Principles of Digital Communication — Theoretical part

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Consider a communication channel with the following input-output relationship (both the input and the output are complex numbers): when the input is $x \in \mathbb{C}$, the output $Y \in \mathbb{C}$ is given by

$$Y = \exp(j\Theta)x + Z \tag{1}$$

where the random phase shift Θ is uniformly chosen between $[0, 2\pi[$, and Z is a 0 mean complex Gaussian — the real and imaginary parts of Z are *i.i.d.* $\mathcal{N}(0, 1)$. Z and Θ are independent of each other.

(a) Show that, given the channel input x, the output Y

- has independent magnitude $U = |Y|$ and argument $\arg(Y)$.
  Hint: recall that $Y = |Y| \exp(j \arg(Y))$.
- $\arg(Y)$ is uniform in $[0, 2\pi[$.
- For any $\rho \geq 0$

  $$\mathbb{P}(U < \rho) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\rho} r \cdot \exp\left(-\frac{1}{2}\{r^2 + |x|^2 - 2r|x|\cos(\theta)\}\right) \, dr d\theta$$

  Conclude that for any $u \geq 0$,

  $$f_{U|X}(u|x) = u \exp\left(-\frac{u^2 + |x|^2}{2}\right) I_0(u|x|)$$

  where $I_0(v) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{-v\cos\theta} \, d\theta$.

  Note : $I_0(v)$ is the 0th order modified Bessel function of the 1st kind. It is well approximated by $\exp(v)/\sqrt{2\pi v}$ when $1 \ll v$.

Solution. First, let's rewrite Y in terms of real and imaginary parts

$$\begin{aligned}
Y &= \exp(j\Theta)x + Z \\
&= (\cos\Theta + j\sin\Theta)(\alpha + j\beta) + (Z_1 + jZ_2), \quad \alpha, \beta \in \mathbb{R}, Z_1, Z_2 \sim \mathcal{N}(0, 1) \\
&= \underbrace{(\alpha\cos\Theta - \beta\sin\Theta + Z_1)}_{Re(Y)} + j\underbrace{(\alpha\sin\Theta + \beta\cos\Theta + Z_2)}_{Im(Y)}
\end{aligned}$$

Hence, Y can be represented as a random vector where

$$Y = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{where} \quad \begin{cases} X_1 = Re(Y) \sim \mathcal{N}(\alpha\cos\Theta - \beta\sin\Theta, 1) \\ X_2 = Im(Y) \sim \mathcal{N}(\alpha\sin\Theta + \beta\cos\Theta, 1) \end{cases}$$

Now, we want to calculate the CDF of $U = |Y|$. Note that in polar coordinates we have

$$X_1 = U\cos\Psi$$
$$X_2 = U\sin\Psi$$

and then for $\rho > 0$

$$\mathbb{P}(U < \rho) = \int_0^{2\pi} \int_0^{\rho} f_{U\Psi}(r, \psi)\, d\psi.$$

Define $\mu_{X_1} = \alpha \cos\Theta - \beta \sin\Theta$ and $\mu_{X_2} = \alpha \sin\Theta + \beta\cos\Theta$, we have that

$$\begin{aligned}
f_{U\Psi}(r, \psi) &= J(r, \Psi) f_{X_1 X_2}(x_1, x_2) \\
&= \frac{u}{2\pi} \exp\left(-\frac{(x_1 - \mu_{X_1})^2 + (x_2 - \mu_{X_2})^2}{2}\right) \\
&= \frac{u}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2 - 2x_1\mu_{X_1} - 2x_2\mu_{X_2} + \mu_{X_1}^2 + \mu_{X_2}^2}{2}\right)
\end{aligned}$$

