

1(a) Show that

$$N(x; a, A) N(x; b, B) = N(a; b, A+B) N(x; (A'+B')^{-1}(A'a+B'b), (A'+B')^{-1})$$

We start by showing that their multiplicative factors are equivalent

LHS

$$(2\pi)^{-D/2} \det(A)^{-1/2} \cdot (2\pi)^{-D/2} \det(B)^{-1/2} = (2\pi)^{-D} \det(A)^{-1/2} \det(B)^{-1/2} \\ = (2\pi)^{-D} \det(AB)^{-1/2}$$

Using $\det(A) \det(B) = \det(AB)$

RHS :

$$(2\pi)^{-D/2} \det(A+B)^{-1/2} (2\pi)^{-D/2} \det((A'+B')^{-1})^{-1/2} \\ = (2\pi)^{-D} \det(A+B)^{-1/2} \det((A'+B')^{-1})^{-1/2} \\ = (2\pi)^{-D} \det((A+B)(A'+B')^{-1})^{-1/2}$$

using the Woodberry identity $(A'+B')^{-1} = A - A(A+B)^{-1}A = A(I - (A+B)^{-1}A)$

$$= (2\pi)^{-D} \left[\det((A+B)A(I - (A+B)^{-1}A)) \right]^{-1/2} \\ = (2\pi)^{-D} \left[\det(A+B) \det(A) \det(I - (A+B)^{-1}A) \right]^{-1/2} \\ = (2\pi)^{-D} \det(A)^{-1/2} \left[\det((A+B) - (A+B)(A+B)^{-1}A) \right]^{-1/2} \\ = (2\pi)^{-D} \det(A)^{-1/2} \left[\det(A+B - A) \right]^{-1/2} \\ = (2\pi)^{-D} \det(A)^{-1/2} \det(B)^{-1/2} \\ = (2\pi)^{-D} \det(AB)^{-1/2}$$

Thus the multiplying exponent in both RHS and LHS are equal.

We now compare the exponent term of LHS & RHS ignoring the $-1/2$ factor for simplicity

$$\text{LHS} : (x-a)^T A^{-1} (x-a) + (x-b)^T B^{-1} (x-b)$$

opening the symmetric multiplicative terms we have.

$$(x-a)^T (A^{-1}x - A^{-1}a) + (x-b)^T (B^{-1}x - B^{-1}b)$$

$$\Rightarrow x^T A^{-1}x - x^T A^{-1}a - a^T A^{-1}x + x^T B^{-1}x - x^T B^{-1}b - b^T B^{-1}x + b^T B^{-1}b$$

\Rightarrow combining $x^T x$ terms using multiplicative & distributive laws.

$$\Rightarrow x^T (A^{-1} + B^{-1})x - x^T (A^{-1}a + B^{-1}b) - (a^T A^{-1} + b^T B^{-1})x + a^T A^{-1}a + b^T B^{-1}b$$

we know that A, B are positive semi-definite & symmetric as they are covariance matrices $\Rightarrow (A^{-1})^T = A^{-1}$

$$\Rightarrow \underbrace{x^T (A^{-1} + B^{-1})x}_{\text{LHS } \#1} - \underbrace{x^T (A^{-1}a + B^{-1}b)}_{\text{LHS } \#2} - \underbrace{(A^T a + B^T b)^T x}_{\text{LHS } \#3} + \underbrace{a^T A^{-1}a + b^T B^{-1}b}_{\text{LHS } \#4 \quad \text{LHS } \#5}$$

We label the terms and argue that the arise equivalently on RHS as well.

RHS : Consider the exponent term ignoring the common $-1/2$ factor of RHS we see we have.

we consider the exponent arising from $N(a; b, A+B)$, with label RHS#1

RHS#1

$$(a-b)^T (A+B)^{-1} (a-b)$$

$$\Rightarrow (a-b)^T \left[(A+B)^{-1} a - (A+B)^{-1} b \right]$$

$$\Rightarrow \underbrace{a^T (A+B)^{-1} a}_{\text{RHS}\#1.1} - \underbrace{a^T (A+B)^{-1} b}_{\text{RHS}\#1.2} - \underbrace{b^T (A+B)^{-1} a}_{\text{RHS}\#1.3} + \underbrace{b^T (A+B)^{-1} b}_{\text{RHS}\#1.4}$$

Next we consider the terms arising in the exponent for $N(x; (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b), (A^{-1}+B^{-1})^{-1})$

RHS#2

$$(x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b))^T (A^{-1}+B^{-1}) (x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b))$$

$$\Rightarrow (x - (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b))^T ((A^{-1}+B^{-1}) x - (A^{-1}a + B^{-1}b))$$

$$\Rightarrow \underbrace{x^T (A^{-1}+B^{-1}) x}_{\text{RHS}\#2.1} - \underbrace{x^T (A^{-1}a + B^{-1}b)}_{\text{RHS}\#2.2} - \underbrace{(A^{-1}a + B^{-1}b)^T x}_{\text{RHS}\#2.3} + \underbrace{(A^{-1}a + B^{-1}b)^T ((A^{-1}+B^{-1})^{-1})^T (A^{-1}a + B^{-1}b)}_{\text{RHS}\#2.4}$$

we note that RHS#2.1 = LHS#1

RHS#2.2 = LHS#2

RHS#2.3 = LHS#3

We first simplify RHS #2.4

$$(A^{-1}a + B^{-1}b)^T \left((A^{-1} + B^{-1})^{-1} \right)^T (A^{-1}a + B^{-1}b)$$

