iALC - ND proofs

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1 Introduction

This document serves as an aid for our proofs of normalisation, soundness and completeness of the labeled Natural Deduction system created for the logic iALC.

2 Full proofs

2.1 Proof of Lemma 3

Lemma 3. *Let δ a formula in iALC, Γ a set of formulas and Π a critical derivation of $\Gamma \vdash_{ND} \delta$. Then, Π reduces to a derivation Π' of $\Delta \subseteq \Gamma \vdash_{ND} \delta$ such that $deg(\Pi') < deg(\Pi)$.*

Proof. The proof follows by induction in the structure of Π. We will use the reductions shown in Definition 7, but assuming they are critical derivations. We will call each derivation before a reduction Π, and Π' after the reduction.

Let us divide the proof by cases:

- $\sqcap$-reduction

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array} \quad \begin{array}{c}\Pi_2\\ x:\beta\end{array}}{x:(\alpha\sqcap\beta)}\sqcap-i}{\begin{array}{c}x:\alpha\\ \Pi_3\end{array}}\sqcap-e_1 \qquad \triangleright \qquad \begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_3\end{array}$$

  We have $deg(\Pi) = deg(x:\alpha\sqcap\beta)$ (according to Item 3 of the definition of a critical formula), which equals $deg(\alpha)+deg(\beta)+3$. When reduced to Pi', we have, as per Lemma 1, that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_3), deg(x:\alpha))\}$. If it is equal to $deg(\Pi_1)$ or $deg(\Pi_3)$, then it is strictly smaller because of item 4 of the definition of a critical formula. If it is $deg(x:\alpha) = deg(\alpha)+2$, then it is strictly smaller than $deg(\Pi) = (deg(\alpha)+2)+(deg(\beta)+1)$.

- $\sqsubseteq$-reduction (and $\neg$-reduction)

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array} \qquad \dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ x:\beta\end{array}}{x:(\alpha\sqsubseteq\beta)}\sqsubseteq -i}{\begin{array}{c}x:\beta\\ \Pi_3\end{array}}\sqsubseteq -e \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_2\\ x:\beta\\ \Pi_3\end{array}$$

  We have $deg(\Pi) = deg(x : \alpha \sqsubseteq \beta)$, which equals $deg(\alpha) + deg(\beta) + 3$. When reduced to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(x : \alpha), deg(x : \beta)\}$. It is one of the sub-derivations, and it is strictly smaller. On the other hand, if it is either $deg(x : \alpha) = deg(\alpha) + 2$ or $deg(x : \beta) = deg(\beta) + 2$ it will be smaller than $deg(\Pi)$ as well.

  Since $\neg$ is defined as a special case of $\sqsubseteq$, the proof for $\neg$-reduction follows directly.

- $\forall$-reduction (and $\exists$-reduction)

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ y:\alpha^{\forall R}\end{array}}{x:\forall R.\alpha}\forall - i}{\begin{array}{c}y:\alpha^{\forall R}\\ \Pi_2\end{array}}\forall - e \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ y:\alpha^{\forall R}\\ \Pi_2\end{array}$$

  We have $deg(\Pi) = deg(x : \forall R.\alpha)$, which equals $deg(\alpha) + 4$. With Π', we have that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(y : \alpha^{\forall R})\}$. The argument for $deg(\Pi') = deg(\Pi_1)$ or $deg(\Pi_2)$ is the same. Since the label does not affect the degree, if $deg(\Pi') = deg(y : \alpha^{\forall R}) = deg(\alpha)+2$, then it is strictly smaller than $deg(\Pi)$.

  The argument for $\exists$-reduction is exactly the same.