Now, note that

$$\begin{aligned}
\mu_{X_1}^2 + \mu_{X_2}^2 &= \alpha^2\cos^2\Theta - \cancel{2\alpha\beta\cos\Theta\sin\Theta} + \beta^2\sin^2\Theta + \alpha^2\sin^2\Theta + \cancel{2\alpha\beta\cos\Theta\sin\Theta} + \beta^2\cos^2\Theta \\
&= \alpha^2 + \beta^2 = |x|^2.
\end{aligned}$$

thus

$$\begin{aligned}
f_{U\Psi}(r, \psi) &= \frac{u}{2\pi} \exp\left(-\frac{u^2 - 2x_1\mu_{X_1} - 2x_2\mu_{X_2} + |x|^2}{2}\right) \\
&= \frac{u}{2\pi} \exp\left(-\frac{u^2 - 2u\cos\Psi\mu_{X_1} - 2u\sin\Psi\mu_{X_2} + |x|^2}{2}\right) \\
&= \frac{u}{2\pi} \exp\left(-\frac{u^2 + |x|^2}{2} - u(\cos\Psi\mu_{X_1} + \sin\Psi\mu_{X_2})\right)
\end{aligned}$$

Now, use the fact that $a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2}\cos(x - \epsilon)$ (see proof in appendix A) and we obtain that

$$f_{U\Psi}(r, \psi) = \frac{u}{2\pi} \exp\left(-\frac{u^2 + |x|^2}{2} - u|x|\cos(\Psi - \epsilon)\right)$$

Replace this in the expression of the CDF of U to find that

$$\begin{aligned}
\mathbb{P}(U < \rho) &= \int_0^{2\pi} \int_0^{\rho} \frac{u}{2\pi} \exp\left(-\frac{u^2 + |x|^2}{2} - u|x|\cos(\psi - \epsilon)\right)\, du d\psi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\rho} u \exp\left(-\frac{u^2 + |x|^2}{2} - u|x|\cos(\theta)\right)\, du d\theta \\
&= \int_0^{\rho} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u \exp\left(-\frac{u^2 + |x|^2}{2} - u|x|\cos(\theta)\right)\, d\theta \right\}\, du
\end{aligned}$$

since $\cos(\psi - \epsilon)$ is a periodic function of ψ. Now, knowing that $f_{U|X}(u|x)$ is nothing but the integrand with respect to θ in the above expression we get

$$\begin{aligned}
f_{U|X}(u|x) &= \frac{1}{2\pi} \int_0^{2\pi} u \exp\left(-\frac{u^2 + |x|^2}{2} - u|x|\cos(\theta)\right)\, d\theta \\
&= u \exp\left(-\frac{u^2 + |x|^2}{2}\right) \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{-u|x|\cos(\theta)}\, d\theta \right\} \\
&= u \exp\left(-\frac{u^2 + |x|^2}{2}\right) I_0(u|x|).
\end{aligned}$$

Q.E.D.

(b) Suppose we have m equally likely hypotheses, where $H = j$ means $x = c_j$ (c_j's are given complex numbers), and suppose that the observation is Y. Show that $U = |Y|$ is a sufficient statistic.

Solution. We can use the Neyman-Fisher theorem for this. We have

$$\begin{aligned}
\mathbb{P}(Y = y|x = c_j) &= \mathbb{P}(|Y|e^{j\arg(Y)} = y|x = c_j) \\
&= \mathbb{P}(|Y| = |y| \cap \arg(Y) = \arg(y)|x = c_j) \\
&= \mathbb{P}(|Y| = |y| \mid x = c_j)\mathbb{P}(\arg(Y) = \arg(y) \mid x = c_j) \\
&= \frac{1}{2\pi} f_{U|X}(|y||c_j) \\
&= \frac{|y|}{2\pi} \exp\left(-\frac{|y|^2 + |c_j|^2}{2}\right) I_0(|y||c_j|)
\end{aligned}$$

Picking $g_j(|y|) = \exp\left(-\frac{|y|^2+|c_j|^2}{2}\right) I_0(|y||c_j|)$ and $h(|y|) = \frac{|y|}{2\pi}$, the Neyman-Fisher factorisation theorem ensures that $|Y| = U$ is a sufficient statistics.

