iALC - ND proofs

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1 Introduction

This document serves as an aid for our proofs of normalisation, soundness and completeness of the labelled Natural Deduction system created for the logic iALC. Since the publishing of the article, we changed the way of presenting the proofs, so the following section contains an overhaul of them.

2 Full proofs

In this section we show that important properties which are expected of ND systems hold in the one here proposed for iALC. In order to improve readability, lists of labels are omitted whenever possible throughout Section 2.1, as they do not interfere with the process of normalisation. When dealing with soundness and completeness in Sections 2.2 and 2.3, however, labels will be explicitly shown.

Our main focus here is to show that these properties still hold when dealing with labels and the rules which introduce and eliminate existential and universal restrictions.

2.1 Normalisation

Normalisation is a crucial step to show that the sub-formula principle holds in our system. This principle states that every step in a derivation contains only sub-formulas of either the conclusion or the premises, which is extremely important in order to do proof search in an efficient way. We will base our normalisation proof on the ones presented in [1, 2, 3, 4, 5].

To reach normalisation, the following definitions are needed.

Definition 2.1 (*Formulas immediately above and below*)**.** *Let δ be the end-formula in derivation Π, as the consequence of an application of rule ρ. Let $\delta_1, \ldots, \delta_n$ be the premises of ρ. Then, δ is immediately below each δ_i, for $i \leq n$, and each of the δ_i's is immediately above δ.*

Definition 2.2 (*Side-connected formulas*)**.** *Let δ be the end-formula in derivation Π, as the consequence of an application of rule ρ. Let $\delta_1, \ldots, \delta_n$ be the premises of ρ, in this order. Then, δ_i is side-connected with δ_j, for $i, j \leq n$.*

Definition 2.3 (*Major premise*)**.** *The major premise is the premise of an elimination rule which contains the operator to be eliminated. Elimination rules are* $\sqcap e$*,* $\sqcup e$*,* $\dashv e$*,* $\forall e$*,* $\exists e$*, and* $\neg e$*.*

Definition 2.4 (*Minor premise*)**.** *Other premises in the previously mentioned rules, if existent, are called minor premises.*

In non-elimination rules, the premises are **not** divided into major and minor.

Definition 2.5 (*Thread*)**.** *In a derivation, a thread is a sequence of formulas starting with one of the premises or discharged hypotheses and ending in the conclusion. Every formula in a thread must be immediately above the next one in the sequence (i.e. given a thread* $\delta_1, \ldots, \delta_n$ *of size* n*, every formula* δ_i*, for* $i < n$*, must be one of the premises of the application of a rule* ρ *which has* δ_{i+1} *as its conclusion).*

Example 2.1 (A thread)**.** *In the following derivation, there are two threads, namely* $x : \alpha \sqcap \beta$*,* $x : \beta$*,* $x : \beta \sqcap \alpha$ *(on the left side) and* $x : \alpha \sqcap \beta$*,* $x : \alpha$*,* $x : \beta \sqcap \alpha$ *(on the right side).*

$$\dfrac{\dfrac{x : \alpha \sqcap \beta}{x : \beta}\,\sqcap e_2 \qquad \dfrac{x : \alpha \sqcap \beta}{x : \alpha}\,\sqcap e_1}{x : \beta \sqcap \alpha}\,\sqcap i$$

So far, we have been talking about discharging formulas in a rather informal fashion, but a formal definition is needed in order to reach normalisation.

Definition 2.6 (*Discharge*)**.** *Let* δ *be an assumption in a deduction* Π *and* τ *be the thread which begins with* δ*. We say that* δ *is discharged in* Π *at* δ' *by an application* ρ' *of rule* ρ *if and only if* δ' *is the first formula occurrence* δ_1 *in* τ *such that one of the following conditions holds:*

1. ρ *is* $\sqcup e$*,* δ *has the form* $x : \alpha$ *for some* x *and* α*, the major premise of* ρ' *has the form* $x : \alpha \sqcup \beta$ *or* $x : \beta \sqcup \alpha$ *for some* β*, and* δ_1 *is the first or second minor premise of* ρ'*, respectively;*

2. ρ *is* $\dashv i$*,* δ_1 *is the premise of* ρ' *and has the form* $x : \beta$ *for some* x *and* β*,* δ *has the form* $x : \alpha$ *for some* α*, and the consequence of* ρ' *has the form* $x : \alpha \dashv \beta$*;*

3. ρ *is* $\neg i$*,* δ_1 *is the premise of* ρ' *and has the form* $x : \bot$ *for some* x*,* δ *has the form* $x : \alpha$ *for some* α*, and the consequence of* ρ' *has the form* $x : \neg\alpha$*.*

Definition 2.7 (*Quasi-deduction*)**.** Π *is a quasi-deduction if it is a formula-tree such that, if* δ *is a formula occurrence in* Π *and* $\delta_1, \ldots, \delta_n$ *are formula occurrences immediately above* δ *in* Π *in their order from left to right, then the following is an application of an inference rule:*

$$\frac{\delta_1 \qquad \ldots \qquad \delta_n}{\delta}$$

Definition 2.8 (*Discharge-function*)**.** *A discharge-function $\mathcal{F}$ for a quasi-deduction Π is a function from a set of assumptions in Π that assigns to a formula δ either δ itself or a formula occurrence in Π below δ.*

Let $\mathcal{F}$ be a discharge-function for a quasi-deduction Π. There are a few ways to talk about how formulas are discharged:

- An assumption δ in Π is discharged with respect to $\mathcal{F}$ at δ' if $\mathcal{F}(\delta) = \delta'$.
- δ' is said to depend on the assumption δ w.r.t $\mathcal{F}$ if δ' belongs the the thread τ in Π that begins with δ and δ is not discharged with respect to $\mathcal{F}$ at a formula occurrence before δ' in τ.
- δ' is sad to depend on the formula δ w.r.t. $\mathcal{F}$ if δ' depends on an assumption of the shape δ.

Definition 2.9 (*Regular Discharge-function*)**.** *$\mathcal{F}$ is a regular discharge-function for a quasi-deduction Π if both the following conditions apply:*

1. *terms in an application of $\forall i$ or $\exists e$ do not occur in any assumption on which the premise of this application depends; and*
2. *$\mathcal{F}(\delta)$ is a premise δ_1 in an application ρ' of a rule ρ satisfying one of the conditions in Definition 2.6.*

Then, by means of a regular discharge-function and a quasi-deduction, we can define a deduction formally.

Definition 2.10 (*Deduction*)**.** *Let Π be a quasi-deduction, δ a formula and Γ, a set of formulas. Then, Π is a deduction of δ from Γ if δ is the end-formula of Π and there exists a regular discharge-function $\mathcal{F}$ for Π such that the end-formula of Π depends only on formulas of Γ with respect to $\mathcal{F}$.*

Definition 2.11 (*Branch*)**.** *A branch in a derivation is a sequence $\delta_1, \ldots, \delta_n$ of formula occurrences such that:*

1. *δ_1 is an assumption that is not discharged by an application of $\sqcup e$;*
2. *δ_i, for all $i < n$, is not a minor premise of an application of $\dashv e$ and:*
   (a) *if δ_i is not a major premise of $\sqcup e$, then δ_{i+1} occurs immediately below δ_i;*
   (b) *if δ_i is a major premise of $\sqcup e$, then δ_{i+1} is an assumption discharged by the same application of $\sqcup e$;*
3. *δ_n is either a minor premise of an application of $\dashv e$, the end formula of the derivation, or a major premise of an application of $\sqcup e$ that does not discharge any assumption.*

Defining a branch is useful to understand the behaviour of a derivation around applications of $\sqcup e$, since, in this rule, the major premise should be seen as happening before or *above* discharged assumptions in the sub-derivations above the minor premises.

Example 2.2 (A branch - left side). *In the following example, the formulas in bold form a branch. i.e. the sequence* $\boldsymbol{x : \alpha \sqcup \beta}$, $\boldsymbol{x : \alpha}$, $\boldsymbol{x : \alpha \sqcap \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$*:*

$$\dfrac{\boldsymbol{x : \alpha \sqcup \beta} \qquad \dfrac{[\boldsymbol{x : \alpha}]^1 \quad x : \beta}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcap i \qquad \dfrac{x : \alpha \quad [x : \beta]^1}{x : \alpha \sqcap \beta}\sqcap i}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcup e(1)$$

Example 2.3 (Another branch - right side). *In this example, we highlight the branch on the right side, namely* $\boldsymbol{x : \alpha \sqcup \beta}$, $\boldsymbol{x : \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$*:*

$$\dfrac{\boldsymbol{x : \alpha \sqcup \beta} \qquad \dfrac{[x : \alpha]^1 \quad x : \beta}{x : \alpha \sqcap \beta}\sqcap i \qquad \dfrac{x : \alpha \quad [\boldsymbol{x : \beta}]^1}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcap i}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcup e(1)$$

Notice that the disjunction always appears before a corresponding discharged formula. In a way, a branch generalises the concept of a thread, as it encompasses derivations with $\sqcup e$ more accurately.

