1. Theory Question : This theory exerise is about the properties of block matrices. That is matrices of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$, where A and D are square blocks of arbitrary size.

You can assume all the inverses below exist.

(a) Inverse of a block matrix: In the lecture, the following statement about inverse of M was provided without proof.

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1} B Q^{-1} C A^{-1} & -A^{-1} B Q^{-1} \\ -Q^{-1} C A^{-1} & Q^{-1} \end{bmatrix}$$

$Q := D - CA^{-1}B$, denotes the so called Schur complement. Prove this statement.

<u>Answer</u> :- We construct a set of reductions E_1, E_2, E_3, E_4 such that

$$(E_4 E_3 E_2 E_1) M = I \Rightarrow M^{-1} = (E_4 E_3 E_2 E_1)$$

① Let $E_1 = \begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix} \Rightarrow E_1 M = \begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ C & D \end{bmatrix} := R_1$

② Let $E_2 = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} \Rightarrow E_2 R_1 = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ C-C & D-CA^{-1}B \end{bmatrix} := R_2$

③ Let $E_3 = \begin{bmatrix} I & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \Rightarrow E_3 R_2 = \begin{bmatrix} I & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & (D-CA^{-1}B) \end{bmatrix}$

$$= \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} := R_3$$

④ Let $E_4 = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \Rightarrow E_4 R_4 = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$$\Rightarrow M^{-1} = (E_4 E_3 E_2 E_1) = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} \right) \begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -A^{-1}B(D-CA^{-1}B)^{-1} \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}B\,Q^{-1}CA^{-1} & -A^{-1}B\,Q^{-1} \\ -Q^{-1}CA^{-1} & Q^{-1} \end{bmatrix}$$

$\blacksquare$

(b) Block LU decomposition: The LU decomposition of M is a decomposition into a lower triangular matrix L and upper triangular matrix U such that $M = LU$

Show that LU decomposition of M is given by.

$$L = \begin{bmatrix} A^{1/2} & 0 \\ C(A^{-1/2})^T & Q^{1/2} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} (A^{1/2})^T & A^{-1/2}B \\ 0 & (Q^{1/2})^T \end{bmatrix}$$

<u>Answer</u> : Forward multiplying LU we get

$$LU = \begin{bmatrix} A^{1/2} & 0 \\ C(A^{-1/2})^T & Q^{1/2} \end{bmatrix} \begin{bmatrix} (A^{1/2})^T & A^{-1/2}B \\ 0 & (Q^{1/2})^T \end{bmatrix}$$

$$= \begin{bmatrix} A^{1/2}(A^{1/2})^T & A^{1/2}A^{-1/2}B \\ C(A^{-1/2})^T(A^{1/2})^T & C(A^{-1/2})^T A^{-1/2}B + Q^{1/2}Q^{1/2^T} \end{bmatrix}$$

Using
① $A^{1/2}A^{1/2^T} = A$
② $Q^{1/2}Q^{1/2^T} = Q$

$$= \begin{bmatrix} A & B \\ C & D - CA^{-1}B + CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

We can also derive the expression for LU using block elimination.

Let $E_1 = \begin{bmatrix} A^{-1/2} & 0 \\ 0 & I \end{bmatrix}$ $\qquad E_1 M = \begin{bmatrix} A^{-1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^{1/2^T} & A^{-1/2}B \\ C & D \end{bmatrix} := R_1$

Let $E_2 = \begin{bmatrix} I & 0 \\ -CA^{-1/2^T} & I \end{bmatrix}$ $\qquad E_1 R_1 = \begin{bmatrix} I & 0 \\ -CA^{-1/2^T} & I \end{bmatrix} \begin{bmatrix} A^{1/2^T} & A^{-1/2}B \\ C & D \end{bmatrix}$

$$= \begin{bmatrix} A^{1/2^T} & A^{-1/2}B \\ -CA^{-1/2^T}A^{1/2^T} + C & D - CA^{-1/2^T}A^{-1/2}B \end{bmatrix}$$

$$= \begin{bmatrix} A^{1/2^T} & A^{-1/2}B \\ 0 & D - CA^{-1}B \end{bmatrix} := R_2$$

Let $E_3 = \begin{bmatrix} I & 0 \\ 0 & (D-CA^{-1}B)^{-1/2} \end{bmatrix} \Rightarrow E_3 R_2 = \begin{bmatrix} I & 0 \\ 0 & \underbrace{(D-CA^{-1}B)^{-1/2}}_{Q^{-1/2}} \end{bmatrix} \begin{bmatrix} A^{1/2 T} & A^{-1/2} B \\ 0 & \underbrace{(D-CA^{-1}B)}_{Q^{1/2} Q^{1/2 T}} \end{bmatrix}$

$$= \begin{bmatrix} A^{1/2 T} & A^{-1/2} B \\ 0 & (D-CA^{-1}B)^{1/2 T} \end{bmatrix}$$

Thus $(E_3 E_2 E_1) M = U$

$\Rightarrow M = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U$

$E_1^{-1} = \begin{bmatrix} A^{1/2} & 0 \\ 0 & I \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} I & 0 \\ CA^{-1/2 T} & I \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} I & 0 \\ 0 & Q^{1/2} \end{bmatrix}$

using block diagonal inverse formula above.

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} A^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ CA^{-1/2 T} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Q^{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} A^{1/2} & 0 \\ CA^{-1/2 T} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Q^{1/2} \end{bmatrix} = \begin{bmatrix} A^{1/2} & 0 \\ CA^{-1/2 T} & Q^{1/2} \end{bmatrix}$$

Which is the Lower triangular matrix as expected.

$\Rightarrow M = LU$ where $L = \begin{bmatrix} A^{1/2} & 0 \\ CA^{-1/2 T} & Q^{1/2} \end{bmatrix} \quad U = \begin{bmatrix} (A^{1/2})^T & A^{-1/2} B \\ 0 & Q^{1/2 T} \end{bmatrix}$

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1(c) Determinant of a block matrix: Show that

$$\det(M) = \det(A)\det(Q)$$

Hint: You can use that

$$\det\begin{pmatrix} P & Q \\ 0 & S \end{pmatrix} = \det\begin{pmatrix} P & 0 \\ R & S \end{pmatrix} = \det(P)\det(S)$$

and multiplicativity of the determinant $\det(P \cdot Q) = \det(P)\det(Q)$

<u>Answer</u>:

We know $M = LU$ is the LU decomposition for M.

$$\det(M) = \det(LU) = \det(L)\det(U)$$

From 1(b) we know

$$\Rightarrow \det\left(\begin{bmatrix} A^{1/2} & 0 \\ CA^{-1/2T} & Q^{1/2} \end{bmatrix}\right)\det\left(\begin{bmatrix} A^{1/2T} & A^{-1/2}B \\ 0 & Q^{1/2T} \end{bmatrix}\right)$$

$$\Rightarrow \det(A^{1/2})\det(Q^{1/2})\det(A^{1/2T})\det(Q^{1/2T})$$

$$\Rightarrow \det(A^{1/2}A^{1/2T})\det(Q^{1/2}Q^{1/2T})$$

$$= \det(A)\det(Q)$$

$\blacksquare$