(c) Continuing with (b) suppose for two different hypotheses $j \neq \ell$, $|c_j| = |c_\ell|$. Can one distinguish between these two hypotheses?

Solution. Let's compute the MAP rule (which is equivalent to the ML rule as the priors are equiprobable) :

$$\frac{f_{U|c_\ell}(u|c_\ell)}{f_{U|c_j}(u|c_j)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \iff \frac{u \exp\left(-\frac{u^2+|c_\ell|^2}{2}\right) I_0(u|c_\ell|)}{u \exp\left(-\frac{u^2+|c_j|^2}{2}\right) I_0(u|c_j|)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1$$

but since $|c_j| = |c_\ell|$, the likelihood ratio evaluates to 1 and the MAP rule simplifies

$$1 \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1$$

and we can not decide between hypotheses j and ℓ.

(d) Suppose a friend has designed a system to transmit k bits over this channel where message j is transmitted as the complex c_j and decoding is performed by the ML decoder. We design a new system by replacing c_j with $|c_j|$. Show that the new system and the old system perform identically.

Solution. We note that the equivalence of the two systems can be expressed as follows:

$$\underset{i}{argmax}\ f_{U|c_i}(u|c_i) = \underset{i}{argmax}\ f_{U||c_i|}(u||c_i|)$$

This entails that the ML rule yields the same result for both systems, i.e.

$$\frac{f_{U|c_\ell}(u|c_\ell)}{f_{U|c_j}(u|c_j)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \iff \frac{f_{U||c_\ell|}(u||c_\ell|)}{f_{U||c_j|}(u||c_j|)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \tag{2}$$

The purpose of this question is to demonstrate the equivalence in Equation 2.
As a side note, the development of the ML rule for the "$|c_i|$ encoding" system gives the following:

$$\frac{f_{U||c_\ell|}(u||c_\ell|)}{f_{U||c_j|}(u||c_j|)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \iff \frac{u \exp\left(-\frac{u^2+||c_\ell||^2}{2}\right) I_0(u||c_\ell||)}{u \exp\left(-\frac{u^2+||c_j||^2}{2}\right) I_0(u||c_j||)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1$$

where $||x||$ denotes the module of the module of x.
We note that $||c_i|| = \sqrt{|c_i|}$. Taking Equation 2 and substituting the expressions of the distributions and $||c_i||$'s gives :

$$\frac{u\exp\left(-\frac{u^2+|c_\ell|^2}{2}\right)I_0(u|c_\ell|)}{u\exp\left(-\frac{u^2+|c_j|}{2}\right)I_0(u|c_j|)} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \iff \frac{u\exp\left(-\frac{u^2+|c_\ell|}{2}\right)I_0(u\sqrt{|c_\ell|})}{u\exp\left(-\frac{u^2+|c_j|}{2}\right)I_0(u\sqrt{|c_j|})} \underset{\hat{H}=j}{\overset{\hat{H}=\ell}{\gtrless}} 1 \tag{3}$$

We note that $\sqrt{|c_i|} = |\sqrt{c_i}|$ where $\sqrt{z} = \sqrt{|z|}\exp\left(j\frac{Arg(z)}{2}\right)$. Equation 3 tells us that encoding on $|c_i|$ is the same as encoding on $\sqrt{c_i}$, which in turn is the same as encoding on c_i.

The last equivalence comes from the fact we apply the same transformation to datapoints, without disrupting the relative placement of points in the complex plane.

(e) Note that in the new system in (c) the transmitted symbols are positive real numbers. Suppose these are $c_0 = a, c_1 = 3a, \dots, c_{m-1} = (2m+1)a$, where $a \gg 1$ is a positive real number and $m = 2k$. Show that the error probability is well approximated by $2Q(a)$.
[Hint: by the note in (a) you can observe that when $||x| - u|$ is small compared to x, $f_{U|X}(u|x)$ is well approximated by $(2\pi)^{-1/2}\exp(-\frac{1}{2}[u-|x|]^2)$ - the pdf of the output of an additive Gaussian channel with input $|x|$.]