Definition 2.12 (*Detour*). *A detour occurs when a formula is introduced via an introduction rule only to be eliminated immediately later as the major premise of an elimination rule.*

Our goal in the end will be to remove such detours via *reductions* and arrive at so called *normal* derivations, to show that the system has normalisation.

Example 2.4 (A detour with $\sqcap$). *Let* Π_1 *and* Π_2 *be sub-derivations. An example of a detour is given below, where the list of labels is empty:*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha\end{matrix} \quad \begin{matrix}\Pi_2 \\ x : \beta\end{matrix}}{x : (\alpha \sqcap \beta)}\sqcap i}{x : \alpha}\sqcap e_1$$

By introducing and eliminating a conjunction, $x : \alpha$ *occurs twice in the same branch of the derivation. This is clearly an unnecessary step, as no progress is made in the derivation.*

Definition 2.13 (*Maximum Formula*). *A formula* δ *in a derivation* Π *is a maximum formula if:*

1. δ *is the conclusion of an application of an introduction rule and the major premise of an application of an elimination rule of the same operator; or*

2. δ *is the conclusion of an application of* $\sqcup e$ *as well as the major premise of an application of an elimination rule.*

Maximum formulas are precisely those that represent a detour.

In order to evaluate how a reduction works in simplifying a derivation, it is necessary to have some kind of measure. We then introduce the concept of degree of a formula, indicating its size by the number of occurrences of logical operators. Not surprisingly, the name *maximum formula* is related to this measurement.

Definition 2.14 (*Degree of a Formula*)**.** *The degree of a formula is given as follows:*

$$
\begin{aligned}
&deg(C) = 0, \text{ for an atomic concept } C \\
&deg(\bot) = 0 \\
&deg(x : \alpha) = deg(\alpha) \\
&deg(\alpha \sqcup \beta) = deg(\alpha) + deg(\beta) + 1 \\
&deg(\alpha \sqcap \beta) = deg(\alpha) + deg(\beta) + 1 \\
&deg(\alpha \dashv \beta) = deg(\alpha) + deg(\beta) + 1 \\
&deg(\neg\alpha) = deg(\alpha) + 1 \\
&deg(\exists R.\alpha) = deg(\alpha) + 2 \\
&deg(\forall R.\alpha) = deg(\alpha) + 2
\end{aligned}
$$

As seen in the second case of Definition 2.13, it is necessary to generalise the notion of maximum formula via defining what a *segment* and a *maximum segment* are, as there may be *hidden* detours in applications of $\sqcup e$.

Definition 2.15 (*Segment*)**.** *A segment σ in a derivation Π is a sequence δ_1, ..., δ_n of consecutive formulas in a branch on Π such that:*

1. *δ_1 is not the consequence of an application of $\sqcup e$;*
2. *δ_i, for all $i < n$, is a minor premise of an application of $\sqcup e$; and*
3. *δ_n is not a minor premise of an application of $\sqcup e$.*

Since the formulas are consecutive in the branch that births the segment, all of the formulas are immediately above the next one (apart, obviously, from the last formula in the segment).

One can see that a segment consists on a sequence of occurrences of the same formula,[1] since all but the last of them are minor premises of applications of $\sqcup e$, and in this rule the conclusion must be the same as the minor premises. As δ_n is the conclusion of the final application of $\sqcup e$, it is as well the same formula as the others.

Example 2.5 (A segment)**.** *Let Π_{1-6} and Π' be sub-derivations. An example of a segment is given below by the formulas in bold (i.e. $\boldsymbol{x : \alpha}$, $\boldsymbol{x : \alpha}$, $\boldsymbol{x : \alpha}$),*

[1] Alternatively: *formulas of the same shape* [1].

where the lists of labels are empty:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha_1\sqcup\beta_1\end{matrix}\qquad \dfrac{\begin{matrix}\Pi_2\\ x:\alpha_2\sqcup\beta_2\end{matrix}\qquad \dfrac{\begin{matrix}\Pi_5\\ x:\alpha\sqcap\beta\end{matrix}}{\boldsymbol{x:\alpha}}\sqcap e_1 \qquad \begin{matrix}\Pi_4\\ x:\alpha\end{matrix}}{\boldsymbol{x:\alpha}}\sqcup e\qquad \begin{matrix}\Pi_3\\ x:\alpha\end{matrix}}{\boldsymbol{x:\alpha}}\sqcup e \qquad \begin{matrix}\Pi_6\\ x:\alpha\dashv\beta\end{matrix}}{\begin{matrix}x:\beta\\ \Pi'\end{matrix}}\dashv e$$

In this example, there are surely at least two other segments, starting in either Π_3 or Π_4 or, if they both do not end in an application of $\sqcup e$, in the formulas shown directly below them. Nothing stops the possibility of existence of there being even another segment in any of the sub-derivations, still.

An important consideration is that, as $\sqcup e$ has two minor premises of equal shape, a derivation containing this rule always spawns at least two segments, which can then be further subdivided depending on the amount of applications of this rule and how they are spaced within the derivation.

Definition 2.16 (*Length of a segment*)**.** *Let $\sigma = \delta_1, \ldots, \delta_n$ be a segment in a derivation. We call n the length of σ.*

Definition 2.17 (*Maximum Segment*)**.** *A maximum segment is a segment σ which begins with an application of an introduction rule (i.e. $\sqcap i$, $\sqcup i$, $\dashv i$, $\neg i$, $\forall i$ or $\exists i$) or rule efq and ends with a major premise of an elimination rule.*

Since the formula δ_n in σ is the major premise of an elimination rule, if the rule that has δ_1 (which is the same as δ_n) was an introduction rule for the corresponding operator, then we have a *hidden detour*.

One can see that a maximum formula is equivalent to a maximum segment of size 1.

Example 2.6 (A maximum segment)**.** *Let Π_{1-6} and Π' be sub-derivations. An example of a maximum segment is given below by the formulas in bold (i.e. $\boldsymbol{x:\neg\alpha}$, $\boldsymbol{x:\neg\alpha}$, $\boldsymbol{x:\neg\alpha}$), where the lists of labels are empty:*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha_1\sqcup\beta_1\end{matrix}\qquad \dfrac{\begin{matrix}\Pi_2\\ x:\alpha_2\sqcup\beta_2\end{matrix}\qquad \dfrac{\begin{matrix}\Pi_5\\ x:\bot\end{matrix}}{\boldsymbol{x:\neg\alpha}}\neg i \qquad \begin{matrix}\Pi_4\\ x:\neg\alpha\end{matrix}}{\boldsymbol{x:\neg\alpha}}\sqcup e\qquad \begin{matrix}\Pi_3\\ x:\neg\alpha\end{matrix}}{\boldsymbol{x:\neg\alpha}}\sqcup e \qquad \begin{matrix}\Pi_6\\ x:\alpha\end{matrix}}{\begin{matrix}x:\bot\\ \Pi'\end{matrix}}\neg e$$

In this example, there may be other maximum segments - even of greater length and degree than the one above.

We also generalise the concept of degree of a formula for a segment:

Definition 2.18 (*Degree of a Segment*)**.** *The degree of a segment $\sigma = \delta_1, \ldots, \delta_n$ is defined as: $deg(\sigma) = deg(\delta_n)$.*

As the formulas in a segment are all the same, it does not matter which of them is chosen.

Definition 2.19 (*Degree of a Derivation*)**.** *The degree of a derivation* Π *is defined as:* $deg(\Pi) = max\{deg(\sigma) : \sigma$ *is a maximum segment in* $\Pi\}$*.*

This definition shows the intuition behind the terms *maximum formula* and *maximum segment.*

Definition 2.20 (*Index of a Derivation*)**.** *The index of a derivation* Π *is a pair* $i(\Pi) = \langle d, l\rangle$*, where* l *is the sum of the lengths of the maximal segments of* Π *with degree* d*. When there are no maximal segments, then* $i(\Pi) = \langle 0, 0\rangle$*.*

Indexes of derivations are compared in lexicographical order.

With these definitions in hand, we can then begin to evaluate derivations in terms of detours.

Definition 2.21 (*Critical Derivation*)**.** *A derivation* Π *is called critical when*

1. Π *ends with an application of an elimination rule* ρ_e*;*
2. *the major premise* δ *of* ρ_e *is in a maximum segment* σ*;*
3. $deg(\Pi) = deg(\sigma)$*; and*
4. *for all* Π' *sub-derivation of* Π*,* $deg(\Pi') < deg(\sigma) = deg(\Pi)$*.*

With this definition, we know that the main detour to be reduced lies (at first) in ρ_e.

