Excercise-11

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Exercise

(a) Derive f, that is a formula for the update mean m_t depending on only the previous filtering mean m_{t-1} and not the prediction mean m_t^-.

Answer: We have the prediction distribution:

$$p(x_t|Y_{0:t-1}) = \mathcal{N}(x_t; Am_{t-1}, AP_{t-1}A^T + Q) := \mathcal{N}(x_t; m_t^-, P_t^-) \quad (1)$$

and the update distribution:

$$p(x_t|Y_{0:t}) = \mathcal{N}(x_t; m_t^- + Kz, (I - KH)P_t^-) := \mathcal{N}(x_t; m_t, P_t) \quad (2)$$

Where we have

$$K := P_t^- H^T (HP_t^- H^T + R)^{-1} \quad (3)$$

$$z := y_t - Hm_t^- \quad (4)$$

Thus substituting into equation 3 for Kalman gain K the value of P_t^- from equation 1 of the predictive distribution we get:

$$\begin{aligned} K &= P_t^- H^T (HP_t^- H^T + R)^{-1} \\ &= \Big((AP_{t-1}A^T + Q)H^T(H(AP_{t-1}A^T + Q)H^T + R)^{-1}\Big) \end{aligned} \quad (5)$$

From the update equation we have :

$$m_t = m_t^- + Kz \quad (6)$$

Using values of $m_t^- = Am_{t-1}$ from the predictive, K and z from the update we get:

$$m_t = Am_{t-1} + \Big((AP_{t-1}A^T + Q)H^T(H(AP_{t-1}A^T + Q)H^T + R)^{-1}\Big)(y_t - HAm_{t-1}) \quad (7)$$

(b) Now we note that the function for g for the filter covariance does not depend on the data. In fact, the Kalman filter covariance taks on the form of the so-called *discrete-time algebraic Riccati equation(DARE)*,. i.e an implicit equation of the form:

$$P_t = C^T P_{t-1} C - (C^T P_{t-1} U)(Z + U^T P_{t-1} U)^{-1} (U^T P_{t-1} C) + N$$

Derive the exact form of the Riccati equation by inserting the predicted covariance matrix into the updated covariance matrix (similar to what you did with the mean). Identify the relationship between (C, U, Z, N) and (A, Q, H, R).

Hints:

- Make sure that there is no m^- or P^- hidden in some auxilliary variables.
- You an write the update step for the covariance matrix $P_t = (I - KH)P_t^-$ where K is the Kalman gain and I is the idntity matrix. This should simplify things.

Answer:

We start with the equations we know that is:

$$P_t^- = AP_{t-1}A^T + Q \tag{8}$$

as well as :

$$P_t = (I - KH)P_t^- \tag{9}$$

We know the kalman gain is given by:

$$K = P_t^- H^t (HP_t^- H^t + R)^{-1} \tag{10}$$

We simplify the equation 9 by first muliplying through then using the Woodburry identity [1] which is given as:

$$(A_w + U_w C_w V_w)^{-1} = A_w^{-1} - A_w^{-1} U_w (C_w^{-1} + V_w A_w^{-1} U_w)^{-1} V_w A_w^{-1}$$

Thus we get:

$$P_t = P_t^- - P_t^- H^T (HP_t^- H^T + R)^{-1} HP_t^- \tag{11}$$

We note that the left hand side follows the form of the Woodburry Identity[2] where $A_w = (P_t^-)^{-1}$, $U_w = H^T$, $V_w = H$, $C = R^{-1}$.

[1] Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831. [Online; accessed 16-July-2023]

[2] Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831. [Online; accessed 16-July-2023]

Thus we can simplify the LHS of 11 to

$$P_t = ((P_t^-)^{-1} + H^T R^{-1} H)^{-1} \tag{12}$$

Now substituting 8 into 12 we get:

$$P_t = ((Q + AP_{t-1}A^T)^{-1} + H^T R^{-1} H)^{-1} \tag{13}$$

Using woodburry identity with $A_w = Q$ $U_w = A$, $C_w = P_{t-1}$, $V = A^T$ simplifying the inner inverse we get:

$$P_t = \left((Q^{-1} - Q^{-1}A(P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1}) + H^T R^{-1} H\right)^{-1} \tag{14}$$

We can re arrange the terms collecting terms without P_{t-1} to get:

$$P_t = \left((Q^{-1} + H^T R^{-1} H) - Q^{-1}A(P_{t-1}^{-1} + A^T Q^{-1} A)^{-1} A^T Q^{-1}\right)^{-1} \tag{15}$$

Let $K_1 = (Q^{-1} + H^T R^{-1} H)$, then we note that we have, $A_w = K_1$, $U_w = Q^{-1}A$, $C_w = -(P_{t-1}^{-1} + A^T Q^{-1} A)^{-1}$ and $V_w = A^T Q^{-1}$, Thus we can open the inverse to get:

$$P_t = K_1^{-1} - K_1^{-1}(Q^{-1}A)\left(-(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A\right)^{-1} (A^T Q^{-1}) K_1^{-1} \tag{16}$$

Consider the inner inverse term:

$$K_2 := (-(P_{t-1}^{-1} + A^T Q^{-1} A) + A^T Q^{-1} K_1^{-1} Q^{-1} A)^{-1} \tag{17}$$

Simplifying it we get:

$$K_2 = -\left(P_{t-1}^{-1} + A^T(Q^{-1} - Q^{-1}K_1^{-1}Q^{-1})A\right)^{-1} \tag{18}$$