Solution. For the decoding we use the sufficient statistic $U = |Y|$ and we do a minimum-distance decoding, i.e we choose $\hat{i}$ as $\arg\min_{i\in\{0,\dots,m-1\}} |U - c_i|$. Hence, since our constellation is pretty much a PAM constellation we have

$$\begin{aligned}
\mathbb{P}_e(0) = \mathbb{P}_e(m-1) &= \mathbb{P}(U - |c_i| > a) \\
&= \mathbb{P}(U > a + c_i) \\
&= \int_{a+c_i}^{\infty} f_{U|X}(u|c_i)\,\mathrm{d}u \\
&\approx \int_{a+c_i}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(u-c_i)^2\right)\,\mathrm{d}u \\
&= \mathbb{P}(K > a + c_i), \quad \text{where } K \sim \mathcal{N}(c_i, 1). \\
&= Q\left(\frac{a + c_i - c_i}{\sigma}\right) = Q(a).
\end{aligned}$$

And for $i \in \{1, m-2\}$ we have

$$\begin{aligned}
\mathbb{P}_c(i) &= \mathbb{P}(c_i - a < U < c_i + a) \\
&= \int_{c_i-a}^{c_i+a} f_{U|X}(u|c_i)\,\mathrm{d}u \\
&\approx \int_{c_i-a}^{c_i+a} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(u-c_i)^2\right)\,\mathrm{d}u \\
&= \mathbb{P}(c_i - a \le K \le c_i + a) \quad \text{where } K \sim \mathcal{N}(c_i, 1) \\
&= 1 - \mathbb{P}(K \le c_i - a) - \mathbb{P}(K \ge c_i + a) \\
&= 1 - 2Q(a)
\end{aligned}$$

and then

$$\mathbb{P}_e(i) = 1 - \mathbb{P}_c(i) = 2Q(a)$$

So, the average error probability is

$$\mathbb{P}_e = \sum_{i=0}^{m-1} \mathbb{P}(i)\mathbb{P}_e(i) = \frac{1}{m}\left(2Q(a) + 2(m-2)Q(a)\right) = \underbrace{\frac{2}{m}}_{\to 0} Q(a) + 2\underbrace{\frac{m-2}{m}}_{\to 1} Q(a) \approx 2Q(a)$$

(f) Continuing with (e) show that the average energy per bit required for an error probability $p_e = 2Q(a)$ approximately equals $a^2(4^{k+1})/(3k)$.

Solution. We compute the overall average energy $\mathcal{E}$:

$$\begin{aligned}
\mathcal{E} &= \sum_{i=0}^{m-1} \frac{|c_i|^2}{m} \\
&= \sum_{i=0}^{m-1} \frac{|a(2i+1)|^2}{m} \\
&= a^2 \sum_{i=0}^{m-1} \frac{(2i+1)^2}{m} \\
&= \frac{a^2}{m} \left(\sum_{i=0}^{m-1} 4i^2 + \sum_{i=0}^{m-1} 4i + \sum_{i=0}^{m-1} 1 \right) \\
&= \frac{a^2}{m} \left(4\frac{(m-1)m(2m-1)}{6} + \frac{4(m-1)m}{2} + m - 1 \right) \\
&= \frac{2a^2}{3}(m-1)(2m-1) + \mathcal{O}(m) \\
&= \frac{2a^2}{3}(2^k-1)(2\times 2^k - 1) + \mathcal{O}(2^k) \\
&= \frac{2a^2}{3}(2^{2k+1} - 2^k - 2^{k+1} - 1) + \mathcal{O}(2^k) \\
&= \frac{2^{2k+2}a^2}{3} + \mathcal{O}(2^{k+2}) \\
&= \frac{a^2 4^{k+1}}{3} + \mathcal{O}(2^{k+2})
\end{aligned}$$

In order to compute the energy per bit, we simply divide by the number of bits k, and obtain the following:

$$\mathcal{E}_{/bit} \approx \frac{a^2 4^{k+1}}{3k}$$

(g) Suppose now, the receiver is provided with the value of Θ. Show that we can transmit $2k$ bits with error probability less than $4Q(a)$ and energy per bit $a^2(4^k - 1)/(3k)$.
[Hint: consider c_j 's of the form $\pm(2p-1) \pm i(2q-1)$ where p, q are in $\{1, \dots, 2^{k-1}\}$, where p and q are integers (perhaps different letters?) with $|2p|, |2q| \leq 2^k$. Note that the design we made here has significant savings in energy compared to the design in (e).]