Definition 2.22 (*Normal Derivation*)**.** *A derivation* Π *is called normal when* $i(\Pi) = \langle 0, 0\rangle$*.*

Now we will see how to remove these unwanted detours.

Definition 2.23 (*Reduction*)**.** *We say a derivation* Π *reduces to another,* Π'*, when we apply zero or more reductions to maximum formulas of* Π *and it becomes equal to* Π'*. We denote by* $\Pi \rhd \Pi'$ *the reduction from* Π *to* Π'*.*

The regular discharge-function $\mathcal{F}$ of Π is changed to a function $\mathcal{F}'$, which is an adaptation of $\mathcal{F}$ with the necessary modifications due to the fact that certain assumptions in Π disappear and that Π' may contain many sub-deductions of the same form, as is the case with, for instance, $\multimap$-reduction if there is more than one discharged hypothesis by the application of the rule to be removed.

A reduction is an operation on derivations that functions as a way to remove maximum formulas locally. The reductions for the *propositionally-based* operators are extensions of the usual Prawitz-reductions. As such, we have a permutation operation in order to arrange a $\sqcup$-maximum segment to be properly reduced. This will be shown after the more basic cases.

As a quick note on reductions, they usually end up copying whole sections of sub-derivations (thus possibly increasing the size of the derivation itself), but they do not increase the complexity of the formulas therein. First we will show some basic cases.

$\sqcap$-reduction Taking the derivation from Example 2.4, we have:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array}\quad\begin{array}{c}\Pi_2\\ x:\beta\end{array}}{x:(\alpha\sqcap\beta)}\,\sqcap i}{x:\alpha}\,\sqcap e_1\\ \Pi_3\end{array}\qquad\rhd\qquad\begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_3\end{array}$$

This way the proof goes directly through $x : \alpha$, without introducing and eliminating an unused operator. The reduction utilises $\sqcap e_1$, but $\sqcap e_2$ is analogous, the only difference being the presence of $x : \beta$ and Π_2 instead of $x : \alpha$ and Π_1 in Π'.

$\sqsupset$-reduction This derivation has a maximum formula whose main operator is $\sqsupset$:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array}\quad\dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ x:\beta\end{array}}{x:(\alpha\sqsupset\beta)}\,\sqsupset i}{x:\beta}\,\sqsupset e\\ \Pi_3\end{array}\qquad\rhd\qquad\begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_2\\ x:\beta\\ \Pi_3\end{array}$$

Note that, in this case, there can be multiple occurrences of the hypothesis discharge over Π_2, so this reduction ends up copying the entirety of Π_1 possibly many times, increasing the (horizontal) size of the derivation itself, but not its complexity, since the maximum formula was removed.

$\neg$-reduction The case of $\neg$-reduction via $\neg i$ and $\neg e$ is analogous, as negation is but a special case of $\sqsupset$:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array}\quad\dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ x:\bot\end{array}}{x:\neg\alpha}\,\neg i}{x:\bot}\,\neg e\\ \Pi_3\end{array}\qquad\rhd\qquad\begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_2\\ x:\beta\\ \Pi_3\end{array}$$

$\forall$-reduction The case for the universal restriction on concepts is rather simple. Let R a role, and x and y VLSs (x does not occur in Π_1):

$$\begin{array}{c}\dfrac{xRy\quad\dfrac{xRy\quad\begin{array}{c}\Pi_1\\ y:\alpha^{\forall R}\end{array}}{x:\forall R.\alpha}\,\forall i}{y:\alpha^{\forall R}}\,\forall e\\ \Pi_2\end{array}\qquad\rhd\qquad\begin{array}{c}\Pi_1\\ y:\alpha^{\forall R}\\ \Pi_2\end{array}$$

The reader may note that the extra premise xRy vanishes when the reduction occurs. This is another reason why the explicit presence of this premise does not interfere directly in the calculus itself, only in the explainability of the proof to human readers.

The case for $\exists$-reduction is analogous, as the rules share the same format:

$\exists$-reduction Let R a role, and x and y VLSs (y does not occur in Π_1):

$$\dfrac{xRy \qquad \dfrac{xRy \quad \begin{matrix}\Pi_1\\ y:\alpha^{\exists R}\end{matrix}}{x:\exists R.\alpha}\,\exists i}{\begin{matrix}y:\alpha^{\exists R}\\ \Pi_2\end{matrix}}\,\exists e \qquad \rhd \qquad \begin{matrix}\Pi_1\\ y:\alpha^{\exists R}\\ \Pi_2\end{matrix}$$

$\sqcup$-reduction This operator, much like $\sqcap$, has two possible reductions, but both are equivalent. Only the one involving $\sqcup i_1$ will be shown.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\end{matrix}}{x:\alpha\sqcup\beta}\,\sqcup i_1 \qquad \begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta\end{matrix} \qquad \begin{matrix}[x:\beta]\\ \Pi_3\\ \delta\end{matrix}}{\begin{matrix}\delta\\ \Pi_4\end{matrix}}\,\sqcup e \qquad \rhd \qquad \begin{matrix}\Pi_1\\ x:\alpha\\ \Pi_2\\ \delta\\ \Pi_4\end{matrix}$$

As was the case with $\dashv$, there can be multiple copies of Π_1 to be added. The case of $\sqcup i_2$ has $x:\beta$ instead of $x:\alpha$ in both Π and Π'.

$\sqcup$-permutation This operator has a peculiarity in that it may *hide* a maximum formula, since δ can be the major premise of another elimination rule. Let us, then, define a $\sqcup$-permutation that allows for a rearrangement of the derivation tree around the $\sqcup e$. Let ρ_e be an elimination rule with major premise δ_1, minor premise δ_2 and conclusion δ_3. We assume that ρ_e has only one minor premise; with more premises the argument is analogous, as the minor premises will just be propagated accordingly. If there are no minor premises, this holds as well.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\sqcup\beta\end{matrix} \qquad \begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta_1\end{matrix} \quad \begin{matrix}[x:\beta]\\ \Pi_3\\ \delta_1\end{matrix}}{\delta_1}\,\sqcup e \qquad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\begin{matrix}\delta_3\\ \Pi_5\end{matrix}}\,\rho_e$$

This derivation is changed via **$\sqcup$-permutation** to:

$$\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\sqcup\beta\end{matrix} \qquad \dfrac{\begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta_1\end{matrix} \quad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\delta_3}\,\rho_e \qquad \dfrac{\begin{matrix}[x:\beta]\\ \Pi_3\\ \delta_1\end{matrix} \quad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\delta_3}\,\rho_e}{\begin{matrix}\delta_3\\ \Pi_5\end{matrix}}\,\sqcup e$$

With this derivation in hand, there may be an introduction rule in either Π_2 or Π_3 which creates an *unseen* detour with ρ_e, since ρ_e is an elimination rule and there may be other detours to be reduced in the sub-derivations. It is important to note that this permutation does not increase the degree of the derivation, since the maximum formula, be it $x : \alpha \sqcup \beta$ or any of the δ's, does not increase in complexity.

Example 2.7 (An example of an unseen detour)**.** *In this example, let ρ_e be an application of $\sqcap e_1$, δ_m be a formula introduced by $\sqcap i$ (and eliminated by ρ_e). Let δ_1 and δ_2 be formulas, and Π_{1-5} be sub-derivations.*

$$\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\sqcup\beta\end{matrix} \qquad \dfrac{\begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta_1\end{matrix}\qquad \begin{matrix}[x:\alpha]\\ \Pi_3\\ \delta_2\end{matrix}}{\delta_m}\,\sqcap i \qquad \begin{matrix}[x:\beta]\\ \Pi_4\\ \delta_m\end{matrix}}{\begin{matrix}\dfrac{\delta_m}{\delta_1}\,\sqcap e_1\\ \Pi_5\end{matrix}}\,\sqcup e$$

After an application of $\sqcup$-permutation, this derivation becomes:

$$\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\sqcup\beta\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta_1\end{matrix}\qquad \begin{matrix}[x:\alpha]\\ \Pi_3\\ \delta_2\end{matrix}}{\delta_m}\,\sqcap i}{\delta_1}\,\sqcap e_1 \qquad \dfrac{\begin{matrix}[x:\beta]\\ \Pi_4\\ \delta_m\end{matrix}}{\delta_1}\,\sqcap e_1}{\begin{matrix}\delta_1\\ \Pi_5\end{matrix}}\,\sqcup e$$

Then, after the detour around the $\sqcap$ operator becomes apparent, this derivation is reduced via $\sqcap$-reduction to:

$$\dfrac{\begin{matrix}\Pi_1\\ x:\alpha\sqcup\beta\end{matrix} \qquad \begin{matrix}[x:\alpha]\\ \Pi_2\\ \delta_1\end{matrix} \qquad \dfrac{\begin{matrix}[x:\beta]\\ \Pi_4\\ \delta_m\end{matrix}}{\delta_1}\,\sqcap e_1}{\begin{matrix}\delta_1\\ \Pi_5\end{matrix}}\,\sqcup e$$

From here on we will refer to simple derivations utilising the following notation: Π/δ represents a derivation Π with δ as a conclusion. Conversely, δ/Π represents a derivation Π with δ as a premise. Note that there may be more copies of δ in the actual derivation Π, as stated previously.