- $\sqcup$-reduction

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\end{array}}{x:\alpha\sqcup\beta}\sqcup - i_1 \qquad \begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta\end{array} \qquad \begin{array}{c}[x:\beta]\\ \Pi_3\\ \delta\end{array}}{\begin{array}{c}\delta\\ \Pi_4\end{array}}\sqcup - e \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ x:\alpha\\ \Pi_2\\ \delta\\ \Pi_4\end{array}$$

  We have $deg(\Pi) = deg(x : \alpha\sqcup\beta)$, which equals $deg(\alpha)+deg(\beta)+3$. When turned to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(\delta), deg(x : \alpha))\}$. The argument for the sub-derivations or $deg(\delta)$ is still due to Π being a critical derivation. If $deg(\Pi') = deg(x : \alpha) = deg(\alpha) + 2$, then it is strictly smaller than $deg(\Pi)$.

$\square$

2.2 Full Proof of Soundness

Theorem 2 (Soundness). *Let δ be a formula in iALC and Γ a set of formulas. Then, $\Gamma \vdash_{ND} \delta$ implies in $\Gamma \models \delta$.*

Proof. The proof follows by showing that each rule preserves soundness. For rules that involve hypothesis discharge, we will use induction on the length of derivations.

The soundness of $p{-}\exists$ is proved as a consequence of the following reasoning in first-order intuitionistic logic used for deriving the semantics of the conclusions from the semantics of the premises:

$$\forall x(A(x) \wedge B(x) \rightarrow C(x)) \models \\ \forall x A(x) \wedge \exists x B(x) \rightarrow \exists x C(x)$$

It is worth noting that $\forall w, \mathcal{I}, w \models x : \alpha$ iff $\forall z, x \preceq z \Rightarrow \mathcal{I}, z \models \alpha$, which includes $z = x$ itself, as $\preceq$ is reflexive. Since most rules involve only x as an outer nominal, when arguing inductively over the formulas in the premises the part $\forall z, x \preceq z$ is considered valid, just omitted to avoid prolixity in the argument.

- $\sqcap - i$

$$\frac{x : \alpha^{L^\forall} \quad x : \beta^{L^\forall}}{x : (\alpha \sqcap \beta)^{L^\forall}} \sqcap - i$$

We have $\mathcal{I} \models x : \alpha^{L^\forall}$ and $\mathcal{I} \models x : \beta^{L^\forall}$, which are the same as having $\mathcal{I}, x \models \alpha^{L^\forall}$ and $\mathcal{I}, x \models \beta^{L^\forall}$. In other words, $x \in (\alpha^{L^\forall})^{\mathcal{I}}$ and $x \in (\beta^{L^\forall})^{\mathcal{I}}$, i.e. $x \in ((\alpha^{L^\forall})^{\mathcal{I}} \cap (\beta^{L^\forall})^{\mathcal{I}})$, exactly the definition of $x \in (\alpha^{L^\forall} \sqcap \beta^{L^\forall})^{\mathcal{I}}$.

Notice how the lists of labels do not interfere at all with the argument for this rule, as they have to all be the same for the rule to work. They will only come into play when the rules involve changing them, i.e. $\forall - i$, $\forall - e$, $\exists - i$, and $\exists - e$.

- $\sqsubseteq -i$

$$\frac{\begin{array}{c}[x : \alpha^{L1}] \\ \vdots \\ x : \beta^{L2}\end{array}}{x : (\alpha^{L1} \sqsubseteq \beta^{L2})} \sqsubseteq -i$$

We have, by induction, that soundness is preserved from $x : \alpha^{L1}$ to $x : \beta^{L2}$. In other words, having $x \in (\alpha^{L1})^{\mathcal{I}}$ implies in $x \in (\beta^{L2})^{\mathcal{I}}$, precisely the definition of $x \in (\alpha^{L1} \sqsubseteq \beta^{L2})^{\mathcal{I}}$, since $\preceq^{\mathcal{I}}$ is reflexive.