Solution. First, let us define correctly the constellation. Using the hint, we have a QAM constellation with $4 \cdot 2^{k-1} \cdot 2^{k-1} = 4^k$ points. Recall, that we receive $Y = c_j e^{j\Theta} + Z$, now let's compute error probability for each points

- **Corners**. Wrongly decoding a corner is equivalent to have $Re(Z) > a$ or $Im(Z) < -a$ (a is half the distance between a point and each of his neighbours). Thus, the error probability for a corner is

$$\mathbb{P}_e(\text{corner}) = \mathbb{P}(Re(Z) > a \cup Im(Z) < -a) \leq \mathbb{P}(Re(Z) > a) + \mathbb{P}(Im(Z) < -a) = 2Q(a)$$

- **Boundary points but not corners**. Wrongly decoding the king of is equivalent to have

$Re(Z) > a$ or $Im(Z) < -a$ or $Im(Z) > a$ and the error probability is

$$\begin{aligned}\mathbb{P}_e(\text{boundary}) &= \mathbb{P}(Re(Z) > a \cup Im(Z) < -a \cup Im(Z) > a)\\ &\leq \mathbb{P}(Re(Z) > a) + \mathbb{P}(Im(Z) < -a) + \mathbb{P}(Im(Z) > a)\\ &= 3Q(a)\end{aligned}$$

- **Other points**. Wrongly decoding a point strictly inside our constellation is equivalent to have $Re(Z) > a$ or $Re(Z) < -a$ or $Im(Z) < -a$ or $Im(Z) > a$

$$\begin{aligned}\mathbb{P}_e(\text{other points}) &= \mathbb{P}(Re(Z) > a \cup Re(Z) < -a \cup Im(Z) < -a \cup Im(Z) > a)\\ &\leq \mathbb{P}(Re(Z) > a) + \mathbb{P}(Re(Z) < -a) + \mathbb{P}(Im(Z) > a) + \mathbb{P}(Im(Z) < -a)\\ &= 4Q(a)\end{aligned}$$

Now, we can see that each point has a probability of being wrongly decoded of at most $4Q(a)$. Hence, we have that

$$\mathbb{P}_e \leq \frac{1}{4^k}\sum_{j=0}^{4^k-1} 4Q(a) = 4Q(a)$$

as desired. Now, we need to compute the average energy per bit of the constellation. We have that

$$|c_j|^2 = a^2[(2p-1)^2 + (2q-1)^2]$$

So, the average energy is

$$\begin{aligned}\mathcal{E} &= \frac{a^2}{4^k}\sum_{p=0}^{2^{k-1}}\sum_{q=0}^{2^{k-1}} |c|^2\\ &= \frac{a^2}{4^k}\sum_{p=0}^{2^{k-1}}\sum_{q=0}^{2^{k-1}} (2p-1)^2 + (2q-1)^2\\ &= \frac{a^2}{4^k}\sum_{p=0}^{2^{k-1}}\left(2^{k-1}(2p-1)^2 + \sum_{q=0}^{2^{k-1}}(2q-1)^2\right)\\ &= \frac{2^{k-1}a^2}{4^k}\sum_{p=0}^{2^{k-1}}(2p-1)^2 + \frac{2^{k-1}a^2}{4^k}\sum_{q=0}^{2^{k-1}}(2q-1)^2\\ &= \frac{2^k a^2}{4^k}\sum_{p=0}^{2^{k-1}}(2p-1)^2 = \frac{a^2}{2^k}\sum_{p=0}^{2^{k-1}}(2p-1)^2\end{aligned}$$