Before reaching normalisation, it is important to take a few preliminary steps. We begin with shortening maximum segments of arbitrary length until they are simply instances of maximum formulas.

Lemma 2.1. *Let Π_1/δ and δ/Π_2 be two derivations in the ND system such that $deg(\Pi_1) = n_1$ and $deg(\Pi_2) = n_2$. Then, $deg(\Pi_1/[\delta]/\Pi_2) = max\{deg(\delta), n_1, n_2\}$.*

Proof. Directly from Definition 2.19: the maximum formula in this derivation is either in Π_1, in Π_2 or is δ itself. $\square$

Lemma 2.2. *Let δ be a formula in iALC, Γ be a set of formulas, $\Delta \subseteq \Gamma$, and Π be a critical derivation of $\Gamma \vdash_{ND} \delta$, such that $i(\Pi) = \langle d, l \rangle$, and there is at least one maximum segment σ in Π of length greater than 1. Then, there is a derivation Π' of $\Delta \vdash_{ND} \delta$ such that $i(\Pi') = \langle d, l' \rangle$, where $l' < l$, and all maximum segments in Π' have length 1.*

Proof. We prove by induction on the length of each segment. The base cases consist of applications of $\sqcup$-permutation on maximum segments of length 2. The following cases show a detour happening at the first minor premise of $\sqcup e$, but the case where it lies above the second minor premise of $\sqcup e$ is analogous. For each case, let δ_m be the maximum formula in question.

- ρ_e is an application of $\sqcap e_i$

  This is covered in Example 2.7, in which the segment $\sigma = \delta_m, \delta_m$ of length 2 is reduced to a segment of length 1. Below, a new segment of length 2 is created. However, as it consists of occurrences of δ_1, which has lower degree than δ_m (our maximum formula), it does not interfere with the index of the derivation itself.

  The case for $\sqcap e_2$ is analogous.

- ρ_e is an application of $\sqcup e$

  It is important to note that this application does not continue the segment since δ_m is the major premise of this application of $\sqcup e$, not one of the minor prmeises.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \dfrac{\delta_1}{\delta_m}\sqcup i\end{array} \quad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_m}\sqcup e \quad \begin{array}{c}[\delta_1]\\ \Pi_4\\ \delta_2\end{array} \quad \begin{array}{c}\Pi_5\\ \delta_2\end{array}}{\begin{array}{c}\delta_2\\ \Pi_6\end{array}}\sqcup e$$

  After applying $\sqcup$-permutation, this becomes:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ \dfrac{\delta_1}{\delta_m}\sqcup i_1\end{array} \quad \begin{array}{c}[\delta_1]\\ \Pi_4\\ \delta_2\end{array} \quad \begin{array}{c}\Pi_5\\ \delta_2\end{array}}{\delta_2}\sqcup e \quad \dfrac{\begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array} \quad \begin{array}{c}[\delta_1]\\ \Pi_4\\ \delta_2\end{array} \quad \begin{array}{c}\Pi_5\\ \delta_2\end{array}}{\delta_2}\sqcup e}{\begin{array}{c}\delta_2\\ \Pi_6\end{array}}\sqcup e$$

  The case for $\sqcap i_2$ is analogous.

- ρ_e is an application of $\dashv e$

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \qquad \dfrac{\begin{array}{c}[x:\alpha][\delta_2]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\dashv i \qquad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_m}\sqcup e}{\delta_1}\dashv e\\ \Pi_5\end{array}$$

After applying $\sqcup$-permutation, this becomes:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \qquad \dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \dfrac{\begin{array}{c}[x:\alpha][\delta_2]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\dashv i}{\delta_1}\dashv e \qquad \dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_1}\dashv e}{\delta_1}\sqcup e\\ \Pi_5\end{array}$$

Thus, the new maximum segment has length 1.

- ρ_e is an application of $\neg e$

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \qquad \dfrac{\begin{array}{c}[x:\alpha][\delta_2]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\neg i \qquad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_m}\sqcup e}{\delta_1}\neg e\\ \Pi_5\end{array}$$

After applying $\sqcup$-permutation, this becomes:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \qquad \dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \dfrac{\begin{array}{c}[x:\alpha][\delta_2]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\neg i}{\delta_1}\neg e \qquad \dfrac{\begin{array}{c}\Pi_4\\ \delta_2\end{array} \qquad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_1}\neg e}{\delta_1}\sqcup e\\ \Pi_5\end{array}$$

Thus, the new maximum segment has length 1.

- ρ_e is an application of $\forall e$

  Let δ_2 be an assertion of the form yRz, where y and z are VLSs and R is a role. It is important to note that there is no sub-derivation above each δ_2. The usual restrictions on variables apply for Π_2 and Π_3.

$$\dfrac{\delta_2 \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\,\forall i \quad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_m}\,\sqcup e}{\begin{array}{c}\delta_1\\ \Pi_4\end{array}}\,\forall e$$

After applying $\sqcup$-permutation, this becomes:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\delta_2 \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\,\forall i}{\delta_1}\,\forall e \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_1}\,\forall e}{\begin{array}{c}\delta_1\\ \Pi_4\end{array}}\,\sqcup e$$

Thus, the new maximum segment has length 1.

- ρ_e is an application of $\exists e$

  This case is similar to the previous one. The same conditions on variables apply.

$$\dfrac{\delta_2 \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\,\exists i \quad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_m}\,\sqcup e}{\begin{array}{c}\delta_1\\ \Pi_4\end{array}}\,\exists e$$

  After applying $\sqcup$-permutation, this becomes:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\delta_2 \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array}}{\delta_m}\,\exists i}{\delta_1}\,\exists e \quad \dfrac{\delta_2 \quad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta_m\end{array}}{\delta_1}\,\exists e}{\begin{array}{c}\delta_1\\ \Pi_4\end{array}}\,\sqcup e$$

  Thus, the new maximum segment has length 1.

- ρ_e is an application of efq

If this is the case, then the formula δ_m present in the maximum segment is of the form $x : \bot$, which has degree 0. Then, this is not a maximum segment.

For the inductive case, we apply consecutive $\sqcup$-permutations to the sub-derivation containing the upper-leftmost maximum segment of length greater than 1, until it has length 1. By upper-leftmost we mean that there is no maximum segment of the same degree above it or contains a formula occurrence side-connected with the last formula occurrence of the segment in question. For each application of $\sqcup$-permutation, the segment in question has its length reduced by 1, as the application of the elimination rule ρ_e swaps place in the respective branch with the lowest application of $\sqcup e$ of the segment, thus staying above it, *breaking* the preexisting segment and reducing its length by 1.

By repeating this procedure, we create a new segment as more and more applications of $\sqcup e$ stack below ρ_e, but this new segment is: (a) of lower degree than the original maximum segment, as its formula is no longer the maximum formula in question, and (b) of strictly lower length than the original maximum segment, as it is limited by the number of applications of $\sqcup e$ in the original segment. Thus, it does not interfere with the index of the derivation.

We repeat this procedure for each maximum segment σ throughout the derivation (choosing always the upper-leftmost one) until all of them have length 1. $\square$

Lemma 2.3. *If* Π *reduces to* Π'*, then* $i(\Pi') \leq i(\Pi)$.

Proof. It follows directly from the reductions in Definition 2.23 and the degree of a formula in Definition 2.14: since a reduction always lowers the degree of a maximum formula, it also lowers the degree of the derivation in which it is contained (as per Definition 2.19). $\square$

Lemma 2.4. *Let* δ *be a formula in iALC,* Γ *be a set of formulas,* $\Delta \subseteq \Gamma$ *and* Π *be a critical derivation of* $\Gamma \vdash_{ND} \delta$*. Then,* Π *reduces to a derivation* Π' *of* $\Delta \vdash_{ND} \delta$ *such that* $deg(\Pi') < deg(\Pi)$.

Proof. The proof follows by induction in the structure of Π. We first apply Lemma 2.2 to turn maximal segments into maximal formulas. We will use the reductions shown in Definition 2.23, but assuming they are critical derivations - starting by the upper-leftmost maximum formulas in Π. We will call each derivation before a reduction Π, and Π' after the reduction.

