CS–308 — Calcul Quantique

Homework II — François Dumoncel — 314420

Exercise 1 *Matrix representations of a few gates*

Consider the following component representation of the canonical computational basis for a quantum bit $|0\rangle = [1\ 0]^T, |1\rangle = [0\ 1]^T$ and for two quantum bits $|00\rangle = [1\ 0\ 0\ 0]^T$ and so on.

(a) Give a matrix representation for the following reversible gates : NOT ; CNOT ; CCNOT.

Solution. The NOT gate is a 1-input gate meaning that its matrix representation is a 2×2 matrix. It is easy to see that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} |0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \,, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} |1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

so the matrix representation of the NOT gate is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Note : More formally, one can notice that to find the matrix we must understand what the NOT gate do. It flips the qbit $|0\rangle$ to give $|1\rangle$ and vice versa. We can write this as

$$\text{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The CNOT gate is a 2-input gate meaning that its matrix representation is a $2^2 \times 2^2$ matrix. Recall that this gate flips his second input if and only if the first bit is equal to $|1\rangle$. This lead to the following truth table

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Using this input-output relationship we can now compute

$$\begin{aligned} CNOT &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

The CCNOT gate is a 3-input gate meaning that its matrix representation is a $2^3 \times 2^3$ matrix. Its action is simple : if the first two bits are both set to 1, it inverts the third bit, otherwise all bits stay the same :

Input	Output
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|001\rangle$
$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|011\rangle$

and

Input	Output
$\|100\rangle$	$\|100\rangle$
$\|101\rangle$	$\|101\rangle$
$\|110\rangle$	$\|111\rangle$
$\|111\rangle$	$\|110\rangle$

So, the CCNOT gate acts as the identity matrix on the first 6 possibles inputs. With this we can already guess that the matrix gonna have the 6 first diagonal coefficient equal to 1. Doing the maths gives us

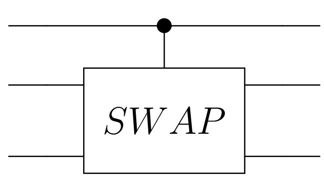
$$\begin{aligned} CCNOT &= |000\rangle\langle 000| + |001\rangle\langle 001| + \cdots + |111\rangle\langle 110| \\ &= \dots \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

(b) Recognize that these are all permutation matrices. What is their inverse matrix ?

Solution. Let's use some abstract algebra to solve this. For the NOT gate we see that this gate permutes the two basis vectors of $\mathbb{C}^2$. If we number the vectors of the basis, we obtain an element $\pi = (1\ 2)$ of the symmetric group S_2. Now, we know that the inverse of a permutation π in $S_n, n \geq 2$, is just the components of the permutation written backwards. So $\pi^{-1} = (2\ 1) = \pi$. In the world of gates this means that the inverse of the NOT gate is the NOT gate itself. This reasoning can also be applied to CNOT and CCNOT since these gates perform only 2 permutations. So, $A^2 = \mathbb{I} \Rightarrow A^{-1} = A$ for $A =$ NOT, CNOT, CCNOT.

Exercise 2 *Fredkin gate*

The SWAP operation takes two input bits and permutes them : $\text{SWAP}|b_1, b_2\rangle = |b_2, b_1\rangle$. The Fredkin gate is a three input controlled SWAP gate and is reversible. The gate swaps the two last bits if the first bit is a 1. Otherwise it leaves the input bits unchanged. One intriguing particularity of the Fredkin gate is that it conserves the number of ones.



(a) Show that the irreversible gates AND, OR can be represented in a reversible way from the Fredkin gate.

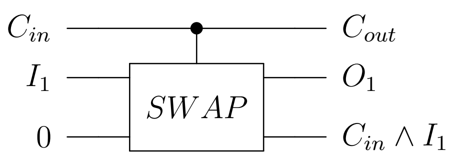
Solution. Using the truth table of the Fredkin gate we have

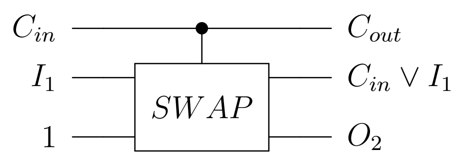
Input	Output
$\vert 000\rangle$	$\vert 000\rangle$
$\vert 001\rangle$	$\vert 001\rangle$
$\vert 010\rangle$	$\vert 010\rangle$
$\vert 011\rangle$	$\vert 011\rangle$

and

Input	Output
$\vert 100\rangle$	$\vert 100\rangle$
$\vert 101\rangle$	$\vert 110\rangle$
$\vert 110\rangle$	$\vert 101\rangle$
$\vert 111\rangle$	$\vert 111\rangle$

Now, look at the 4 red rows. We can see that in the input the third bit is always 0. Looking at the output and we found that the last bit, let's name it O_2, is exactly equal to the logical conjunction between the first two bits of the input, let's name them C_{in} and I_1. Looking at the blue rows this time and see that we have the third input bit set to 1 all the time and one can remark that the middle bit of the output, let's name it O_1, is exactly equal to the logical disjunction between the first two bits of the inputs. Let's summarize, if the input is $|C_{\text{in}}I_1I_2\rangle$ and the output is $|C_{\text{out}}O_1O_2\rangle$ then we have the following relationship





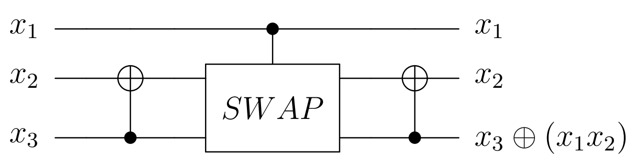
(b) Give the matrix representation of the Fredkin gate.