Now, using the following results

$$\begin{aligned}\sum_{k=1}^{m} k &= \frac{m(m+1)}{2} = \frac{m^2}{2} + \frac{m}{2}\\ \sum_{k=1}^{m} k^2 &= \frac{m(m+1)(2m+1)}{6} = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}\end{aligned}$$

we have that

$$\begin{aligned}
\sum_{p=1}^{2^{k-1}}(2p-1)^2 &= \sum_{p=0}^{2^{k-1}-1} 4p^2-4p+1 \\
&= 4\sum_{p=0}^{2^{k-1}-1} p^2 + 4\sum_{p=0}^{2^{k-1}-1} p + \sum_{p=0}^{2^{k-1}-1} 1 \\
&= 4\left(\frac{m^3}{6}+\frac{m^2}{4}+\frac{m}{12}\right)+4\left(\frac{m^2}{4}+\frac{m}{4}\right)+\frac{m}{2}, \quad m=2^k \\
&\approx \frac{2m^3}{3}-\frac{2m}{3}
\end{aligned}$$

and then

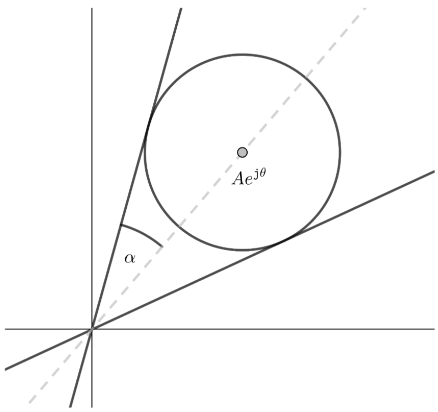
$$\mathcal{E}=\frac{a^2}{2^k}\sum_{p=0}^{2^{k-1}}(2p-1)^2 \approx \frac{a^2}{2^k}\left(\frac{2m^3}{3}-\frac{2m}{3}\right)=\frac{2a^2(m^2-1)}{3}=\frac{2a^2(4^k-1)}{3}.$$

Now, since we are sending $2k$ bit we have that

$$\mathcal{E}_b=\frac{\mathcal{E}}{2k}=\frac{2a^2(4^k-1)}{3}.$$

as expected.

Suppose now that when a sequence of complex numbers $x_1,\dots,x_n$ is transmitted over the channel the received complex numbers $Y_1,\dots,Y_n$ are given by $Y_j = x_j\exp(j\Theta)+Z_j$ where $Z_1,\dots,Z_n$, are i.i.d. each distributed as in Equation (1), and Θ is chosen uniformly in $[0,2\pi[$, independent from $(Z_1,\dots,Z_n)$. Observe that there is only a single phase shift Θ that affects all the transmitted symbols. Consequently, a reasonable communication strategy is to first estimate the value of Θ and then use a strategy as in (g) to communicate.



(h) Suppose we set $x_1 = A$, with A a positive real number and $\Theta \in [0,2\pi[$. Suggest a method to estimate Θ from Y_1. Can you come up with a method that makes sure $\mathbb{P}(|\hat{\Theta}-\Theta|>\alpha) \leq \exp(-A^2\sin^2(\alpha)/2)$ for $0 \leq \alpha < \frac{\pi}{2}$?
[Hint: The following figure gives a visual representation of the problem. Also if M and N are two independent $\mathcal{N}(0,1)$ variables, then $\mathbb{P}(M^2+N^2>r^2)=\exp(-r^2/2)$]