Let us divide the proof by cases:

- $\sqcap$-reduction

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha\end{matrix} \quad \begin{matrix}\Pi_2 \\ x : \beta\end{matrix}}{x : (\alpha \sqcap \beta)}\sqcap i}{\begin{matrix}x : \alpha \\ \Pi_3\end{matrix}}\sqcap e_1 \qquad \rhd \qquad \begin{matrix}\Pi_1 \\ x : \alpha \\ \Pi_3\end{matrix}$$

We have $deg(\Pi) = deg(x : \alpha \sqcap \beta)$ (according to Item 3 of the definition of a critical formula), which equals $deg(\alpha) + deg(\beta) + 1$. When reduced to Pi', we have, as per Lemma 1, that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_3), deg(x : \alpha))\}$. If it is equal to $deg(\Pi_1)$ or $deg(\Pi_3)$, then it is strictly smaller because of item 4 of the definition of a critical formula. If it is $deg(x : \alpha) = deg(\alpha)$, then it is strictly smaller than $deg(\Pi) = deg(\alpha) + deg(\beta) + 1$.

- $\dashv$-reduction (and $\neg$-reduction)

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\Pi_1\\ x : \alpha\end{array} \quad \dfrac{\begin{array}{c}[x : \alpha]\\ \Pi_2\\ x : \beta\end{array}}{x : (\alpha \dashv \beta)}\,{\dashv}i}{x : \beta}\,{\dashv}e\\
\Pi_3
\end{array}
\qquad \triangleright \qquad
\begin{array}{c}\Pi_1\\ x : \alpha\\ \Pi_2\\ x : \beta\\ \Pi_3\end{array}
$$

We have $deg(\Pi) = deg(x : \alpha \dashv \beta)$, which equals $deg(\alpha) + deg(\beta) + 1$. When reduced to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(x : \alpha), deg(x : \beta)\}$. It is one of the sub-derivations, and it is strictly smaller. On the other hand, if it is either $deg(x : \alpha) = deg(\alpha)$ or $deg(x : \beta) = deg(\beta)$ it will be smaller than $deg(\Pi)$ as well.

Since $\neg$ is defined as a special case of $\dashv$, the proof for $\neg$-reduction follows directly.

- $\forall$-reduction (and $\exists$-reduction)

$$
\begin{array}{c}
\dfrac{xRy \quad \dfrac{xRy \quad \begin{array}{c}\Pi_1\\ y : \alpha^{\forall R}\end{array}}{x : \forall R.\alpha}\,\forall i}{y : \alpha^{\forall R}}\,\forall e\\
\Pi_2
\end{array}
\qquad \triangleright \qquad
\begin{array}{c}\Pi_1\\ y : \alpha^{\forall R}\\ \Pi_2\end{array}
$$

We have $deg(\Pi) = deg(x : \forall R.\alpha)$, which equals $deg(\alpha) + 2$. With Π', we have that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(y : \alpha^{\forall R})\}$. The argument for $deg(\Pi') = deg(\Pi_1)$ or $deg(\Pi_2)$ is the same. Since the label does not affect the degree, if $deg(\Pi') = deg(y : \alpha^{\forall R}) = deg(\alpha)$, then it is strictly smaller than $deg(\Pi)$.

The argument for $\exists$-reduction is exactly the same.

- $\sqcup$-reduction

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x : \alpha\end{array}}{x : \alpha \sqcup \beta}\,\sqcup i_1 \quad \begin{array}{c}[x : \alpha]\\ \Pi_2\\ \delta\end{array} \quad \begin{array}{c}[x : \beta]\\ \Pi_3\\ \delta\end{array}}{\delta}\,\sqcup e\\
\Pi_4
\end{array}
\qquad \triangleright \qquad
\begin{array}{c}\Pi_1\\ x : \alpha\\ \Pi_2\\ \delta\\ \Pi_4\end{array}
$$

We have $deg(\Pi) = deg(x : \alpha \sqcup \beta)$, which equals $deg(\alpha) + deg(\beta) + 1$. When turned to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(\delta), deg(x : \alpha))\}$. The argument for the sub-derivations or $deg(\delta)$ is still due to Π being a critical derivation. If $deg(\Pi') = deg(x : \alpha) = deg(\alpha)$, then it is strictly smaller than $deg(\Pi)$.

□

Lemma 2.5. *Let δ be a formula in iALC, Γ be a set of formulas, $\Delta \subseteq \Gamma$ and Π be a derivation of $\Gamma \vdash_{ND} \delta$ such that $i(\Pi) > \langle 0, 0\rangle$. Then, Π reduces to a derivation Π' of $\Delta \vdash_{ND} \delta$ such that $i(\Pi') < i(\Pi)$.*

Proof. It follows directly from Lemmas 2.2, 2.3, and 2.4 using induction on the length of Π. □

Theorem 2.1 (Normalisation)**.** *Let Π be a derivation of $\Gamma \vdash_{ND} \delta$. Then, Π reduces to a normal derivation Π' of $\Delta \subseteq \Gamma \vdash_{ND} \delta$.*

Proof. This follows directly from Lemmas 2.2 and 2.5 by applying the permutations to remove maximum segments and reductions to remove maximum formulas inductively over the index of Π.

It is important to note that, once one removes the maximum formulas of degree d from a derivation with index $\langle d, l\rangle$, a new derivation with index $\langle d', l'\rangle$ is generated, where $d' < d$, but there may be the case where there is one - or more - maximum segment σ' such that its length is greater than 1, so the permutations of Lemma 2.2 have to be reapplied for the new maximum segments of this new derivation. This process is then repeated until the index of the derivation becomes $\langle 0, 0\rangle$, i.e. it is normal. □

Corollary 2.1 (Termination)**.** *The normalisation process is terminating, since it is done inductively on the size of the derivation.*

Corollary 2.2 (Sub-concept Principle)**.** *Let Π be a normal derivation of $\Gamma \vdash_{ND} \delta$, where Γ is a set of formulas and δ, a formula. Then, every concept occurrence in Π is a sub-concept of concepts either in δ or in formulas of Γ.*

From normalisation, one arrives at the sub-concept principle (our version of the sub-formula principle, as we deal with TBox reasoning) by the same steps made by Prawitz in [1].

2.2 Soundness

Theorem 2.2 (Soundness)**.** *Let δ be a formula in iALC and Γ a set of formulas. Then, $\Gamma \vdash_{ND} \delta$ implies $\Gamma \models \delta$.*

Proof. We start by showing soundness for the rule $p{-}\exists$. Its soundness is proved as a consequence of the following reasoning in first-order intuitionistic logic used for deriving the semantics of the conclusions from the semantics of the premises:

$$\forall x(A(x) \wedge B(x) \rightarrow C(x)) \models \\ \forall x A(x) \wedge \exists x B(x) \rightarrow \exists x C(x)$$

The proof for the rest of the calculus follows by induction on the size of the derivation, focusing on the last formula applied.

For the base case, $\delta \in \Gamma$. Since δ is a premise, we have $\mathcal{I} \models \delta$, given any interpretation $\mathcal{I}$. Then, we trivially have $\mathcal{I} \models \delta$, which is the conclusion.

For the inductive cases, we will consider derivations ending in the different rules of our system, assume that the inductive hypothesis works for the sub-derivations above them (as they have smaller size), and will show that the rules preserve soundness, given any interpretation $\mathcal{I}$ - we will not assume anything else on $\mathcal{I}$ other than the assumptions needed in each step.

It is worth noting that $\mathcal{I} \models x : \alpha$ if and only if $\forall z, x^{\mathcal{I}} \preceq z \Rightarrow z \Vdash_{\mathcal{I}} \alpha^{\mathcal{I}}$, which includes $z = x^{\mathcal{I}}$ itself, as $\preceq$ is reflexive. Since most rules involve only x as an outer nominal, when arguing inductively over the formulas in the premises the part $\forall z, x \preceq z$ is considered valid, just omitted to avoid prolixity in the argument.

Another note based on the semantic definitions given in Section 4.1: an assertion of the form $x : \alpha^L$, where concept α is a labelled concept with L as its list of labels, behaves the same way as if α were not labelled w.r.t. satisfiability, i.e. $\mathcal{I} \Vdash x : \alpha^L$ happens if and only if $\mathcal{I} \models x : \alpha^L$ does, as well. This happens because, given function σ, a labelled concept is interpreted as an unlabelled concept restricted by different roles (either universally or existentially, depending on what is present in the corresponding list of labels), and these restrictions do not alter the assertion for x - one can view α^L as $\sigma(\beta)$, for some unlabelled concept β.

In the subsequent derivations, sub-derivations will be named Π_i, where $i > 0$, each having their own corresponding set of assumptions Γ_i.