- ① A^{-1} is symmetric positive definite
- ② B^{-1} is symmetric positive definite
- ③ $(A^{-1} + B^{-1})$ is symmetric positive definite.
- ④ $\left((A^{-1} + B^{-1})^{-1} \right)^T = (A^{-1} + B^{-1})^{-1}$

$$\Rightarrow (A^{-1}a + B^{-1}b)^T (A^{-1} + B^{-1})^{-1} (A^{-1}a + B^{-1}b)$$

$$\Rightarrow (a^T A^{-1} + b^T B^{-1}) (A^{-1} + B^{-1})^{-1} (A^{-1}a + B^{-1}b)$$

$$\Rightarrow \left[a^T A^{-1} (A^{-1} + B^{-1})^{-1} + b^T B^{-1} (A^{-1} + B^{-1})^{-1} \right] (A^{-1}a + B^{-1}b)$$

Simplifying using the Woodbury identities.

$$(A^{-1} + B^{-1})^{-1} = A - A(A+B)^{-1}A = B - B(A+B)^{-1}B$$

$$\Rightarrow \left[a^T A^{-1} (A - A(A+B)^{-1}A) + b^T B^{-1} (B - B(A+B)^{-1}B) \right] (A^{-1}a + B^{-1}b)$$

$$= \left[(a^T - a^T (A+B)^{-1}A) + (b^T - b^T (A+B)^{-1}B) \right] (A^{-1}a + B^{-1}b)$$

$$= \left[(a^T (A^{-1}a + B^{-1}b)) - a^T (A+B)^{-1}(A^{-1}a + B^{-1}b) - b^T (A+B)^{-1}B (A^{-1}a + B^{-1}b) + b^T (A^{-1}a + B^{-1}b) \right]$$

Simplifying,

$$= \left[\underbrace{a^T A^{-1} a}_{\text{RHS 2.4.1}} + \underbrace{a^T B^{-1} b}_{\text{RHS 2.4.3}} - \underbrace{a^T (A+B)^{-1} a}_{\text{RHS 2.4.5}} - \underbrace{a^T (A+B)^{-1} AB^{-1} b}_{\text{RHS 2.4.7}} \right. \\ \left. + \underbrace{b^T B^{-1} B}_{\text{RHS 2.4.2}} + \underbrace{b^T A^{-1} a}_{\text{RHS 2.4.4}} - \underbrace{b^T (A+B)^{-1} b}_{\text{RHS 2.4.6}} - \underbrace{b^T (A+B)^{-1} BA^{-1} a}_{\text{RHS 2.4.8}} \right]$$

We note that $\text{RHS } 2 \cdot 4 \cdot 1 = \text{LHS } 4$

$\text{RHS } 2 \cdot 4 \cdot 2 = \text{LHS } 5$

We next note that : $\text{RHS } 1 \cdot 1 + \text{RHS } 2 \cdot 4 \cdot 5 = 0 \quad \left\{ a^T (A+B)^{-1} a \right\}$
 $\text{RHS } 1 \cdot 4 + \text{RHS } 2 \cdot 4 \cdot 6 = 0 \quad \left\{ b^T (A+B)^{-1} b \right\}$

We collect the remaining RHS terms and show they sum to zero :

$$\Rightarrow \left[- \underbrace{a^T (A+B)^{-1} b}_{\text{RHS } 1 \cdot 2} - \underbrace{b^T (A+B)^{-1} a}_{\text{RHS } 1 \cdot 3} \right. \\ \left. + \underbrace{a^T B^{-1} b}_{\text{RHS } 2 \cdot 4 \cdot 3} - \underbrace{\overbrace{a^T (A+B)^{-1} A B^{-1} b}^{\text{RHS } 2 \cdot 4 \cdot 7}}_{\text{RHS } 2 \cdot 4 \cdot 8} \right. \\ \left. + \underbrace{b^T A^{-1} a}_{\text{RHS } 2 \cdot 4 \cdot 4} - \underbrace{b^T (A+B)^{-1} B A^{-1} a}_{\text{RHS } 2 \cdot 4 \cdot 8} \right]$$

collecting terms $a^T \cdot * b$ and $b^T \cdot * a$
matching.

$$\Rightarrow a^T [B^{-1} - (A+B)^{-1} - (A+B)^{-1} A B^{-1}] b \\ + b^T [A^{-1} - (A+B)^{-1} - (A+B)^{-1} B A^{-1}] a$$

$$\Rightarrow a^T [B^{-1} - (A+B)^{-1} [I + A B^{-1}]] b + b^T [A^{-1} - (A+B)^{-1} (I + B A^{-1})] a$$

$$\Rightarrow a^T [B^{-1} - (A+B)^{-1} (B + A) B^{-1}] b + b^T [A^{-1} - (A+B)^{-1} (A+B) A^{-1}] a$$

$$\Rightarrow a^T [B^{-1} - B^{-1}] b + b^T [A^{-1} - A^{-1}] a$$

$$\Rightarrow 0 + 0 = 0$$

Thus we have shown that both the exponent and multiplicative factors of LHS and RHS are equivalent.