- $\sqcup - e$

$$\dfrac{x : (\alpha \sqcup \beta)^{L^{\exists}} \quad \begin{matrix}\left[x : \alpha^{L^{\exists}}\right] \\ \vdots \\ \delta\end{matrix} \quad \begin{matrix}\left[x : \beta^{L^{\exists}}\right] \\ \vdots \\ \delta\end{matrix}}{\delta} \ \sqcup - e$$

By induction, we have that soundness is preserved from $x : \alpha^{L^{\exists}}$ to δ. We also have that $x \in (\alpha^{L^{\exists}})^{\mathcal{I}}$ (without loss of generality). From the major premise of the rule, we have $x \in (\alpha^{L^{\exists}} \cup \beta^{L^{\exists}})^{\mathcal{I}}$. Since $(\alpha^{L^{\exists}})^{\mathcal{I}} \subseteq (\alpha^{L^{\exists}} \cup \beta^{L^{\exists}})^{\mathcal{I}}$, we have $\mathcal{I} \models \delta$.

- $\forall - i$

$$\dfrac{y : \alpha^{\forall R,L}}{x : (\forall R.\alpha)^{L}} \ \forall - i$$

As always, we assume xRy. y also does not appear in any other formula other than $y : \alpha^{\forall R,L}$. Since the premise is valid we have that if $\forall z_x \forall z_y, x^{\mathcal{I}} \preceq z_x$, $y^{\mathcal{I}} \preceq z_y$, and $z_x R^{\mathcal{I}} z_y$, then $z_y \in (\alpha^{\forall R,L})^{\mathcal{I}}$ and $z_x \in (\forall R.\alpha^{L})^{\mathcal{I}}$. Since $\preceq$ is reflexive, we have $x^{\mathcal{I}} \in (\forall R.\alpha^{L})^{\mathcal{I}}$. As y does not appear anywhere else, this lets us conclude that the conclusion is valid.

- $\forall - e$

$$\dfrac{x : (\forall R.\alpha)^{L}}{y : \alpha^{\forall R,L}} \ \forall - e$$

We assume xRy. We have that $x \in (\forall R.\alpha^{L})^{\mathcal{I}}$. Then, by the definition of $(\forall R.\alpha^{L})^{\mathcal{I}}$, we have to find a certain individual to which x is related via $\preceq$ and at the same time is related to y via R. Since, $\preceq$ is reflexive, this individual can be x itself. Then, we have $y \in (\alpha^{\forall R,L})^{\mathcal{I}}$, which is what is stated in the conclusion.

$\square$

2.3 Full Proof of Completeness

Theorem 3 (Completeness)**.** *Let δ be a formula in iALC and Γ a set of formulas. Then, $\Gamma \models \delta$ implies in $\Gamma \vdash_{ND} \delta$.*

Proof. In [1, 2], the authors provide a Hilbert system that implements TBOX reasoning for iALC as per [3, 4, 5, 6] consisting in:

(IPL) all axioms of intuitionistic propositional logic

($\forall$K) $(\forall R.(\alpha \sqsubseteq \beta)) \sqsubseteq (\forall R.\alpha \sqsubseteq \forall R.\beta)$

($\exists$K) $(\forall R.(\alpha \sqsubseteq \beta)) \sqsubseteq (\exists R.\alpha \sqsubseteq \exists R.\beta)$

(DIST) $\exists R.(\alpha \sqcup \beta) \sqsubseteq (\exists R.\alpha \sqcup \exists R.\beta)$

(DIST0) $\exists R.\bot \sqsubseteq \bot$

(DISTm) $(\exists R.\alpha \sqsubseteq \forall R.\beta) \sqsubseteq \forall R.(\alpha \sqsubseteq \beta)$

(Nec) If α is a theorem, then $\forall R.\alpha$ is a theorem too.

(MP) If α and $\alpha \sqsubseteq \beta$ are theorems,
then β is a theorem too.

In [6], the authors prove that this system is sound and complete for TBOX reasoning. So, in order to prove completeness of our ND system, all we have to do is prove each of these axioms. Axiom **(IPL)** is easily proven since all substitution instances of IPL theorems can be proven in our ND system using only propositional rules. **(MP)** is covered by our $\sqsubseteq -e$ rule, and **(Nec)**, by the *Gen* rule.

We, then, only have to prove the remaining five axioms.