Finally, in this proof, whenever one sees an interpretation of a labelled concept, e.g. $(\alpha^L)^{\mathcal{I}}$, one must assume that the function σ defined in Section 4.1.1 is being applied. The explicit application of this function is omitted to minimise visual clutter in the proof.

- $\sqcap i$

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ x : \alpha^{L^\forall} & x : \beta^{L^\forall}\end{array}}{x : (\alpha \sqcap \beta)^{L^\forall}} \sqcap i$$

  We have $\Gamma_1 \vdash_{ND} x : \alpha^{L^\forall}$ and $\Gamma_2 \vdash_{ND} x : \beta^{L^\forall}$. Then, by inductive hypothesis, we have $\Gamma_1 \models x : \alpha^{L^\forall}$ and $\Gamma_2 \models x : \beta^{L^\forall}$. We wish to prove $\Gamma_1 \cup \Gamma_2 \models x : (\alpha \sqcap \beta)^{L^\forall}$.

  First, suppose $\mathcal{I} \Vdash \Gamma_1 \cup \Gamma_2$. Then, we also have $\mathcal{I} \Vdash x : \alpha^{L^\forall}$ and $\mathcal{I} \Vdash x : \beta^{L^\forall}$, since $\Gamma_1 \models x : \alpha^{L^\forall}$ and $\Gamma_2 \models x : \beta^{L^\forall}$. These are the same as having

$\mathcal{I}, x \Vdash \alpha^{L^\forall}$ and $\mathcal{I}, x \Vdash \beta^{L^\forall}$. From both, we have $\mathcal{I}, x \Vdash (\alpha \sqcap \beta)^{L^\forall}$. Hence, $\mathcal{I} \Vdash x : (\alpha \sqcap \beta)^{L^\forall}$, concluding the proof for $\sqcap i$.

Notice how the lists of labels do not interfere at all with the argument for this rule, as they have to all be the same for the rule to work. They will only come into play when the rules involve changing them, i.e. $\forall i$, $\forall e$, $\exists i$, and $\exists e$.

- $\sqcap e$

$$\dfrac{\begin{matrix}\Pi_1\\ x : (\alpha \sqcap \beta)^{L^\forall}\end{matrix}}{x : \alpha^{L^\forall}} \sqcap e_1$$

Here, we provide the argument for $\sqcap e_1$ only, as $\sqcap e_2$ is analogous.

We have $\Gamma_1 \vdash_{ND} x : (\alpha \sqcap \beta)^{L^\forall}$. Then, by inductive hypothesis, we have $\Gamma_1 \models x : (\alpha \sqcap \beta)^{L^\forall}$. We wish to prove $\Gamma_1 \models x : \alpha^{L^\forall}$.

We start by assuming $\mathcal{I} \Vdash \Gamma_1$. Then, we have $\mathcal{I} \Vdash x : (\alpha \sqcap \beta)^{L^\forall}$. From this we have also $\mathcal{I}, x \Vdash \alpha^{L^\forall}$, as $(\alpha^{L^\forall})^{\mathcal{I}} \cap (\beta^{L^\forall})^{\mathcal{I}} \subseteq (\alpha^{L^\forall})^{\mathcal{I}}$, which is equivalent to $\mathcal{I} \Vdash x : \alpha^{L^\forall}$.

- $\sqcup i$

$$\dfrac{\begin{matrix}\Pi_1\\ x : \alpha^{L^\exists}\end{matrix}}{x : (\alpha \sqcup \beta)^{L^\exists}} \sqcup i_1$$

Here follows the proof only for $\sqcup i_1$, as the reasoning for $\sqcup i_2$ is analogous.

We have $\Gamma_1 \vdash_{ND} x : \alpha^{L^\exists}$. By inductive hypothesis, $\Gamma_1 \models x : \alpha^{L^\exists}$. We wish to prove $\Gamma_1 \models x : (\alpha \sqcup \beta)^{L^\exists}$.

Start by assuming $\mathcal{I} \models \Gamma_1$. This leads us to $\mathcal{I} \Vdash x : \alpha^{L^\exists}$ which is the same as $\mathcal{I}, x \Vdash \alpha^{L^\exists}$. Since $(\alpha^{L^\exists})^{\mathcal{I}} \subseteq (\alpha^{L^\exists})^{\mathcal{I}} \cup (\beta^{L^\exists})^{\mathcal{I}}$, we have, then, $\mathcal{I}, x \Vdash (\alpha \sqcup \beta)^{L^\exists}$, which is the same as $\mathcal{I} \Vdash x : (\alpha \sqcup \beta)^{L^\exists}$.

- $\sqcup e$

$$\dfrac{\begin{matrix}\Pi_1\\ x : (\alpha \sqcup \beta)^{L^\exists}\end{matrix} \qquad \begin{matrix}\left[x : \alpha^{L^\exists}\right]\\ \Pi_2\\ \delta\end{matrix} \qquad \begin{matrix}\left[x : \beta^{L^\exists}\right]\\ \Pi_3\\ \delta\end{matrix}}{\delta} \sqcup e$$

We have $\Gamma_1 \vdash_{ND} x : (\alpha \sqcup \beta)^{L^\exists}$, $\Gamma_2 \cup \{x : \alpha^{L^\exists}\} \vdash_{ND} \delta$ and $\Gamma_3 \cup \{x : \beta^{L^\exists}\} \vdash_{ND} \delta$. Then, by inductive hypothesis, $\Gamma_1 \models x : (\alpha \sqcup \beta)^{L^\exists}$, $\Gamma_2 \cup \{x : \alpha^{L^\exists}\} \models \delta$ and $\Gamma_3 \cup \{x : \beta^{L^\exists}\} \models \delta$. We want to prove $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \models \delta$.

We assume $\mathcal{I} \Vdash \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Then, we have $\mathcal{I} \Vdash \Gamma_1$, which leads to $\mathcal{I} \Vdash x : (\alpha \sqcup \beta)^{L^\exists}$. So, we either have $\mathcal{I} \Vdash x : \alpha^{L^\exists}$ or $\mathcal{I} \Vdash x : \beta^{L^\exists}$, since $(\alpha^{L^\exists})^{\mathcal{I}} \subseteq (\alpha^{L^\exists})^{\mathcal{I}} \cup (\beta^{L^\exists})^{\mathcal{I}}$ and $(\beta^{L^\exists})^{\mathcal{I}} \subseteq (\alpha^{L^\exists})^{\mathcal{I}} \cup (\beta^{L^\exists})^{\mathcal{I}}$.

Consider the first case: $\mathcal{I} \Vdash x : \alpha^{L^\exists}$. We have $\mathcal{I} \Vdash \Gamma_2$. Then, $\mathcal{I} \Vdash \delta$, for we have $\Gamma_2 \cup \{x : \alpha^{L^\exists}\} \models \delta$.

Now for the second case: $\mathcal{I} \Vdash x : \beta^{L^\exists}$. We have $\mathcal{I} \Vdash \Gamma_3$. Then, $\mathcal{I} \Vdash \delta$, for we have $\Gamma_3 \cup \{x : \beta^{L^\exists}\} \models \delta$.

- $\dashv i$

$$\dfrac{\begin{array}{c}[x : \alpha^{L1}] \\ \Pi_1 \\ x : \beta^{L2}\end{array}}{x : (\alpha^{L1} \dashv \beta^{L2})} \dashv i$$

We have $\Gamma_1 \cup \{x : \alpha^{L1}\} \vdash_{ND} x : \beta^{L2}$. Then, by inductive hypothesis, $\Gamma_1 \cup \{x : \alpha^{L1}\} \models x : \beta^{L2}$. We wish to prove $\Gamma_1 \models x : (\alpha^{L1} \dashv \beta^{L2})$, i.e. $\forall x'$ such that $x \preceq x'$, if $\mathcal{I} \Vdash x' : \alpha^{L1}$, then $\mathcal{I} \Vdash x' : \beta^{L2}$.

We assume $\mathcal{I} \Vdash \Gamma_1$ and $\mathcal{I}, x' \Vdash \alpha^{L1}$, for any x' such that $x \preceq x'$. Hence, $\mathcal{I}, x' \Vdash \beta^{L2}$, since $\Gamma_1 \cup \{x : \alpha^{L1}\} \models x : \beta^{L2}$ and $x \preceq x'$, giving β^{L2} to x' by heredity. Thus, we have $\mathcal{I} \Vdash x' : \beta^{L2}$, and, finally, $\mathcal{I} \Vdash x : (\alpha^{L1} \dashv \beta^{L2})$.

