(c) Does this mean we can turn any discriminative model (predicting classlabels from inputs, $p(\mathcal{C}_k \mid \boldsymbol{x})$) into a generative one (predicting inputs from class labels, $p(\boldsymbol{x} \mid \mathcal{C}_k)$) by "going backward" through the model? Unfortunately, the answer is *no*, because continuous distributions for inputs are more complex than binary distributions. To make this clear, consider a special case: Assume that the class distributions are both univariate Gaussians with different means and variances: $p(\boldsymbol{x} \mid \mathcal{C}_k) = \mathcal{N}(\boldsymbol{x}; \mu_k, \sigma_k^2)$. From your answer above, construct the explicit form of θ, θ_0 (that's two real numbers) as a function of $\mu_1, \mu_2, \sigma_1, \sigma_2, p(\mathcal{C}_1)$. Assume someone has performed logistic regression and given you the parameters θ, θ_0. Can you recover the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2, p(\mathcal{C}_1)$? Could you do so if we set $p(C_1) = p(C_2) = \frac{1}{2}$ and $\sigma = 1$?

Answer: $p(C_k \mid x)$ into $p(x \mid C_k)$

inputs → class labels. class labels → inputs

* continuous distributions for inputs are more complex than binary -distributions.

We assume that class distributions are both univariate Gaussians with different mean and variances.

$p(x \mid C_1) = \mathcal{N}(x; \mu_1, \sigma_1^2)$ $\quad$ $p(x \mid C_2) = \mathcal{N}(x; \mu_2, \sigma_2^2)$

we know

$$p(x \mid \omega) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$= \exp\left[-\frac{x^2 - 2\mu x + \mu^2}{2\sigma^2} - \log\sqrt{2\pi\sigma^2}\right]$$

$$= \exp\left[\begin{bmatrix} x & -\frac{1}{2}x^2 \end{bmatrix}\begin{bmatrix} \mu/\sigma^2 \\ 1/\sigma^2 \end{bmatrix} - \left(\frac{\mu^2}{2\sigma^2} + \log\sqrt{2\pi\sigma^2}\right)\right]$$

if $\omega_1 = \mu/\sigma^2$ $\quad$ $\omega_2 = 1/\sigma^2$

$$= \exp\left[\begin{bmatrix} x & -\frac{1}{2}x^2 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} - \underbrace{\frac{1}{2}\left(\frac{\omega_1^2}{\omega_2} - \log\omega_2 + \log(2\pi)\right)}_{\log Z(\omega)}\right]$$

$$p(x|C_1) = \exp\left[[x \quad -1/x^2] \begin{bmatrix} \mu_1/\sigma_1^2 \\ 1/\sigma_1^2 \end{bmatrix} - \frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \log 2\pi\sigma_1^2\right)\right.$$

$$p(x|C_2) = \exp\left[[x \quad -1/x^2] \begin{bmatrix} \mu_2/\sigma_2^2 \\ 1/\sigma_2^2 \end{bmatrix} - \frac{1}{2}\left(\frac{\mu_2^2}{\sigma_2^2} - \log 2\pi\sigma_2^2\right)\right]$$

$$\frac{p(C_1|x)}{p(C_2|x)} = \frac{p(C_1)}{p(C_2)} \frac{\exp\left[[x \quad -1/x^2] \begin{bmatrix} \mu_1/\sigma_1^2 \\ 1/\sigma_1^2 \end{bmatrix} - \frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \log 2\pi\sigma_1^2\right)\right]}{\exp\left[[x \quad -1/x^2] \begin{bmatrix} \mu_2/\sigma_2^2 \\ 1/\sigma_2^2 \end{bmatrix} - \frac{1}{2}\left(\frac{\mu_2^2}{\sigma_2^2} - \log 2\pi\sigma_2^2\right)\right]}$$

$$= \frac{p(C_1)}{p(C_2)} \exp\left[[x \quad -1/x^2] \begin{bmatrix} \mu_1/\sigma_1^2 - \mu_2/\sigma_2^2 \\ 1/\sigma_1^2 - 1/\sigma_2^2 \end{bmatrix} - \frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_2^2}{\sigma_2^2} - \log 2\pi\sigma_1^2 + \log 2\pi\sigma_2^2\right)\right]$$

$$= \exp\left([x \quad -1/x^2] \begin{bmatrix} \mu_1/\sigma_1^2 - \mu_2/\sigma_2^2 \\ 1/\sigma_1^2 - 1/\sigma_2^2 \end{bmatrix} - \frac{1}{2}\left[\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_2^2}{\sigma_2^2} - \log \sigma_1^2 + \log \sigma_2^2\right] + \ln\left[\frac{p(C_1)}{p(C_2)}\right]\right)$$

Consider ratio of class prior probabilities.

$\ln \dfrac{p(C_1)}{p(C_2)}$ and variances $-\dfrac{1}{2} \log \dfrac{\sigma_1^2}{\sigma_2^2}$

for all values where these terms coincide $a(x)$ will be the same thus $p(C_1), p(C_2)$ σ_1, σ_2 will not be recoverable.

if we have $\sigma_1 = \sigma_2 = 1$ and $p(c_1) = p(c_2) = 1/2$

$$a(x) = \begin{bmatrix} x & -1/x^2 \end{bmatrix} \begin{bmatrix} \mu_1 - \mu_2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \mu_1^2 - \mu_2^2 \end{bmatrix}$$

we have been gin. that θ, θ_0 are the the result of performing logistic regression then we have.

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ 0 \end{bmatrix} \quad \text{and} \quad \theta_0 = -\frac{1}{2} \mu_1^2 - \mu_2^2$$

$$\theta_1 = \mu_1 - \mu_2 \qquad \theta_0 = \frac{1}{2}(\mu_1 + \mu_2)(\theta_1)$$

$$\frac{\theta_0}{\theta_2} = \frac{1}{2}(\mu_1 + \mu_2)$$

Thus μ_1 and μ_2 are recoverable in this scenaro