Solution. We see that the output differ from the input only in two cases ($|101\rangle$ and $|110\rangle$) meaning that the Fredkin acts as the identity for all other possible inputs. Thus we have

$$FREDKIN = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Represent the Toffoli (CCNOT) gate in terms of {Fredkin, CNOT}. *Hint* : You can achieve with at most one Fredkin gate and two CNOT gates

Solution. After trying multiple things, we find that the following circuit implements the CCNOT gate



We prove this by looking at the state of the inputs after each gate. After the first CNOT gate we have

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \oplus x_3 \\ x_3 &= x_3 \end{aligned}$$

After the SWAP gate we have

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_1 x_3 \oplus \bar{x}_1 (x_2 \oplus x_3) \\ x_3 &= x_1 (x_2 \oplus x_3) \oplus \bar{x}_1 x_3 \end{aligned}$$

Using that $ab \oplus \bar{a}b \equiv b$ we have at the end (after the second CNOT gate)

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_1 x_3 \oplus \bar{x}_1 (x_2 \oplus x_3) \oplus x_1 (x_2 \oplus x_3) \oplus \bar{x}_1 x_3 \\ &= x_1 x_3 \oplus (x_2 \oplus x_3) \oplus \bar{x}_1 x_3 \\ &= x_3 \oplus x_2 \oplus x_3 \\ &= x_2 \\ x_3 &= x_1 (x_2 \oplus x_3) \oplus \bar{x}_1 x_3 \\ &= x_1 x_2 \oplus x_1 x_3 \oplus \bar{x}_1 x_3 \\ &= x_3 \oplus x_1 x_2 \end{aligned}$$

Exercise 3 : *The Mach-Zehnder interferometer*

Consider the following matrix product $H(NOT)H$.

(a) Is the product unitary ? Why ?

Solution. Hadamard and NOT gates are unitary. Computing the product gives

$$\begin{aligned} H(NOT)H(H(NOT)H)^\dagger &= H(NOT)HH^\dagger NOT^\dagger H^\dagger \\ &= H(NOT)NOT^\dagger H^\dagger \\ &= HH^\dagger \\ &= \mathbb{I}. \end{aligned}$$

Same reasoning can be used to show that $(H(NOT)H)^\dagger H(NOT)H = \mathbb{I}$. Hence $H(NOT)H$ is unitary.

(b) Compute the output when the input is $|0\rangle$, $|1\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Solution.

$$H(NOT)H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$H(NOT)H|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle .$$

$$H(NOT)H|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|1\rangle .$$

$$H(NOT)H\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

$$H(NOT)H\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

~~(c) Draw the circuit and interpret it as a quantum interferometer.~~

(d) Describe the measurement outcomes at the output of the circuit (interferometer) when we measure in the computational basis.

Solution. $\mathcal{H} = \mathbb{C}^2 = \left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$. Let $|\Psi\rangle \in \mathcal{H}$ be the input. Then we have

$$H(NOT)H|\Psi\rangle = H(NOT)H(\alpha|0\rangle) + H(NOT)H(\beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle, \quad \forall |\Psi\rangle \in \mathcal{H}. \tag{1}$$

Exercise 4 *Production of Bell states*

(a) Compute the four Bell states using the following identity using Dirac's notation. Do not use the component and matrix representations.

$$|B_{xy}\rangle = (CNOT)(H \otimes \mathbb{I})|x\rangle \otimes |y\rangle .$$

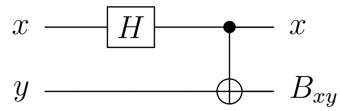
where $x, y \in \{0, 1\}$ and $|B_{xy}\rangle$ are the Bell states.

Solution.

$$\begin{aligned} |B_{xy}\rangle &= (CNOT)(H|x\rangle \otimes |y\rangle) \\ &= (CNOT)\left(\frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}} \otimes |y\rangle\right) \\ &= \frac{1}{\sqrt{2}}(CNOT)(|0\rangle \otimes |y\rangle) + \frac{1}{\sqrt{2}}(CNOT)((-1)^x|1\rangle \otimes |y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0 \oplus y\rangle) + \frac{(-1)^x}{\sqrt{2}}(|1\rangle \otimes |1 \oplus y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle) + \frac{(-1)^x}{\sqrt{2}}(|1\rangle \otimes |\bar{y}\rangle) \end{aligned}$$

(b) Represent the corresponding circuit.

Solution.



(c) Represent the circuit corresponding to the inverse identity :

$$|x\rangle \otimes |y\rangle = (H \otimes \mathbb{I})(CNOT)\,|B_{xy}\rangle$$

Solution.