- $\dashv e$

$$\dfrac{\begin{array}{c}\Pi_1 \\ x : \alpha^{L1}\end{array} \quad \begin{array}{c}\Pi_2 \\ x : (\alpha^{L1} \dashv \beta^{L2})\end{array}}{x : \beta^{L2}} \dashv e$$

We have $\Gamma_1 \vdash_{ND} x : \alpha^{L1}$ and $\Gamma_2 \vdash_{ND} x : (\alpha^{L1} \dashv \beta^{L2})$. By inductive hypothesis, $\Gamma_1 \models x : \alpha^{L1}$ and $\Gamma_2 \models x : (\alpha^{L1} \dashv \beta^{L2})$. We wish to prove $\Gamma_1 \cup \Gamma_2 \models x : \beta^{L2}$.

We assume $\mathcal{I} \Vdash \Gamma_1 \cup \Gamma_2$. We have $\mathcal{I} \Vdash \Gamma_1$ and $\mathcal{I} \Vdash \Gamma_2$, leading us to $\mathcal{I} \Vdash x : \alpha^{L1}$ and $\mathcal{I} \Vdash x : (\alpha^{L1} \dashv \beta^{L2})$, respectively. They are equivalent to $\mathcal{I}, x \Vdash \alpha^{L1}$ and $\mathcal{I}, x \Vdash \alpha^{L1} \dashv \beta^{L2}$. Since $x \preceq x$, then we also have $\mathcal{I}, x \Vdash \beta^{L2}$, by the definition of the interpretation of $\dashv$, which is equivalent to $\mathcal{I} \Vdash x : \beta^{L2}$.

- $\neg i$

$$\dfrac{\begin{array}{c}[x : \alpha^{L}] \\ \Pi_1 \\ x : \bot\end{array}}{x : (\neg\alpha)^{\neg L}} \neg i$$

We have $\Gamma_1 \cup \{x : \alpha^{L}\} \vdash_{ND} x : \bot$. Then, by inductive hypothesis, $\Gamma_1 \cup \{x : \alpha^{L}\} \models x : \bot$. We wish to prove $\Gamma_1 \models x : (\neg\alpha)^{\neg L}$, i.e. $\forall x'$ such that $x \preceq x'$, $\mathcal{I} \nVdash x' : \alpha^{L}$.

We assume $\mathcal{I} \Vdash x : \alpha^{L}$. If it is the case that $\mathcal{I} \nVdash x : (\neg\alpha)^{\neg L}$, then there must exist a world x' such that $x \preceq x'$, $\mathcal{I} \Vdash \Gamma_1$ and $\mathcal{I}, x' \Vdash \alpha^{L}$. We can take x itself. This would mean that $\mathcal{I} \Vdash x : \bot$, since $\Gamma_1 \cup \{x : \alpha^{L}\} \models x : \bot$. However, this is the same as having $\mathcal{I}, x \Vdash \bot$, an impossibility. Thus, $\mathcal{I} \Vdash x : (\neg\alpha)^{\neg L}$.

- $\neg e$

$$\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ x : \alpha^L & x : (\neg\alpha)^{\neg L}\end{matrix}}{x : \bot}\ \neg e$$

We have $\Gamma_1 \vdash_{ND} x : \alpha^L$ and $\Gamma_2 \vdash_{ND} x : (\neg\alpha)^{\neg L}$. By inductive hypothesis, we have $\Gamma_1 \models x : \alpha^L$ and $\Gamma_2 \models x : (\neg\alpha)^{\neg L}$, respectively. We wish to prove $\Gamma_1 \cup \Gamma_2 \models x : \bot$.

Let us assume that $\mathcal{I} \Vdash \Gamma_1 \cup \Gamma_2$. Then, we have both $\mathcal{I} \Vdash x : \alpha^L$ and $\mathcal{I} \Vdash x : (\neg\alpha)^{\neg L}$. From the definition of $((\neg\alpha)^{\neg L})^{\mathcal{I}}$, we have that, $\forall x'$ such that $x \preceq x'$, $\mathcal{I} \nVdash x : \alpha^L$. Since $\preceq$ is reflexive, it must be the case for x as well. However, we have $\mathcal{I} \Vdash x : \alpha^L$, a contradiction. Thus, x cannot belong to any concept, i.e. $\mathcal{I} \Vdash x : \bot$.

- $\forall i$

$$\dfrac{\begin{matrix} & \Pi_1 \\ xRy & y : \alpha^{\forall R,L}\end{matrix}}{x : (\forall R.\alpha)^L}\ \forall i$$

Let R represent any role. y does not appear in any formula in Γ_1. We have that $\Gamma_1 \vdash_{ND} y : \alpha^{\forall R,L}$. By the inductive hypothesis, we have $\Gamma_1 \models y : \alpha^{\forall R,L}$. We want to prove $\Gamma_1 \cup \{xRy\} \models x : (\forall R.\alpha)^L$.

We start by assuming $\mathcal{I} \Vdash \Gamma_1$. We assume xRy, as well. Then, we have $\mathcal{I} \Vdash y : \alpha^{\forall R,L}$, which is equivalent to $\mathcal{I}, y \Vdash \alpha^{\forall R,L}$. Since xRy, from the frame conditions of **IK**, we have that (**F1**) $\forall x'$ such that, if $x \preceq x'$, then $\exists y'$ such that $x'Ry'$ and $y \preceq y'$, as well as (**F2**) $\forall y'$ such that, if $y \preceq y'$, then $\exists x'$ such that $x'Ry'$ and $x \preceq x'$. From this, we have $\mathcal{I}y' \Vdash \alpha^{\forall R,L}$ for each y', fitting precisely the criteria needed for $\mathcal{I}, x \Vdash (\forall R.\alpha)^L$, since the frame conditions cover any x' such that $x \preceq x'$. Thus, $\mathcal{I} \Vdash x : (\forall R.\alpha)^L$.

- $\forall e$

$$\dfrac{\begin{matrix} & \Pi_1 \\ xRy & x : (\forall R.\alpha)^L\end{matrix}}{y : \alpha^{\forall R,L}}\ \forall e$$

Let R represent any role. We have that $\Gamma_1 \vdash_{ND} x : (\forall R.\alpha)^L$. By the inductive hypothesis, we have $\Gamma_1 \models x : (\forall R.\alpha)^L$. We want to prove $\Gamma_1 \cup \{xRy\} \models y : \alpha^{\forall R,L}$.

We start by assuming $\mathcal{I} \Vdash \Gamma_1$. Then, we have $\mathcal{I} \Vdash x : (\forall R.\alpha)^L$, which is equivalent to $\mathcal{I}, x \Vdash (\forall R.\alpha)^L$. We assume xRy. Then, by the definition of $(\forall R.\alpha^L)^{\mathcal{I}}$, $\forall x'$ such that $x \preceq x'$, if it is the case that $x'Ry$, then $\mathcal{I} \Vdash y : \alpha$ - and, since α has the same semantics as $\alpha^{\forall R,L}$, we will arrive at $\mathcal{I} \Vdash y : \alpha^{\forall R,L}$. Since $\preceq$ is reflexive, we may take $x' = x$ itself. Then, from xRy we conclude the proof.

- $\exists i$

$$\dfrac{xRy \quad \begin{array}{c}\Pi_1\\ y : \alpha^{\exists R,L}\end{array}}{x : (\exists R.\alpha)^L}\ \exists i$$

Let R represent any role. We have that $\Gamma_1 \vdash_{ND} y : \alpha^{\exists R,L}$. By the inductive hypothesis, we have $\Gamma_1 \models y : \alpha^{\exists R,L}$. We want to prove $\Gamma_1 \cup \{xRy\} \models x : (\exists R.\alpha)^L$.

We start by assuming $\mathcal{I} \Vdash \Gamma_1$. Then, we have $\mathcal{I} \Vdash y : \alpha^{\exists R,L}$, which is equivalent to $\mathcal{I}, y \Vdash \alpha^{\exists R,L}$. We assume xRy. This fits precisely the definition of $x : (\exists R.\alpha)^L$, since $\exists x'$ and $\exists y'$ such that $x \preceq x'$ and $x'Ry'$, with $y' \in \alpha^{\mathcal{I}}$, one just has to take $x' = x$ and $y' = y$. Thus, $\mathcal{I}, x \Vdash (\exists R.\alpha)^L$, which is the same as $\mathcal{I} \Vdash x : (\exists R.\alpha)^L$.

- $\exists e$

$$\dfrac{xRy \quad \begin{array}{c}\Pi_1\\ x : (\exists R.\alpha)^L\end{array}}{y : \alpha^{\exists R,L}}\ \exists e$$

Let R represent any role. y does not appear in any formula in Γ_1. We have that $\Gamma_1 \vdash_{ND} x : (\exists R.\alpha)^L$. Then, by inductive hypothesis, $\Gamma_1 \models x : (\exists R.\alpha)^L$. We want to prove $\Gamma_1 \cup \{xRy\} \models y : \alpha^{\exists R,L}$.