Where $A_w = P_{t-1}^{-1}$, $U_w = A^T$, $C_w = (Q^{-1} - Q^{-1}K_1^{-1}Q^{-1})$ and $V_w = A$, using woodburry identity we get:

$$K_2 = -\left(P_{t-1} - P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1}\right) \tag{19}$$

$$K_2 = -P_{t-1} + P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1} \tag{20}$$

Substituting 20 into 16 we get:

$$P_t = K_1^{-1} - K_1^{-1}(Q^{-1}A)\left(-P_{t-1} + P_{t-1}A^T((Q^{-1} - Q^{-1}K_1Q^{-1})^{-1} + AP_{t-1}A^T)^{-1}AP_{t-1}\right)(A^T Q^{-1})K_1^{-1} \tag{21}$$

Thus P_t becomes:

$$
\begin{aligned}
P_t = \\
& K_1^{-1}(Q^{-1}A)P_{t-1}(A^TQ^{-1})K_1^{-1} \\
& -(K_1^{-1}(Q^{-1}A)P_{t-1}A^T)((Q^{-1}-Q^{-1}K_1Q^{-1})^{-1}+AP_{t-1}A^T)^{-1}(AP_{t-1}(A^TQ^{-1})K_1^{-1}) \\
& + K_1^{-1} \qquad (22)
\end{aligned}
$$

Comparing with the equation

$$P_t = C^TP_{t-1}C - (C^TP_{t-1}U)(Z+U^TP_{t-1}U)^{-1}(U^TP_{t-1}C) + N$$

We see that $C = A^TQ^{-1}K_1^{-1}$ and $C^T = K_1^{-1}Q^{-1}A$, $N = K_1^{-1}$, $Z = (Q^{-1} - Q^{-1}K_1Q^{-1})^{-1}$, $U^T = A$ and $U = A^T$

Substituting the value of $K_1 = (Q^{-1} + H^TR^{-1}H)$

We get:

- $C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1}$
- $C^T = (Q^{-1} + H^TR^{-1}H)^{-1}Q^{-1}A$
- $Z = (Q^{-1} - Q^{-1}(Q^{-1} + H^TR^{-1}H)Q^{-1})^{-1}$
- $U^T = A, U = A^T$
- $N = K_1^{-1} = (Q^{-1} + H^TR^{-1}H)^{-1}$

We still need to show that the terms C and C^T are transposes of each other Consider, $C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1}$

Let $A_w = Q^{-1}$, $U_w = H^T$, $C_w = R^{-1}$, $V_w = H$ then we get

$$
\begin{aligned}
C = A^TQ^{-1}(Q^{-1}+H^TR^{-1}H)^{-1} &= A^TQ^{-1}(Q - QH^T(R+HQH^T)^{-1}HQ) \\
&= A^T - A^TH^T(R+HQH^T)^{-1}HQ
\end{aligned}
$$

Consider $C^T = (Q^{-1} + H^TR^{-1}H)^{-1}Q^{-1}A$, where

$$
\begin{aligned}
C^T = (Q^{-1}+H^TR^{-1}H)^{-1}Q^{-1}A &= (Q - QH^T(R+HQH^T)^{-1}HQ)Q^{-1}A \\
&= A - QH^T(R+HQH^T)^{-1}HA \qquad (23)
\end{aligned}
$$

We note that R, Q is are PSD as they are covariance matrices of normal distributions, with $R = R^T$ and $Q = Q^T$ respectively. Moreover the matrix HQH^T is also symmetric as because assuming an eigendecompistion [3] $H^T(L\Lambda L^T)H$ of Q then

$$
\begin{aligned}
HQH^T = (H^T(L\Lambda L^T)H)^T &= (H^T(L\Lambda L^T)^T(H^T)^T \\
&= H(L\Lambda L^T)H^T) = HQH^T \qquad (24)
\end{aligned}
$$

[3] Wikipedia contributors. Definite matrix — Wikipedia, the free encyclopedia, 2023b. URL `https://en.wikipedia.org/w/index.php?title=Definite_matrix&oldid=1158373968`. [Online; accessed 16-July-2023]

Thus we see that $(R + HQH^T)^{-1}$ is symmetric as :

$$((R + HQH^T)^{-1})^T = (R^T + HQH^T)^{-1} = (R + HQH^T)^{-1} \quad (25)$$

Thus taking the transpose of C^T in equation 23 yields

$$\begin{aligned}(C^T)^T &= (A - QH^T(R + HQH^T)^{-1}HA)^T \\ &= A^T - A^TH^T(R + HQH^T)^{-1}HQ^T \\ &= A^T - A^TH^T(R + HQH^T)^{-1}HQ = C \quad (26)\end{aligned}$$

Thus the final mapping we get is

1. $C = A^TQ^{-1}(Q^{-1} + H^TR^{-1}H)^{-1} = A^T - A^TH^T(R + HQH^T)^{-1}HQ$
2. $C^T = (Q^{-1} + H^TR^{-1}H)^{-1}Q^{-1}A = A - QH^T(R + HQH^T)^{-1}HA$
3. $Z = (Q^{-1} - Q^{-1}(Q^{-1} + H^TR^{-1}H)Q^{-1})^{-1}$
4. $U^T = A, U = A^T$
5. $N = (Q^{-1} + H^TR^{-1}H)^{-1}$

References

Wikipedia contributors. Woodbury matrix identity — Wikipedia, the free encyclopedia, 2023a. URL `https://en.wikipedia.org/w/index.php?title=Woodbury_matrix_identity&oldid=1158363831`. [Online; accessed 16-July-2023].

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