I + BA^{-1})\right]a$$

$$\Rightarrow a^T\left[B^{-1} - (A+B)^{-1}(B+A)B^{-1}\right]b + b^T\left[A^{-1} - (A+B)^{-1}(A+B)A^{-1}\right]a$$

$$\Rightarrow a^T\left[B^{-1} - B^{-1}\right]b + b^T\left[A^{-1} - A^{-1}\right]a$$

$$\Rightarrow 0 + 0 = 0$$

Thus we have shown that bothe the exponent and multiplicative factors of LHS and RHS are equivalent.

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