1. **($\forall$K) $\boldsymbol{x : (\forall R.(\alpha \sqsubseteq \beta) \sqsubseteq (\forall R.\alpha \sqsubseteq \forall R.\beta))}$**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x:\forall R.\alpha]^2}{y:\alpha^{\forall R}}\ \forall-e \qquad \dfrac{[x:\forall R.(\alpha\sqsubseteq\beta)]^1}{y:\alpha^{\forall R}\sqsubseteq\beta^{\forall R}}\ \forall-e}{y:\beta^{\forall R}}\ \sqsubseteq -e}{x:\forall R.\beta}\ \forall-i(3)}{x:(\forall R.\alpha\sqsubseteq\forall R.\beta)}\ \sqsubseteq -i(2)}{x:(\forall R.(\alpha\sqsubseteq\beta)\sqsubseteq(\forall R.\alpha\sqsubseteq\forall R.\beta))}\ \sqsubseteq -i(1)$$

2. **($\exists$K) $\boldsymbol{x : (\forall R.(\alpha \sqsubseteq \beta)) \sqsubseteq (\exists R.\alpha \sqsubseteq \exists R.\beta)}$**

$$\dfrac{\dfrac{\dfrac{[x:\exists R.\alpha]^2 \quad [x:\forall R.(\alpha\sqsubseteq\beta)]^1}{x:\exists R.\beta}\ p-\exists}{x:(\exists R.\alpha\sqsubseteq\exists R.\beta)}\ \sqsubseteq -i(2)}{x:(\forall R.(\alpha\sqsubseteq\beta))\sqsubseteq(\exists R.\alpha\sqsubseteq\exists R.\beta)}\ \sqsubseteq -i(1)$$

3. **(DIST) $\boldsymbol{x : \exists R.(\alpha \sqcup \beta) \sqsubseteq (\exists R.\alpha \sqcup \exists R.\beta)}$**

$$\Pi_1 : \quad \dfrac{\dfrac{[y:\alpha^{\exists R}]^2}{x:\exists R.\alpha}\ \exists-e}{x:\exists R.\alpha\sqcup\exists R.\beta}\ \sqcup - i_1$$

$$\Pi_2 : \quad \dfrac{\dfrac{[y : \beta^{\exists R}]^2}{x : \exists R.\beta}\ \exists - e}{x : \exists R.\alpha \sqcup \exists R.\beta}\ \sqcup - i_2$$

$$\dfrac{\dfrac{\dfrac{[x : \exists R.(\alpha \sqcup \beta)]^1}{y : (\alpha \sqcup \beta)^{\exists R}}\ \exists - e \qquad \Pi_1 \quad \Pi_2}{x : \exists R.\alpha \sqcup \exists R.\beta}\ \sqcup - e(2)}{x : \exists R.(\alpha \sqcup \beta) \sqsubseteq (\exists R.\alpha \sqcup \exists R.\beta)}\ \sqsubseteq -i(1)$$

4. **(DIST0)** $\boldsymbol{x : (\exists R.\bot \sqsubseteq \bot)}$

$$\dfrac{\dfrac{\dfrac{[x : \exists R.\bot]^1}{y : \bot^{\exists R}}\ \exists - e}{x : \bot}\ efq}{x : (\exists R.\bot \sqsubseteq \bot)}\ \sqsubseteq -i(1)$$

5. **(DISTm)** $\boldsymbol{x : (\exists R.\alpha \sqsubseteq \forall R.\beta) \sqsubseteq \forall R.(\alpha \sqsubseteq \beta))}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : (\exists R.\alpha \sqsubseteq \forall R.\beta)]^1 \qquad \dfrac{[y : \alpha^{\exists R}]^2}{x : \exists R.\alpha}\ \exists - i}{x : \forall R.\beta}\ \sqsubseteq -e}{y : \beta^{\forall R}}\ \forall - e}{y : (\alpha^{\exists R} \sqsubseteq \beta^{\forall R})}\ \sqsubseteq -i(2)}{x : \forall R.(\alpha \sqsubseteq \beta)}\ \forall - i}{x : (\exists R.\alpha \sqsubseteq \forall R.\beta) \sqsubseteq \forall R.(\alpha \sqsubseteq \beta))}\ \sqsubseteq -i(1)$$

□

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