We start by assuming $\mathcal{I} \Vdash \Gamma_1$. We also assume xRy. Then, we have $\mathcal{I} \Vdash x : (\exists R.\alpha)^L$, which is equivalent to $\mathcal{I}, x \Vdash (\exists R.\alpha)^L$. Then, it must be the case that $\exists x'$ and $\exists y'$ such that $x \preceq x'$ and $x'Ry'$, with $y' \in \alpha^{\mathcal{I}}$, one just has to take $x' = x$ and $y' = y$. Then, we arrive at $\mathcal{I}, y \Vdash \alpha^{\exists R,L}$, which is the same as $\mathcal{I} \Vdash y : \alpha^{\exists R,L}$.

- *Gen*

$$\dfrac{\begin{array}{c}\Pi_1\\ x : \alpha^L\end{array}}{x : \alpha^{L,\forall R}}\ Gen$$

Let R represent any role. In this case, $\Gamma_1 = \varnothing$, as is required by the application of this rule. Then, we have $\vdash_{ND} x : \alpha^L$. By inductive hypothesis, $\models x : \alpha^L$. We wish to prove $\models x : \alpha^{L,\forall R}$. This is trivial, since $(\alpha^L)^{\mathcal{I}} = (\alpha^{L,\forall R})^{\mathcal{I}}$.

- *efq*

$$\dfrac{\begin{array}{c}\Pi_1\\ x : \bot\end{array}}{\delta}\ efq$$

We have $\Gamma_1 \vdash_{ND} x : \bot$. By inductive hypothesis, we have $\Gamma_1 \models x : \bot$. We want to prove $\Gamma_1 \models \delta$.

If it is the case that $\Gamma_1 \not\models \delta$, then there exist an interpretation $\mathcal{I}$ and a world x' such that $\mathcal{I}, x' \Vdash \Gamma_1$ and $\mathcal{I}, x' \not\Vdash \delta$. Since $\Gamma_1 \models x : \bot$, then

$\mathcal{I} \Vdash x : \bot$. As x is a generic world, let us take $x = x'$. Then, $\mathcal{I} \Vdash x' : \bot$, leading to $\mathcal{I}, x' \Vdash \bot$. This is impossible, since, by definition, $\mathcal{I}, x' \nVdash \bot$ (x' cannot be an element of an empty set). Thus, $\mathcal{I}, x' \Vdash \delta$, leading to $\Gamma_1 \models \delta$.

$\square$

2.3 Completeness

Theorem 2.3 (Completeness)**.** *Let δ be a formula in iALC and Γ a set of formulas. Then, $\Gamma \models \delta$ implies $\Gamma \vdash_{ND} \delta$.*

Proof. In [6, 7], the authors provide a Hilbert system that implements TBox reasoning for iALC as per [8, 9, 10, 11] consisting in:

$$
\begin{array}{rl}
\text{(IPL)} & \text{all axioms of intuitionistic propositional logic} \\
(\forall\text{K}) & (\forall R.(\alpha \sqsupset \beta)) \sqsupset (\forall R.\alpha \sqsupset \forall R.\beta) \\
(\exists\text{K}) & (\forall R.(\alpha \sqsupset \beta)) \sqsupset (\exists R.\alpha \sqsupset \exists R.\beta) \\
\text{(DIST)} & \exists R.(\alpha \sqcup \beta) \sqsupset (\exists R.\alpha \sqcup \exists R.\beta) \\
\text{(DIST0)} & \exists R.\bot \sqsupset \bot \\
\text{(DISTm)} & (\exists R.\alpha \sqsupset \forall R.\beta) \sqsupset \forall R.(\alpha \sqsupset \beta) \\
\text{(Nec)} & \text{If } \alpha \text{ is a theorem, then } \forall R.\alpha \text{ is a theorem too.} \\
\text{(MP)} & \text{If } \alpha \text{ and } \alpha \sqsupset \beta \text{ are theorems,} \\
 & \text{then } \beta \text{ is a theorem too.}
\end{array}
$$

In [11], the authors prove that this system is sound and complete for TBox reasoning. So, in order to prove completeness of our ND system, all we have to do is prove each of these axioms. Axioms in **(IPL)** are easily proven since all substitution instances of IPL theorems can be proven in our ND system using only propositional rules. **(MP)** is covered by our $\sqsupset e$ rule, and **(Nec)**, by the following derivation:

$$
\dfrac{xRx \quad \dfrac{x : \alpha}{x : \alpha^{\forall R}}\,Gen}{x : \forall R.\alpha}\,\forall i
$$

We, then, only have to prove the remaining five axioms.

1. **($\forall$K) $\boldsymbol{x : (\forall R.(\alpha \sqsupset \beta) \sqsupset (\forall R.\alpha \sqsupset \forall R.\beta))}$**

$$
\Pi_1 : \quad \dfrac{xRy \quad [x : \forall R.\alpha]^2}{y : \alpha^{\forall R}}\,\forall e
$$

$$
\Pi_2 : \quad \dfrac{xRy \quad [x : \forall R.(\alpha \sqsupset \beta)]^1}{y : \alpha^{\forall R} \sqsupset \beta^{\forall R}}\,\forall e
$$

$$\dfrac{\dfrac{\dfrac{xRy \qquad \dfrac{\Pi_1 \quad \Pi_2}{y : \beta^{\forall R}}\ \dashv e}{x : \forall R.\beta}\ \forall i(3)}{x : (\forall R.\alpha \dashv \forall R.\beta)}\ \dashv i(2)}{x : (\forall R.(\alpha \dashv \beta) \dashv (\forall R.\alpha \dashv \forall R.\beta))}\ \dashv i(1)$$

2. **($\exists$K) $\boldsymbol{x : (\forall R.(\alpha \dashv \beta)) \dashv (\exists R.\alpha \dashv \exists R.\beta)}$**

$$\dfrac{\dfrac{\dfrac{[x : \exists R.\alpha]^2 \quad [x : \forall R.(\alpha \dashv \beta)]^1}{x : \exists R.\beta}\ p-\exists}{x : (\exists R.\alpha \dashv \exists R.\beta)}\ \dashv i(2)}{x : (\forall R.(\alpha \dashv \beta)) \dashv (\exists R.\alpha \dashv \exists R.\beta)}\ \dashv i(1)$$

3. **(DIST) $\boldsymbol{x : \exists R.(\alpha \sqcup \beta) \dashv (\exists R.\alpha \sqcup \exists R.\beta)}$**

$$\Pi_1 : \quad \dfrac{\dfrac{xRy \quad [y : \alpha^{\exists R}]^2}{x : \exists R.\alpha}\ \exists e}{x : \exists R.\alpha \sqcup \exists R.\beta}\ \sqcup i_1$$

$$\Pi_2 : \quad \dfrac{\dfrac{xRy \quad [y : \beta^{\exists R}]^2}{x : \exists R.\beta}\ \exists e}{x : \exists R.\alpha \sqcup \exists R.\beta}\ \sqcup i_2$$

$$\dfrac{\dfrac{\dfrac{xRy \quad [x : \exists R.(\alpha \sqcup \beta)]^1}{y : (\alpha \sqcup \beta)^{\exists R}}\ \exists e \qquad \Pi_1 \quad \Pi_2}{x : \exists R.\alpha \sqcup \exists R.\beta}\ \sqcup e(2)}{x : \exists R.(\alpha \sqcup \beta) \dashv (\exists R.\alpha \sqcup \exists R.\beta)}\ \dashv i(1)$$

4. **(DIST0) $\boldsymbol{x : (\exists R.\bot \dashv \bot)}$**

$$\dfrac{\dfrac{\dfrac{xRy \quad [x : \exists R.\bot]^1}{y : \bot^{\exists R}}\ \exists e}{x : \bot}\ efq}{x : (\exists R.\bot \dashv \bot)}\ \dashv i(1)$$

5. **(DISTm) $\boldsymbol{x : (\exists R.\alpha \dashv \forall R.\beta) \dashv \forall R.(\alpha \dashv \beta))}$**

$$
\dfrac{\dfrac{xRy \qquad \dfrac{\dfrac{xRy \qquad \dfrac{[x : (\exists R.\alpha \dashv \forall R.\beta)]^1 \qquad \dfrac{xRy \quad [y : \alpha^{\exists R}]^2}{x : \exists R.\alpha}\,\exists i}{x : \forall R.\beta}\,\dashv e}{y : \beta^{\forall R}}\,\forall e}{y : (\alpha^{\exists R} \dashv \beta^{\forall R})}\,\dashv i(2)}{x : \forall R.(\alpha \dashv \beta)}\,\forall i}{x : (\exists R.\alpha \dashv \forall R.\beta) \dashv \forall R.(\alpha \dashv \beta))}\,\dashv i(1)
$$

□

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