Theory:

$$p(r_{oil}, r_{air}) = \mathcal{N}\left[\begin{pmatrix} r_{oil} \\ r_{air} \end{pmatrix}; \begin{pmatrix} 0.05 \\ 0.02 \end{pmatrix}, \underbrace{\begin{pmatrix} 0.080 & -0.045 \\ -0.045 & 0.040 \end{pmatrix}}_{\Sigma}\right]$$

$$R = x\, r_{oil} + (B-x)\, r_{air}$$

a) $$R = \underbrace{[x \;\; (B-x)]}_{A} \underbrace{\begin{bmatrix} r_{oil} \\ r_{air} \end{bmatrix}}_{z}$$

$$p(R) = p(Az) = \mathcal{N}(Az; A\mu, A\Sigma A^T)$$

$$= \mathcal{N}\left(R; 0.05x + 0.02(B-x), [x, B-x]\,\Sigma \begin{bmatrix} x \\ B-x \end{bmatrix}\right)$$

$$p(R) = \mathcal{N}\left(R; 0.05x + 0.02(B-x), [x, B-x]\begin{bmatrix} 0.125x - 0.045B \\ -0.085x + 0.040B \end{bmatrix}\right)$$

$$= \mathcal{N}(R; 0.05x + 0.02(B-x), 0.125x^2 - 0.045Bx + 0.085x^2 - 0.085Bx + 0.040B^2 - 0.040Bx)$$

$$= \mathcal{N}(R; \underbrace{0.03x + 0.02B}_{\mu_R}, \underbrace{0.210x^2 - 0.170Bx + 0.040B^2}_{\sigma_R^2})$$

b) $$E(R) = \int_{-\infty}^{\infty} p(R)\, R\, dR$$

$\star$ Expectation of a variable with Gaussian pdf is its mean

$$= \mu_R = \underbrace{0.03x + 0.02B}_{} \Rightarrow x_R^* = 0.05B$$

Maximization of this expression is possible with $x = B$ (i.e. all of the budget is spent on oil futures)

c) $$Var(R) = E(R^2) - E(R)^2 = E((R-\mu_R)^2) \overset{\star\star}{=} \sigma_R^2$$

By definition, the variance of a random variable with Gaussian pdf

$$\frac{\partial \sigma_R^2}{\partial x} = 0.42x - 0.17B = 0$$

$$0.42x = 0.17B$$

$$\boxed{x_0 = \frac{17}{42}B}$$

To minimize variance

d) We would like to maximize $E(R)$ given that

$$Var(R) = E(R^2) - E(R)^2 \leq 0.03$$

$$\sigma_R^2 = 0.210\,x^2 - 0.170\,x + 0.040 \leq 0.03$$

$$0.21\,x^2 - 0.17x + 0.04 + t^2 = 0.03 \quad \text{where } t \geq 0$$

$$E(R) = \mu_R = 0.03x + 0.028 = 0.03x + 0.02$$

We can find optimal x given the condition using the method of Lagrange multipliers. Imposing the condition:

$$\mathcal{L}(x, \lambda) = 0.03x + 0.02 - \lambda(0.210x^2 - 0.170x + 0.01 + t^2)$$

t^2 is for slackness introduced by the inequality

$$\frac{\partial \mathcal{L}}{\partial x} = 0.03 - 0.42\lambda x + 0.17\lambda = 0$$

$$x^* = \frac{17}{42} + \frac{3}{42\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -0.21\,x^2 + 0.17x + 0.01 - t^2 = 0$$

$$0.21\,x^2 - 0.17x + 0.01 \leq 0$$

$$21\,x^2 - 17x + 1 \leq 0$$

$$x_{\mp} = \frac{17 \mp \sqrt{205}}{42}$$

Since λ is positive by our initial choice:

$$x^* = \frac{\sqrt{205} + 17}{42} \approx 0.746$$

e) Maximize the probability that the return is positive

$$P(R>0) = 1 - P(R<0)$$

↓ Converting into normal distribution and writing as cumulative

$$P(R>0) = 1 - F\left(-\frac{\mu_R}{\sigma_R}\right)$$

$$\frac{dP}{dx} = -\underbrace{F'\left(-\frac{\mu_R}{\sigma_R}\right)}_{\text{Positive}}\left(\sigma_R \frac{d\mu_R}{dx} - \mu_R \frac{d\sigma_R}{dx}\right) \Big/ \underbrace{\sigma_R^2}_{\text{Positive}} = 0$$

Reduces to $\sigma_R \frac{d\mu_R}{dx} = \mu_R \frac{d\sigma_R}{dx}$

$$(0.21x^2 - 0.17Bx + 0.01B^2)(0.03) = (0.03x + 0.02B)(0.42x - 0.17B)$$

$$3(21x^2 - 17Bx + B^2) = (3x + 2B)(42x - 17B)$$

$$63x^2 - 51Bx + 12B^2 = 126x^2 - 51Bx + 84Bx - 34B^2$$

e) continued:

$\Rightarrow 63x^2 - 84Bx - 46B^2 = 0$

The equation has solutions $x_1 \approx 0.42B$
$x_2 \approx -1.75B$

since budget cannot be negative $x_1 = x^* = 0.42B$ is the optimal split.

f) $p'(r_{air} \mid \mu'_{air} = 0.03, \sigma'^2_{air} = 0.02) = \mathcal{N}(r_{air} \mid \mu'_{air} = 0.03, \sigma'^2_{air} = 0.04)$

$p(r_{oil} \mid r_{air}) = \dfrac{p(r_{air}, r_{oil})}{p(r_{air})}$ → First find the conditional

$= \mathcal{N}\left(r_{oil} ; \mu_{oil} + \sigma^2_{oa} \dfrac{1}{\sigma'^2_{air}} (r_{air} - \mu_{air}), \sigma^2_{oil} - \sigma^2_{oa} \dfrac{1}{\sigma'^2_{air}} \sigma^2_{oa}\right)$

where $\sigma^2_{oa} = \Sigma_{1,2} = \Sigma_{2,1} = -0.045$

$\mu_{oil} = 0.05$, $\mu_{air} = 0.02$

$\sigma^2_{oil} = 0.08$, $\sigma^2_{air} = 0.04$

See Lecture 6 slides 30-31 for the formula

Inserting the values in their place:

$p(r_{oil} \mid r_{air}) = \mathcal{N}(r_{oil} ; 0.05 + (-0.045) \dfrac{1}{0.04} (r_{air} - 0.02), \backslash$

$0.08 - (0.045)^2 \dfrac{1}{0.04})$

$= \mathcal{N}(r_{oil} ; -1.125\, r_{air} + 0.0725, 0.029375)$

We know the conditional and the prior (new data)

Using the info in L8, slide 29:

If $p(r_{air}) = \mathcal{N}(r_{air} ; \mu'_{air}, \Sigma'_{air})$

and $p(r_{oil} \mid r_{air}) = \mathcal{N}(r_{oil} ; A r_{air} + b, \Lambda)$

then $p(r_{oil}) = \mathcal{N}(r_{oil} ; A\mu'_{air} + b, \Lambda + A \Sigma'_{air} A^T)$

$= \mathcal{N}(r_{oil} ; -1.125(0.03) + 0.0725, 0.029375 + (-1.125)^2 (0.04))$

$= \mathcal{N}(r_{oil} ; \underbrace{0.03875}, 0.08)$

Mean is the rise in oil futures