Solution. One way to estimate Θ from Y_1 would be to send a signal to the channel, and upon receiving the output Y, where $Y = Ae^{j\Theta}+Z$, we would compute its argument. The recieved argument would serve as an estimation of the phase shift Θ, since the phase difference between the input and the output would be $\arg(Y)$ ($x = A$ has argument 0). Let $Y_1 = Ae^{j\Theta} + Z_1$, therefore Y_{11} and Y_{12} respectively the x and y axis components of Y_1. Moreover let Z_{11} and Z_{12} be the components of Z_1 as $Z_1 = Z_{11} + jZ_{12}$. Then we have

$$\begin{cases} Y_{11} = A\cos(\Theta) + Z_{11} \\ Y_{12} = A\sin(\Theta) + Z_{12} \end{cases} \quad \text{and } C \text{ is the circle centered at } e^{j\Theta} \text{ of radius } r = A\sin(\alpha).$$

Observing the figure, we can see

$$\begin{aligned} \mathbb{P}(|\hat{\Theta} - \Theta| > \alpha) &= \int_{y \notin \text{Cone}} f_{Y_1|X_1}(y|a)\, \mathrm{d}y \\ &\le \int_{Z_1 \notin C} f(z_{11}, z_{12})\, \mathrm{d}z_{11}\mathrm{d}z_{12} \\ &= \mathbb{P}(|z_{11}^2 + z_{12}^2| \ge r^2) \\ &= \exp(-A^2\sin^2(\alpha)/2) \end{aligned}$$

where we used in the last inequality the fact that Z_{11} and $Z_{12} \sim \mathcal{N}(0,1)$ and are independent.

(i) Suppose we set $x_1 = \cdots = x_{n_0} = A$. Devise a method to estimate Θ with $\mathbb{P}(|\hat{\Theta} - \Theta| > \alpha) \le \exp(-n_0A^2\sin^2(\alpha)/2)$ again with $0 \le \alpha < \frac{\pi}{2}$
[Hint: consider $Y = (Y_1 + \cdots + Y_{n_0})/\sqrt{n_0}$.]

Solution. We have that $Y_i = Ae^{j\Theta} + Z_i$, so

$$\begin{aligned} Y &= \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Y_i \\ &= \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Ae^{j\Theta} + Z_i \\ &= \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Ae^{j\Theta} + \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Z_i \\ &= A\sqrt{n_0}e^{j\Theta} + \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Z_i. \end{aligned}$$

Now, let us define $Z = \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Z_i$ and recall that $Z_i = Z_{i1} + jZ_{i2}$. We compute

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Z_i\right] = \frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} \mathbb{E}[Z_i] = 0$$

since the Z_i's have mean 0, and for the variance we have

$$\mathbb{V}\text{ar}[Z] = \mathbb{V}\text{ar}\left[\frac{1}{\sqrt{n_0}} \sum_{i=1}^{n_0} Z_i\right] = \frac{1}{n_0} \sum_{i=1}^{n_0} \mathbb{V}\text{ar}[Z_i] = \frac{n_0}{n_0} = 1.$$

since the Z_i's have unit variance, and are *i.i.d.* Thus, Y can be written as $Y = A\sqrt{n_0}e^{j\Theta} + N$ where $N = Z_1 + jZ_2$ with $Z_1, Z_2 \sim \mathcal{N}(0,1)$ iid.
Hence, we found ourselves in a similar scenario to that of question (h). In this case however, x is scaled by a $\sqrt{n_0}$ factor. If we proceed as in the previous question, but with a circle centered at $A\sqrt{n_0}e^{j\Theta}$ this time, the new radius is

$$\frac{r}{|A\sqrt{n_0}e^{j\Theta}|} = \frac{r}{A\sqrt{n_0}} = \sin(\alpha) \iff r = A\sqrt{n_0}\sin(\alpha).$$

And we can bound the error probability of our estimation

$$
\begin{aligned}
\mathbb{P}(|\hat{\Theta} - \Theta| > \alpha) &\leq \mathbb{P}(|N|^2 > r^2) \\
&= \mathbb{P}(|Z_1 + \dots Z_{n_0}|^2 > r^2) \\
&\leq \exp(-A^2 n_0 \sin^2(\alpha)/2)
\end{aligned}
$$

as desired.