ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 Principles of Digital Communications
Project Description May. 3, 2022

Consider a communication channel with the following input-output relationship (both the input and the output are complex numbers): when the input is $x \in \mathbb{C}$, the output $Y \in \mathbb{C}$ is given by

$$Y = \exp(\mathrm{j}\Theta)x + Z \tag{1}$$

where the random phase shift Θ is uniformly chosen between $[0, 2\pi[$, and Z is a 0 mean complex Gaussian — the real and imaginary parts of Z are *i.i.d.* $\mathcal{N}(0, 1)$. Z and Θ are independent of each other.

(a) Show that, given the channel input x, the output Y

- has independent magnitude $U = |Y|$ and argument $\arg(Y)$.
  [Hint: recall that $Y = |Y| \exp(\mathrm{j} \arg(Y))$.]
- $\arg(Y)$ is uniform in $[0, 2\pi[$
- For any $\rho \geq 0$

$$\Pr(U < \rho) = (2\pi)^{-1} \int_0^{2\pi} \int_0^{\rho} \exp\left(-\frac{1}{2}[r^2 + |x|^2 - 2r|x|\cos(\theta)]\right) r dr d\theta$$

Conclude that for any $u \geq 0$,

$$f_{U|X}(u|x) = u \exp\left(-\frac{u^2 + |x|^2}{2}\right) I_0(u|x|)$$

where $I_0(v) = (2\pi)^{-1} \int_0^{2\pi} \exp(-v \cos\theta) d\theta$.

[Note: $I_0(v)$ is the 0th order modified Bessel function of the 1st kind. It is well approximated by $\exp(v)/\sqrt{2\pi v}$ when $1 \ll v$.]

(b) Suppose we have m equally likely hypotheses, where $H = j$ means $x = c_j$ (c_j's are given complex numbers), and suppose that the observation is Y. Show that $U = |Y|$ is a sufficient statistic.

(c) Continuing with (b) Suppose for two different hypotheses $j \neq \ell$, $|c_j| = |c_\ell|$. Can one distinguish between these two hypotheses?

(d) Suppose a friend has designed a communication system to transmit k bits over this channel where message j is transmitted as the complex number c_j, and decoding is performed by the ML decoder. We design a new system by replacing c_j with $|c_j|$. Show that the new system and the old system performs identically.

(e) Note that in the new system in (c) the transmitted symbols are positive real numbers. Suppose these are $c_0 = a$, $c_1 = 3a$, $\dots$, $c_{m-1} = (2m+1)a$, where $a \gg 1$ is a positive real number and $m = 2^k$. Show that the error probability is well approximated by $2Q(a)$.

[Hint: by the note in (a) you can observe that when $||x| - u|$ is small compared to x, $f_{U|X}(u|x)$ is well approximated by $(2\pi)^{-1/2} \exp\left(-\frac{1}{2}[u - |x|]^2\right)$ — the pdf of the output of an additive Gaussian channel with input $|x|$.]

(f) Continuing with (e) show that the average energy per bit required for an error probability $p_e = 2Q(a)$ approximately equals $a^2 4^{(k+1)}/(3k)$.

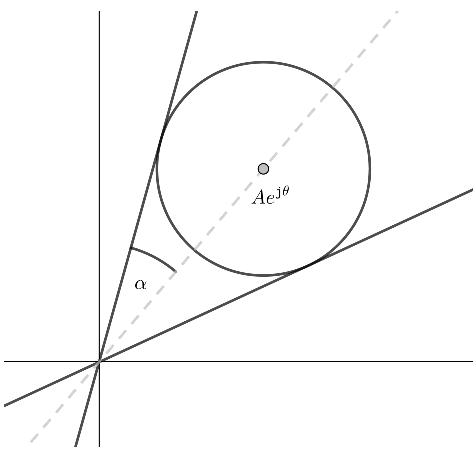
(g) Suppose now, the receiver is provided with the value of Θ. Show that we can transmit $2k$ bits with error probability less than $4Q(a)$ and energy per bit $a^2(4^k - 1)/(3k)$.

[Hint: consider c_j's of the form $\pm(2p-1) \pm i(2q-1)$ where p,q are in $\{1, \dots, 2^{k-1}\}$. where p and q are integers (perhaps different letters?) with $|2p|$, $|2q| \le 2^k$. Note that the design we made here has significant savings in energy compared to the design in (e).]

Suppose now that when a sequence of complex numbers $x_1,\dots,x_n$ is transmitted over the channel the received complex numbers $Y_1, \dots, Y_n$ are given by $Y_j = x_j \exp(\mathrm{j}\theta) + Z_j$ where $Z_1,\dots,Z_n$, are *i.i.d.* each distributed as in Equation (1), and Θ is chosen uniformly in $[0, 2\pi[$, independent from $(Z_1, .., Z_n)$. Observe that there is only a single phase shift Θ that affects all the transmitted symbols. Consequently, a reasonable communication strategy is to first estimate the value of Θ and then use a strategy as in (g) to communicate.

(h) Suppose we set $x_1 = A$, with A a positive real number and $\theta \in [0, 2\pi[$. Suggest a method to estimate Θ from Y_1. Can you come up with a method that makes sure $\Pr(|\hat{\Theta} - \Theta| > \alpha) \le \exp(-A^2 \sin^2(\alpha)/2)$ for $0 \le \alpha < \frac{\pi}{2}$?

[Hint: The following figure gives a visual representation of the problem. Also if M and N are two independent $\mathcal{N}(0,1)$ variables, then $\Pr(M^2 + N^2 > r^2) = \exp(-r^2/2)$]



(i) Suppose we set $x_1 = \dots = x_{n_0} = A$. Devise a method to estimate Θ with $\Pr(|\hat{\Theta} - \theta| > \alpha) \le \exp(-n_0 A^2 \sin^2(\alpha)/2)$ again with $0 \le \alpha < \frac{\pi}{2}$.

[Hint: consider $Y = (Y_1 + \dots + Y_{n_0})/\sqrt{n_0}$.]